

A Fredholm determinant formula for Toeplitz determinants

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1 Introduction

The purpose of this note is to explain how the results of [13] apply to a question raised by A. Its and, independently, P. Deift during the MSRI workshop on Random Matrices in June 1999. The question was whether there exists a general formula expressing a Toeplitz determinant

$$D_n(\varphi) = \det (\varphi_{i-j})_{1 \leq i, j \leq n}, \quad \varphi(\zeta) = \sum_{k \in \mathbb{Z}} \varphi_k \zeta^k,$$

as the Fredholm determinant of an operator $1 - \mathcal{K}$ acting on $\ell_2(\{n, n+1, \dots\})$, where the kernel $\mathcal{K} = \mathcal{K}(\varphi)$ admits an integral representation in terms of φ . The answer is affirmative and the construction of the kernel is explained below.

We give two versions of the result: an algebraic one, which holds in the suitable algebra of formal power series, and an analytic one. In order to minimize the amount of analysis, we make a rather restrictive analyticity assumption on the function φ . One should be able to relax this assumption, see Remark 3. Our proof is a direct application of two results due to I. Gessel [7] and one of the authors [13], respectively.

We also consider 3 examples in which the kernel \mathcal{K} can be expressed in classical special functions. First two of them have been worked out in [1, 15, 2, 10].

2 Statement of the result

We assume that φ is an exponential of a Laurent series $\varphi(\zeta) = \exp V(\zeta)$ and we assume that φ is normalized in such a way that $V(\zeta)$ has no constant

term. Let us separate positive and negative powers in this series

$$V(\zeta) = V^+(\zeta) + V^-(\zeta),$$

where $V^\pm(\zeta) = \sum_{k=1}^{\infty} v_k^\pm \zeta^{\pm k}$.

First, consider the algebraic situation. Introduce a \mathbb{Z}_+ -grading in the polynomial algebra $\mathbb{C}[v_k^\pm]_{k \geq 1}$ by setting

$$\deg v_k^\pm = k.$$

For an arbitrary $f \in \mathbb{C}[v_k^\pm]_{k \geq 1}$ we denote by $\deg f$ the *minimal* degree of all terms in f . Let \mathfrak{B} be the completion of $\mathbb{C}[v_k^\pm]_{k \geq 1}$ with respect to this grading. Recall that $f_n \rightarrow f$ in \mathfrak{B} by definition means that

$$\deg(f - f_n) \rightarrow \infty, \quad n \rightarrow \infty.$$

Because V has no constant term it follows that $\varphi(\zeta) = \exp V(\zeta)$ is a well-defined Laurent series in ζ with coefficients in \mathfrak{B} and, moreover,

$$\deg [\zeta^k] \varphi(\zeta) \geq |k|,$$

where $[\zeta^k] \varphi(\zeta)$ stands for the coefficient of ζ^k in φ .

By definition, set

$$V^*(\zeta) = V^-(-\zeta) - V^+(-\zeta),$$

and define the kernel \mathcal{K} by the following generating function

$$\sum_{i,j \in \mathbb{Z}} \zeta^i \eta^{-j} \mathcal{K}(i, j) = \frac{\exp(V^*(\zeta) - V^*(\eta))}{\zeta/\eta - 1}, \quad |\zeta| > |\eta|. \quad (2.1)$$

Here the condition $|\zeta| > |\eta|$ means that $\frac{1}{\zeta/\eta - 1}$ is to be expanded in powers of $\frac{\eta}{\zeta}$. In other words,

$$\mathcal{K}(i, j) = \sum_{l \geq 1} [\zeta^{i+l} \eta^{-j-l}] \exp(V^*(\zeta) - V^*(\eta)),$$

where $[\zeta^{i+l} \eta^{-j-l}]$ stands for the coefficient of $\zeta^{i+l} \eta^{-j-l}$. Since

$$\deg [\zeta^k] \exp(V^*(\zeta)) \geq |k|$$

we conclude that $\mathcal{K}(i, j)$, $i, j \geq 0$, is a well defined element of \mathfrak{A} and

$$\deg \mathcal{K}(i, j) \geq i + j + 2.$$

In the analytic situation, we will assume that the Laurent series $V(\zeta)$ converges in some nonempty annular neighborhood of the unit circle

$$r^{-1} < |\zeta| < r, \quad \zeta \in \mathbb{C}, \quad r > 1.$$

In this case the kernel \mathcal{K} can be expressed by the following contour integral

$$\mathcal{K}(i, j) = \frac{1}{(2\pi\sqrt{-1})^2} \iint_{|\zeta| > |\eta|} \frac{\exp(V^*(\zeta) - V^*(\eta))}{\zeta - \eta} \frac{d\zeta}{\zeta^{i+1}} \frac{d\eta}{\eta^{-j}}. \quad (2.2)$$

Here the integral is taken, for example, over the torus $|\zeta| = |\eta|^{-1} = \rho$, $1 < \rho < r$, in \mathbb{C}^2 . It is clear that

$$\sum_{i, j \geq 0} |\mathcal{K}(i, j)| \leq \text{const} \sum_{i, j \geq 0} \frac{1}{\rho^{i+j+1}} < \infty$$

and therefore \mathcal{K} defines a trace class operator on $\ell_2(\{0, 1, \dots, \})$.

Finally, set

$$Z = \exp \left(\sum_{k=1}^{\infty} k v_k^+ v_k^- \right).$$

This is easily seen to be well defined in both algebraic and analytic case. Our main result is the following

Theorem 1 *We have the following identity*

$$D_n(\varphi) = Z \det(1 - \mathcal{K})_{\ell_2(\{n, n+1, \dots\})}, \quad (2.3)$$

where $D_n(\varphi)$ is the $n \times n$ Toeplitz determinant with symbol $\varphi(\zeta) = \exp V(\zeta)$ and the Fredholm determinant in the right-hand side is defined by

$$\det(1 - \mathcal{K})_{\ell_2(\{n, n+1, \dots\})} = \sum_{m=0}^{\infty} (-1)^m \sum_{n \leq l_1 < \dots < l_m} \det [\mathcal{K}(l_i, l_j)]_{i, j=1}^m. \quad (2.4)$$

The convergence in (2.4) is the usual convergence in the case when $V(\zeta)$ converges in some neighborhood of the unit circle in \mathbb{C} or else the convergence in the algebra \mathfrak{A} of formal power series.

Remark 1 The $n \rightarrow \infty$ limit corresponds to the situation of the strong Szegö limit theorem [14]

$$D_n(\varphi) \rightarrow Z, \quad n \rightarrow \infty.$$

In the algebraic situation, this convergence means that for any fixed k the terms of degree $\leq k$ in D_n will coincide with those of Z for large enough n .

Remark 2 Replacing ζ and η in the right-hand side of (2.2) by $t\zeta$ and $t\eta$, respectively, and applying $\frac{d}{dt}\Big|_{t=1}$ yields

$$\begin{aligned} & (i-j) \mathcal{K}(i, j) \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \iint_{|\zeta| > |\eta|} \frac{\zeta \frac{d}{d\zeta} V^*(\zeta) - \eta \frac{d}{d\eta} V^*(\eta)}{\zeta - \eta} e^{V^*(\zeta) - V^*(\eta)} \frac{d\zeta}{\zeta^{i+1}} \frac{d\eta}{\eta^{-j}}. \end{aligned} \quad (2.5)$$

The advantage of the formula (2.5) over the formula (2.2) is that often the division in (2.5) can be carried out explicitly and this leads to an expression of the type

$$(i-j) \mathcal{K}(i, j) = \sum_{k=1}^m f_k(i) g_k(j),$$

for some m and some functions f_1, \dots, f_m and g_1, \dots, g_m . This means that the kernel \mathcal{K} is *integrable* in the sense of [9], see also [6].¹

The main disadvantage of the formula (2.5) is that it does not allow to compute the diagonal entries $\mathcal{K}(i, i)$.

Remark 3 The analyticity assumption on $V(z)$ can, presumably, be seriously weakened. One obvious remark is that if we have a sequence of convergent Laurent series $V_n(\zeta)$ such that the coefficients of $e^{V_n(z)}$ converge as $n \rightarrow \infty$ and the associated kernels \mathcal{K}_n converge in the trace norm in $\ell_2(\{0, 1, \dots\})$, then Theorem 1 is satisfied for the limit series $V(z)$ which may not represent an analytic function.

Remark 4 Finally notice that the left-hand side in (2.3) obviously depends only on $\varphi_{-n+1}, \dots, \varphi_{n-1}$, which is not so clear from the right-hand side.

¹Note, however, that the notion of an integrable operator was introduced and extensively used for kernels with *continuous* arguments while in our case the set of arguments is *discrete*.

3 Proof

The proof of Theorem 1 in the analytic and algebraic setup is identical, with the understanding that the meaning of convergence of infinite series in these two cases is very different. Theorem 1 will be established by applying the results of [13] to a formula, due to Gessel [7], expressing $D_n(\varphi)$ as a series in Schur functions. Although neither side of the equality in the theorem involves Schur functions, they will be needed in our proof.

Recall [12], I.3, that Schur functions s_λ are certain very distinguished symmetric polynomials in auxiliary variables $x = (x_1, x_2, \dots)$. For our purposes and notational compatibility with [13], it is more convenient to regard s_λ as polynomials in the power-sum symmetric functions

$$t_k = k^{-1} \sum x_i^k, \quad k = 1, 2, \dots,$$

which are free commutative generators of the algebra of symmetric functions. Since we will never use the original variables x , we will abuse the notation and write $s_\lambda(t)$ for the polynomial in the t_k 's which gives the corresponding Schur function. We do not need the explicit form of these polynomials but it is worth mentioning that their coefficients are the characters of symmetric groups, see [12], Section I.7.

We now recall a result of Gessel [7] or, more precisely, its dual version [15]. Consider the following sum:

$$\sum_{\lambda, \lambda_1 \leq n} s_\lambda(t^+) s_\lambda(t^-),$$

where

$$t^\pm = (t_1^\pm, t_2^\pm, \dots)$$

are two sets of variables and the sum is taken over all partitions λ with first part λ_1 less or equal to n .

Gessel's theorem asserts that ²

$$\sum_{\lambda, \lambda_1 \leq n} s_\lambda(t^+) s_\lambda(t^-) = D_n \left(\exp \left[-T^+(-\zeta) - T^-(-\zeta) \right] \right) \quad (3.1)$$

² To see the connection with Section 4 of [15] note that if $t_k = k^{-1} \sum x_i^k$ then $\sum_{k \geq 1} t_k \zeta^k = \prod_i (1 - x_i \zeta)^{-1}$.

where $T^\pm(z) = \sum_{k=1}^{\infty} t_k^\pm \zeta^{\pm k}$. Let us identify the parameters v_k^\pm and t_k^\pm by setting

$$V(\zeta) = -T^+(-\zeta) - T^-(-\zeta). \quad (3.2)$$

That is, $v_k^\pm = (-1)^{k+1} t_k^\pm$.

Following [13], consider the following *Schur measure* on partitions

$$\mathfrak{M}(\lambda) = \frac{1}{Z} s_\lambda(t^+) s_\lambda(t^-)$$

where Z is the sum in the Cauchy identity for the Schur functions

$$Z = \sum_{\lambda} s_\lambda(t^+) s_\lambda(t^-) = \exp\left(\sum_k k t_k^+ t_k^-\right) = \exp\left(\sum_{k=1}^{\infty} k v_k^+ v_k^-\right).$$

Note that although we call \mathfrak{M} a measure it does not have to be positive or even real for our purposes. It is, however, positive if $t_k^- = \overline{t_k^+}$, $k = 1, 2, \dots$, which is the case if $V(z)$ is real on the unit circle.

Given a partition λ , set $S(\lambda) = \{\lambda_i - i\} \subset \mathbb{Z}$. The Gessel theorem can be rewritten as follows

$$D_n(\varphi) = Z \sum_{\lambda, \lambda_1 \leq n} \mathfrak{M}(\lambda) = Z \sum_{\lambda, S(\lambda) \cap \{n, n+1, \dots\} = \emptyset} \mathfrak{M}(\lambda).$$

It was proved in [13], Theorems 1 and 2, that for any finite subset $X \subset \mathbb{Z}$

$$\sum_{\lambda, X \subset S(\lambda)} \mathfrak{M}(\lambda) = \det \left[\mathcal{K}(x_i, x_j) \right]_{x_i, x_j \in X}. \quad (3.3)$$

Here the kernel \mathcal{K} is given by the generating function³

$$\sum_{i, j \in \mathbb{Z}} \zeta^i \eta^{-j} \mathcal{K}(i, j) = \frac{\exp(T^+(\zeta) - T^-(\zeta) - T^+(\eta) + T^-(\eta))}{\zeta/\eta - 1}, \quad |\eta| < |\zeta|,$$

which is immediately seen to be identical to (2.1) after the identification (3.2).

³Note that our kernel \mathcal{K} differs from the kernel K used in [13] by a shift of variables $\mathcal{K}(x, y) = K(x + \frac{1}{2}, y + \frac{1}{2})$.

Also note that we identify the variables t^\pm with the variables t and t' used in [13] as follows: $t = t^-$, $t' = t^+$.

By the usual inclusion–exclusion principle we have

$$\begin{aligned} \mathfrak{M}(\{\lambda, S(\lambda) \cap \{n, n+1, \dots\} = \emptyset\}) = \\ \sum_{m=0}^{\infty} (-1)^m \sum_{n \leq l_1 < \dots < l_m} \mathfrak{M}(\{\lambda, \{l_1, \dots, l_m\} \subset S(\lambda)\}) \end{aligned}$$

This formula together with (3.3) concludes the proof of (2.3).

4 Examples

Example 1 We start with the simplest nontrivial example when $v_k^\pm = 0$ for all $k > 1$. Then $\varphi(\zeta) = \exp(v_1^+ \zeta + v_1^- \zeta^{-1})$. Since Toeplitz determinants $D_n(\varphi)$ do not change under the transformation $\varphi(\zeta) \rightarrow \varphi(a\zeta)$, $a \in \mathbb{C} \setminus \{0\}$, all that matters in our case is the product $v_1^+ v_1^-$. Thus, we may assume that $v_1^+ = v_1^- = \theta \in \mathbb{C} \setminus \{0\}$. Note that the asymptotics of Toeplitz determinants with symbol $\varphi(\zeta) = \exp(\theta(\zeta + \zeta^{-1}))$ played a central role in [1]. We have

$$V(\zeta) = \theta(\zeta + \zeta^{-1}), \quad V^*(\zeta) = \theta(\zeta - \zeta^{-1}).$$

To compute the kernel \mathcal{K} we will employ (2.5). We get

$$\frac{\zeta \frac{d}{d\zeta} V^*(\zeta) - \eta \frac{d}{d\eta} V^*(\eta)}{\zeta - \eta} = \theta - \frac{\theta}{\zeta\eta},$$

$$\begin{aligned} (i-j) \mathcal{K}(i, j) \\ = \frac{\theta}{(2\pi\sqrt{-1})^2} \iint_{|\zeta|=c_1, |\eta|=c_2} \left(\frac{e^{\theta(\zeta-\zeta^{-1})}}{z^{i+1}} \frac{e^{\theta(\eta^{-1}-\eta)}}{\eta^{-j}} - \frac{e^{\theta(\zeta-\zeta^{-1})}}{z^{i+2}} \frac{e^{\theta(\eta^{-1}-\eta)}}{\eta^{-j+1}} \right) d\zeta d\eta \quad (4.1) \end{aligned}$$

Since

$$\exp(\theta(\xi - \xi^{-1})) = \sum_{k=-\infty}^{\infty} \xi^k J_k(2\theta),$$

where $J_\nu(x)$ is the Bessel function, see [8], 7.2.4, we can rewrite (4.1) as follows

$$\mathcal{K}(i, j) = \theta \frac{J_i(2\theta) J_{j+1}(2\theta) - J_{i+1}(2\theta) J_j(2\theta)}{i-j}. \quad (4.2)$$

It can be shown, for example, by expanding $(\zeta - \eta)^{-1}$ in (2.2) as the sum of a geometric progression, that the diagonal entries $\mathcal{K}(i, i)$ are given by the limit of the right-hand side of (4.2) as $j \rightarrow i$. The l'Hopital rule yields

$$\mathcal{K}(i, i) = \theta \left(\frac{d}{di} J_i(2\theta) J_{i+1}(2\theta) - \frac{d}{di} J_{i+1}(2\theta) J_i(2\theta) \right).$$

The kernel \mathcal{K} has been obtained in [2] and, independently, in [10] in connection with asymptotics of the Plancherel measures for symmetric groups.

The constant Z in this example equals $\exp(\theta^2)$, and Theorem 1 reads

$$D_n(\exp(\theta(\zeta + \zeta^{-1}))) = \exp(\theta^2) \det(1 - \mathcal{K})_{\ell_2(\{n, n+1, \dots\})} \quad (4.3)$$

where \mathcal{K} is as above.

This formula also follows from the results of [7, 2, 10].

Example 2 In this example we will consider Toeplitz determinants with symbols of the form $\varphi(\zeta) = (1 + \theta_1 \zeta)^\kappa \exp(\theta_2/\zeta)$. Similarly to the previous example, since dilations do not change Toeplitz determinants, we may assume that $\theta_1 = \theta_2 = \theta$. Then

$$V(\zeta) = \ln \varphi(\zeta) = \kappa \ln(1 + \theta \zeta) + \theta \zeta^{-1}.$$

Note that in order to satisfy the conditions of Theorem 1 we have to assume that $|\theta| < 1$. We have

$$V^*(\zeta) = -\kappa \ln(1 - \theta \zeta) - \theta \zeta^{-1}, \quad \zeta \frac{d}{d\zeta} V^*(\zeta) = \frac{\kappa \theta \zeta}{1 - \theta \zeta} + \theta \zeta^{-1},$$

$$\frac{\zeta \frac{d}{d\zeta} V^*(\zeta) - \eta \frac{d}{d\eta} V^*(\eta)}{\zeta - \eta} = \frac{\kappa \theta}{(1 - \theta \zeta)(1 - \theta \eta)} - \frac{\theta}{\zeta \eta}.$$

By (2.5) we get, cf. (4.1),

$$\begin{aligned} & (i - j) \mathcal{K}(i, j) \\ &= \frac{\theta}{(2\pi\sqrt{-1})^2} \iint_{|\zeta|=c_1 < \frac{1}{|\theta|}, |\eta|=c_2} \left(\kappa \frac{(1 - \theta \zeta)^{-\kappa-1} e^{-\theta \zeta^{-1}}}{\zeta^{i+1}} \frac{(1 - \theta \eta)^{\kappa-1} e^{\theta \eta^{-1}}}{\eta^{-j}} \right. \\ & \quad \left. - \frac{(1 - \theta \zeta)^{-\kappa} e^{-\theta \zeta^{-1}}}{\zeta^{i+2}} \frac{(1 - \theta \eta)^\kappa e^{\theta \eta^{-1}}}{\eta^{-j+1}} \right) d\zeta d\eta. \quad (4.4) \end{aligned}$$

The contour integrals can be easily computed in terms of the confluent hypergeometric function $\Phi(a, c; x) = {}_1F_1(a; c; x)$: for any positive integer m

$$\frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=c_1 < \frac{1}{|\theta|}} (1 - \theta\zeta)^{-\alpha} e^{-\theta\zeta^{-1}} \frac{d\zeta}{\zeta^{m+1}} = \frac{(\alpha)_m}{m!} e^{-\theta^2} \Phi(1 - \alpha, m + 1; \theta^2),$$

$$\frac{1}{2\pi\sqrt{-1}} \int_{|\eta|=c_2} (1 - \theta\eta)^\beta e^{\theta\eta^{-1}} \eta^{m-1} d\eta = \frac{1}{m!} \Phi(-\beta, m + 1; \theta^2)$$

where $(a)_k = a(a+1)\cdots(a+k-1)$, $(a)_0 = 1$, is the Pochhammer symbol.

Hence, (4.4) yields

$$\mathcal{K}(i, j) = \frac{(\kappa)_{i+1} e^{-\theta^2}}{i-j} \left(\frac{\Phi(-\kappa, i+1; \theta^2)}{i!} \frac{\Phi(-\kappa+1, j+2; \theta^2)}{(j+1)!} - \frac{\Phi(-\kappa+1, i+2; \theta^2)}{(i+1)!} \frac{\Phi(-\kappa, j+1; \theta^2)}{j!} \right). \quad (4.5)$$

As in the previous example, it can be shown that the diagonal entries $\mathcal{K}(i, i)$ can be obtained as the limit of the right-hand side of (4.5) when $j \rightarrow i$.

The constant Z in this example equals $\exp(\kappa\theta^2)$, and Theorem 1 gives

$$D_n((1 + \theta\zeta)^\kappa \exp(\theta\zeta^{-1})) = \exp(\kappa\theta^2) \det(1 - \mathcal{K})_{\ell_2(\{n, n+1, \dots\})} \quad (4.6)$$

where $|\theta| < 1$ and \mathcal{K} is as above.

Toeplitz determinants from the left-hand side of (4.6) with positive integral κ have been studied in [15] in connection with asymptotics of the length of longest increasing subsequences in random words.

On the other hand, if κ is a positive integer, the kernel \mathcal{K} described above turns into a kernel conjugate to the Christoffel–Darboux kernel of κ th order for Charlier polynomials with parameter θ^2 , see [8] for definitions. This *Charlier kernel* appeared in [10] in the study of the same combinatorial problem on random words.

For positive integral κ 's (4.6) follows from the results of [15, 10].

The main result of the previous example (relation (4.3)) can be obtained from (4.6) by a simple limit transition.

Indeed, using the invariance of Toeplitz determinants with respect to dilations, for any positive integer k we can write that

$$D_n((1 + \tilde{\theta}\zeta/k)^k \exp(\tilde{\theta}\zeta^{-1})) = D_n((1 + \tilde{\theta}\zeta/\sqrt{k})^k \exp(\tilde{\theta}\zeta^{-1}/\sqrt{k})). \quad (4.7)$$

As $k \rightarrow \infty$, the left-hand side of (4.7) tends to the left-hand side of (4.3) with $\theta = \tilde{\theta}$. On the other hand, one can check that the Charlier kernel of k th order with parameter $\theta^2 = \tilde{\theta}^2/k$ converges, as $k \rightarrow \infty$, to the kernel of the previous example; [10] contains a detailed discussion of this limit transition, see also [5].

Example 3 Here we will be interested in Toeplitz determinants with symbols of the form $\varphi(\zeta) = (1 + \xi_1\zeta)^z(1 + \xi_2\zeta^{-1})^{z'}$ where ξ_1, ξ_2, z, z' are complex parameters. Invariance of Toeplitz determinants with respect to dilations implies that it suffices to consider the case when $\xi_1 = \xi_2 = \xi$. Then

$$V(\zeta) = z \ln(1 + \xi\zeta) + z' \ln(1 + \xi\zeta^{-1})$$

and conditions of Theorem 1 require that $|\xi| < 1$. We have

$$V^*(\zeta) = -z \ln(1 - \xi\zeta) + z' \ln(1 - \xi\zeta^{-1}), \quad \zeta \frac{d}{d\zeta} V^*(\zeta) = \frac{z\xi\zeta}{1 - \xi\zeta} + \frac{z'\xi\zeta^{-1}}{1 - \xi\zeta^{-1}},$$

$$\frac{\zeta \frac{d}{d\zeta} V^*(\zeta) - \eta \frac{d}{d\eta} V^*(\eta)}{\zeta - \eta} = \frac{z\xi}{(1 - \xi\zeta)(1 - \xi\eta)} - \frac{z'\xi}{\zeta\eta(1 - \xi\zeta^{-1})(1 - \xi\eta^{-1})}.$$

By (2.5) we get, cf. (4.1), (4.4),

$$\begin{aligned} (i-j)\mathcal{K}(i,j) &= \frac{1}{(2\pi\sqrt{-1})^2} \\ &\times \iint_{|\xi| < |\zeta|=c_1, |\eta|=c_2 < \frac{1}{|\xi|}} \left(z \frac{(1 - \xi\zeta)^{-z-1}(1 - \xi\zeta^{-1})^{z'}}{\zeta^{i+1}} \frac{(1 - \xi\eta)^{z-1}(1 - \xi\eta^{-1})^{-z'}}{\eta^{-j}} \right. \\ &\quad \left. - z' \frac{(1 - \xi\zeta)^{-z}(1 - \xi\zeta^{-1})^{z'-1}}{\zeta^{i+2}} \frac{(1 - \xi\eta)^z(1 - \xi\eta^{-1})^{-z'-1}}{\eta^{-j+1}} \right) d\zeta d\eta. \quad (4.8) \end{aligned}$$

This time the contour integrals can be expressed via the Gauss hypergeometric function: for any integer $m \geq 0$

$$\begin{aligned} \frac{1}{2\pi\sqrt{-1}} \int_{|\xi| < |\zeta| = c_1 < \frac{1}{|\xi|}} (1 - \xi\zeta)^{-\alpha} (1 - \xi\zeta^{-1})^{\alpha'} \frac{d\zeta}{\zeta^{m+1}} \\ = \frac{(\alpha)_m}{m!} \xi^m (1 - \xi^2)^{-\alpha'} F\left(1 - \alpha, \alpha'; m + 1; \frac{\xi^2}{\xi^2 - 1}\right), \\ \frac{1}{2\pi\sqrt{-1}} \int_{|\xi| < |\eta| = c_2 < \frac{1}{|\xi|}} (1 - \xi\eta)^\beta (1 - \xi\eta^{-1})^{-\beta'} \eta^{m-1} d\eta \\ = \frac{(\beta')_m}{m!} \xi^m (1 - \xi^2)^{-\beta} F\left(\beta, 1 - \beta'; m + 1; \frac{\xi^2}{\xi^2 - 1}\right). \end{aligned}$$

Substituting these formulas in (4.8), we get

$$\begin{aligned} \mathcal{K}(i, j) &= \frac{(z)_{i+1} (z')_{j+1}}{i! j!} \xi^{i+j+2} (1 - \xi^2)^{z+z'-1} \\ &\times \frac{1}{i-j} \left(F\left(-z, -z'; i+1; \frac{\xi^2}{\xi^2-1}\right) \frac{F\left(1-z, 1-z'; j+2; \frac{\xi^2}{\xi^2-1}\right)}{j+1} \right. \\ &\quad \left. - \frac{F\left(1-z, 1-z'; i+2; \frac{\xi^2}{\xi^2-1}\right)}{i+2} F\left(-z, -z'; j+1; \frac{\xi^2}{\xi^2-1}\right) \right) \end{aligned} \quad (4.9)$$

As in two examples above, the diagonal values $\mathcal{K}(i, i)$ can be computed as limits of the right-hand side of (4.9) as $j \rightarrow i$.

The kernel thus obtained is a part of the *hypergeometric kernel* obtained in [4]. The hypergeometric kernel describes a remarkable 3-parametric family of measures on partitions called *z-measures* which are closely related to so-called generalized regular representations of the infinite symmetric group, see [11, 3, 4, 5]. It lies on top of a hierarchy of kernels which are expressible in classical special functions, see [5] for details.

Let us compute the constant Z . We have

$$v_k^+ = \frac{z(-\xi)^k}{k}, \quad v_k^- = \frac{z'(-\xi)^k}{k}.$$

Hence,

$$Z = \exp \left(zz' \sum_{k=1}^{\infty} \frac{\xi^{2k}}{k} \right) = (1 - \xi^2)^{-zz'}.$$

Then Theorem 1 gives

$$D_n((1 + \xi\zeta)^z(1 + \xi\zeta^{-1})^{z'}) = (1 - \xi^2)^{-zz'} \det(1 - \mathcal{K})_{\ell_2(\{n, n+1, \dots\})} \quad (4.10)$$

where $|\xi| < 1$ and \mathcal{K} is as above.

When one of parameters z, z' , say z , is a positive integer, the kernel \mathcal{K} turns into the Christoffel–Darboux kernel of order z for Meixner polynomials with parameters $(z' - z, \xi^2)$, see [4], §4, for details.

Formulas (4.3), (4.6) can be obtained as certain limits of (4.10). To get (4.3) we just take $\xi = \theta/k$, $z = z' = k$ with positive integers k and send k to the infinity.

To get (4.6) we employ the same trick as was used in the previous example. Using invariance of Toeplitz determinants with respect to dilations, we look at Toeplitz determinants with symbols $(1 + \theta\zeta)^\kappa(1 + \theta\zeta^{-1}/k)^k$ for positive integers k and take the limit $k \rightarrow \infty$.

A detailed discussion of these limit transition in the language of kernels and underlying combinatorial problems can be found in [10, 5].

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