

# The length of a typical Huffman codeword\*

Rüdiger Schack<sup>†</sup>

Department of Physics and Astronomy, University of New Mexico  
Albuquerque, NM 87131–1156

January 11, 1993

## Abstract

If  $p_i$  ( $i = 1, \dots, N$ ) is the probability of the  $i$ -th letter of a memoryless source, the length  $l_i$  of the corresponding binary Huffman codeword can be very different from the value  $-\log p_i$ . For a *typical* letter, however,  $l_i \approx -\log p_i$ . More precisely,  $P_m^- = \sum_{j \in \{i | l_i < -\log p_i - m\}} p_j < 2^{-m}$  and  $P_m^+ = \sum_{j \in \{i | l_i > -\log p_i + m\}} p_j < 2^{-c(m-2)+2}$  where  $c \approx 2.27$ .

## Introduction

Concepts from information theory gained new importance in physics [1, 2] when Bennett [3] realized that Landauer's principle [4], which specifies the unavoidable energy cost  $k_B T \ln 2$  for the erasure of a bit of information, is the clue to the solution of the problem posed by Maxwell's demon. This problem can be summarized as follows: A demon knows initially that a system is in the  $i$ -th possible state ( $i = 1, \dots, N$ ) with probability  $p_i$ . The demon then finds the actual state of the system—thereby lowering the system's entropy by the amount  $H = -\sum p_i \log p_i$ . This is in apparent violation of the second law of thermodynamics, since the entropy decrease corresponds to a free-energy increase  $\Delta F = H k_B T \ln 2$  that can be extracted as work. Bennett solved this inconsistency by noting that in order to return to its original configuration the demon must erase its record of the system state. The second law is saved since, due to Shannon's noiseless coding theorem, the average length of the demon's record cannot be smaller than  $H$ . Therefore, the Landauer erasure cost cancels the extracted work on the average.

If the demon wants to operate with maximum efficiency, it must use an optimal coding procedure, i. e., Huffman coding [5]. In this context, the question arises as to how the record length  $l_i$  for the  $i$ -th state can be interpreted. Zurek [1] discusses two alternative (sub-optimal) coding procedures for the demon: minimal programs for a universal

---

\*Submitted to IEEE Transactions on Information Theory.

<sup>†</sup>Supported by a fellowship from the Deutsche Forschungsgemeinschaft.

computer, where the record length is the algorithmic complexity [6] of the state; and Shannon-Fano coding, where the record length is determined by the state's probability through the inequality  $-\log p_i \leq l_i < -\log p_i + 1$ . The length of a Huffman codeword, on the other hand, is neither determined by the state's complexity nor by its probability. Given  $p_i$ , the Huffman codeword length can, in principle, be as small as 1 bit and as large as  $[\log((\sqrt{5} + 1)/2)]^{-1} \approx 1.44$  times  $-\log p_i$  [7].

In this correspondence, we show that the lengths of both Huffman and Shannon-Fano codewords have a similar interpretation. The probability of the states for which the Huffman codeword length differs by more than  $m$  bits from  $-\log p_i$  decreases exponentially with  $m$ . In this sense, one can say that, for a *typical* state, the Huffman codeword satisfies  $l_i \approx -\log p_i$ , just as for Shannon-Fano coding. This is especially relevant in a thermodynamic context where entropies are of the order of  $2^{80}$  bits and where an error of a few hundred bits in the length of a typical record would be unnoticeable.

## Result

In this section we return to the terminology of the abstract and consider a discrete memoryless  $N$ -letter source ( $N \geq 2$ ) to which a binary Huffman code is assigned. The  $i$ -th letter has probability  $p_i < 1$  and codeword length  $l_i$ . The Huffman code can be represented by a binary tree having the *sibling property* [8] defined as follows: The number of links leading from the root of the tree to a node is called the *level* of that node. If the level- $n$  node  $a$  is connected to the level- $(n + 1)$  nodes  $b$  and  $c$ , then  $a$  is called the *parent* of  $b$  and  $c$ ;  $a$ 's *children*  $b$  and  $c$  are called *siblings*. There are exactly  $N$  terminal nodes or *leaves*, each leaf corresponding to a letter. Each link connecting two nodes is labeled 0 or 1. The sequence of labels encountered on the path from the root to a leaf is the codeword assigned to the corresponding letter. The codeword length of a letter is thus equal to the level of the corresponding leaf. Each node is assigned a probability such that the probability of a leaf is equal to the probability of the corresponding letter and the probability of each non-terminal node is equal to the sum of the probabilities of its children. A tree has the sibling property iff each node except the root has a sibling and the nodes can be listed in order of nonincreasing probability with each node being adjacent to its sibling in the list [8].

*Definition:* A level- $l$  node with probability  $p$ —or, equivalently, a letter with probability  $p$  and codeword length  $l$ —has the property  $X_m^+$  ( $X_m^-$ ) iff  $l > -\log p + m$  ( $l < -\log p - m$ ).

*Theorem 1:*  $P_m^- = \sum_{j \in I_m^-} p_j < 2^{-m}$  where  $I_m^- = \{i | l_i < -\log p_i - m\}$ , i. e., the probability that a letter has property  $X_m^-$  is smaller than  $2^{-m}$ . (This is true for any prefix-free code.)

*Proof:*  $P_m^- = 2^{-m} \sum_{j \in I_m^-} 2^{\log p_j + m} < 2^{-m} \sum_{j \in I_m^-} 2^{-l_j} \leq 2^{-m}$ . The last inequality follows from the Kraft inequality.

*Lemma:* Any node with property  $X_m^+$  has probability  $p < 2^{-c(m-1)}$  where  $c = (1 - \log g)^{-1} - 1 \approx 2.27$  with  $g = (\sqrt{5} + 1)/2$ .

*Proof:* Property  $X_m^+$  implies  $l > \lfloor -\log p + m \rfloor$  where  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ . It is shown in Ref. [7] that, if  $p$  and  $l$  are the probability and level of a given

node,  $p \geq 1/F_n$  implies  $l \leq n-2$  for  $n \geq 3$  where  $F_n = [g^n - (-g)^{-n}]/\sqrt{5} \geq g^{n-2}$  is the  $n$ -th Fibonacci number ( $n \geq 1$ ). Therefore, if  $\lfloor -\log p + m \rfloor \geq 1$ , the inequality  $l > \lfloor -\log p + m \rfloor$  implies  $p < (F_{\lfloor -\log p + m \rfloor + 2})^{-1} \leq g^{-\lfloor -\log p + m \rfloor} \leq g^{\log p - m + 1}$ . For  $\lfloor -\log p + m \rfloor < 1$ ,  $p < g^{\log p - m + 1}$  holds trivially. Solving for  $p$  proves the lemma.

*Theorem 2:*  $P_m^+ = \sum_{j \in I_m^+} p_j < 2^{-c(m-2)+2}$  where  $I_m^+ = \{i | l_i > -\log p_i + m\}$ , i. e., the probability that a letter has property  $X_m^+$  is smaller than  $2^{-c(m-2)+2}$ .

*Proof:* Suppose there is at least one letter—and hence a corresponding leaf—having the property  $X_m^+$ . Then, among all nodes having the property  $X_m^+$ , there is a nonempty subset with minimum level  $n_0 > 0$ . In this subset, there is a node having maximum probability  $p_0$ . In other words, there is no node having property  $X_m^+$  on a level  $n < n_0$ , and on level  $n_0$ , there is no node with probability  $p > p_0$ . Thus property  $X_m^+$  implies

$$p_0 > 2^{-n_0+m} .$$

Now let  $k_0$  be the number of nodes on level  $n_0 - 1$ , and define the integer  $l_0 < n_0$  such that  $2^{l_0} \leq k_0 < 2^{l_0+1}$ . Then the number of level- $n_0$  nodes is less than  $2^{l_0+2}$ . Since all nodes having property  $X_m^+$  are on levels  $n \geq n_0$ , it follows that

$$P_m^+ < 2^{l_0+2} p_0 .$$

In order to turn this into a useful bound, note the following. The sibling property or, more directly, the optimality of a Huffman code implies that all level- $(n_0 - 1)$  nodes have probability  $p \geq p_0$ . Since there are at least  $2^{l_0}$  level- $(n_0 - 1)$  nodes, it is again a consequence of the sibling property that there exists a level- $(n_0 - 1 - l_0)$  node with probability  $p_1 \geq 2^{l_0} p_0 > 2^{-n_0+m+l_0}$  and thus having property  $X_{m-1}^+$ . Using the lemma, one finds  $p_1 < 2^{-c(m-2)}$  and therefore

$$P_m^+ < 2^{l_0+2} p_0 \leq 2^2 p_1 < 2^{-c(m-2)+2} .$$

## Acknowledgements

The author profited much from frank discussions with Carlton M. Caves and Christopher Fuchs.

## References

- [1] W. H. Zurek, “Algorithmic randomness and physical entropy”, *Phys. Rev. A*, vol. 40, pp. 4731–4751, Oct. 1989.
- [2] R. Schack and C. M. Caves, “Information and entropy in the baker’s map”, *Phys. Rev. Lett.*, vol. 69, pp. 3413–3416, Dec. 1992.
- [3] C. H. Bennett, “The thermodynamics of computation - a review”, *International Journal of Theoretical Physics*, vol. 21, pp. 905–940, Dec. 1982.

- [4] R. Landauer, “Irreversibility and heat generation in the computing process”, *IBM J. Res. Develop.*, vol. 5, pp. 183–191, July 1961.
- [5] D. A. Huffman, “A method for the construction of minimum-redundancy codes”, *Proceedings of the I.R.E.*, vol. 40, pp. 1098–1101, Sep. 1952.
- [6] G. J. Chaitin, *Algorithmic Information Theory*. Cambridge: Cambridge University Press, 1987.
- [7] G. O. H. Katona and T. O. H. Nemetz, “Huffman codes and self-information”, *IEEE Trans. Inform. Theory*, vol. IT-22, pp. 337–340, May 1976.
- [8] R. G. Gallager, “Variations on a theme by Huffman”, *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 668–674, Nov. 1978.