Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I

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1 Introduction

This is the first of a series of papers devoted to the study of how Gromov-Witten invariants of Calabi-Yau 3-folds transform under surgery. In [R1], [RT1],[RT2], Ruan-Tian established the mathematical foundations of the theory of quantum cohomology or Gromov-Witten invariants for semipositive symplectic manifolds. Recently, semipositivity condition has been removed by the work of many authors: [LT2], [B], [FO],[LT3], [R5], [S]. The focus now is on calculation and applications. As quantum cohomology was developed, many examples were computed. They are all Fano manifolds. One of most important classes of examples is the Calabi-Yau manifolds. The famous mirror symmetry principle asserts that there is a duality among Calabi-Yau 3-folds which interchanges $h^{1,1}$ with $h^{2,1}$ (classical level) and quantum cohomology with the variation of Hodge structures (quantum level). The most general form of mirror symmetry is false due to the existence of rigid Calabi-Yau 3-folds where $h^{2,1} = 0$. The rigid CY-3-folds are closely related to surgery operation on CY-3-folds. We believe that the most difficult problem in mirror symmetry is to find the precise range where mirror symmetry holds. However, a lot is known for the complete intersections of smooth toric variety.

Unfortunately, the results we have so far from quantum cohomology give little insight for the larger problem: the classification of CY 3-folds. Moreover, we almost know nothing about higher genus case. It seems to us that the difficulty in determining the precise range of CY-3-folds where mirror symmetry holds is due in part to our incomplete understanding of the classification of CY 3-folds. It would be fruitful to make a connection between the mirror symmetry and the classification problem. One should mention that the classification problem has been investigated long before the discovery of mirror symmetry because of its connection with Mori's program. Let's give a quick review of this subject.

A first step in the classification of CY 3-folds is to classify CY 3-folds in the same birational class. Any two CY 3-folds are birationally equivalent to each other iff they can be connected by a sequence of *flops* [Ka], [K]. A flop is a kind of surgery: two CY 3-folds M and M_f are said to be related by a flop if there is a a singular CY 3-fold M_s obtained from M by a "small contraction" (contracting finitely many rational curves) with M_f obtained from M_s by blowing up these singularities differently. For our purpose, it is also important that the singularities of M_s are rational double points which can be deformed to ordinary double points (ODP) locally. A conjecture by Morrison [Mo1] is that flop induces an isomorphism on quantum cohomology. It is well-known that flop does not induce an isomorphism on ordinary cohomology. Birational geometry is a central topic in algebraic geometry. It is often difficult to construct birational invariants. A proof of Morrison's conjecture would provide the first truly quantum birational invariant. It seems to us that it will be an important problem to study quantum cohomology under birational transformation over more general manifolds, for example flip. We shall leave this to the future research.

For nonbirationally equivalent CY 3-folds, an influential conjecture by Miles Reid (Reid's fantasy) states that any two Calabi-Yau 3-folds are connected to each other by a sequence of so called contract-deform or deform-resolve surgery. Contract-deform means that we contract certain curves or divisors to obtain a singular Calabi-Yau 3-fold and deform the result to a smooth CY 3-fold. Deform-resolve is the reverse. Reid's original conjecture was stated in the category of non-Kahler manifolds. To stay in the Kahler category, the contractions must be *extremal* in the sense of Mori. The smoothing theory for the case of ordinary double points was treated in [F], [Ti1]: In the general case, we refer to [Gro] for references. An extremal contraction-smoothing or its opposite surgery is called *extremal transition or transition*. A modified version of Reid's conjecture is that any two smooth CY 3-folds are connected to each other by a sequence of flops or extremal transitions. Large classes of CY 3-folds are indeed connected to each other this way.

For this paper, we also need the classifcation of extremal transitions. Here is a a brief summary. An extremal transition consists of two parts, extremal contraction and smoothing. We shall restrict ourself to *primitive extremal contractions*, which contract a Mori-extremal ray. We call the corresponding transition *a primitive transition*. Wilson [Wi1] classified primitive extremal contractions into three types: Type I is the small contraction (same as the case of flop). Type II contracts a del Pezzo surface to a point. Type III contracts a divisor to a curve. Certain cases of type III contraction can be deformed to type I by deforming the complex structure. Following Wilson, we choose a generic complex structure and define type according to this generic complex structure. The smoothing theory of ODP was understood in [F], [Ti1]. In general, an extremal contraction of a smooth CY 3-fold defines a singular CY 3-fold with canonical singularities. The smoothing theory of type II primitive transition has been worked out by M. Gross [Gro]. The smoothing theory of type III transition is still under development. We denote the type of transition by its type of contraction. In general, an extremal contraction (crepant contraction) could contract a higher dimensional boundary (for example, an extremal face) of the Mori effective cone (composition of primitive extremal contractions), One may be able to use our results to show that any transition can be decomposed into a sequence of primitive transitions. We call the compostion of several primitive transitons of the type I *a general type I transition*.

Type I primitive transitions exhibit an interesting phenomenon. From [F], [Ti1], in the Kahler category these transitions require a linear homological condition on the exceptional rational curves to ensure the existence of homologically nontrivial vanishing cycle. In particular, a type I primitive transition in the holomorphic category will decrease b_2 by one and increase b_3 by two. However, one can always obtain a symplectic manifold without imposing any condition. In that case, one can decrease b_2 without increasing b_3.

The classification of CY 3-folds involves the study of surgeries. It would be desirable to study the effect of these surgeries on mirror symmetry. This paper makes a step towards this direction. An interesting speculation [Mo2] here was that, except for the obvious counterexamples, each extremal transition has a mirror surgery which preserves mirror symmetry. It can be summaried as following *Mirror surgery conjecture: Every extremal transition L has a mirror surgery L_m with following property: If we have a mirror pair (X, Y) and perform an extremal transition L on X and obtain $\tilde{X}$, then one of following is true: (1) $\tilde{X}$ has no large complex limit. In this case, X has no mirror. (2) The mirror of $\tilde{X}$ can be obtained by performing L_m on Y.* Indeed, all the known examples of rigid CY-3-folds can be obtained by extremal transitions. This gives a nice explanation of the failure of mirror symmetry for rigid CY 3-folds. This conjecture grows from a discussion with P. Aspinwall. A closely related conjecture was also proposed by D. Morrison [Mo2]. Once we understand how relevant invariants change under extremal transitions, we can extend mirror symmetry to a large classes of CY 3-folds, and hopefully find the precise category where mirror symmetry holds. One can view mirror surgery conjecture as a combination of classification problem and mirror symmetry. Any results on mirror surgery conjecture would yield deeper understanding to both classification problem and mirror symmetry. If we want to prove Morrison's conjecture or mirror surgery conjecture, it is clear that we have to calculate the change of GW-invariants under flops and transitions. This is the main topic of the current and subsequent papers.

We take a symplectic approach towards this question. It is well-known in symplectic geometry that we can glue two symplectic manifold with same contact boundary, provided that one is convex and another one is concave. We call such a surgery *a contact surgery*. A closely related symplectic surgery is the cutting and gluing along a hypersurface admitting a local S^1-hamiltonian action. This is called symplectic normal sum or symplectic cutting. We first interpret flops and extremal transitions as instances of these two types of surgeries. The main part of this paper establishes a gluing theory for pseudo-holomorphic curves in these two types of surgeries. We know of two instances where surgeries were used to calculated GW-invariants. In an approach quite different from ours, G. Tian studied the degeneration of rational curves under symplectic degeneration [Ti] which is an analogous of the degeneration in algebraic geometry. The method McDuff used in [M2], [Lo] is similar to that of this paper. However, she studied the holomorphic curves completely inside the symplectic manifold with contact boundary, which is much easier to deal with. The main difficulty in this paper is to handle holomorphic curves which intersect the boundary.

For our point of view, the degree of difficulty increases as the singularity in the surgery becomes more complicated. One advantage of the symplectic approach is the flexibility of almost complex deformation. By choosing a generic almost complex deformation, one can often simplify the intermediate singularity. The simplest singularity is ODP, which occurs in flops and type I primitive transitions. In both cases, the exceptional loci are curves. Naively, holomorphic curves should be disjoint in CY 3-folds for a "generic" complex structure. One can easily guess the change of GW-invariant by assuming that all the holomorphic curves stay away from the exceptional loci. Under such an assumption, Morrison [Mo1] derived a formula for the change of quantum cohomology under a flop. Tian [Ti] derived a formula for the change of quantum cohomology under type I primitive transition. However, such an assumption is not true in general CY-3-folds. One can attempt to deform the complex structure to a generic almost complex structure, but then we don't know how to define extremal contraction, which is a complex operation. However, an easy application of our gluing theory handles to both cases completely.

The invariant we consider here are primitive GW-invariants $\Psi^M_{(A,g,k)}(\overline{\mathcal{M}}_{g,k};\{\alpha_i\})$ for stable range $2g+k\geq 3$. It was conjectured [RT2] that any GW-invariants can be reduced to primitive ones. We shall drop $\overline{\mathcal{M}}_{g,k}$ to simplify the notation. If $k=0$, We will drop k as well. For primitive invariant, it is also convenient to drop stable range condition. The construction is standard. Let us briefly outline the case of $g=0, k=0$ and the constructions for other cases are the same.

Suppose that (M,ω) is a symplectic manifold of dimension and J is a tamed almost complex structure. Let $\tilde{\mathcal{M}}_A(J)$ be the moduli space of genus zero pseudo-holomorphic maps with fundamental class A. Consider the moduli space of pseudo-holomorphic curves $\mathcal{M}_A(J)=\tilde{\mathcal{M}}_A(J)/PSL_2\mathbf{C}$. We can also construct its stable compactification $\overline{\mathcal{M}}_A(J)$ where we simply treat a principal component as a bubble and use the ordinary definition of stable maps. We still call it stable map without any confusion. Now one can construct a virtual neighborhood and define an invariant $\Phi^M_{(A,0,0)}$. One can eliminate divisor class $\alpha\in H^2(M,\mathbf{R})$ by the relation

$$\Psi^M_{(A,g,k+1)}(\alpha,\alpha_1,\cdots\alpha_k)=\alpha(A)\Psi^M_{(A,g,k)}(\alpha_1,\cdots,\alpha_k), \tag{1.1}$$

for $A\neq 0$. We shall omit zero in the case $\Psi^M_{(A,g,0)}$.

Choose a basis $A_1,\cdots,A_k$ of $H_2(M,\mathbf{Q})$. For $A=\sum_i a_iA_i$, we define the formal product $q^A=(q_{A_1})^{a_1}\cdots(q_{A_k})^{a_k}$. We can define a quantum 3-point function

$$\Psi^M_w(\alpha_1,\alpha_2,\alpha_3)=\sum_k\frac{1}{k!}\sum_A\Psi^M_{(A,0,k+3)}(\alpha_1,\alpha_2,\alpha_3,w,\cdots,w)q^A, \tag{1.7}$$

where w appears k-times. Here, we view Ψ^M as a power series of formal variable $p_i=q^{A_i}$. Clearly, an isomorphism on H_2 will induce a change of variables p_i. To define quantum product, we fix the symplectic class $[\omega]$ (a polarization) and require q^A to an element of Novikov ring $\wedge_{[\omega]}$. Formally, we can define quantum multiplication by the formula

$$\alpha\times^w_Q\beta\cup\gamma[M]=\Psi^M_w(\alpha,\beta,\gamma). \tag{1.8}$$

Clearly, the quantum 3-point function contains the same information as quantum product. For our purpose, it is convenient to work directly on quantum 3-point function.

Definition 1.1: *Suppose that*

$$\varphi: H_2(X,\mathbf{Z})\to H_2(Y,\mathbf{Z}),\ H^{even}(Y,\mathbf{R})\to H^{even}(X,\mathbf{R})$$

are group homomorphisms such that the maps over H_2, H^2 are dual to each other. We call φ is natural with respect (big) quantum cohomology if $\varphi^*\Psi^Y_0=\Psi^X_0|_{Im(\varphi^*)}$ ($\varphi^*\Psi^Y_w=\Psi^X_{\varphi^*w}|_{Im(\varphi^*)}$) up to a change of formal variable $q^A\to q^{\varphi(A)}$. If φ is also an isomorphism, we say φ induces an isomorphism on (big) quantum cohomology or they have the same (big) quantum cohomology.

Here, two power series F, G are treated as the same if $F = H + F', G = H + G'$ such that F', G' are power series expansion of the same rational function at the different points.

For example, we can expand $\frac{1}{1-t} = \sum_{i=0} t^i$ at $t = 0$ or $\frac{1}{1-t} = \frac{1}{-t(1-t^{-1})} = -\sum_{i=0} t^{-i-1}$ at $t = \infty$. Hence, we will treat $\sum_{i=0} t^i, \sum_{i=0} t^{-i-1}$ as the same power series.

When X, Y are 3-folds, such a φ is completely decided by maps on H_2. For example, the dual map of $\varphi : H_2(X, \mathbf{Z}) \to H_2(Y, \mathbf{Z})$ gives a map $H^2(Y, \mathbf{R}) \to H^2(X, \mathbf{R})$. A map $H^4(Y, \mathbf{R}) \to H^4(X, \mathbf{R})$ is Poincare dual to a map $H_2(Y, \mathbf{R}) \to H_2(X, \mathbf{R})$. In the case of flop, the natural map $H_2(X, \mathbf{Z}) \to H_2(Y, \mathbf{Z})$ is an isomorphism. Therefore, we can take the map $H_2(Y, \mathbf{Z}) \to H_2(X, \mathbf{Z})$ as its inverse. The maps on H^0, H^6 are obvious.

We remark that flop and type I extremal transition can be performed for any 3-folds. In particular, they will preserve Calabi-Yau condition. Although the focus of this paper is CY-3-folds, the main theorems actually hold for arbitrary 3-folds. Let's state our main theorem on this generality. Suppose that M_f is obtained after a flop on M. There is a natural isomorphism (see section 2)

$$\varphi : H_2(M, \mathbf{Z}) \to H_2(M_f, \mathbf{Z}). \tag{1.3}$$

The manifolds M and M_f have the same set of exceptional curves. Suppose that Γ is an exceptional curve and Γ_f is the corresponding exceptional curve on M_f. Then,

$$\varphi([\Gamma]) = -[\Gamma_f]. \tag{1.4}$$

Our first theorem is that

Theorem A *If $A \neq n[\Gamma]$ for any exceptional curve Γ, then*

$$\Psi^M_{(A,g)}(\{\varphi^* \alpha_i\}) = \Psi^{M_f}_{(\varphi(A),g)}(\{\alpha_i\}). \tag{1.5}$$

Moreover,

$$\Psi^M_{(n[\Gamma],g)} = \Psi^{M_f}_{(n[\Gamma_f],g)}. \tag{1.6}$$

When M is a Calabi-Yau 3-fold, Theorem A takes a particular simple form.

Corollary A.1: *Suppose that M is a Calabi-Yau 3-fold. If $A \neq n[\Gamma]$ for any exceptional curve Γ, then*

$$\Psi^M_{(A,g)} = \Psi^{M_f}_{(\varphi(A),g)}.$$

Moreover,

$$\Psi^{M}_{(n[\Gamma],g)} = \Psi^{M_f}_{(n[\Gamma_f],g)}. \tag{1.6}$$

Using our formula in genus zero case and Morrison's argument [Mo1], we have following corollary.

Corollary A.2: *φ induces an isomorphism on quantum cohomology.*

Recall that a minimal model is a projective variety with terminal singularities and nef canonical bundle. It was known that in higher dimension there are many different minimal models in the same birational class. However, in dimension three, they are related by flops. Then above corollary yields

Corollary A.3: *Any two smooth minimal models in dimension three have the same quantum cohomology.*

The second author conjectures that *two smooth minimal model over any dimension have isomorphic quantum cohomology [R6].* When M is a CY-3-fold, above corollary implies that Morrison's conjecture. In the case of CY-3-fold, Corollary A.3 admits another interpretation. Instead of considering M, M_f as different manifolds, we can use ϕ to map cohomology of different birational models into a single space called movable cone. Then, the Corollary A.3 can be restated as that quantum 3-point function extends analytically over the movable cone. In fact, this was how Morrison stated his conjecture.

Let M_e be the 3-fold obtained after a general type I transition. The extremal transition induces a surjective map

$$\varphi_e : H_2(M, \mathbf{Z}) \to H_2(M_e, \mathbf{Z}). \tag{1.9}$$

We choose a right inverse, which is a map $H_2(M_e, \mathbf{R}) \to H_2(M, \mathbf{R})$. Using the general gluing theory we established, we obtain following formula:

Theorem B *Let M_e be obtained by a general type I-transition. Then,*

$$\Psi^{M_e}_{(B,g)}(\{\alpha_i\}) = \sum_{\varphi_e(A)=B} \Psi^{M}_{(A,g)}(\{\varphi^*(\alpha_i)\}). \tag{1.10}$$

Again, Theorem B takes a simple form for Calabi-Yau 3-fold.

Corollary B.1 *Let M_e be obtained by a general type I-transition performed on a Calabi-Yau 3-fold M. Then,*

$$\Psi^{M_e}_{(B,g)} = \sum_{\varphi_e(A)=B} \Psi^M_{(A,g)}.$$

Then, we can use genus zero case of Theorem B and generalize Tian's argument [Ti] to compare quantum cohomology.

Corollary B.2:φ *is natural with respect to big quantum cohomology.*

We remark that in the case of Calabi-Yau 3-folds, the genus zero case of both Theorems A and B have been studied in physics related to black holes. However, higher genus GW-invariants are not enumerative invariants. The nature of these invariants is quite mysterious even today. We were surprised that the same formula is also true for higher genus GW-invariants. Several years ago, there was a mysterious Kodara-Spencer quantum field theory in physics which dealt with the higher genus case. We hope that our calculation will shed some light on the structure of higher genus invariants.

For type II, III primitive transitions, a holomorphic curve will always intersect the divisor. As far as the authors know, nobody has even conjectured the formula of GW-invariant under these transitions. We shall lay an analytic foundation for the study of other type primitive transitions in this paper and leave the specific computation to a subsequent paper [LQR]. When we search for the formula for general surgery, Theorems A and B are rather misleading. The natural invariants appeard in general surgery are *relative GW-invariants* instead of *absolute GW-invariants*. To describe relative GW-invariant, let's first review the definition of symplectic cutting [L].

Suppose that $H : M \to \mathbf{R}$ is a periodic Hamiltonian function such that the Hamiltonian vector field X_H generates a circle action. By adding a constant, we can assume that 0 is a regular value. Then, $H^{-1}(0)$ is a smooth submanifold preserved by circle action. The quotient $Z = H^{-1}(0)/S^1$ is the famous symplectic reduction. Namely, it has an induced symplectic structure.

For simplicity, we assume that M has a global Hamiltonian circle action. Once we write down the construction, we then observe that a local circle Hamiltonian action is enough to define a symplectic cutting.

Consider the product manifold $(M \times \mathbf{C}, \omega \oplus -idz \wedge d\bar{z})$. The moment map $H - |z|^2$ generates a Hamiltonian circle action $e^{i\theta}(x, z) = (e^{i\theta}x, e^{-i\theta}z)$. Zero is a regular value and we have symplectic reduction

$$\overline{M}^+ = \{H = |z|^2\}/S^1. \tag{1.11}$$

We have decomposition

$$\overline{M}^+ = \{H = |z|^2\}/S^1 = \{H = |z|^2 > 0\}/S^1 \cup H^{-1}(0)/S^1. \tag{1.12}$$

Furthermore,

$$\phi^+ : \{H > 0\} \to \{H = |z|^2 > 0\}/S^1$$

by

$$\phi^+(x) = (x, \sqrt{H(x)}). \tag{1.13}$$

is a symplectomorphism. Let

$$M_b^+ = H^{-1}(\geq 0). \tag{1.14}$$

Then, M_b^+ is a manifold with boundary and there is a map

$$M_b^+ \to \overline{M}^+. \tag{1.15}$$

Clearly, $\overline{M}^+$ is obtained by collapsing the S^1 action of the $H^{-1}(0)$. Clearly, we only need a local S^1 Hamiltonian action. To obtain $\overline{M}^-$, we consider circle action $e^{i\theta}(x, z) = (e^{i\theta}x, e^{i\theta}z)$ with the moment map $H + |z|^2$. $\overline{M}^+, \overline{M}^-$ are called symplectic cutting of M. We define M_b^- similarly. By the construction, $Z = H^{-1}(0)/S^1$ with induced symplectic structure embedded symplectically into $\overline{M}^{\pm}$. Moreover, its normal bundles have opposite first Chern classes. The virtual neighborhood construction requires that the symplectic forms $\omega, \omega^+, \omega^-$ over $M, \overline{M}^+, \overline{M}^-$ have integral periods. Notes that $\omega^+|_Z = \omega^-|_Z$. By Mayer-Vietoris sequence, (ω^+, ω^-) defines a cohomology class $\omega^+ \cup_Z \omega^-$ of $\overline{M}^+ \cup_Z \overline{M}^-$. The latter is the quotient of M by circle action on level set $H^{-1}(0)$. Hence, there is a map

$$\pi : M \to \overline{M}^+ \cup_Z \overline{\mathcal{M}}^-.$$

It induces a homomorphism

$$\pi^* : H^*(\overline{M}^+ \cup_Z \overline{M}^-, \mathbf{Q}) \to H^*(M, \mathbf{Q}). \tag{1.15A}$$

It is easy to observe that $\pi^*(\omega^+ \cup_Z \omega^-) = \omega$. We can deform ω^+, ω^- slightly (still denoted by ω^+, ω^-) such that $\omega^+|_Z = \omega^-|_Z$ and $\omega^\pm$ has rational period. Then, we use Gompf-Macarthy-Wolfsen gluing construction (an inverse operation of symplectic cutting (see section 2)) to obtain ω. Then, ω has rational period. To get integral period, we just multiple ω by a large integer.

Suppose that B is a real codimension two symplectic submanifold of M. By a result of Guillemin and Sternberg [GS], the symplectic structure of a tubular neighborhood of B is modeled on a neighborhood of Z in either $\overline{M}^+$ or $\overline{M}^-$.

We can define *a relative GW-invariant* $\Psi^{(M,Z)}$ by counting the number of relative stable holomorphic maps intersecting Z at the finite many points with prescribed tangency. Let $\mathbf{k} = \{k_1, \cdots, k_l\}$ be a set of positive integers. Consider the moduli space $\mathcal{M}_A(g, m, \mathbf{k})$ of genus g pseudo-holomorphic maps f such that f has marked points $(x_1, \cdots, x_m; y_1, \cdots, y_l)$ with the property that f is tangent to Z at y_i with order k_i. Then, we compatify $\mathcal{M}_A(g, m, \mathbf{k})$ by $\overline{\mathcal{M}}_A(g, m, \mathbf{k})$, *the space of relative stable maps*. We have two maps

$$\Xi_{g,m} : \overline{\mathcal{M}}_A(g, m, \mathbf{k}) \to M^m.$$

and

$$P : \overline{\mathcal{M}}_A(g, m, \mathbf{k}) \to B^l. \tag{1.16}$$

Roughly, the relative GW-invariants is defined as

$$\Psi^{(M,B)}_{(A,g)}(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_l) = \int_{\overline{\mathcal{M}}_A(g,m,\mathbf{k})} \Xi^*_{g,m} \prod_i \alpha_i \wedge P^* \prod_j \beta_j. \tag{1.17}$$

To be precise, we need to construct a virtual neighborhood (7.1) of $\overline{\mathcal{M}}_A(g, m, \mathbf{k})$. Then, we take integrand (1.17) over the virtual neighborhood.

To justify our notation, we remark that when $\mathbf{k} = \{1, \cdots, 1\}$, $\Psi^{(M,Z)}$ is different from ordinary or absolute GW-invariant in general. So $\Psi^{(M,Z)}$ is not a "generalized" GW-invariant. The relation between relative and absolute GW-invariants is one of main topics of [LQR].

Theorem C(Theorem 7.6):

(i). $\Psi^{(M,Z)}_{(A,g,m,\mathbf{k})}(\alpha_1, ..., \alpha_m; \beta_1, ..., \beta_l)$ *is well-defined, multilinear and skew symmetry.*

(ii). $\Psi^{(M,Z)}_{(A,g,m,\mathbf{k})}(\alpha_1, ..., \alpha_m; \beta_1, ..., \beta_l)$ *is independent of the choice of forms* α_i, β_j *representing the cohomology classes* $[\beta_j], [\alpha_i]$*, and the choice of virtual neighborhoods.*

(iii). $\Psi^{(M,Z)}_{(A,g,m,\mathbf{k})}(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_l)$ *is independent of the choice of almost complex structures on* Z *and on the normal bundle* ν_Z*, and hence an invariant of* (M, Z)*.*

Theorem D(Theorem 7.9, 7.10): *Suppose that* $\alpha_i^+|_Z = \alpha_i^-|_Z$ *and hence* $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{M}^+ \cup_Z \overline{M}^-, \mathbf{R})$*. Let* $\alpha_i = \pi^*(\alpha_i^+ \cup_Z \alpha_i^-)$ *(1.15A). There is a gluing formula to relate* $\Psi^M(\{\alpha_i\})$ *to relative invariants* $\Psi^{(\overline{M}^+, Z)}(\{\alpha_i^+\})$ *and* $\Psi^{(\overline{M}^-, Z)}(\{\alpha_i^-\})$*.*

We remark that different $\alpha_i^{\pm}$ may give the same α_i. Then we obtain different gluing formula for the same invariant. For example, if α_i is Poincare dual to a point, its support could be in $\overline{V}^+$ or $\overline{V}^-$. This is very important in the applications.

The exact statement of the general formula is rather complicated. We refer reader to section 7.2 for the details.

In a subsequent paper [LQR], we will apply our general gluing formula to study the change of GW-invariants for type II transition. For dimensional reasons, all the relative GW-invariants with higher tangency condition on a Calabi-Yau 3-folds vanish. However, the relative GW-invariant without the tangency condition is still different from absolute GW-invariants in general. We shall first establish a comparison theorem between relative GW-invariant without tangency condition and corresponding absolute GW-invariants. In the case of extremal transition, such a comparison theorem works the best for genus zero invariants. We will focus on the case of genus zero invariants. Then, we will derive a recursive formula for the change of genus zero GW-invariant in the case of type II transition. Such a formula needs the GW-invariants of two 3-folds associated with each singularity, corresponding to resolution and smoothing of the singularity. There are eight different singularities corresponding to eight different type II transitons. We finish our program by computing the GW-invariants of 16 different 3-folds associated with these eight different sigularities.

Let's briefly describe the idea of the proof. The first step is to reinterpret the flop and extremal transiton as a combination of contact surgery and symplectic cutting (section 2). Then, we stretch symplectic manifolds along either contact hypersurface or the hypersurfaces admitting a local S^1-hamiltonian action. Let's focus on the case of contact hypersurface; the other case is similar. Every contact manifold with a fixed contact form possesses a unique vector field called Reeb vector field. The existence of periodic orbits of Reeb vector field has been under intensive study in hamiltonian dynimics. Hofer observed that the boundary of a finite energy pseudoholomorphic curve will converge to a periodic orbit of Reeb vector field. Furthermore, Hofer and his collabrators established the analysis of the moduli space of pseudo-holomorphic curves whose ends converge to periodic orbits in the case that the periodic orbits are nondegenerate. In our case, the periodic orbits are known. However, they are only nondegenerate in the sense of Bott, so we have to generalize Hofer's analysis to the case of Bott-type nondegenerate. We will do so by casting it into the language of stable maps (section 3-6). Furthermore, we establish the gluing theorem which is the reverse of stretching construction (section 6). The last piece of information we need is a vanishing theorem of certain GW-invariants. This is done by a simple index calculation.

We should point out that the relative GW-invariants also appeard in the work of Caporaso-Harris [CH]. They are closely related to the blow-up formula for Seiberg-Witten invariants.

We would like to mention some results related to our work. There are many papers in physics to discuss both flop and extremal transitions . We refer to [Mo2] for the relevant references. But they discuss only genus zero invariants. The paper by P. Wilson [Wi3] calculates GW-invariants of extremal rays. His paper is complementary to ours. During the preparation of this paper, we received an article [BCKS] which is related to our results. The local hamiltonian S^1-action plays an important role in our work. The general gluing formula over a contact boundary requires a definition of contact Floer homology, which is developed by Eliashberg and Hofer. We are also informed that Ionel and Parker are developing a gluing theory independently using a different approach.

This paper is organized as follows. We will review the constructions of various surgery operations in symplectic geometry and interpret both flop and extremal transition as symplectic surgeries. The gluing theory will be established in sections 3-7. Theorems A and B will be proved in section 8.

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2 Symplectic surgery, flops and extremal transition

Symplectic surgeries have been extensively studied by a number of authors. Many such surgeries already appeared in [Gr1]. One of oldest one is the gluing along contact boundary. We do not know who was the first to propose the contact surgeries, however, the second author benefited from a number of stimulating discussions with Y. Eliashberg on contact surgeries over the years. During the last ten years, symplectic surgeries have been successfully used to study symplectic topology, for example, symplectic blow-up and blow-down by McDuff [MS1] and symplectic norm sum by Gompf [Go2] and McCarthy and Wolfson [MW]. Very recently, Lerman [L] introduced an operation called "symplectic cutting" which plays an important role in this article. For the reader's convienence, we will give a fairly detailed review of each of these operations. The second half of section is devoted to study algebro-geometric surgeries of flop and extremal transition in terms of symplectic surgery.

Definition 2.1 *A contact structure ξ of an odd dimensional manifold N^{2n+1} is a codimension one distribution defined globably by a one form λ ($Ker\lambda = \xi$) such that $\lambda \wedge (d\lambda)^n$ is a volume form. We call λ a contact form.*

Any contact manifold has a canonical orientation defined by volume form $\lambda \wedge (d\lambda)^n$. If λ is a contact form, so is $f\lambda$ for a positive function f. Any contact form defines the Reeb vector field X_λ by the equation

$$i_{X_\lambda}\lambda = 1, i_{X_\lambda}d\lambda = 0. \tag{2.1}$$

In general, the dynamics of X_λ depends on λ. There is a version of Moser's theorem for contact manifolds as follows. Let λ_t be a family of contact forms. There is a family of diffeomorphisms

$$\varphi_t : N \to N$$

such that $\varphi_0 = Id$ and $\varphi_t^*\lambda_t = f_t\lambda$, where f_t is a family of functions such that $f_0 = 1$.

Example The sphere $S^{2n-1} \subset \mathbf{C}^n$ with standard symplectic structure $\omega = \sum_i dx_i \wedge dy_i$ is a contact hypersurface. The contact form is the restriction of $\lambda = \sum(x_i dy_i - y_i dx_i)$. The closed orbit of Reeb vector field is generated by complex multiplication $e^{i\theta}$. Furthermore, if Q is a symplectic submanifold intersecting transversely to S^{2n-1}, $Q \cap S^{2n-1}$ is a contact hypersurface of Q.

Definition 2.2 *(M, ω) is a compact symplectic manifold with boundary such that $\partial M = N$. M has a contact boundary N if in a tubular neighborhood $N \times [0, \epsilon)$ of N, $\omega = d(f\lambda)$, where f is a function. Suppose that X is a outward transverse vector field on N. We say that N is a convex contact boundary if $\omega(X_\lambda, X) > 0$. otherwise, we call N a concave contact boundary. In another word, the induced orientation of convex contact boundary coincides with the canonical orientation. But the induced orientation of a concave contact boundary is the opposite of canonical orientation.*

Let $(M^+,\omega^+),(M^-,\omega^-)$ be symplectic manifolds with contact boundary N^+, N^-. Furthermore, suppose that N^+ is a convex boundary of M^+ and N^- is a concave boundary of M^-. For any contact diffeomorphisms $\varphi : N^+ \to N^-$, we can glue them together to form a closed symplectic manifold $M_{gl} = M^+ \cup_\varphi M^-$. For the reader's convienence, we reproduce here the details of the gluing construction from [E] (Proposition 3.1).

By adjusting the contact form, there exist a tubular neighborhood $U_+ = N^+ \times [1, 1+\epsilon)$ of N^+ and a tubular neighborhood $U_- = N^- \times (1-\epsilon, 1]$ of N^- such that

$$\omega^\pm = d(t\lambda^\pm) \tag{2.2}$$

for contact form $\lambda_\pm$ on $N^\pm$. Let $f : N^+ \to N^-$ be the contact diffeomorphisms. Then, $(f^{-1})^*(\lambda^+) = h\lambda^-$ for a positive function h. By rescaling ω_-, we can assume that $h \le 1$. Now we enlarge M^- to $\tilde{M}^-$ by attaching

$$U_{gl} = \{(x,t); x \in N^-, 1 \le t \le \frac{1}{h(x)}\}. \tag{2.3}$$

We can define a symplectomorphism $F : U_+ \to U_{gl}$ by the formula

$$F(x,t) = (f(x), \frac{t}{h(f(x))}),\ 1 \le t \le 1+\epsilon, x \in N^+. \tag{2.4}$$

F allows us to glue N^+ and $\tilde{N}^-$. We denote the resulting manifold as M_{gl}.

Recall that GW-invariants only depend on the symplectic deformation class.

Definition 2.3 Two symplectic manifolds $(M,\omega),(M'\omega')$ with contact boundaries $(N,\lambda),(N',\lambda')$ are deformation equivalent if there is a diffeomorphism $\varphi : M \to M'$, a family of symplectic structures ω_t and a family of contact structures λ_t such that (M,ω_t) is a symplectic manifold with contact boundary (N,λ_t) and

$$\omega_0 = \omega,\ \ \omega_1 = \varphi^*\omega',\ \ \lambda_0 = \lambda,\ \ \lambda_1 = \varphi^*\lambda'. \tag{2.5}$$

The proof of following Lemma is trivial. We omit it.

Lemma 2.4 *If we deform the symplectic structures of $M^\pm$ according to the Definition 2.3, the glued-up manifold M_{gl} are deformation equivalent.*

Another closed related surgery is symplectic cutting described in the introduction. The converse gluing process has been studied previously by McCarthy and Wolfson [MW] in connection to symplectic norm sum.

Suppose that $H : M \to \mathbf{R}$ is a periodic Hamiltonian function. The Hamiltonian vector field X_H generates a circle action. By adding a constant, we can assume that 0 is a regular value. Then, $H^{-1}(0)$ is a smooth submanifold preserved by circle action. The quotient $H^{-1}(0)/S^1$ is the famous symplectic reduction. Namely, it has an induced symplectic structure. Let

$$\pi : H^{-1}(0) \to Z = H^{-1}(0)/S^1. \tag{2.6}$$

Z admits a natural symplectic structure τ_0 such that

$$\pi^*\tau_0 = i_0^*\omega, \tag{2.7}$$

where $i_0 : H^{-1}(0) \to M$ is the inclusion. $H^{-1}(0)$ plays the role of contact boundary in contact surgery and X_H plays the role of Reeb vector field. We remark that Z is a symplectic orbifold in general. Furthermore, it is enough that H is defined in a neighborhood of $H^{-1}(0)$. (2.6) is a circle bundle. In the special case that τ_0 represents the first chern class of (2.7). There is a connection one form α over $H^{-1}(0)$ such that

$$d\alpha = \pi^*\tau_0 = i_0^*\omega. \tag{2.8}$$

Moreover, α is a contact form. To show that $H^{-1}(0)$ is a contact boundary, we have to study the symplectic structure on the tubular neighborhood of $H^{-1}(0)$. This is done by using famous Duistermaat-Heckmann theorem [DH].

Now we presuit over general ground without assumption that τ_0 represents the first Chern class of circle bundle (2.6). Following argument is due to McDuff [M1], McCarthy-Wolfson [MW]. Since 0 is a regular value, there is a small interval $I = (-\epsilon, \epsilon)$ of regular values. We use a S^1-invariant connection on the fibration $H^{-1}(I) \to I$ to show that there is a S^1-diffeomorphism $H^{-1}(I) \cong N \times I$. We will identify $H^{-1}(I)$ with $N \times I$ without any confusion. Then, the Hamiltonian function is simply the projection onto the second factor. In such way, we also identify the symplectic reduction $H^{-1}(t)/S^1$ with Z. Suppose that its symplectic form is τ_t. A celebrated theorem of Duistermaat-Heckman [DH] says that

$$[\tau_t] = [\tau_0] + tc, \tag{2.9}$$

where c is the first Chern class of circle bundle (2.6).

McDuff, McCarthy-Wolfson show that the symplectic form over $N \times I$ is standard up to diffeomorphism. In fact, one can choose

$$\omega = \pi^*(\tau_t) - \alpha \wedge dt. \tag{2.10}$$

The formula (2.10) only depends on τ_0, c. Hence, if the boundary components of two symplectic manifolds have the same τ_0, c, we can glue them together.

When $[\tau_0] = c$, $[\tau_t] = (1+t)[\tau_0]$. By Moser's theorem, we can assume that

$$\tau_t = (1+t)\tau_0. \tag{2.11}$$

Therefore,

$$\omega = (1+t)d(\alpha) - \alpha \wedge dt = d((1+t)\alpha) \tag{2.12}$$

so N is a contact boundary.

Conversely, if $H^{-1}(I)$ is symplectically embedded in a symplectic manifold, by [L] we can also cut the symplectic manifold along N and collapse the S^1-action on N to form two closed symplectic manifolds. This is the symplectic cutting described in the introduction. It is obvious that we have a map

$$\pi : M \to \overline{M}^+ \cup_Z \overline{M}^-.$$

It induces a map

$$\pi_* : H_i(M, \mathbf{Z}) \to H_i(\overline{M}^+ \cup_Z \overline{M}^-, \mathbf{Z}).$$

A class in $\ker \pi_*$ is called *a vanishing cycle.*

Examples:

1). We consider the Hamiltonian action of S^1 on $(\mathbf{C}^n, \omega_0)$ corresponding to multiplication by e^{it} with the Hamiltonian function $H(z) = |z|^2 - \epsilon$. Then we have a Hamiltonian action of S^1 on $(\mathbf{C}^n \times \mathbf{C}, -i(dw \wedge d\bar{w} + \Sigma dz_j \wedge d\bar{z}_j))$ given by

$$e^{it}(z, w) = \left(e^{it}z, e^{it}w\right) \tag{2.16}$$

with momentum map

$$\mu(z, w) = H(z) + |w|^2. \tag{2.17}$$

We have

$$\mu^{-1}(0) = \left\{(z, w) | 0 < |w|^2 \leq \epsilon, H(z) = -|w|^2\right\} \bigcup \{(z, 0) | H(z) = 0\} \tag{2.18}$$

$$\overline{M}^+ = \mu^{-1}(0)/S^1. \tag{2.19}$$

where $M^+ = \{z||z| \leq \epsilon\}$ and $\overline{M}^+$ is corresponding symplectic cut. Choose the canonical complex structure i in $\mathbf{C}^{n+1} = \mathbf{C}^n \times \mathbf{C}$. The following fact is well-known (see [MS1]) :

For $\mathbf{C}^n$ the symplectic cut $\overline{M}^+$ is $\mathbf{P}^n$ with symplectic form

$$-i\epsilon d\left(\frac{zd\bar{z} - \bar{z}dz}{1 + \|z\|^2}\right). \tag{2.20}$$

2). We consider $\mathbf{O}(-1) + \mathbf{O}(-1)$ over $\mathbf{P}^1$ with the symplectic form

$$\omega + dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 \tag{2.21}$$

where ω is the symplectic form on the rational curve, z_i are the coordinates in $\mathbf{C}$. The Hamiltonian function is

$$H(x, z_1, z_2) = |z_1|^2 + |z_2|^2 - \epsilon, \tag{2.22}$$

and the S^1-action is given by

$$e^{it}(x, z_1, z_2) = \left(x, e^{it}z_1, e^{it}z_2\right). \tag{2.23}$$

We perform the symplectic cutting along the hypersurface $\widetilde{M} = H^{-1}(0)$. By example 1, we conclude that

For $\mathbf{O}(-1) + \mathbf{O}(-1)$ over $\mathbf{P}^1$ the symplectic cut $\overline{M}^+$ is $P(\mathbf{O}(-1) + \mathbf{O}(-1) + \mathbf{O})$.

3). Consider the algebraic variety M defined by a homogeneous polynomial F in $\mathbf{C}^n$. Let $S^{2n-1}(\epsilon)$ be the sphere of radius ϵ. Let $\widetilde{M} = S^{2n-1}(\epsilon) \cap \{F = 0\}$. Consider the the Hamiltonian action of S^1 on $(\mathbf{C}^n, \omega_0)$ corresponding to multiplication by e^{it} with the Hamiltonian function

$$H(z) = |z|^2 - \epsilon. \tag{2.24}$$

Since F is a homogeneous polynomial, S^1 act on $S^{2n-1}(\epsilon) \cap \{F = 0\}$. By example 1 we have

For $M = \{F = 0\}$ the symplectic cut $\overline{M}^+$ is the variety defined by $F = 0$ in $\mathbf{P}^n$.

We have finished our digression about symplectic surgery. Next we study flop and extremal transition from a symplectic point of view. Recall that two projective manifolds M, M' are birational equivalent iff some Zariski open sets are isomorphic. If M, M' are smooth CY-3-folds, M, M' are related by a sequence of flops [Ka], [K]. Namely, there is a sequence of smooth CY-3-folds

$M_1, \ldots, M_k$ such that M_1 is obtained by a flop from M, M_{i+1} is obtained by a flop from M_i, and M' is obtained by a flop from M_k. Furthermore, we can choose M_i and corresponding singular CY-3-folds $(M_i)_s$ to be projective. Moreover, we can assume that the contractions are primitive. The extremal transition starts from an extremal contraction. In this paper, we only deal with the a general type I-transition, which starts from a small contraction. The singularities are rational double points. Wilson observed that by choosing a local complex deformation around singularities together with their simultaneous resolution, we can reduce them to the case of ordinary double points (ODP). We shall give a symplectic description for the case of ODP, and sketch Wilson's argument for general case.

We embed M_s into a projective space $\mathbf{P}^m$. Up to an analytic change of coordinates, we can write a neighborhood of singular point in M_s as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, x_5 = 0 \cdots x_m = 0. \tag{2.25}$$

where the origin is the singular point. Let's assume $m = 4$ to simplify the notation. To get a small resolution, we blow up the origin and obtain M_b. The exceptional locus is a quadric defined by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$. The blow-up M_b is not a Calabi-Yau 3-fold in general. However, any quadric surface is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Then, we contract one of rulings to obtain a small resolution M with an exceptional smooth rational curve with normal bundle $\mathbf{O}(-1) + \mathbf{O}(-1)$. If we contract the other ruling, we obtain the flop M_f. This is the complex description of flop. If M_s has more than one singular point, we have to repeat above construction for every singular point.

It is much more subtle to put a Kahler (hence symplectic) structure on M, M_f. The blow-up M_b always has a Kahler structure. However, blowing-down along a symplectic submanifold is still not well understood. In general, some global condition must be imposed in order to contract a ruling of $\mathbf{P}^1 \times \mathbf{P}^1$. For example, if there is any other rational curve l homologous to $n[l_0]$ where l represents the ruling we contract, we have to contract l as well. This implies that we have to flop all the rational curves in the same Mori extremal ray. Clearly, the same thing is still true in symplectic category. So, we assume that M, M_f indeed have Kahler structures. Then we obtain the same symplectic manifold M_b (up to deformation equivalence) by blowing up M, M_f along the exceptional curves.

As we mentioned in the introduction, there is a homomorphism

$$\varphi : H_2(M, \mathbf{Z}) \to H_2(M_f, \mathbf{Z}) \tag{2.26}$$

such that ϕ flips the sign of fundamental class of exceptional rational curves. Topologically, it can be understood as follow. For any $A \in H_2(M, \mathbf{Z})$, it is represented by a 2-dimensional pseudo-submanifold Σ of M. Since the exceptional set is a union of finitely many curves, we can perturb Σ so that Σ does not intersect any exceptional curve. Any two different perturbations are pseudo-submanifold cobordant. Moreover, the cobordism can also be pushed off the exceptional curves. Hence, two different perturbations represent the same homologous homology class. Let $\{\Gamma_k\}$ be the set of exceptional curves. We have showed that the inclusion

$$i : M - \cup\Gamma_k \to M \tag{2.27}$$

induces an isomorphism on H_2. The same thing is true for M_f. Moreover, $M - \cup\Gamma_k$ is the same as $M_f - \cup(\Gamma_f)_k$ where $\{(\Gamma_f)_k\}$ is the corresponding set of exceptional rational curves of M_f. Hence, $H_2(M, \mathbf{Z})$ is isomorphic to $H_2(M_f, \mathbf{Z})$ and φ is the isomorphism. Let $\Gamma \subset M$, $\Gamma_f \subset M_f$ be a pair of exceptional curves obtained by blowing down the same exceptional divisor of M_b over different ruling. We claim

$$\varphi([\Gamma]) = -[\Gamma_f]. \tag{2.28}$$

Note that the normal bundle of an exceptional divisor of M_b is $\mathbf{N} = \mathbf{O}_1(-1) \otimes \mathbf{O}_2(-1)$ over $\mathbf{P}^1 \times \mathbf{P}^1$, where $\mathbf{O}_i(-1)$ means the $\mathbf{O}(-1)$ over the i-th factor of $\mathbf{P}^1 \times \mathbf{P}^1$. Let $\tau : \mathbf{P}^1 \to \mathbf{P}^1$ be the antipodal map which reverses the orientation. Then, the restriction of $\mathbf{N}$ over $E = \{(x, \tau(x))\}$ is trivial. One can push E off the $\mathbf{P}^1 \times \mathbf{P}^1$. We obtain the push-off of exceptional curve of M by projecting the perturbation of E to M. By the construction, $\Gamma, -\Gamma_f$ have the same push-off, where $-\Gamma_f$ means the opposite orientation. Hence

$$\phi([\Gamma]) = -[\Gamma_f].$$

It was already observed in [L] that the blow-up along a symplectic submanifold can be viewed as the symplectic cutting. For the reader's convenience, let's construct the hamiltonian S^1-action explicitly. By the symplectic neighborhood theorem, the symplectic structure of a neighborhood of Γ is uniquely determined by the symplectic structure of Γ and the almost complex structure of symplectic normal bundle. Moreover, a symplectic structure of Γ is uniquely determined by its volume. It is enough to construct a specific symplectic structure on the total space of $\mathbf{O}(-1) \oplus \mathbf{O}(-1)$ tamed to the complex structure and having the same volume as that of Γ. For reader's convenience, we sketch the construction.

Choose a hermittian metric on $\mathbf{O}(-1) \oplus \mathbf{O}(-1)$. The S^1-hamiltonian function is given by $||z||^2$, where the S^1 action is given by complex multiplication. We perform the symplectic cutting along hypersurface $N_\epsilon = \{||z||^2 = \epsilon\}$ and obtain two symplectic manifolds $\overline{M}^+$ and $\overline{M}^-$. One is M_b. By the example 2, another one is the compactication of $\mathbf{O}(-1)+\mathbf{O}(-1)$ at infinity by $P(\mathbf{O}(-1)+\mathbf{O}(-1))$, which can be identified as the projectivization $P(\mathbf{O}(-1)+\mathbf{O}(-1)+\mathbf{O})$.

Remark 2.A: *In the case of blow-up over a complex codimension two submanifolds, there is no vanishing 2-cycle. For any 2-cycle $\Sigma \subset M$, using PL-transversality, we can assume that $\Sigma \subset \overline{M}^-$. Hence, Σ defines a homology class $\Sigma_b \in H_2(\overline{M}^-, \mathbf{Z})$. However, $\overline{M}^- = M_b$ and there is a map $R: M_b \to M$. Moreover, $R_*(\Sigma_b) = \Sigma$. Hence, $\Sigma \neq 0$ implies $\Sigma_b \neq 0$.*

Next, we discuss the general type I-transition. Again, we first focus on the case of ODP. Let $U_s \subset M_s$ be a neighborhood of singularity and $U_b \subset M_b$ be the its resolution. We first standardize the symplectic form over $0 \in U_s \subset M_s$. In the standard affine coordinates, Fubini-Study form can be written as

$$\omega_0 = \frac{i}{4\pi} d\left(\frac{\sum_i (x_i d\bar{x}_i - \bar{x}_i dx_i)}{1+|x|^2}\right). \tag{2.29}$$

However, we must choose an analytic change of coordinates to obtain the standard form of the singularity (2.27). Suppose that the change of coordinates are

$$x_i = f_i(z_1, \cdots, z_4), \tag{2.30}$$

where $f_i(0) = 0$ and $(\frac{\partial f_i}{\partial z_j}(0))$ is nondegenerate. We use $L(f_i)$ to denote the linear term of f_i and $L(\omega_0)$ to denote the two form obtained from ω_0 by linear change of coordinates $(\frac{\partial f_i}{\partial z_j}(0))$. Under such a change of coordinates,

$$\omega = L(\omega_0) + d(\tilde{\alpha}), \tag{2.31}$$

where $\tilde{\alpha} = O(|z|^2)(dz_i + d\bar{z}_i)$. Let β_r be a cut-off function which equals to 1 when $|z| > 2r$ and equal to zero when $|z| < r$. Let

$$\omega_r = \omega - d(\beta_r \tilde{\alpha}). \tag{2.32}$$

ω_r is closed by construction and equal to $L(\omega_0)$ when $|z| > 2r$ and equal to ω when $|z| < r$. Moreover,

$$d(\beta_r \tilde{\alpha}) = (d\beta_r)\tilde{\alpha} + \beta_r d\tilde{\alpha} = O(r)(dz_i \wedge d\bar{z}_j). \tag{2.33}$$

Therefore, ω_r is nondegenerate for small r. Furthermore, by Moser's theorem, it is the same symplectic structure as ω on $\mathbf{P}^N$. Hence, we can assume that $\omega = \omega_r$. We use the inverse of

$(\frac{\partial f_i}{\partial z_j}(0))$ to change ω to the standard form ω_0 and the equation to some homogeneous polynomial F_2 of degree 2.

Let S_r be the sphere of radius r. Here we choose r small enough such that $\omega = \omega_0$. Then, S_r is a contact hypersurface of $\mathbf{P}^N$. Let $N = S_r \cap \{F_2 = 0\}$. N is a contact hypersurface of M_s. Consider a S^1-hamiltonian function $H = ||z||^2$, where the S^1-action is given by complex multiplication. Since M_s is smooth near N, we can perform the symplectic cutting to obtain a closed symplectic manifold (aways from the singularity). It is easy to observe that it is precisely M_b.

We can also describe the closed orbits of Reeb vector field on N. It is well-known that the closed orbits on S_r is generated by multiplication $e^{i\theta}$. The space of closed orbits is $\mathbf{P}^3$. Since F_2 is a homogeneous equation, F_2 preserves the complex multiplication.. It is easy to check that the closed orbits of N is also generated by complex multiplication and the space of orbits of N is a quadric surface of $\mathbf{P}^3$. Any quadric surface is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.

To obtain M_t, we have to deform the defining equation of M_s to smooth the singularity. This step can be described by a contact surgery. Let $M_s^- = D_r \cap \{F_2 = 0\}$ where D_r is the ball of radius r and $M_s^+ = \overline{M_s - M_s^-}$. The local smoothing is simply

$$\{F_2 = t\}. \tag{2.34}$$

Let

$$M_{t_0}^- = D^r \cap \{F_2 = t_0\}$$

for small $t_0 \neq 0$. The global smoothing is well-understood by [F], [Ti1] which requires some linear conditions on homology classes of rational curves contracted. Here, we assume that the global smoothing exists and smoothing is M_t. We use the previous construction to decompose

$$M_t = M_t^+ \cup_{id} M_t^-, \tag{2.35}$$

where $M_t^- = D_r \cap M_t$ and M_t^+ is the complement. However, M_t^+ is symplectic deformation equivalent to M_s^+. M_t^- is deformation equivalent to $M_{t_0}^-$. Hence, M_t is symplectic deformation equivalent to

$$M_e = M_s^+ \cup M_{t_0}^-. \tag{2.36}$$

This is the symplectic model which we will use to calculate GW-invariants. Clearly, if we choose r' slightly larger than r. The manifolf M_e also admits a S^1-hamiltonian function near $M_e \cap S^{r'}$.

Then, we can perform the symplectic cutting to obtain two symplectic manifolds. As we observed before, one of them is precisely M_b. Another one is symplectic deformation equivalent to

$$\{F_2 - t_0 x_5^2 = 0\} \subset \mathbf{P}^4.$$

By the previous argument, we can push any 2-dimensional homology class of M off the exceptional loci, and hence off the singular points of M_s. Therefore, we have a map

$$\varphi : H_2(M, \mathbf{Z}) \to H_2(M_e, \mathbf{Z}). \tag{2.37}$$

Clemens showed that $M_{t_0}^-$ is homeomorphic to $S^3 \times D^3$. It doesn't carry any nontrivial 2-dimensional homology class in the interior. Therefore, φ (2.37) is surjective.

Remark 2.B: *The previous argument also shows that there is no vanishing 2-cycle for type I transition.*

In general, the singularities of M_s could be rational double points. In this case, Wilson [Wi2] used a local complex deformation argument to reduce it to the case of ODP. For reader's convenience, let's sketch his argument.

Recall that there are local complex deformations

$$\pi : Y \to \Delta, \bar{\pi} : \bar{Y} \to \Delta, \tag{2.38}$$

where Δ is ball in $\mathbf{C}^k$ and

$$\pi^{-1}(0) = U_s \subset M_s, \bar{\pi}^{-1}(0) = U \subset M.$$

Moreover, $\mu : \bar{Y} \to Y$ is the simultaneous small resolution. Furthermore, there is $t \in \Delta$ such that $\pi^{-1}(t)$ only has ODP singularities and hence $\bar{\pi}^{-1}(t)$ has only rational curves of normal bundle $\mathbf{O}(-1) + \mathbf{O}(-1)$. Furthermore, $\bar{Y}$ is a trivial smooth fibration. Therefore, we can view $\bar{\pi}^{-1}(t)$ as a different complex structure on U. Choose a generic small value t_0. The induced complex structure on U is still tamed and close to the original complex structure. Hence, we can patch it with the complex structure on the complement of U to define a tamed almost complex structure. For this almost complex structure, all the exceptional curves are rational curve with normal bundle $\mathbf{O}(-1) + \mathbf{O}(-1)$. This enables Wilson to compute GW-invariants associated with the exceptional class. It is clear that one can also use this complex structure to construct flop. When we deform U_s to $\pi^{-1}(t)$, we also deform the complex structure on the neighborhood U_f of exceptional curve

Γ_f. In fact, it is isomorphic to that of U. So, we repeat the same construction for M_f to obtain a tamed almost complex structure. They give the same blow-up M_b again.

The symplectic structure of M_b can also be understood by the gluing construction we described previously. First of all, we can choose m large enough to contain all the deformation parameters of Δ. Namely, it is enough to use the local coordinates of $\mathbf{P}^m$ to construct deformations in Friedman's theorem. Let $F = 0$ be the local equation of $U_s \subset \mathbf{C}^m \subset \mathbf{P}^m$. By Friedman's argument, there are holomorphic function g_t such that

$$\{F + g_t = 0\} \cap D_r = \pi^{-1}(t). \tag{2.39}$$

Let $N_t = \{F + g_t = 0\} \cap S^r$. N_t is a contact boundary of $\pi^{-1}(t)$. Fix a generic value t_0 and glue $\pi^{-1}(t_0)$ with $M^+ = M - \pi^{-1}(t_0)$. $\pi^{-1}(t_0)$ contains only ODP. If we blow up the singular points of $\pi^{-1}(t_0)$, we obtain M_b. To obtain smoothing M_e, we cut a disjoint union of small balls around ODP's inside D_r and perform the contact surgery as before. Therefore, we can reduce the proof of the theorem to the case of ODP. To summary our result

Theorem 2.5: *Suppose that V_s is projective with rational double points. We always have a symplectic manifold V_e obtained by gluing with local smoothing of singularities. In the holomorphic case, V_e is symplectic deformation equivalent to holomophic type I transition.*

Recall that a holomorphic type I transition exists iff $\Sigma_i a_i[\Gamma_i] = 0$ for $a_i \neq 0$, where $\{\Gamma_i\}$ is the set of exceptional curves [F], [Ti1].

The situation become considerably easier for type II primitive transiton. In this case, we can also find appropriate local hamiltonian cycle action such that $V_b = V$. The divisor contracted is a appropriate blow-up of singular point of V_s [Gr]. Moreover, $b_2(V^-_{t_0}) = 1$ and the C_1 is the twice of the generator. We will give a detailed descriptiop in a subsequent paper (part II).

Remark 2.6: Finally, we make a remark about general extremal tansitions. Many interesting examples are constructed by nonprimitive extremal transitions. A natural question is if we can decompose an extremal transition as a sequence of primitive transitions. Our theorems actually gives a criterior to this question. After we perform a primitive extremal transition, we need to study the change of Kahler cone. Suppose that E gives a different extremal ray. Then, it is easy to observe that E will remain to be an extremal ray if its GW-invariants are not zero. Our theorem shows that the GW invariant can be calculated using the GW-invariant of Calabi-Yau 3-fold before the transition. Moreover, GW-invariants of a crepant resolution depend only on a neighborhood of exceptional loci and can be calculated indepedently from Calabi-Yau 3-folds.

Remark 2.7: We would like to make another remark about the smoothing of singular Calabi-Yau 3-folds. One can study the smoothing theory by studying the local smoothings of singularities and the global extension of a local smoothing. The singular CY-3-folds obtained by contracting a smooth CY-3-fold can only has so called canonical singularities.

The smoothing theory of general singular CY-3-folds is under intensive study. There are many interesting phenomena in the smoothing theory of CY-3-folds with canonical singularities. For examples, M. Gross showed that the same singular CY-3-fold could have two different smoothings [Gro1]. They give a pair of examples where two diffeomorphic CY-folds carries different quantum cohomology structures [R1]. By our arguments, it is clear that if a singular Calabi-Yau 3-fold has a local smoothing for its singularity, it has a global symplectic smoothing. This is not true in algebro-geometric category by Friedman's results. This indicates an exciting possibility that there is perhaps a theory of symplectic Calabi-Yau 3-folds, which is broader than algebro-geometric theory. Such a theory would undoubtedly be important to the classification of CY-3-folds itself.

3 Convergence to Periodic Orbits

Let $(\widetilde{M}, \lambda)$ be a $(2n+1)$-dimensional compact manifold equipped with a contact form λ. We recall that a contact form λ is a 1-form on $\widetilde{M}$ such that $\lambda \bigwedge (d\lambda)^n$ is a volume form. Associated to $(\widetilde{M}, \lambda)$ we have the contact structure $\xi = \ker(\lambda)$, which is a $2n$-dimensional subbundle of $T\widetilde{M}$, and $(\xi, d\lambda|_\xi)$ defines a symplectic vector bundle. Furthermore, there is a unique nonvanishing vector field $X = X_\lambda$, called the Reeb vector field, defined by the condition

$$i_X \lambda = 1, \quad i_X d\lambda = 0.$$

We have a canonical splitting of $T\widetilde{M}$,

$$T\widetilde{M} = \mathbf{R}X \oplus \xi,$$

where $\mathbf{R}X$ is the line bundle generated by X. We choose a compatible complex structure $\widetilde{J}$ for the symplectic vector bundle $(\xi, d\lambda) \to \widetilde{M}$ such that

$$g_{\widetilde{J}(x)}(h,k) = d\lambda(x)(h, \widetilde{J}(x)k), \tag{3.1}$$

for all $x \in \widetilde{M}$, $h, k \in \xi_x$, defines a smooth fibrewise metric for ξ. Denote by $\Pi : T\widetilde{M} \to \xi$ the projection along X. We define a Riemannian metric $\langle \, , \, \rangle$ on $\widetilde{M}$ by

$$\langle h, k \rangle = \lambda(h)\lambda(k) + g_{\widetilde{J}}(\Pi h, \Pi k) \tag{3.2}$$

for all $h, k \in T\widetilde{M}$. For $p, q \in \widetilde{M}$, let

$$\Omega_{pq} = \{\gamma \in C^\infty([0,1], \widetilde{M}) | \gamma(0) = p,\ \gamma(1) = q\}. \tag{3.3}$$

We define a function $\widetilde{d} : \widetilde{M} \times \widetilde{M} \to \mathbf{R}$ by

$$\widetilde{d}(p,q) = \inf_{\gamma \in \Omega_{pq}} \left\{ \int_0^1 \langle \Pi\dot{\gamma}, \Pi\dot{\gamma} \rangle^{\frac{1}{2}} dt \right\}. \tag{3.4}$$

It is easy to see that $\widetilde{d}(\, , \,)$ satisfies

i) $\widetilde{d}(p,q) \geq 0$

ii) $\widetilde{d}(p,q) = \widetilde{d}(q,p)$

iii) $\widetilde{d}(p,r) \leq \widetilde{d}(p,q) + \widetilde{d}(q,r)$

iv) $\widetilde{d}(p,q) = 0 \Longleftrightarrow p, q$ lie in an integral curve of the Reeb vector field.

Given a $\widetilde{J}$ as above there is an associated almost complex structure J on $\mathbf{R} \times \widetilde{M}$ defined by

$$J(a, \widetilde{u})(h,k) = (-\lambda(\widetilde{u})(k), \widetilde{J}(\widetilde{u})(\Pi k) + h\, X(\widetilde{u})) \tag{3.5}$$

for all $(h,k) \in T(\mathbf{R} \times \widetilde{M})$. Let (Σ, i) be a compact Riemannian surface and $P \subset \Sigma$ be a finite collection of points. Denote $\overset{\circ}{\Sigma} = \Sigma \backslash P$. Let $u : \overset{\circ}{\Sigma} \to \mathbf{R} \times \widetilde{M}$ be a J-holomorphic curve, i.e., u satisfies

$$du \circ i = J \circ du. \tag{3.6}$$

Following [HWZ1] we impose an energy condition on u. Let $\delta_1 < \delta_2$ be two real numbers and Φ be the set of all smooth functions $\phi : \mathbf{R} \to [\delta_1, \delta_2]$ satisfying

$$\begin{aligned} &\phi' > 0 \\ &\phi(a) \to \delta_2 \quad \text{as } a \to \infty \\ &\phi(a) \to \delta_1 \quad \text{as } a \to -\infty. \end{aligned}$$

For any $\phi \in \Phi$ we equip the tube $\mathbf{R} \times \widetilde{M}$ with a symplectic form $d(\phi\lambda)$. It is easy to see that J is compatible with the symplectic form $d(\phi\lambda)$. We require that

$$Sup_{\phi \in \Phi} \left\{ \int_{\overset{\circ}{\Sigma}} u^* d(\phi\lambda) \right\} < \infty \tag{3.7}$$

and call such a map a finite energy J-holomorphic curve. We shall see later that the condition is natural in view of our surgery.

We are interested in the behaviors of the finite energy J-holomorphic curves near a puncture. As is shown in [HWZ2], there are two different types of puncture : the removable singularities and the non-removable singularities. If u is bounded near a puncture then this puncture is a removable singularity. In the following we assume that all punctures in P are non-removable. Then u is unbounded near the punctures. Hofer, Wysocki and Zehnder [HWZ1] proved the following results.

Theorem 3.1 *Assume $u = (a, \widetilde{u}) : \mathbf{C} \to \mathbf{R} \times \widetilde{M}$ is a nonconstant finite energy J-holomorphic curve. Then there exists a sequence $R_i \to \infty$ such that $\lim_{i\to\infty} \widetilde{u}(R_i e^{2\pi it}) = x(Tt)$ in $C^\infty(S^1, \widetilde{M})$ for a T-periodic solution $x(t)$ of the Reeb vector field $\dot{x}(t) = X(x(t))$. If this solution is nondegenerate then*

$$\lim_{R\to\infty} \widetilde{u}(R\, e^{2\pi it}) = x(Tt), \tag{3.8}$$

in $C^\infty(S^1, \widetilde{M})$.

In this section we shall generalize Theorem 3.1 to periodic solutions of Bott type.

3.1 The first and second variational formulas.

Let $C^\infty(S^1, \widetilde{M})$ be the space of C^∞-loops in $\widetilde{M}$ which are represented by smooth curves

$$\gamma : S^1 \to \widetilde{M}.$$

The tangent space $T_\gamma C^\infty(S^1, \widetilde{M})$ is the space of vector fields $\eta \in C^\infty(\gamma^* T\widetilde{M})$ along γ. There is a natural Riemannian metric (,) on $C^\infty(S^1, \widetilde{M})$ defined by

$$(\eta_1, \eta_2) := \int_{S^1} \langle \eta_1, \eta_2 \rangle dt \tag{3.9}$$

for all $\eta_1, \eta_2 \in T_\gamma C^\infty(S^1, \widetilde{M})$. We consider the action functional $\mathcal{A} : C^\infty(S^1, \widetilde{M}) \to \mathbf{R}$ defined by

$$\mathcal{A}(\gamma) = \int_{S^1} \gamma^* \lambda. \tag{3.10}$$

The gradient vector $grad\mathcal{A}(\gamma)$ in $T_\gamma C^\infty(S^1, \widetilde{M})$ is defined as a representation of the 1-form $d\mathcal{A}(\gamma)$, i.e.,

$$(grad\ \mathcal{A}(\gamma), \eta) = d\mathcal{A}(\gamma)\eta \quad \forall \eta \in T_r C^\infty(S^1, \widetilde{M}). \tag{3.11}$$

Suppose γ_p is a smooth curve in $C^\infty(S^1, \widetilde{M})$, with $\gamma_0 = \gamma$. Then

$$\eta = \left(\frac{d\gamma_p}{dp}\right)_{p=0}$$

is a vector field along γ. We have the first variational formula:

$$d\mathcal{A}(\gamma)\eta = \left.\frac{d\mathcal{A}(\gamma_p)}{dp}\right|_{p=0} = -\int_{S^1} d\lambda(\dot{\gamma}, \eta)dt = \int_{S^1} \langle \Pi\dot{\gamma}, \widetilde{J}\Pi\eta\rangle dt \tag{3.12}$$

where $\dot{\gamma} := \frac{d\gamma}{dt}$. So the condition for γ to be a critical point of A is that $\dot{\gamma}$ is parallel to X everywhere. Let $x(t)$ be a T-periodic solution of $\dot{x} = X(x)$, then $x_T(t) := x(Tt) \in C^\infty(S^1, \widetilde{M})$ is a critical point of $\mathcal{A}$. The period T need not be the minimal period of $x(t)$. Let κ be the positive minimal period. Then $T = k\kappa$ for some integer k, and $\langle \dot{x}_T, X\rangle = T$.

We now calculate the second variation for $\mathcal{A}$. Consider a 2-parameter variation (t, w, v) of γ, i.e., a smooth map $\alpha : Q \to \widetilde{M}$ such that $\alpha(t, 0, 0) = \gamma(t)$, where $Q = S^1 \times (-\epsilon, \epsilon) \times (-\delta, \delta)$. Let τ, η, ζ be vector fields corresponding to the first, second and third variable of α, respectively. Note that $\tau(t, 0, 0)$ is parallel to X. Put

$$\mathcal{A}(w, v) = \int_{S^1} \alpha^*\lambda.$$

Then

$$\frac{d\mathcal{A}}{\partial w} = \int_{S^1} \langle \Pi\tau, \widetilde{J}\Pi\eta\rangle dt$$

$$\left.\frac{\partial^2\mathcal{A}}{\partial v\partial w}\right|_{(0,0)} = \int_{S^1} \langle \nabla_\zeta(\Pi\tau), \widetilde{J}\Pi\eta\rangle dt$$

where ∇ denotes the covariant derivative with respect to the Levi-Civita connection of $\langle\ ,\ \rangle$ on $\widetilde{M}$. Here we have used the fact that $d\lambda(\dot{\gamma}, \cdot) = 0$. Denote $\zeta^\perp := \Pi\zeta$, the normal component of ζ, and $T_\gamma^\perp := \{\zeta | \langle \zeta, \dot{\gamma}\rangle = 0\}$. Note that at $(t, 0, 0)$

$$\begin{aligned}\nabla_\zeta(\Pi\tau) &= \nabla_\zeta\tau - \nabla_\zeta(\langle\tau, X\rangle X) \\ &= \nabla_\tau\zeta - \langle\tau, X\rangle\nabla_\zeta X \mod X \\ &= \nabla_\tau\zeta^\perp - \langle\tau, X\rangle\nabla_{\zeta^\perp} X \mod X.\end{aligned} \tag{3.13}$$

Define the linear transformation $S : T_\gamma^\perp \to T_\gamma^\perp$ by $\zeta \to \widetilde{J}\nabla_\zeta X$. We get the following variational formula:

$$\left.\frac{\partial^2\mathcal{A}}{\partial v\partial w}\right|_{(0,0)} = -\int_{S^1} \langle (\widetilde{J}\Pi\frac{D}{dt} + \langle\dot{\gamma}, X\rangle S)\zeta^\perp, \eta^\perp\rangle dt \tag{3.14}$$

where D/dt denotes the covariant derivative along γ. It is easy to show that the second variation formula is a symmetric bilinear form on the space of smooth vector fields along γ, denoted by $I(\zeta, \eta)$. Let $P = -\widetilde{J}\Pi\frac{D}{dt} - \langle\dot{\gamma}, X\rangle S$. We have

$$I(\zeta, \eta) = (P\zeta, \eta). \tag{3.15}$$

Remark 3.2 *Let $\varphi : S^1 \to S^1$ be a diffeomorphism . Then $x_T \circ \varphi$ is also a critical point of $\mathcal{A}$, and the second variational formula (3.14) remains valid.*

Now we consider a smooth map $\alpha : S^1 \times (-\epsilon, \epsilon) \to \widetilde{M}$ such that $\alpha(t, 0) = \gamma(t)$ and such that at every point (t, w), $\Pi\tau = 0$, i.e, for every w, $\alpha(\cdot, w)$ is a critical point of $\mathcal{A}$. It follows from (3.13) that ξ satisfies the following equation:

$$\nabla_\tau \zeta^\perp - \langle\tau, X\rangle \nabla_{\zeta^\perp} X = 0 \mod X. \tag{3.16}$$

With the usual C^∞-topology $C^\infty(S^1, \widetilde{M})$ is not a Banach manifold. Hence we shall consider the Sobolev space $W^2_r(S^1, \widetilde{M})$, where r is a large positive integer. It is easy to see that $\mathcal{A}$ is a C^∞ function on $W^2_r(S^1, \widetilde{M})$ and the first and the second variation formulas (3.12),(3.14) hold on $W^2_r(S^1, \widetilde{M})$. Let $S_T = \{x_T(t)|x(t)$ is a T-periodic solution of $\dot{x}(t) = X(x(t))\}$. The proof of the following proposition is standard.

Proposition 3.3 S_T is compact.

3.2 Exponential decay

In order to generalize Theorem 3.1, we introduce the notion of Bott-type periodic solutions. Let $x_T(t)$ be the loop which corresponds to a T-periodic solution $x(t)$ of the Reeb vector field X. We choose an ϵ-ball $O_{x,\epsilon}$ of 0 in the Hilbert space $T_{x_T}W^2_r(S^1, \widetilde{M})$ such that the exponential map

$$\exp : O_{x,\epsilon} \to W^2_r(S^1, \widetilde{M})$$

identifies $O_{x,\epsilon}$ with a neighbourhood of x_T in $W^2_r(S^1, \widetilde{M})$. Since $x_T : S^1 \to \widetilde{M}$ is an immersion and the points in $O_{x,\epsilon}$ are near x_T with respect to the norm W^2_r, we can assume that for any $y \in O_{x,\epsilon}$, $y : S^1 \to \widetilde{M}$ is an immersion. Let

$$T^\perp_{x_T} = \{\eta \in T_{x_T}W^2_r(S^1, \widetilde{M})|\eta \perp \dot{x}_T\}.$$

Then $T_{x_T}^{\perp}$ is a closed subspace of $T_{x_T}W_r^2(S^1, \widetilde{M})$. Let $C_r(\mathcal{A})$ denote the set of critical points of $\mathcal{A}$.

Definition 3.4 *We say S_T is of Bott-type at x_T if*

1) $C_r(\mathcal{A}) \bigcap O_{x,\epsilon} \bigcap T_{x_T}^{\perp}$ is a smooth submanifold in $T_{x_T}^{\perp}$,

2) The restriction of the index $I(\ ,\ )$ to the normal direction of the submanifold at x_T in $T_{x_T}^{\perp}$ is nondegenerate.

S_T is said to be of Bott-type if it is of Bott-type at every point $x_T \in S_T$.

Remark 3.5 *A nondegenerate periodic solution can be viewed as a special Bott-type periodic solution.*

Proposition 3.6 *Let $x(t)$ be a T-periodic solution of $\dot{x}(t) = X(x(t))$. Suppose that S_T is of Bott-type at x_T. Then there exists a neighbourhood O of $x_T(t+c)$, $0 \le c \le 1$, in $W_r^2(S^1, \widetilde{M})$ and a constant $C > 0$ such that the inequality*

$$\|\nabla \mathcal{A}\|_{L^2(S^1)} \geq C|\mathcal{A} - T|^{\frac{1}{2}} \tag{3.17}$$

holds in O.

Proof: We abbreviate

$$H = W_r^2({x_T}^* T\widetilde{M}),\ Q = T_{x_T}^{\perp}$$

and identify an ϵ-ball O_ϵ of 0 in H with a neighbourhood of x_T in $W_r^2(S^1, \widetilde{M})$. We first prove that (3.17) holds in a neighbourhood of 0 in Q. Denote $K = C_r(\mathcal{A}) \cap O_\epsilon \cap Q$. By definition K is a smooth submanifold in Q, and on K

$$\mathcal{A} = T$$

$$\|\nabla \mathcal{A}\|_{L^2(S^1} = 0.$$

Moreover, for any $x \in K$, denoting by N_x the normal space of K in Q, the restriction of $I(\ ,\ )$ to N_x is nondegenerate. Using the Morse lemma with parameters we can find a neighbourhood $O_{\epsilon'} \subset O_\epsilon$ of 0 such that the following holds: there are diffeomorphisms $\varphi_x : N_x \to N_x$, which depend continuously on $x \in K \cap O_{\epsilon'}$, such that under the diffeomorphisms the function $\mathcal{A}|_{N_x}$ has the form

$$\mathcal{A}(y) - T = (P'(x)y, y)$$

where P' denotes the restriction of $P(x)$ to N_x. Since φ_x is under control it suffices to prove the inequality for the quadratic function $\mathcal{A}(y) - T = (P'y, y)$. By the nondegeneraty of $I(\ ,\ )$ we can find a constant $C_2 > 0$ such that $|\lambda_i| \geq C_2$ for all $x \in K \cap O_{\epsilon'}$ and all eigenvalues λ_i. Then

$$\|\nabla \mathcal{A}\|_{L^2(S^1)} \geq \|\nabla_y \mathcal{A}\|_{L^2(S^1)} = 2(P'(x)y, P'(x)y)^{\frac{1}{2}}$$

$$\geq \sqrt{C_2}(P'(x)y, y)^{\frac{1}{2}} = \sqrt{C_2}|\mathcal{A} - T|^{\frac{1}{2}}.$$

Now let $\eta \in H$ be a vector field along x_T such that $\|\eta\|_r$ is very small. For any $t \in S^1$ we can find a unique $t' \in S^1$ and a vector $\eta' \in T^{\perp}_{x_T(t')}\widetilde{M}$ such that

$$\exp_{x_T(t)} \eta = \exp_{x_T(t')} \eta'.$$

This induces a map $\varphi : S^1 \to S^1,\ t \longmapsto t'$. When $\|\eta\|_r$ is very small φ is a diffeomorphism with $\varphi' \approx 1$. Since $\mathcal{A}$ is invariant under the diffeomorphism group $S^1 \to S^1$ we can find a neighbourhood O of $x_T(t+c),\ \ 0 \leq c \leq 1$, in $W^2_r(S^1, \widetilde{M})$ such that (3.17) holds. □

Theorem 3.7 *Assume $u = (a, \widetilde{u}) : \mathbf{C} \to \mathbf{R} \times \widetilde{M}$ is a J-holomorphic curve with finite energy. Then there exists a sequence $s_i \to \infty$ such that $\widetilde{u}(s_i, t) \to x(Tt)$ in $C^\infty(S^1, \widetilde{M})$ for a T-periodic solution $x(t)$ of $\dot{x}(t) = X(x(t))$. If, in addition, S_T is of Bott type then*

$$lim_{s\to\infty}\widetilde{u}(s,t) = x(Tt).$$

The convergence is to be understood as exponential decay uniformly in t.

Proof: The first part has been proved in [HWZ1]. We prove the second part. Denote

$$\widetilde{E}(s) = \int_s^\infty \int_{S^1} \widetilde{u}^* d\lambda.$$

Then

$$\widetilde{E}(s) = \int_s^\infty \int_{S^1} |\Pi \widetilde{u}_t|^2 ds dt,$$

(3.18)
$$\frac{d\widetilde{E}(s)}{ds} = -\int_{S^1} |\Pi \widetilde{u}_t|^2 dt.$$

By the Stokes theorem we obtain

(3.19)
$$\widetilde{E}(s) - \widetilde{E}(s') = |\mathcal{A}(s) - \mathcal{A}(s')|$$

where $s < s'$ and $\mathcal{A}(s) := \mathcal{A}(\widetilde{u}(s,\cdot))$. There follows the existence of the limit $lim_{s\to\infty}\mathcal{A}(s)$. Let s_i be a sequence such that $\widetilde{u}(s_i, t) \to x(Tt)$ for some T-periodic orbit $x(t)$. Let $i \to \infty$, then

(3.20)
$$\widetilde{E}(s) = |\mathcal{A}(s) - T|.$$

By Proposition 3.6 and compactness of S_T , there is a neighbourhood O of S_T in which the inequality (3.17) holds. We show that there is $N > 0$ such that if $s > N$ then $\widetilde{u}(s, \cdot) \in O$. If not, we could find a sequence $s_i \to \infty$ such that $\widetilde{u}(s_i, \cdot) \notin O$. By Hofer's result we can choose a subsequence, still denoted by s_i, such that

$$\widetilde{u}(s_i, t) \to x'(Tt) \qquad \text{in } C^\infty(S^1, \widetilde{M})$$

for some T-periodic solution $x'(t)$. This is a contradiction. So the inequality (3.17) holds for $\widetilde{u}(s, \cdot), s > N$. We first prove the exponential decay of $\widetilde{E}(s)$. If there is some s_0 such that $\widetilde{E}(s_0) = 0$, then $|\Pi\widetilde{u}_t|^2 = |\Pi\widetilde{u}_s|^2 = 0 \quad \forall s \geq s_0$. In this case we are done. In the following we assume that $\widetilde{E}(s) \neq 0 \;\; \forall s > N$. For $s > s_0 > N$ we have

$$\frac{d\widetilde{E}(s)}{ds} = -\int_{S^1} |\Pi\widetilde{u}_t|^2 dt = -\| \bigtriangledown \mathcal{A}(s)\|^2_{L^2(S^1)} \leq -C^2|\mathcal{A}(s) - T| \leq -C^2\widetilde{E}(s) \tag{3.21}$$

$$\frac{d\widetilde{E}(s)}{ds} \leq -C\|\Pi\widetilde{u}_t\|_{L^2(S^1)}\widetilde{E}(s)^{\frac{1}{2}}. \tag{3.22}$$

It follows that

$$\widetilde{E}(s) \leq \widetilde{E}(s_0)e^{-C^2(s-s_0)} \tag{3.23}$$

$$\int_s^{s_i} \|\Pi\widetilde{u}_t\|_{L^2(S^1)} ds \leq \frac{1}{C}\left(\widetilde{E}(s)^{1/2} - \widetilde{E}(s_i)^{1/2}\right) \tag{3.24}$$

for all $s_i \geq s \geq N$. Then

$$\begin{aligned} \int_{S^1} \widetilde{d}(\widetilde{u}(s,t), \widetilde{u}(s_i,t))dt &\leq \int_s^{s_i} \|\Pi\widetilde{u}_t\|_{L^2} ds \\ &\leq \frac{1}{C}\left(\widetilde{E}(s)^{1/2} - \widetilde{E}(s_i)^{1/2}\right) \leq \frac{1}{C}\left(\widetilde{E}(s) - \widetilde{E}(s_i)\right)^{1/2}. \end{aligned} \tag{2.25}$$

Taking the limit $i \to \infty$ we get

$$\int_{S^1} \widetilde{d}(\widetilde{u}(s,t), x(Tt))dt \leq \frac{1}{C}\widetilde{E}(s)^{1/2}. \tag{3.26}$$

Let $s_k \to \infty$ be an arbitary sequence such that $\widetilde{u}(s_k, t)$ converges to a T-periodic orbit $x'(Tt)$. Then, from (3.26) we obtain

$$\int_{S^1} \widetilde{d}(x'(Tt), x(Tt))dt = 0.$$

It follows that

$$x'(Tt) = x(Tt + \theta_0)$$

for some constant θ_0. We choose local Darboux coordinate system $(\theta, x_1, \cdots, x_n, y_1, \cdots, y_n)$ near $x(t)$ (see [HWZ1]) such that

$$\lambda = f\lambda_0, \quad \lambda_0 = d\theta + \sum x_i dy_i \tag{3.27}$$

and

$$f(\theta, 0, 0) = \kappa, \quad df(\theta, 0, 0) = 0. \tag{3.28}$$

Denote $w = (x, y)$. From above discuss we conclude that for s large enough, $\widetilde{u}(s,t)$ lies in a Darboux coordinate domain of $x(Tt)$. We write

$$u(s,t) = (a(s,t), \theta(s,t), w(s,t)).$$

Then $a(s,t), \theta(s,t), w(s,t)$ satisfy the following equations (see [HWZ1]):

$$a_s = (\theta_t + \sum x_i y_{it})f \tag{3.29}$$

$$a_t = -(\theta_s + \sum x_i y_{is})f \tag{3.30}$$

$$w_s + J_0 w_t + S(s,t)w = 0. \tag{3.31}$$

The following estimates are derived in [H] , [HWZ1]:

$$sup_{\mathbf{R}\times\widetilde{M}} |\partial^r \widetilde{u}(s,t)| \leq \infty, \ \ |r| \geq 1, \tag{3.32}$$

$$sup_{\mathbf{R}\times\widetilde{M}} |\partial^r S(s,t)| \leq \infty, \ \ \forall r, \tag{3.33}$$

$$|\partial^r [a(s,t) - Ts]| \to 0, \ \ |r| \geq 1 \tag{3.34}$$

$$|\partial^r [\theta(s,t) - kt]| \to 0, \ \ |r| \geq 1 \tag{3.35}$$

$$|\partial^r w(s,t)| \to 0, \ \ \forall r. \tag{3.36}$$

The convergence is to be understood as uniformly in t. At the points $(\theta, 0, 0)$ the contact plane is the w-plane and the metric $g_{\widetilde{J}}$ is just the Euclidean metric. Denote by Π_0 the projection into w-plane along X. Since $\xi_{(\theta,w)} \to \xi_{(\theta,0)}$ as $w \to 0$, we can find a constant $C_1 > 0$ such that

$$< \Pi v, \Pi v > \geq C_1 < \Pi_0 v, \Pi_0 v >, \tag{3.37}$$

for $|w|$ small enough and for all $v \in \mathbf{R}^{2n+1}$. Then (3.26) together with (3.37) give the following estimate for $\|w\|_{L^2(S^1)}$:

$$\|w(s,t)\|_{L^2(S^1)} \leq \frac{1}{C}\widetilde{E}(s)^{1/2}. \tag{3.38}$$

Then, by using (3.38) and the same argument as in [HWZ1] one can show that the meanvalues of the periodic function $\theta(s,t) - kt$ over S^1 satisfies the Cauchy condition. It follows that there is constant θ_0 such that

$$|\partial^r[\theta(s,t) - kt - \theta_0] \to 0 \tag{3.39}$$

as $s \to \infty$. Thus we have

$$lim_{s\to\infty}\widetilde{u}(s,t) = x(Tt).$$

Futhermore, using the estimate (3,38) and the same argument in [HWZ1] we can show that there is constants $\delta > 0$, ℓ_0 such that for all $r = (r_1, r_2) \in \mathbf{Z^2}$

$$|\partial^r[a(s,t) - Ts - \ell_0]| \leq c_r e^{-\delta|s|} \tag{3.40}$$

$$|\partial^r[\theta(s,t) - kt - \theta_0]| \leq c_r e^{-\delta|s|} \tag{3.41}$$

$$|\partial^r w(s,t)| \leq c_r e^{-\delta|s|} \tag{3.42}$$

where c_r are constants. $\square$

Remark 3.8 *We can also use standard elliptic estimates on the band $(s, s+1) \times S^1$ and the exponential decay (3.38) to derive the exponential estimates (3.40)-(3.42).*

4 Compactness Theorems

In this section we shall prove a compactness theorem for stable holomorphic curves in a symplectic manifold with cylindrical end. We shall also prove a convergence theorem for stable holomorphic curves when a closed symplectic manifold is stretched to infinity along a contact hypersurface. There have been various compactness theorems proved (see [RT1],[FO],[MS],[Ye]). In our situation the manifold is not compact, so we have to analyse the behaviour of sequences of holomorphic curves at infinity. For simplicity, in this section we will assume that all periodic orbits are contractible. We remark that if we don't assume the contractibility of periodic orbits, our compactness theorems and convergence theorem remain valid.

4.1 Almost complex manifolds with cylindrical end

Let (M,ω) be a compact symplectic manifold, and $(\widetilde{M},\lambda)$ a compact contact hypersurface of (M,ω). We assume that $\widetilde{M}$ divides M into two parts M^+ and M^-. In a neighborhood of $\widetilde{M}$ we can express ω as the form

$$d(e^t\lambda), \qquad -\delta < t < \delta.$$

We can write

$$M^+ = M_0^+ \bigcup \left\{[-\delta, 0) \times \widetilde{M}\right\},$$
$$M^- = M_0^- \bigcup \left\{(0, \delta] \times \widetilde{M}\right\},$$
$$M = M_0^+ \bigcup \left\{[-\delta, \delta] \times \widetilde{M}\right\} \bigcup M_0^-$$

where M_0^+ and M_0^- are compact manifolds with boundary. For any $r > 0$ we stretch the tube

$$\left\{[-\delta, \delta] \times \widetilde{M}\right\}$$

to

$$\left\{[0, 4r] \times \widetilde{M}\right\}$$

through a diffeomorphism. Let Φ_r be the set of smooth functions $\phi_r : [0,4r] \to [-\delta, \delta]$ satisfying

$$\phi_r' > 0, \quad \phi_r(0) = -\delta, \quad \phi_r(2r) = 0, \quad \phi_r(4r) = \delta.$$

Choose any function $\phi_r \in \Phi_r$, which gives a diffeomorphism

$$\phi_r : [0, 4r] \times \widetilde{M} \to [-\delta, \delta] \times \widetilde{M}.$$

Put

$$M_r = M_0^+ \bigcup \left\{[0, 4r] \times \widetilde{M}\right\} \bigcup M_0^-$$
$$M_r^+ = M_0^+ \bigcup \left\{[0, 2r) \times \widetilde{M}\right\}$$
$$M_r^- = \left\{(2r, 4r] \times \widetilde{M}\right\} \bigcup M_0^-.$$

Then ϕ_r extends naturally to a diffeomorphism $\phi_r : M_r \to M$. Let $\omega_{\phi_r} = \phi_r^*\omega$. Then $\phi_r : (M_r, \omega_{\phi_r}) \to (M, \omega)$ is a symplectic diffeomorphism. Denote by a^+ the coordinate on $\mathbf{R}$ such that ∂M_0^+ corresponds to $\{0\} \times \widetilde{M}$. Now let $r \to \infty$; the limit of M_r breaks into two open manifolds $M^+ \bigcup M^-$. We may consider M_∞ to be a smooth manifold glued from M^+ and M^- along $widetildeM$ at $+\infty$ and $-\infty$. By introducing a new coordinate

$$a^- = a^+ - 4r \tag{4.1}$$

we may consider M_r^- as

$$M_r^- = \left\{(-2r, 0] \times \widetilde{M}\right\} \bigcup M_0^- .$$

Let $r \to \infty$. We have

$$M_r^+ \to M^+ := M_0^+ \bigcup \left\{[0, \infty) \times \widetilde{M}\right\}$$

$$M_r^- \to M^- = \left\{(-\infty, 0] \times \widetilde{M}\right\} \bigcup M_0^- .$$

Moreover, if we choose the origin on the tube, we obtain $\mathbf{R} \times \widetilde{M}$ in the limit. Each piece is obviously diffeomorphic to an open subset of M. We fix a small $\delta_0 < \delta$, and put

$$\Phi^+ = \{\phi^+ : [1, \infty) \to [-\delta_0, 0) | (\phi^+)' > 0, \phi^+(1) = -\delta_0, \phi^+(a^+) \to 0 \ \ as \ \ a^+ \to \infty\} . \tag{4.2}$$

$$\Phi^- = \{\phi^- : (-\infty, -1] \to (0, \delta_0] | (\phi^-)' > 0, \phi^-(-1) = \delta_0, \phi^-(a^-) \to 0 \ \ as \ \ a^- \to -\infty\} . \tag{4.3}$$

We may assume that for any $R > 0$

$$\phi_r(a^+)|_{[1,R]} \to \phi^+, \quad \phi_r(a^-)|_{[-R,-1]} \to \phi^-$$

for some $\phi^+ \in \Phi^+, \quad \phi^- \in \Phi^-$ uniformly. We will fix ϕ_r and $\phi^\pm$. Let

$$\omega_{\phi^\pm} = (\phi^\pm)^* \omega. \tag{4.4}$$

We choose a compatible complex structure $\widetilde{J}$ for the symplectic vector bundle $(\xi, d\lambda) \to \widetilde{M}$, then we have a natural almost complex structure J on $\mathbf{R} \times \widetilde{M}$ such that

$$J \frac{\partial}{\partial a} = X, \quad JX = -\frac{\partial}{\partial a} \tag{4.5}$$

$$J\xi = \xi, \quad J|_\xi = \widetilde{J}. \tag{4.6}$$

It is obvious that J is compatible with the symplectic forms $d(e^{\phi^\pm}\lambda)$ for any $\phi^\pm$ over the tube.

Definition 4.1 *An almost complex structure J on $M^\pm$ (resp.M_r) is said to be S-compatible with $\omega_{\phi^\pm}$ (resp. ω_{ϕ_r}) if*
i) J is compatible with $\omega_{\phi^\pm}$ (resp. ω_{ϕ_r}) in the usual sense,
ii) Over the tube $\mathbf{R} \times \widetilde{M}$, (4.5) and (4.6) hold.

It is easy to show the existence of S-compatible almost complex structures.

Let J be an S-compatible almost complex structure on $M^\pm$. Then

$$\langle v, w \rangle_{\omega_{\phi^\pm}} = \omega_{\phi^\pm}(v, Jw), \qquad \forall \ \ v, w \in TM^\pm \tag{4.7}$$

defines a Riemannian metric on $M^{\pm}$. Note that $\langle\ ,\ \rangle_{\omega_{\phi^{\pm}}}$ is not complete. We choose another metric $\langle\ ,\ \rangle$ on $M^{\pm}$ such that

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle_{\omega} \quad on \ \ M_0^{\pm}$$

and over the tubes $(-\infty,-1)\times\widetilde{M}$ and $(1,\infty)\times\widetilde{M}$

$$\langle(a,v),(b,w)\rangle = \Big(ab+\lambda(v)\lambda(w)+g_{\widetilde{J}}(\Pi v,\Pi w)\Big)\,. \tag{4.8}$$

It is easy to see that $\langle\ ,\ \rangle$ is a complete metric on $M^{\pm}$ and there is a constant C such that

$$\langle v,w\ \rangle_{\omega_{\phi^{\pm}}} \leq C\langle v,w\ \rangle, \quad \forall v,w\in TM^{\pm}. \tag{4.9}$$

For any J-holomorphic curve $u:\Sigma\to M^{\pm}$ the energy $E(u)_{\phi^{\pm}}$ is defined by

$$E(u)_{\phi^{\pm}} = \int_{\Sigma} u^*\omega_{\phi^{\pm}}. \tag{4.10}$$

The notion of the finite energy J-holomorphic curves is extended to the maps $u:\Sigma\to M^{\pm}$ in a natural way.

Remark 4.2 There is another way of looking at the stretching: We leave (M,ω) invariant, and change the almost complex structure $J_r=\phi_r^*J$ on (M,ω). Then we can view M_∞ and M as the same symplectic manifold. Note that the almost complex structure J_∞ degenerates along the contact hypersurface $\widetilde{M}$.

Let $\mathcal{M}_{g,m}$ be the moduli space of Riemann surfaces of genus g and with m marked points, and $\overline{\mathcal{M}}_{g,m}$ its Deligne-Mumford compactification. Then $\overline{\mathcal{M}}_{g,m}$ consists of all stable curves of genus g and with m marked points. It is well-known that $\overline{\mathcal{M}}_{g,m}$ is a Kahler orbifold.

Let $(\Sigma;y_1,...,y_m,p_1,...,p_\nu)\in\mathcal{M}_{g,m+\nu}$, $\overset{\circ}{\Sigma}=\Sigma-\{p_1,...,p_\nu\}$. $(\overset{\circ}{\Sigma};\mathbf{y},\mathbf{p})$ is a Riemann surface of genus g with m marked points $\mathbf{y}=\{y_1,...,y_m\}$ and ν ends $\mathbf{p}=\{p_1,...,p_\nu\}$. Let $u:\overset{\circ}{\Sigma}\to M^+$ be a finite energy J-holomorphic curve. Suppose that $u(z)$ converges to a T_i-periodic orbit x_i of the Reeb vector field with $T_i>0$ as z tends to p_i. Denote by S_{T_i} the set of T_i-periodic orbits. We call u a J-holomorphic curve with $\mathbf{T}$-ends. For each puncture point p_i we choose a holomorphic cylindrical coordinate system $e^{2\pi(s+it)}$ such that p_i corresponds to ∞. Choose $s_0>0$ large enough such that u maps $(s_0,\infty)\times S^1$ into the cylindrical part $\mathbf{R}\times\widetilde{M}$. If we contract each circle $u(\{s_0\}\times S^1)$ to a point, we get a closed surface $u_*(\Sigma^*)$ in M^+. Denote $A=[u_*(\Sigma^*)]$. We say u *represents the homology class* A. Since each periodic orbit is contractible, the homology class A is independent of the choice of each s_0. By using the Stokes Theorem we have

$$E_{\phi^+}(u)=\omega_{\phi^+}(A)+\sum T_i. \tag{4.11}$$

Choose Darboux coordinates (θ, w) on $\widetilde{M}$ near each $x_i(t)$ and write

$$u(s,t) = (a(s,t), \widetilde{u}(s,t))\,.$$

Then

$$\widetilde{u}(s,t) \to x_i(T_i t)$$

exponentially as $s \to \infty$. We denote by $\mathcal{M}_A(M^+, g, m, \mathbf{T})$ the moduli space of all J-holomorphic curves representing the homology class A and converging to $\mathbf{T}$-periodic orbits. There is a natural map

$$P^+ : \mathcal{M}_A(M^+, g, m, \mathbf{T}) \to \bigoplus_{i=1}^{\nu} S_{T_i}$$

given by

$$P^+(u, \overset{\circ}{\Sigma}; \mathbf{y}, \{p_1, ..., p_\nu\}) = \{x_{T_1}, ..., x_{T_\nu}\}\,. \tag{4.12}$$

Similarly, for M^- we define $\mathcal{M}_A(M^-, g, m, \mathbf{T})$, and the map P^- . Note that

$$E_{\phi^-}(u) = \omega_{\phi^-}(A) - \sum T_i.$$

We need also to consider J-holomorphic curves without end. Denote by $\mathcal{M}_A(M^\pm, g, m)$ the moduli space of all J-holomorphic curves representing the homology class A. We fix a homology class $A \in H_2(M^\pm, \mathbf{Z})$ and a fixed set $\{T_1, ..., T_\nu\}$. For any J-holomorphic curves in $\mathcal{M}_A(M^\pm, g, m, \mathbf{T})$ there is a uniform bound on the energy.

We want to study the convergence properties for sequences of J-holomorphic maps belonging to a fix homology class A. We first prove some lemmas which are needed later.

4.2 Some technical lemmas

We shall mainly discuss M^+ and use M, ϕ, etc. instead of M^+, ϕ^+ etc. when there is no difference in the phenomenon for $M^\pm$. We denote $M_l = M_0 \cup \{[0, l] \times \widetilde{M}\}$. It is obvious that for any $\phi \in \Phi$ the sectional curvature and the injectivity radius with respect to the metric $<,>_{\omega_\phi}$ are bounded, if we restrict to M_l. Hence the following two lemmas can be proved in the same way as in [MS] and [Pan] with only minor changes.

Lemma 4.3 *For any $l > 0$ there exist constants $C_{\phi,l} > 0, \hbar_{\phi,l} > 0$ and $H_{\phi,l} > 0$ such that the following holds. Let $A(R,r)$ denote the annulus $D(R) - D(r)$ in $\mathbf{C}$ for $r < R$. If $u : A(R,r) \to M_l$ is a J-holomorphic curve such that*

$$E_\phi(u; A(R,r)) < \hbar_{\phi,l}$$

then

$$E_\phi(u; A(e^H r, e^{-H} R)) \le \frac{C_{\phi,l}}{H} E_\phi(u; A(R,r)) \tag{4.13}$$

and

$$\int_0^{2\pi} d(u(re^{H+i\theta}), u(Re^{-H+i\theta}))d\theta \le \frac{C_{\phi,l}}{\sqrt{H}}\sqrt{E_\phi(u; A(R,r))} \tag{4.14}$$

for $H \ge H_{\phi,l}$*, where* $d(,)$ *denotes the distance function on* M *defined by the metrc* $<,>_{\omega_\phi}$ *in the usual way.*

Lemma 4.4 *There exist constants* $\varepsilon_0 > 0$ *and* $C > 0$ *independent of* ϕ *such that the following holds, for each* $\varepsilon < \varepsilon_0$ *and each metric ball* $D_p(\varepsilon)$ *centered at* $p \in M_0$ *and of radius* ε*. Let* $u' : \Sigma' \to M$ *be a J-holomorphic curve. Suppose* $u'(\Sigma') \subseteq D_p(\varepsilon), u'(\partial\Sigma') \subset \partial D_p(\varepsilon), \partial\Sigma' \neq \emptyset$ *and* $p \in u'(\Sigma')$*.Then*

$$\int_{\Sigma'} u'^* \omega_\phi > C\varepsilon^2. \tag{4.15}$$

Now we prove

Lemma 4.5 *There exists* $\hbar > 0$ *independent of* ϕ *such that*

$$\int_{\mathbf{C}} u^* \omega_\phi > \hbar \tag{4.16}$$

for any nonconstant, finite energy J-holomorphic curve

$$u : \mathbf{C} \to M(\text{ or } \mathbf{R} \times \widetilde{M}).$$

Proof: Since the set of periodic orbits of the Reeb vector field X of $\widetilde{M}$ is compact, there is a minimal period $\tau > 0$. Let ε_0 be a small number such that $D_p(\varepsilon_0) \subset M_1, \forall p \in M_0$. Denote by $inj(M_1)$ the injectivity radius of M_1. We choose

$$\hbar = min\{\frac{1}{2}C\varepsilon_0^2, \frac{1}{2}C(inj(M_1))^2, \frac{1}{2}\tau\},$$

where C is the constant in Lemma 4.4. Let $u : \overset{\circ}{\Sigma} \to M$ be a J-holomorphic curve with $E_\phi(u) \le \hbar$. We are going to prove u=const. We consider two cases.

Case 1 $u(\overset{\circ}{\Sigma}) \subset (0,\infty) \times \widetilde{M}$.

If there is $l > 0$ such that $u(\overset{\circ}{\Sigma}) \subset (0,l] \times \widetilde{M}$ then $\{\infty\}$ is removable (here we use the terminology "removable singularity" and "nonremovable singularity" as in [HWZ2]). By using the Stokes

Theorem we have

$$\int_{\overset{\circ}{\Sigma}} u^*\omega_\phi = \int_{\Sigma} u^* d(e^\phi \lambda) = 0$$

Hence u=const. We assume now that $\{\infty\}$ is nonremovable. We introduce holomorphic coordinates $(s,t) \in [l,\infty) \times S^1$ near $\{\infty\}$ and write

$$u(s,t) = (a(s,t), \widetilde{u}(s,t)).$$

Suppose that u is non-constant. By using Theorem 3.7 in Section 3 we have

$$\widetilde{u}(s,t) \to x(Tt) \text{ as } s \to \infty$$

and

$$\frac{\partial a(s,t)}{\partial s} \to T \text{ uniformly } in\ t.$$

Hence $T > 0$ and

$$\int_{\mathbf{C}} u^*\omega_\phi = \int_{x(Tt)} \widetilde{u}^*\phi\lambda > \frac{1}{2}T \geq \hbar$$

which contradicts our assumption.

We remark that for M^- this can not take place.

Case 2. $u(\overset{\circ}{\Sigma}) \bigcap M_0 \neq \emptyset$. Take any $p \in u(\overset{\circ}{\Sigma}) \cap M_0$ and let $p_0 \in \Sigma$ such that $u(p_0) = p$. Choose $\varepsilon = \frac{1}{\sqrt{2}}\varepsilon_0$; then $\varepsilon < \varepsilon_0$. Let Σ' be the connected component of $u^{-1}(D_p(\varepsilon))$ containing p_0. By perturbing ε a bit we may assume that Σ' is a Riemann surface with smooth boundary. If $\partial\Sigma' \neq \emptyset$, by Lemma 4.4 we have

$$\int_{\overset{\circ}{\Sigma}} u^*\omega_\phi \geq \int_{\Sigma'} u^*\omega > C\varepsilon^2 > \hbar.$$

We get a contradiction. Hence $u(\Sigma) \subset D_p(\varepsilon)$. Since $\varepsilon < inj(M_1), u$ must be homotopic to zero. It follows that u is a constant. $\square$

Following McDuff and Salamon [MS] we introduce the notion of singular points for a sequence u_i and the notion of mass of singular points. Suppose that $(\Sigma_i; \mathbf{y}_i, \mathbf{p}_i)$ are stable curves and converge to $(\Sigma; \mathbf{y}, \mathbf{p})$ in $\overline{\mathcal{M}}_{g,m+\nu}$. Let U be a small neighborhood of $(\Sigma; \mathbf{y}, \mathbf{p})$ in $\overline{\mathcal{M}}_{g,m+\nu}$ and $\tilde{U}$ be the uniformization of U. Let $\tilde{\mathcal{U}}$ be the universal family of curves over $\tilde{U}$. We may choose a metric h on $\tilde{\mathcal{U}}$ such that h is flat near each singular point. Let $(u_i, \Sigma_i; \mathbf{y}_i, \mathbf{p}_i) \in \mathcal{M}_A(M^+, g, m, \mathbf{T})$ be a sequence. By compactness of the sets of periodic orbits we may assume that the periodic orbits $\mathbf{x}_{\mathbf{T}i}$ converge to periodic orbits $\mathbf{x}_{\mathbf{T}}$. There is a constant $C > 0$ such that $E_\phi(u_i) \leq C$ for all i and any $\phi \in \Phi$.

Denote by h_i the restriction of h to Σ_i. A point $q \in \Sigma - \{double\ points\}$ is called regular for u_i if there exist $q_i \in \Sigma_i$, $q_i \to q$, and $\epsilon > 0$ such that the sequence $|du_i|_{h_i}$ is uniformly bounded on $D_{q_i}(\epsilon, h_i)$, where $|du_i|_{h_i}$ denotes the norm with respect to the metric $<,>$ on M and the metric h_i on Σ_i. A point $q \in \Sigma - \{double\ points\}$ is called singular for u_i if it is not regular. A singular point q for u_i is called rigid if it is singular for every subsequence of u_i. It is called tame if it is isolated and the limit

$$m_\epsilon(q) = \lim_{i\to\infty} E_\phi(u_i; D_{q_i}(\epsilon, h_i))$$

exists for every sufficiently small $\epsilon > 0$. The mass of the singular point q is defined to be

$$m(q) = \lim_{\epsilon\to 0} m_\epsilon(q).$$

Lemma 4.6 *Every rigid singular point q for u_i has mass*

$$m(q) \geq \hbar. \tag{4.17}$$

Proof: By using the standard rescaling argument (see [WS]) we may construct for every rigid singular a nonconstant, finite energy J-holomorphic curve $v : \mathbf{C} \to M($ *or* $\mathbf{R}\times\widetilde{M})$. By using lemma 4.5, one can complete the proof. □

Denote by $P \subset \Sigma$ the set of the singular points for u_i, the double points and the puncture points. By Lemma 4.6, P is a finite set. By definition $|du_i|_{h_i}$ is uniformly bounded on every compact subset of $\Sigma - P$. We first assume that

$$u_i(\Sigma_i) \cap M_l \neq \emptyset \tag{4.18}$$

for some positive number l independent of i. For the case that there is no l satisfying (4.18), we will make a translation along $\mathbf{R} \times \widetilde{M}$, then the discusses are similar (see Remark 4.10). It follows from (4.18) that u_i maps every compact subset of $\Sigma - P$ uniformly into a bounded subset of M. By passing to a further subsequence we may assume that u_i converges uniformly with all derivatives on every compact subset of $\Sigma - P$ to a J-holomorphic curve $u : \Sigma - P \to M$.

We need to study the behaviour of the sequence u_i near each singular point for u_i and near each double point. We first consider a singular point for u_i. Let $q \in \Sigma$ be a rigid singular point for u_i. We may identify each neighborhood of q_i in Σ_i with a neighborhood of 0 in $\mathbf{C}$, said D(1). The sequence u_i is considered to be a sequence of (J, i)-holomorphic maps from $D(1)$ into M, and $q_i \in D(1)$, $q_i \to 0$. We may assume that 0 is the unique singular point and is tame. Denote the mass of the singular point 0 by m_0. By Lemma 4.6 we have $m_0 \geq \hbar$. The point 0 is a puncture for u. There are two types of punctures: removable and nonremovable. If $u(D(1))$ lies in M_l for some $l > 0$, then 0 is a removable singularity. Otherwise 0 is nonremovable. Since for any $\phi \in \Phi$, $E_\phi(u_i) \leq C$, u is a finite energy J-holomorphic curve. The set of nonremovable singular points can be split again into two subsets (see [HWZ2]):

1) negative singularity : the $\mathbf{R}-$components tend to $-\infty$;

2) positive singularity : the $\mathbf{R}-$components tend to ∞. Note that $\widetilde{u}$ converges to a T-periodic orbit $x(Tt)$; the sign of T determines the type of singularity.

Lemma 4.7 *Let p be a singular point for u_i. Then p is either a removable singular point of u or a negative singular point of u.*

Proof: Suppose that $p \in P$ is a positive singular point of u.

$$\lim_{\varepsilon \to 0} \int_{\partial D_{q_i}(\varepsilon)} u^* \lambda = T > 0.$$

It follows that

$$\int_{\partial D_{q_i}(\varepsilon)} {u_i}^* \lambda > \frac{1}{2} T > 0$$

for i large enough and ε small enough. By using the Stokes Theorem we have

$$E(u_i; D_{q_i}(\varepsilon)) = -\int_{\partial D_{q_i}(\varepsilon)} {u_i}^* \lambda < 0;$$

this is impossible. $\square$

We assume $u_i(\Sigma_i) \cap M_l \neq \emptyset$ and consider two cases separately

Case 1 0 is a removable singular point of u. Then $u(z)$ is bounded near 0. We use the cylindrical coordinates (s, t), where $z - z(q_i) = e^{s+2\pi i t}$. Since u_i converges to u uniformly on any compact subset of $D(1) - \{0\}$, we may assume that there is $l > 0$ such that $u_i(\{log\varepsilon\} \times S^1) \subset M_l$ for i large enough. We fix $H > C_{\phi,3l}$. Let $r < \log \varepsilon$ be a number such that $u_i((\log r, \log \varepsilon) \times S^1) \subset M_{3l}$. Then the following estimates hold:

$$|du_i(s,t)|^2 < C_1 E_\phi(u_i; (\log r, \log \varepsilon) \times S^1) \tag{4.19}$$

for $\log r < s < log\varepsilon$ and

$$\int_{S^1} d_\phi(u_i(\log r + H, t), u(\log \varepsilon - H, t) dt \leq C_2 \sqrt{E_\phi(u_i; (\log r, \log \varepsilon) \times S^1)} \tag{4.20}$$

where C_1 and C_2 are constants independent of i, but depending on H, l. It follows from (4.19) and (4.20) that there is a constant $C > 0$ independent of i such that for any $t_1, t_2 \in S^1$

$$d_\phi(u_i(logr, t_1), u_i(\log\varepsilon, t_2)) \le C\sqrt{E_\phi(u_i; (logr, \log\varepsilon) \times S^1)}.$$

We choose a positive number $\hbar'$ such that $\hbar' < \frac{1}{2C} d_\phi(\partial M_l, \partial M_{2l})$. Then it is easy to show that for any $s < \log\varepsilon$, if $E_\phi(u_i; (s, \log\varepsilon) \times S^1) \le \hbar'$, then $u_i((s, \log\varepsilon) \times S^1) \subset M_{2l}$. Put $\hbar_0 = min\{\hbar, \hbar'\}$.

Let S^2 denote the standard sphere in $\mathbf{R}^3$ with two distinguished antipodal points $0 = (0, 0, 0)$, the bubble point, and the north pole $(0, 0, 1)$. It was shown by Parker and Wolfson [PW] that, for any $\varepsilon > 0$, through a conformal transformation of S^2 we may assume that u_i are defined in the disk $D(\varepsilon) = \{z||z| < \varepsilon\}$ such that the center of mass of the measure $|du_i|^2$ is on the z-axis . For every i there exists a number $\delta_i > 0$ such that

$$E_\phi(u_i; D(\delta_i)) = m_0 - \frac{1}{2}\hbar_0.$$

Then $\delta_i \to 0$. We put

$$z = \delta_i w$$

and consider the J-holomorphic curve v_i defined by

$$v_i(w) = u_i(\delta_i w).$$

Lemma 4.8 *Let u_i, u, v_i, be as above. Then there exists a subsequence (still denoted by v_i) such that*

(1) *The set of singular points $\{w_1, \cdots, w_N\}$ for v_i is finite and tame, and is contained in the open disc $D(1) = \{w \mid\mid w \mid< 1\}$;*

(2) *The subsequence v_i converges with all derivatives uniformly on every compact subset of $\mathbf{C}\backslash\{w_1, \cdots, w_N\}$ to a (J,i)-holomorphic curve $v : \mathbf{C} \to M$;*

(3) $E(v) + \sum_1^N m(w_i) = m_0;$

(4) $v(\infty) = u(0).$

Proof:

(1) By definition of m_0 and δ_i we have

$$\lim_{i\to\infty} \sup E(v_i; D(R) - D(1)) \le \frac{\hbar_0}{2}$$

for any $R > 1$. Hence there is no singular point on $\mathbf{C} - D(1)$.

(2) It is standard to show that v_i converges to a J-holomorphic curve $v : \mathbf{C} \to M$ in the C^∞ topology on every compact subset of $\mathbf{C} - \{w_1, \cdots, w_l\}$.

(3) Since

$$u_i(D(1) - D(\delta_i)) \subseteq M_{2l}$$

and

$$E_\phi(u_i; D(1) - D(\delta_i)) < \hbar_{\phi,2l}$$

we can use the formulas (4.13) and (4.14) in Lemma 4.3 with constant $C_{\phi,2l}$. We choose $H > H_{\phi,2l}$ so large that $H > C_{\phi,2l}$. Then for $R > 0$ large enough and $\varepsilon > 0$ small enough we have

$$E_\phi\left(u_i; D(\varepsilon) - D(R\delta_i)\right) \tag{4.21}$$

$$\leq \frac{1}{1 - \frac{C_{\phi,2l}}{H}} \left(E_\phi(u_i; D(\varepsilon) - D(\varepsilon e^{-H})) + E_\phi(u_i; D(R\delta_i e^H) - D(R\delta_i))\right).$$

Since $u_i \to u$ and $v_i \to v$ uniformly on compact sets the above inequality implies that

$$\begin{aligned} 0 &\leq m_\varepsilon - E_\phi(v; D(R)) - \sum_j m_\varepsilon(w_j) \\ &\leq \frac{1}{1 - \frac{C_{\phi,2l}}{H}} \left(E_\phi(u; D(\varepsilon) - D(\varepsilon e^{-H})) + E_\phi(v; D(Re^H) - D(R))\right). \end{aligned}$$

Taking the limits $\varepsilon \to 0$ and $R \to \infty$ we get the equality in (3).

(4) By using the formula (4.14) in Lemma 4.3 we get

$$\int_0^{2\pi} d(u_i(\varepsilon e^{-H+it}), u_i(R\delta_i e^{H+it}))dt \leq \frac{1}{\sqrt{H}} C_{\phi,2l} \sqrt{E(u_i; D(\varepsilon) - D(R\delta_i))}.$$

Let $i \to \infty, \varepsilon \to 0$ and $R \to \infty$. Using (4.21) we obtain $v(\infty) = u(0)$. $\Box$

Case 2 0 is a nonremovable singular point of u. Then for $|z|$ sufficiently small $u(z)$ lies in the cylindrical part. For any $\varepsilon > 0$ we may assume as above that the u_i are defined in the disk $D(\varepsilon) = \{z||z| < \varepsilon\}$ such that the center of mass of the measure $|\Pi du_i|^2$ is on the z-axis . It is convenient to use cylindrical coordinates

$$z = e^{s+2\pi it}.$$

We write

$$u_i(s,t) = (a_i(s,t), \widetilde{u}_i(s,t))$$

$$u(s,t) = (a(s,t), \widetilde{u}(s,t)).$$

Define the mass $\widetilde{m}_0$ by

$$\widetilde{m}_0(\epsilon) = \lim_{i\to\infty} \int_{D(\epsilon)} u_i^* d\lambda, \tag{4.22}$$

$$\widetilde{m}_0 = \lim_{\epsilon\to 0} \widetilde{m}_0(\epsilon). \tag{4.23}$$

By Theorem 3.7 in Section 3 we have

$$\lim_{s\to-\infty} \widetilde{u}(s,t) = x(Tt),$$

where $x(\ .\ )$ is a T-periodic orbit of the Reeb vector field on $\widetilde{M}$. There is $\hbar' > 0$ such that for any period T' either $T = T'$ or $|T - T'| > \hbar'$. Put

$$\hbar_0 = \min\{\hbar, \hbar'\}.$$

For every i there exists $\delta_i > 0$ such that

$$E_\phi(u_i; D_{q_i}(\delta_i)) = m_0 - \frac{1}{2}\hbar_0.$$

Then $\delta_i \to 0$. Put $z = \delta_i w = \delta_i e^{r+2\pi i t}$. Define the J-holomorphic curve v_i by

$$v_i(r,t) = (b_i(r,t), \widetilde{v}_i(r,t)) = \left(a_i(\log\delta_i + r, t) - a_i(\log\delta_i, t_0), \widetilde{u}_i(\log\delta_i + r, t)\right). \tag{4.24}$$

In terms of the cylindrical coordinates, the construction of $\Sigma_i \#_{\sqrt{\delta_i}} S^2$ becomes the following. The part $\{(s,t)|s \le \frac{1}{2}\log\delta_i\}$ is cut out from Σ_i, the part $\{(r,t')|r \ge -\frac{1}{2}\log\delta_i\}$ is cut out from S^2, and the remainder is glued along a collar of length $2\log 2$ of the cylinders

$$\frac{1}{2}\log\delta_i - \log 2 < s < \frac{1}{2}\log\delta_i + \log 2$$

$$-\frac{1}{2}\log\delta_i - \log 2 < r < -\frac{1}{2}\log\delta_i + \log 2$$

by the gluing formula

$$t = t'$$

$$r + \log\delta_i = s.$$

When we take $i \to \infty$ we get $\Sigma \wedge S^2$ with cuspidal point $s = -\infty, r = \infty$.

Lemma 4.9 *Suppose that 0 is a nonremovable singular point of u. Define the J-holomorphic curve v_i as above. Then there exists a subsequence (still denoted by v_i) such that*

(1) *The set of singular points* $\{w_1, \cdots, w_N\}$ *for* v_i *is finite and tame, and is contained in the open disc* $D(1) = \{w \mid\mid w \mid< 1\}$;

(2) *The subsequence* v_i *converges with all derivatives uniformly on every compact subset of* $\mathbf{C}\backslash\{w_1, \cdots, w_N\}$ *to a nonconstant (J,i)-holomorphic curve* $v : \mathbf{C}\backslash\{w_1, \cdots, w_N\} \to \mathbf{R} \times \widetilde{M}$;

(3) $\widetilde{E}(v) + \sum_1^N \widetilde{m}(w_i) = \widetilde{m}_0$;

(4) ∞ *is a nonremovable singular point of* v. *Moreover,* $\widetilde{u}$ *and* $\widetilde{v}$ *converge to the same* T*-periodic orbit.*

Proof: The proof of **(1)** and **(2)** is easy. We prove **(3)** and **(4)** .

(3) For any $\phi \in \Phi$, we choose $\phi_i \in \Phi$ defined by

$$\phi_i(a) := \phi(a + a_i(log\delta_i, t_0)).$$

Then

$$E_\phi(v_i; (-\infty, \log \varepsilon - \log \delta_i) \times S^1) = E_{\phi_i}(u_i; (-\infty, \log \varepsilon) \times S^1) \le C$$

for some constant $C > 0$. It follows that $E_\phi(v; (-\infty, \infty) \times S^1) \le C$. Since ϕ is arbitary, v is a finite energy J-holomorphic curve. Suppose that v converges to a T'-periodic orbit as $r \to \infty$. For ε small and R large we have

$$\frac{2}{3}\hbar_0 > \widetilde{E}(u_i; (\log \varepsilon, \log R + \log \delta_i) \times S^1)$$

$$= \left| \int_{\{\log \varepsilon\} \times S^1} u_i^* \lambda - \int_{\{\log R\} \times S^1} v_i^* \lambda \right|.$$

Taking $i \to \infty, \varepsilon \to 0, R \to \infty$, we obtain

$$\hbar_0 > |T - T'|.$$

It follows that $T = T'$. Therefore

$$\lim_{\varepsilon \to 0, R \to \infty} lim_{i \to \infty} \widetilde{E}(u_i; (\log \varepsilon, \log R + \log \delta_i) \times S^1) = 0. \tag{4.25}$$

Then one can easily derive **(3)** from (4.25).

(4) Recall that there is a neighborhood O of the set S_T of the T-periodic orbits in $W_r^2(S^1, \widetilde{M})$ such that the formula (3.17) holds in O. We prove now that there exist $i_0 > 0, R_0 > 0$ such that

$\widetilde{v}_i(r,t) \in O$ for all $i > i_0, |\log\sqrt{\delta_i}| > r > R_0$. Otherwise, we may find a sequence $r_i \to \infty, r_i < \frac{1}{2}|log\delta_i|$, such that $\widetilde{v}_i(r_i,t_i) \notin O, \forall i$. Consider the sequence of J-holomorphic map

$$g_i : (-\infty, -\log\sqrt{\delta_i}) \times S^1 \to R \times \widetilde{M}$$

defined by

$$g_i(r,t) = (b_i(r+r_i,t) - b(r_i,t_i), \widetilde{v}(r+r_i,t)).$$

By (4.25), $|dg_i|$ is bounded uniformly on $[l_1,l_2] \times S^1$ for any $l_1 < l_2$. After taking a subsequence we may assume that g_i converges to a J-holomorphic curve

$$g : \mathbf{R} \times S^1 \to \mathbf{R} \times \widetilde{M}$$

For any $R > 0$ we have

$$\int_{[-R,R]\times S^1} \widetilde{g}_i^* d\lambda = \int_{[r_i-R,r_i+R]\times S^1} \widetilde{v}_i^* d\lambda \to 0 \ as \ i \to \infty.$$

Here we used (4.25). It follows that

$$\int_{\mathbf{R}\times S^1} \widetilde{g}^* d\lambda = 0$$

By using the Stokes Theorem we have

$$\int_{\{0\}\times S^1} \widetilde{g}^* \lambda = \lim_{i\to\infty} \int_{\{r_i\}\times S^1} \widetilde{v}_i^* \lambda = \lim_{i\to\infty} \int_{(R,r_i)\times S^1} \widetilde{v}_i^* d\lambda$$

$$+ \int_{\{R\}\times S^1} \widetilde{v}^* \lambda.$$

Let $R \to \infty$; we get

$$\int_{\{0\}\times S^1} \widetilde{g}^* \lambda = T.$$

By a result in [HWZ2] we have

$$g(r,t) = (Tr + c, x'(Tt + d)),$$

where c, d are constants, and $x'(\ .\ )$ is a T-periodic orbit of the Reeb vector field. Hence $\widetilde{v}_i(r_i,t)$ converges to a T-periodic orbit $x'(\ .\ )$ in $C^\infty(S^1, \widetilde{M})$ as $i \to \infty$. But $\widetilde{v}_i(r_i,t_i) \notin O$, so we get a contradiction.

Similarly, we can prove that there exist $i_0' > 0, R_0' < 0$ such that $\widetilde{u}_i(s,t) \in O$ for all

$$\log\sqrt{\delta_i} \le s \le R_0', \quad i > i_0'.$$

If $\mathcal{A}(u_i(s,\cdot)) \ne T$ for any $s \in (\log R + \log\delta_i, \log\varepsilon)$, then by using (3.25) in Section 3 we get

$$\int\limits_{S^1} \widetilde{d}(\widetilde{v}_i(\log R, t), \widetilde{u}_i(\log\varepsilon, t))dt \le C\sqrt{\widetilde{E}\,(\widetilde{u}_i; (\log R + \log\delta_i, \log\varepsilon) \times S^1)}, \tag{4.26}$$

If there are $s_1, s_2 \in (\log R + \log\delta_i, \log\varepsilon), s_1 < s_2$ such that $\mathcal{A}(u_i, s_1, \cdot)) = \mathcal{A}(u_i(s_2,\cdot)) = T$, then $\Pi(\frac{\partial u_i}{\partial t}) = 0$ holds for $s \in (s_1, s_2)$. It follows from the generalized similarity principle that $\Pi(\frac{\partial u_i}{\partial t}) = 0$ for $s \in (\log R + \log\delta_i, \log\varepsilon)$. The inequality (4.26) is trivial. In the following we assume that there is $s_i \in (\log R + \log\delta_i, \log\varepsilon)$ such that $\mathcal{A}(u_i(s_i,\cdot)) = T$ and $\mathcal{A}(u_i(s,\cdot)) \ne T \ \ \forall s \in (\log R + \log\delta_i, \log\varepsilon) - \{s_i\}$. The inequality (3.25) holds in both $(\log R + \log\delta_i, s_i)$ and $(s_i, \log\varepsilon)$. Therefore the inequality (4.26) with constant $2C$ holds in $(\log R + \log\delta_i, \log\varepsilon)$.

Suppose that v converges to a T-periodic orbit x'. ¿From (4.25) and (4.26) we obtain

$$\int\limits_{S^1} \widetilde{d}(x(Tt), x'(Tt))dt = 0.$$

This means $x(\ .\ )$ and $x'(\ .\ )$ are the same periodic orbit. □

Remark 4.10 *If there is no l satisfying (4.18), then there is a subsequence, still denoted by u_i, and $z_i \in \Sigma_i$ such that*

$$z_i \to z_0 \in \Sigma - P$$

and

$$a_i(z_i) \to \infty.$$

Consider the sequence

$$v_i(z) = (a_i(z) - a_i(z_i), \widetilde{u}_i(z)).$$

Then v_i converges uniformly with all derivatives on every compact subset of $\Sigma - P$ to a J-holomorphic curve $u : \Sigma - P \to \mathbf{R} \times \widetilde{M}$. We fix a small $\delta_0 < \delta$ and choose a function ϕ satisfying

$$(\phi)' > 0, \phi(a) \to -\delta_0 \ \ as \ \ a \to -\infty, \phi(a) \to \delta \ \ as \ \ a \to \infty.$$

Define $\phi_i(a) = \phi(a + a_i(z_i))$. Consider the sequence of J-holomorphic curves

$$v_i : \Sigma - P \to (\mathbf{R} \times \widetilde{M}, d(\phi\lambda)).$$

The similar formulas as (4.13) and (4.14) hold in $[l_1, l_2] \times \widetilde{M}$ *for any* $l_1, l_2 \in \mathbf{R}$. *We can construct bubbles in the same way as above and show that the Lemma 4.8 and the Lemma 4.9 still hold in this case.*

It is possible for v to have $E(v) = 0$ or $\widetilde{E}(v) = 0$. In both case we call v a ghost bubble.

Lemma 4.11 *If* $E(v) = 0$ *or* $\widetilde{E}(v) = 0$ *then* $N \geq 2$.

Proof: Use Lemmas 4.8, 4.9 and the same argument as in [PW]. Note that in our case there is no energy loss. □

Remark 4.12 *We can repeat the above renormalization procedure for each* w_i *to obtain bubbles on bubbles. The Lemma 4.10 and Lemma 4.6 implies that there are only finitely many ghost bubbles. In the nonremovable singularity case, it follows from Lemma 4.7 and the Stokes Theorem that all the singular* w_i *must be negative singularities and*

$$T \geq \sum_{1}^{N} T_i. \tag{4.27}$$

Therefore the periods of the later orbits are uniformly bounded. Hence there is a constant $\hbar_0 > 0$ *such that every later nonghost bubble has* $\widetilde{E} > \hbar_0$. *This together with Lemma 4.7 implies that the bubble tree is finite.*

We next consider the double points of Σ. Without loss of generality we assume that $\Sigma = \Sigma_1 \wedge \Sigma_2$ with double point y, and Σ_i is obtained from Σ by resolving the singularity using the parameter δ_i, $\delta_i \to 0$, i.e., $\Sigma_i = \Sigma_1 \#_{\sqrt{\delta_i}} \Sigma_2$. In terms of local cylindrical coordinates

$$z = e^{s+2\pi i t}$$

$$w = e^{r+2\pi i t'}$$

near $y(z = 0, w = \infty)$ the construction of $\Sigma_i = \Sigma_1 \#_{\sqrt{\delta_i}} \Sigma_2$ is as above with gluing formulas

$$t = t'$$

$$r + \log \delta_i = s.$$

Suppose that $u_i \to (u, v)$. We put

$$m_0 = \lim_{\varepsilon \to 0, R \to \infty} lim_{i \to \infty} E(u_i; (\log \varepsilon, \log R + \log \delta_i) \times S^1), \tag{4.28}$$

$$\widetilde{m}_0 = \lim_{\varepsilon \to 0, R \to \infty} lim_{i \to \infty} \widetilde{E}(u_i; (\log \varepsilon, \log R + \log \delta_i) \times S^1). \tag{4.29}$$

Remark 4.13 *If $m(p) \neq 0$ or $\widetilde{m}(p) \neq 0$ for a double point y we can use a similar method as we did before to construct the bubbles. Then Lemma 4.8 and Lemma 4.9 still hold . We can also show that the bubble tree is finite.*

Lemma 4.14 *Let $u_i : \Sigma_i \to M^+$ (or M^-) be a sequence of J- holomorphic curves, Σ, u, v be as above. Suppose that $m_0 = 0$. Then y is a removable singular point of u and v, and $u(0) = v(\infty)$.*

Proof: It suffices to prove that y is a removable singular point; the remaining part of the proof is the same as that of Lemma 4.8 and 4.9. We prove this for M^+, the proof for M^- is identical. Suppose that $\widetilde{u} \to x(Tt)$ *as* $s \to -\infty$ where $T > 0$. Let $v_i(r,t) = u_i(r + \log \delta_i, t)$. Choose Darboux coordinates $(\theta, w) = (\theta, x_1, \cdot, x_n, y_1, \cdot, y_n)$ on $\widetilde{M}$ near the periodic orbit $x(\ . \ )$ as in [HWZ1] such that

$$\lambda = f\lambda_0,$$

where $\lambda_0 = d\theta + \sum x_i dy_i$ is the standard contact form and $f(\theta, w)$ is a smooth positive function satisfying

$$f(\theta, 0) = \kappa > 0, \tag{4.30}$$

$$df(\theta, 0) = 0. \tag{4.31}$$

We write

$$u_i(s,t) = (a_i(s,t), \theta_i(s,t), z_i(s,t)),$$

$$v_i(r,t) = (b_i(r,t), \vartheta_i(r,t), w_i(r,t)).$$

Introduce the t-periodic functions $\bar{a}_i, \bar{b}_i, \bar{\theta}_i, \bar{\vartheta}_i$ by

$$\bar{a}_i(s,t) = a_i(s,t) - \frac{T}{\kappa}\int_0^s f(\theta_i(\mu,t), z_i(\log\varepsilon, t))d\mu, \tag{4.32}$$

$$\bar{\theta}_i(s,t) = \theta_i(s,t) - \frac{Tt}{\kappa}; \tag{4.33}$$

$$\bar{b}_i(s,t) = b_i(s,t) + \frac{T}{\kappa}\int_0^r f(\theta_i(\mu,t), z_i(\log R, t))d\mu, \tag{4.34}$$

$$\bar{\vartheta}_i(s,t) = \vartheta_i(s,t) - \frac{Tt}{\kappa}. \tag{4.35}$$

It is easy to see that $a_i(s,t)$ and $\theta_i(s,t)$ satisfy

$$\frac{\partial a_i}{\partial s} = \left(\frac{\partial \theta_i}{\partial t} + \sum_j x_{ij} \frac{\partial y_{ij}}{\partial t} \right) f \tag{4.36}$$

$$\frac{\partial a_i}{\partial t} = - \left(\frac{\partial \theta_i}{\partial s} + \sum_j x_{ij} \frac{\partial y_{ij}}{\partial s} \right) f, \tag{4.37}$$

where (x_{ij}, y_{ij}) denote the z-coordinates of u_i. We have

$$\begin{aligned} \frac{\partial \bar{a}_i(s,t)}{\partial s} &= \left(\frac{\bar{\theta}_i(s,t)}{\partial t} + \sum_j x_{ij}(s,t) \frac{\partial y_{ij}(s,t)}{\partial t} \right) f(\theta_i(s,t), z_i(s,t)) \\ &+ \frac{T}{\kappa} \left(f(\theta_i(s,t), z_i(s,t)) - f(\theta_i(s,t), z_i(\log \varepsilon, t)) \right). \end{aligned} \tag{4.38}$$

By using (3.25) and (3.37) we obtain the following estimates

$$\begin{aligned} \left| \int_{\log \varepsilon}^{s} \int_{S^1} \left(\sum_j x_{ij} \frac{\partial y_{ij}}{\partial t} \right) f ds dt \right| &\le C \int_{\log \varepsilon}^{s} \int_{S^1} \|\Pi \frac{\partial \widetilde{u}_i}{\partial t}\| ds dt \\ &\le C \sqrt{\widetilde{E}(u_i; (\log \varepsilon, s) \times S^1)}, \end{aligned} \tag{4.39}$$

$$\begin{aligned} &\left| \int_{\log R + \log \delta_i}^{\log \varepsilon} \int_{S^1} \left(f(\theta_i(s,t), z_i(s,t)) - f(\theta_i(s,t), z_i(\log \varepsilon, t)) \right) ds dt \right| \\ &\le C \int_{\log R + \log \delta_i}^{\log \varepsilon} \int_{s}^{\log \varepsilon} \|\Pi \frac{\partial \widetilde{u}_i}{\partial s}\|_{L^2(S^1)} d\mu dt \\ &\le C(\widetilde{E}(u_i; (\log R + \log \delta_i, \log \varepsilon) \times S^1). \end{aligned} \tag{4.40}$$

In the last inequality we have used the exponential decay of $\widetilde{E}(u_i)$. Since $\bar{\theta}_i$ is a t-periodic function we have

$$\int_{S^1} \frac{\partial \bar{\theta}_i}{\partial t} \kappa dt = 0,$$

Since $m_0 = 0$, from the proof of Lemma 4.9 (**4**) we may assume that for i large enough, $\widetilde{u}_i$ lies in a small W^2_r-neighbourhood O of S_T, where the inequality (3.17) holds. Hence there is a uniform bound on $|\frac{\partial \bar{\theta}_i}{\partial t}|$. Moreover, as $f(\theta, 0) = \kappa$ we have

$$\int_{S^1} |f(\bar{\theta}_i(s,t), z_i(s,t)) - \kappa| dt < C \|z_i\|_{L^2}. \tag{4.41}$$

Therefore

$$
(4.42) \qquad \left| \int\limits_{\log R+\log \delta_i}^{\log \varepsilon} \int_{S^1} \frac{\partial \bar{\theta}_i}{\partial t} f ds dt \right|
$$

$$
\leq C \left(\widetilde{E}(u_i; (\log R + \log \delta_i, \log \varepsilon) \times S^1) \right)^{1/2}.
$$

Then, from (4.38), (4.39), (4.40) and (4.42) we get

$$
(4.43) \qquad \left| \int_{S^1} \bar{a}_i(\log \varepsilon, t) dt - \int_{S^1} \bar{a}_i(\log R + \log \delta_i, t) dt \right| = \left| \int\limits_{\log R+\log \delta_i}^{\log \varepsilon} \int_{S^1} \frac{\partial \widetilde{a}_i}{\partial s} dt ds \right|
$$

$$
\leq C \widetilde{E}(u_i; (\log R + \log \delta_i, \log \varepsilon) \times S^1) + C \left(\widetilde{E}(u_i; (\log R + \log \delta_i, \log \varepsilon) \times S^1) \right)^{1/2}.
$$

We fix R and ε. From (4.32) we have

$$
(4.44) \qquad \int_{S^1} \bar{a}_i(\log R + \log \delta_i, t) dt = \int_{S^1} a_i(\log R + \log \delta_i, t) dt
$$

$$
+ \frac{T}{\kappa} \int_{S^1} \int\limits_{0}^{\log R+\log \delta_i} f(\theta_i(\mu, t), z_i(\log \varepsilon, t)) d\mu dt.
$$

Let $i \to \infty$. Note that $\bar{a}_i(\log \varepsilon, t)$ and $a_i(\log R + \log \delta_i, t) = b_i(\log R, t)$ are uniformly bounded. Since

$$
f(\theta_i, z_i(\log \varepsilon, t)) > \frac{\kappa}{2}
$$

for i large enough, we have

$$
(4.45) \qquad \frac{T}{\kappa} \int\limits_{0}^{\log R+\log \delta_i} f(\theta_i(\mu, t), z_i(\log \varepsilon, t)) d\mu \to -\infty \quad \text{as} \quad i \to \infty.
$$

¿From (4.43), (4.45) and $m_0 = 0$ we get a contradiction. □

Now we consider a sequence of J-holomorphic curves u_i from Σ_i into M_i. The sequence M_i converges to $M^+ \cup M^-$ as $i \to \infty$. Supoose that $\Sigma_i \to \Sigma = \Sigma^+ \cup \Sigma^-$, $u_i \to u = (u^-, u^+)$. Let p be the intersecting point of Σ^+ and Σ^-. From the proof of Lemma 4.9 we immediately obtain

Lemma 4.15 *If $\widetilde{m}(p) = 0$, then $\widetilde{u}^+$ and $\widetilde{u}^-$ converge to the same T-periodic orbit.*

4.3 Compactness theorems

Definition 4.16 *Let $(\overset{\circ}{\Sigma}; \mathbf{y}, \mathbf{p})$ be a Riemann surface of genus g with m marked points $\mathbf{y}$ and ν ends $\mathbf{p}$. A relative stable holomorphic map with $\{T_1, ..., T_\nu\}$-ends from $(\overset{\circ}{\Sigma}; \mathbf{y}, \mathbf{p})$ into $M^\pm$ is an* *equivalence class of continuous maps* u *from* $\overset{\circ}{\Sigma}{}'$ *into* $(M^{\pm})'$*, modulo the automorphism group* stb_u *and the translations on* $\mathbf{R}\times\widetilde{M}$*, where* $\overset{\circ}{\Sigma}{}'$ *is obtained by joining chains of* $\mathbf{P^1}$*s at some double points of* Σ *to separate the two components, and then attaching some trees of* $\mathbf{P^1}$*s;* $(M^{\pm})'$ *is obtained by attaching some* $\mathbf{R}\times\widetilde{M}$ *to* $M^{\pm}$*. We call components of* $\overset{\circ}{\Sigma}$ *principal components and others bubble components. Furthermore,*

(1) *If we attach a tree of* $\mathbf{P^1}$ *at a marked point* y_i *or a puncture point* p_i*, then* y_i *or* p_i *will be replaced by a point different from intersection points on a component of the tree. Otherwise, the marked points or puncture points do not change;*

(2) $\overset{\circ}{\Sigma}{}'$ *is a connected curve with normal crossings ;*

(3) *Let* m_j *be the number of points on* Σ_j *which are nodal points or marked points or puncture points. Then either* $u|_{\Sigma_j}$ *is not a constant or* $m_j + 2g_j \geq 3$*;*

(4) *The restriction of* u *to each component is J-holomorphic.*

(5) u *converges exponentially to some periodic orbits* $(x_{T_1},...,x_{T_\nu})$ *as the variable tends to the puncture* $(p_1,...,p_\nu)$*; more precisely,* u *satisfies (3.34)-(3.36);*

(6) *Let* q *be a nodal point of* Σ' *. Suppose* q *is the intersection point of* Σ_i *and* Σ_j*. If* q *is a removable singular point of* u*, then* u *is continuous at* q*; If* q *is a nonremovable singular point of* u*, then* Σ_i *and* Σ_j *are mapped into* $\mathbf{R}\times\widetilde{M}$*. Furthermore,* $u|_{\Sigma_i}$ *and* $u|_{\Sigma_j}$ *converge exponentially to the same periodic orbit of the Reeb vector field* X *on* $\widetilde{M}$ *as the variables tend to the nodal point* q*.*

If we drop the condition (4), we simply call u a relative stable map. Let $\overline{\mathcal{M}}_A(M^{\pm},g,m,\mathbf{T})$ be the space of the equivalence class of stable holomorphic curves with ends, and $\overline{\mathcal{B}}_A(M^{\pm},g,m,\mathbf{T})$ be the space of stable maps with ends. By using Lemmas 4.5 - 4.14 and induction one immediately obtains the following theorem.

Theorem 4.17 *Let* $\Gamma_i = (u_i,\overset{\circ}{\Sigma}_i;\mathbf{y}_i,\mathbf{p}_i)\in\mathcal{M}_A(M^{\pm},g,m,\mathbf{T})$ *be a sequence. Then there is a subsequence, which "weakly converges" to a stable J-holomorphic curve in* $\overline{\mathcal{M}}_A(M^{\pm},g,m,\mathbf{T})$*. Here, by weak convergence, we mean the Gromov-Uhlenbeck convergence with possible translation on* $\mathbf{R}\times\widetilde{M}$*.*

Corollary 4.18 $\overline{\mathcal{M}}_A(M^{\pm},g,m,\mathbf{T})$ *is compact.*

We next consider J-holomorphic curves without ends from Σ into $M^{\pm}$. Denote by $\mathcal{M}_A(M^{\pm}, g, m)$ the moduli space of all J-holomorphic curves representing the homology A. The notion of stable holomorphic maps is almost the same as definition 4.15: we just drop the condition (5). The following theorem and corollary also follows immediately from lemmas in subsection 4.2.

Theorem 4.19 *Let $\Gamma_i = (u_i, \Sigma_i; \mathbf{y}_i) \in \mathcal{M}_A(M^{\pm}, g, m)$ be a sequence. Suppose that $(\Sigma_i; \mathbf{y}_i)$ converges to $(\Sigma; \mathbf{y})$ in $\overline{\mathcal{M}}_{g,m}$. Then there is a subsequence, which weakly converges to a stable J-holomorphic curve in $\overline{\mathcal{M}}_A(M^{\pm}, g, m)$.*

Corollary 4.20 *$\overline{\mathcal{M}}_A(M^{\pm}, g, m)$ is compact.*

Next we stretch M_r along the contact hypersurface $\widetilde{M}$ to M_∞, and consider the convergence of sequences of J-holomorphic curves. Note that M_∞ is a compact symplectic manifold with the symplectic form ω; the almost complex structure J is degenerate along the contact hypersurface. To define stable J-holomorphic curves in M_∞ we need to extend $\overline{\mathcal{M}}_A(M^{\pm}, g, m, \mathbf{T})$ to include nonconnected holomorphic curves. Suppose that $\Sigma^{\pm}$ has $l^{\pm}$ connected components $\Sigma_i^{\pm}, i = 1, ..., l^{\pm}$ of genus $g_i^{\pm}$ with $m_i^{\pm}$ marked points and $\nu_i^{\pm}$ ends, $\sum \nu_i^{\pm} = \nu$. Put

$$\overline{\mathcal{M}}_{\mathbf{A}^{\pm}}(M^{\pm}, \mathbf{g}^{\pm}, \mathbf{m}^{\pm}, \mathbf{T}) = \bigoplus_{i=1}^{l^{\pm}} \overline{\mathcal{M}}_{A_i^{\pm}}(M^{\pm}, g_i^{\pm}, m_i^{\pm}, \mathbf{T}_i),$$

where $\mathbf{A}^{\pm} = \left\{A_1^{\pm}, ..., A_{l^{\pm}}^{\pm}\right\}, \mathbf{g}^{\pm} = \left\{g_1^{\pm}, ..., g_{l^{\pm}}^{\pm}\right\}, \mathbf{m}^{\pm} = \left\{m_1^{\pm}, ..., m_{l^{\pm}}^{\pm}\right\}, \mathbf{T} = \{\mathbf{T}_1, ..., \mathbf{T}_\nu\}$. The maps $P^{\pm}$ are extended to $\overline{\mathcal{M}}_{\mathbf{A}^{\pm}}(M^{\pm}, g, m, \mathbf{T})$ in a natural way.

Definition 4.21 *A stable J-holomorphic curve of genus g and class A in M_∞ is a triple $(\Gamma^-, \Gamma^+, \rho)$, where*

$$(\Gamma^-, \Gamma^+) \in \overline{\mathcal{M}}_{\mathbf{A}^-}(M^-, \mathbf{g}^-, \mathbf{m}^-, \mathbf{T}) \times \overline{\mathcal{M}}_{\mathbf{A}^+}(M^+, \mathbf{g}^+, \mathbf{m}^+, \mathbf{T}),$$

and $\rho : \{p_1^+, ..., p_\nu^+\} \to \{p_1^-, ..., p_\nu^-\}$ is a one-to-one map satisfying

(1) *If we identify p_i^+ and $\rho(p_i^+)$ then $\Sigma^+ \bigcup \Sigma^-$ forms a closed Riemann surface of genus g;*

(2) $P^+\left(u^+, \Sigma^+; \mathbf{y}^+, (p_1^+, ..., p_\nu^+)\right) = P^-\left(u^-, \Sigma^-; \mathbf{y}^-, (\rho(p_1^+), ..., \rho(p_\nu^+))\right)$*;*

(3) $A = \Sigma A_i^+ + \Sigma A_j^-$.

Remark 4.22 *If $\{p_1^{\pm}, ..., p_\nu^{\pm}\}$ in the above definition is the empty set then (Σ, u) is a stable J-holomorphic curve in M^+ or M^-.*

Denote by $\overline{\mathcal{M}}_A(M_\infty, g, m)$ the moduli space of stable J-holomorphic curves in M_∞. Using Lemmas in subsection 4.2 we immediately obtain the following convergence theorem:

Theorem 4.23 *Let $\Gamma_r \in \overline{\mathcal{M}}_A(M_r, g, m)$ be a sequence. Then there is a subsequence, which* weakly converges *to a stable J-holomorphic curve in $\overline{\mathcal{M}}_A(M_\infty, g, m)$.*

5 Symplectic cutting

This section deals with symplectic cutting, i.e., a surgery along a hypersurface which admits a local S^1-hamiltonian action. By performing the symplectic cutting we get two closed symplectic manifolds $\overline{M}^+$ and $\overline{M}^-$. The symplectic quotient Z is embedded in both $\overline{M}^+$ and $\overline{M}^-$ as symplectic submanifolds of codimension 2. We will show that the main results we obtained for contact surgery such as convergence to periodic orbits and compactness theorems remain true under the present surgery. Moreover, we may define the relative invariants for the pair $(M^\pm, Z)$.

5.1 Convergence to periodic orbits

Let (M, ω) be a compact symplectic manifold of dimension $2n+2$, $H : M \to \mathbf{R}$ a local hamiltonian function such that there is a small interval $I = (-\delta, \delta)$ of regular values. Denote $\widetilde{M} = H^{-1}(0)$. Suppose that the hamiltonian vector field X_H generates a circle action on $H^{-1}(I)$. There is a circle bundle

$$\pi : \widetilde{M} \to Z = \widetilde{M}/S^1 \tag{5.1}$$

and a natural symplectic form τ_0 on Z (see [MS]). We can choose a connection 1-form α on $\widetilde{M}$ such that $\alpha(X_H) = 1$ and $d\alpha$ represents the first Chern class for the circle bundle. Denote $\xi = ker(\alpha)$. Then ξ is a S^1-invariant distribution and $(\xi, \pi^*\tau_0) \to \widetilde{M}$ is a $2n$-dimensional symplectic vector bundle. We have a canonical splitting of $T\widetilde{M}$:

$$T\widetilde{M} = \mathbf{R}X_H \oplus \xi, \tag{5.2}$$

where $\mathbf{R}X_H$ is the line bundle generated by X_H. We choose an S^1- invariant, compatible complex structure $\widetilde{J}$ for the symplectic vector bundle $(\xi, \pi^*\tau_0) \to \widetilde{M}$ such that

$$g_{\widetilde{J}(x)}(h, k) = \pi^*\tau_0(x)(h, \widetilde{J}(x)k), \tag{5.3}$$

for all $x \in \widetilde{M}$, $h, k \in \xi_x$, defines an S^1-invariant, smooth fibrewise metric for ξ. Denote by $\Pi : T\widetilde{M} \to \xi$ the projection along X_H. We define a Riemannian metric $\langle \; , \; \rangle$ on $\widetilde{M}$ as follows. For any $k_1, k_2 \in T\widetilde{M}$ we can write $k_i = a_i X_H + \Pi k_i$. We define

$$\langle k_1, k_2 \rangle = a_1 a_2 + g_{\widetilde{J}}(\Pi k_1, \Pi k_2). \tag{5.4}$$

We now define an action functional on the loop space $C^\infty(S^1, \widetilde{M})$. For a given homotopy class we choose a fixed loop γ_0. Then for any loop γ and any annulus $W : S^1 \times [0,1] \to \widetilde{M}$ satisfying $W|_{S^1\times\{0\}} = \gamma_0$ and $W|_{S^1\times\{1\}} = \gamma$, we define an action functional by

$$\mathcal{A}(\gamma_0, \gamma) = -\int_{S^1\times[0,1]} W^*\pi^*\tau_0. \tag{5.5}$$

If we choose another annulus W' satisfying $W'|_{S^1\times\{0\}} = \gamma_0$ and $W'|_{S^1\times\{1\}} = \gamma$, the two annuli glue together to form a torus T_2. Then $\mathcal{A}(\gamma_0, \gamma, W)$ and $\mathcal{A}(\gamma_0, \gamma, W')$ differ by $\tau_0([T_2])$. So the functional $\mathcal{A}(\gamma_0, \gamma)$ depends on γ_0 and is well defined up to $\tau_0[T_2]$. We have the following first variational formula:

$$\begin{aligned} d\mathcal{A}(\gamma_0, \gamma)\eta &= -\int_{S^1} \pi^*\tau_0(\dot{\gamma}, \eta)dt \\ &= \int_{S^1} \langle \Pi\dot{\gamma}, \widetilde{J}\Pi\eta\rangle dt. \end{aligned} \tag{5.6}$$

It follows that γ is a critical point if and only if $\dot{\gamma}//X_H$ everywhere. Every orbit of the circle action is a critical point of $\mathcal{A}$, which is a 1-periodic orbit. Let $x(t) \in Z$ be an orbit of the circle action. Then for any integer k the loop $x_k(t) := x(kt)$ is also a critical point. Denote by S_k the set of all such critical points and call it the set of k-periodic orbits. It is standard to prove that S_k is compact. Since $\pi^*\tau_0$ is vanishing along Hamiltonian vector field, it follows that $\mathcal{A}(\gamma_0, \gamma)$ is a constant up to $\tau_0[T_2]$ on the set S_k of k-periodic orbits.

By our choice of the Riemannian metric and the almost complex structure

$$\frac{D\zeta}{dt} \in T^\perp_{x_k} \quad \forall \zeta \in T^\perp_{x_k},$$

where $T^\perp_{x_k} = \{\zeta | \langle \zeta, \dot{x}_k \rangle = 0\}$. Define the linear transformation $S : T^\perp_{x_k} \to T^\perp_{x_k}$ by $\zeta \to \widetilde{J}\nabla_\zeta X_H$. By using the same method as in Section 3 we can derive the second variational formula :

$$\left.\frac{\partial^2 \mathcal{A}}{\partial v \partial w}\right|_{(0,0)} = -\int_{S^1} \langle (\widetilde{J}\frac{D}{dt} + \langle \tau, X_H \rangle S)\zeta^\perp, \eta^\perp \rangle dt. \tag{5.7}$$

Let $P = -\widetilde{J}\Pi\frac{D}{dt} - \langle \dot{\gamma}, X_H \rangle S$. We define

$$I(\zeta, \eta) = (P\zeta, \eta)$$

for any $\zeta, \eta \in T^{\perp}_{x_k}$. Since the inclusion $W^2_1(S^1) \to L^2(S^1)$ is compact, and P is a compact perturbation of the selfadjoint operator $J_0 \frac{d}{dt}$, the spectrum $\sigma(P)$ consists of isolated eigenvalues, which accumulate at $\pm\infty$.

Proposition 5.1 *S_k is of Bott-type.*

Proof: Let $x_k \in S_k$. It is easy to show that $C_r(A) \bigcap O_{x,\epsilon} \bigcap T^{\perp}_{x_k}$ is a smooth submanifold of dimension $2n$ in $T^{\perp}_{x_k}$. We now prove that the restriction of $I(\ ,\ )$ to the normal direction of the submanifold at x_k is nondegenerate. Since any solution of the equation

$$\widetilde{J}\frac{D\zeta}{dt} + kS\zeta = 0 \tag{5.8}$$

is determined by its value $\xi(0)$ at $t = 0$, $dimker(\widetilde{J}\frac{D}{dt} + kS) \leq 2n$. On the other hand, consider a smooth map $\beta : S^1 \times (-\varepsilon, \varepsilon) \to \widetilde{M}$ such that $\beta(t, 0) = x_k(t)$ and $\Pi\tau = 0\ \ \forall (t, w)$, where we denote by τ and ζ the vector fields corresponding to the first and the second variable, respectively. Then $\langle \tau, X_H \rangle = k$ at $w = 0$. A simple calculation such as we did in Section 3 gives us

$$\nabla_\tau \xi^{\perp} - k\nabla_{\xi^{\perp}} X_H = 0. \tag{5.9}$$

It follows that any tangent vector ζ to the submanifold $C_r(\mathcal{A}) \bigcap O_{x,\epsilon} \bigcap T^{\perp}_{x_k}$ at x_T satisfies (5.8). Hence this tangent space coincides with $ker(\widetilde{J}\frac{D}{dt} + kS)$. The conclusion follows. □

For each homotopy class we fix a loop γ_0. By connectedness of S_k, all k-periodic orbits belong to the same homotopy class. We fix an arbitary periodic orbit $y_k \in S_k$ for each S_k, and fix an annulus W bounded by γ_0 and y_k. Then $\mathcal{A}(\gamma_0, \gamma)$ is a constant on S_k. We now determine a single-valued function $\mathcal{A}(\gamma_0, \gamma)$ in a neighborhood of S_k. For any $x_k \in S_k$ we choose an ϵ-ball O_ϵ of 0 in the Hilbert space $T_{x_k} W^2_r(S^1, \widetilde{M})$ such that the exponential map

$$\exp : O_\epsilon \to W^2_r(S^1, \widetilde{M})$$

identifies O_ϵ with a neighbourhood of x_k in $W^2_r(S^1, \widetilde{M})$. For any vector field $\eta \in O_\epsilon$ we choose the annulus $\{\exp t\eta | 0 \leq t \leq 1\}$. Then $\mathcal{A}(\gamma_0, \gamma)$ is a single-valued function in a neighborhood of S_k. By using the same method as in the proof of Proposition 3.6 one can show that there is a $W^2_r(S^1, \widetilde{M})$ neighbourhood O of S_k, for r large enough, such that the following inequality holds

$$\|\nabla \mathcal{A}(\gamma_0, \gamma)\|_{L^2} \geq C|\mathcal{A}(\gamma_0, \gamma) - \mathcal{A}(\gamma_0, x_k)|^{\frac{1}{2}}. \tag{5.10}$$

We identify $H^{-1}(I)$ with $I \times \widetilde{M}$. By a uniqueness theorem on symplectic forms (see McDuff, McCarthy- Wolfson [MW]) we may assume that the symplectic form on $\widetilde{M} \times I$ is expressed by

$$\omega = \pi^*(\tau_0 + t\Omega) - \alpha \wedge dt \tag{5.11}$$

where $\Omega := d\alpha$ is the curvature form, which is a 2-form on Z. We assume that the hypersurface $\widetilde{M} = H^{-1}(0)$ devides M into two parts M^+ and M^-. As in section 3, through a diffeomorphism we may consider $M^\pm$ as a manifold with cylindrical end:

$$M^+ = M_0^+ \cup \{[0,\infty) \times \widetilde{M}\}$$

$$M^- = M_0^- \cup \{(-\infty, 0] \times \widetilde{M}\}$$

with symplectic forms $\omega_{\phi^\pm}|_{M_0^\pm} = \omega$ and over the cylinder

$$\omega_{\phi^\pm} = \pi^*(\tau_0 + \phi^\pm\Omega) + (\phi^\pm)'\alpha \wedge da \tag{5.12}$$

where $\phi^+ : [1,\infty) \to [-\delta_0, 0)$ and $\phi^- : (-\infty, -1] \to (0, \delta_0]$ are functions such that

$$\phi^{\pm\prime} > 0,$$

$$\phi^+(1) = -\delta_0, \quad \lim_{a\to\infty} \phi^+(a) = 0,$$

$$\phi^-(-1) = \delta_0, \quad \lim_{a\to-\infty} \phi^-(a) = 0.$$

Denote by $\mathbf{\Phi}^\pm$ the set of all such functions $\phi^\pm$. Let $r > 0$. We construct M_r as in Section 4. Choose $\delta > \delta_0$, and a function

$$\phi_r : [0, 4r] \to [-\delta, \delta]$$

such that

$$\phi_r' > 0, \quad \phi_r(0) = -\delta, \quad \phi_r(2r) = 0, \quad \phi_r(4r) = \delta,$$

and for any $R > 0$

$$\phi_r(a^+)|_{[1,R]} \to \phi^+, \quad \phi_r(a^-)|_{[-R,-1]} \to \phi^-$$

uniformly, where $a^+ = 4r + a^-$. Let

$$\omega_{\phi_r} = \phi_r^*\omega. \tag{5.13}$$

Given an S^1-invariant compatible complex structure $\widetilde{J}$ for the symplectic vector bundle ξ we define an almost complex structure J on $\mathbf{R} \times \widetilde{M}$ as follows:

$$J\frac{\partial}{\partial a} = X_H, \quad JX_H = -\frac{\partial}{\partial a}, \tag{5.14}$$

$$J\xi = \xi \quad J|_{\xi} = \widetilde{J}. \tag{5.15}$$

By choosing δ small enough we may assume that $\widetilde{J}$ is tamed by $\tau_0 + t\Omega$, and there is a constant $C > 0$ such that

$$\tau_0(v, \widetilde{J}v) \leq C\left(\tau_0(v, \widetilde{J}v) + t\Omega(v, \widetilde{J}v)\right) \tag{5.16}$$

for all $v \in TZ, |t| \leq \delta$. Then J is ω-tamed. An almost complex structure J on $M^{\pm}$ is called S-ω-tamed if it is ω-tamed in the usual sense and (5.14) and (5.15) hold over the cylinder. We choose a S-ω-tamed almost complex structure J on $M^{\pm}$. Then for any $\phi^{\pm} \in \Phi^{\pm}$

$$\langle v, w\rangle_{\omega_{\phi^{\pm}}} = \frac{1}{2}\left(\omega_{\phi^{\pm}}(v, Jw) + \omega_{\phi^{\pm}}(w, Jv)\right) \qquad \forall \;\; v, w \in TM^{\pm}$$

defines a Riemannian metric on $M^{\pm}$. Note that $\langle \; , \; \rangle_{\omega_{\phi^{\pm}}}$ is not complete. We choose another metric $\langle \; , \; \rangle$ on $M^{\pm}$ such that

$$\langle \; , \; \rangle = \langle \; , \; \rangle_{\omega} \quad \textit{on} \;\; M_0^{\pm}$$

and over the tubes $(-\infty, -1) \times \widetilde{M}$ and $(1, \infty) \times \widetilde{M}$

$$\langle (a, v), (b, w)\rangle = \left(ab + \alpha(v)\alpha(w) + g_{\widetilde{J}}(\Pi v, \Pi w)\right). \tag{5.17}$$

It is easy to see that $\langle \; , \; \rangle$ is a complete metric on $M^{\pm}$. For any J-holomorphic curve $u : \Sigma \to M^{\pm}$ the energy $E(u)$ is defined by

$$E_{\phi^{\pm}}(u) = \int_{\Sigma} u^*\omega_{\phi^{\pm}}.$$

For any J-holomorphic curve $u : \Sigma \to \mathbf{R} \times \widetilde{M}$ we write $u = (a, \widetilde{u})$ and define

$$\widetilde{E}_{\phi^{\pm}}(u) = \int_{\Sigma} \widetilde{u}^*(\pi^*\tau_0). \tag{5.18}$$

One computes over the cylindrical part

$$u^*\omega_{\phi^{\pm}} = \phi^{\pm}\left(\tau_0 + \phi^{\pm} d\alpha\right)\left((\pi\widetilde{u})_s, (\pi\widetilde{u})_t)\right) + \phi^{\pm\prime}(a_s^2 + a_t^2), \tag{5.19}$$

which is a nonnegative integrand . The notion of finite energy J-holomorphic curves is defined as in Section 3. We can prove the following

Theorem 5.2 Let $u = (a, \widetilde{u}) : \mathbf{C} \to \mathbf{R} \times \widetilde{M}$ *be a J-holomorphic curve with finite energy. Then*

$$\lim_{s \to \infty} \widetilde{u}(s, t) = x(kt)$$

in $C^\infty(S^1)$ for some k-periodic orbit x. Moreover, we have exponential decay, i.e, the inequalities (3.40),(3.41) and (3.42) hold

Sketch of the Proof:

Step 1 By using the same method as in [H] and the estimate (5.16), one can prove the following two lemmas:

Lemma 5.3 Let $u = (a, \widetilde{u}) : \mathbf{C} \to \mathbf{R} \times \widetilde{M}$ be a J-holomorphic curve with finite energy. If $\int_\Sigma \widetilde{u}^*(\pi^*\tau_0) = 0$, Then u is a constant.

Lemma 5.4 Let $u = (a, \widetilde{u}) : \mathbf{C} \to \mathbf{R} \times \widetilde{M}$ be a nonconstant J-holomorphic curve with finite energy. Then for any sequence $s_i \to \infty$, there is a subsequence, still denoted by s_i, such that

$$lim_{i\to\infty} \widetilde{u}(s_i, t) = x(kt)$$

for some k-periodic orbit x.

Step 2 By repeating the same argument as in the proof of Theorem 3.7 , one can prove

$$\widetilde{E}(s) \leq \widetilde{E}(s_0)e^{-C^2(s-s_0)}, \tag{5.20}$$

$$\int_s^{s_i} \|\Pi u_t\|_{L^2} ds \leq \frac{1}{C}\left(\widetilde{E}(s)^{1/2} - \widetilde{E}(s_i)^{1/2}\right) \tag{5.21}$$

for all $s_i \geq s$ large enough.

Step 3 By Step 1, $\widetilde{u}(s_i, t)$ converges to a k-periodic orbit x. We may choose a local coordinate system $(\theta, y_1, ..., y_{2n})$, $0 \leq \theta \leq 1$, around x such that

$$x_k = \{0 \leq \theta \leq 1, y = 0\}$$

and

$$\alpha = d\theta + \beta, \tag{5.22}$$

where $\beta = \sum b_i dy_i$ is a 1-form on Z with $\beta(0) = 0$. In terms of the coordinates (a, θ, y), we have

$$\bar{a}_s - \bar{\theta}_t = h_1, \tag{5.23}$$

$$\bar{\theta}_s + \bar{a}_t = h_2 \tag{5.24}$$

where $\bar{a}(s,t) = a(s,t) - ks$, $\bar{\theta}(s,t) = \theta(s,t) - kt$, and h_1 and h_2 satisfy the exponential estimate

$$\|h_i\|_{L^2(S^1)} < C_3 e^{-C^2 s} \qquad for\ i = 1, 2. \tag{5.25}$$

By using the same argument as in Section Section 3 (see also [HWZ1]), one completes the proof of Theorem 5.2 $\square$

5.2 Compactness theorems

Recall that if we collapse the S^1-action on $\widetilde{M} = H^{-1}(0)$ we obtain symplectic cuts $\overline{M}^+$ and $\overline{M}^-$. The reduced space Z is a codimension 2 symplectic submanifold of both $\overline{M}^+$ and $\overline{M}^-$ (see [L]). There is another way to look at this. The length of every orbit of the S^1 action on $\widetilde{M}$ with respect to the metric $\langle\ ,\ \rangle_{\omega_{\phi^\pm}}$ is $\phi^{\pm\prime}(a)$, which converges to zero as $a \to \pm\infty$. Hence we can view the symplectic cuts $\overline{M}^+$ and $\overline{M}^-$ as the completions of $M^\pm$ with respect to this metric. Note that the almost complex structure on $M^\pm$ is invariant.

Let $(\Sigma; y_1, ..., y_m, p_1, ..., p_\nu) \in \mathcal{M}_{g,m+\nu}$, and $u : \overset{\circ}{\Sigma} \to M^\pm$ be a finite energy J-holomorphic curve. Suppose that $u(z)$ converges to a k_i-periodic orbit x_{k_i} as z tends to p_i. By using the removable singularity theorem we get a J-holomorphic curve $\bar{u}$ from Σ into $\overline{M}^\pm$. Let $A = [\bar{u}(\Sigma)]$. It is obvious that

$$E_{\phi^\pm}(u) = \omega(A) \tag{5.26}$$

which is independent of $\phi^\pm$. For a map u from Σ into $\mathbf{R} \times \widetilde{M}$, we let $A = [\pi u(\Sigma)]$. Then

$$\widetilde{E}_{\phi^\pm}(u) = \tau_0(A). \tag{5.27}$$

We say u *represents the homology class* A.

We fix a homology class $A \in H_2(M^\pm, \mathbf{Z})$ and a fixed set $\{k_1, ..., k_\nu\}$. For any J-holomorphic curves in $\mathcal{M}_A(M^\pm, g, m, \mathbf{k})$ there is a uniform bound on the energy E.. Similarly, for any J-holomorphic curves in $\mathcal{M}_A(\mathbf{R} \times \widetilde{M}, g, m, \mathbf{k}^+, \mathbf{k}^-)$ there is a uniform bound on the energy $\widetilde{E}$.

Next we consider the compactness theorems. The main part is the same as in Section 4, here we only explain differences. There is a difference between contact surgery and the present surgery: In case of contact surgery there is no closed J-holomorphic curve in the cylindrical part, because the symplectic form is exact in that part; while in case of the present surgery it does exist. Consider a sequence of J-holomorphic curves $(u_i, \Sigma_i; \mathbf{y}_i, \mathbf{p}_i) \in \mathcal{M}_A(M^\pm, g, m, \mathbf{k})$. We may assume that $(\Sigma_i; \mathbf{y}_i, \mathbf{p}_i)$ are smooth curves and converge to $(\Sigma; \mathbf{y}, \mathbf{p})$ in $\overline{\mathcal{M}}_{g,m+\nu}$. Denote by $P \subset \Sigma$ the set of singular points for u_i , the puncture points and the double points of Σ. The same argument as in Section 4 shows that u_i (with possible translation) converges uniformly with all derivatives on every compact subset of $\Sigma - P$ to a finite energy J-holomorphic curve $u : \Sigma - P \to M^{\pm}$(*or* $\mathbf{R}\times\widetilde{M}$). To carry out the bubbling analysis we must check which lemmas in Section 4 remain valid in the present surgery. Lemma 4.3 and Lemma 4.4 are obviously valid. For removable puncture points of u we construct the bubbles as in Section 4, and Lemma 4.8 is obviously valid. Since we have to consider curves in $\mathbf{R} \times \widetilde{M}$ Lemma 4.5 should be replaced by the following lemma:

Lemma 5.5 *There is a constant $\hbar_0 > 0$ such that for every finite energy J-holomorphic curve $u = (a, \widetilde{u}) : \overset{\circ}{\Sigma} \to \mathbf{R} \times \widetilde{M}$ with $\widetilde{E}(u) \neq 0$ we have $\widetilde{E}(u) \geq \hbar_0$.*

Proof: Consider the $\widetilde{J}$-holomorphic curve

$$\widetilde{v} = \pi\widetilde{u} : \overset{\circ}{\Sigma} \to Z.$$

$\widetilde{v}$ extends to a $\widetilde{J}$-holomorphic curve from Σ to Z. Then the assertion follows from a standard result for compact symplectic manifolds. □

Lemma 5.6 *There is a constant $\hbar > 0$ such that every rigid singular point y for u_i has mass*

$$m(y) \geq \hbar.$$

Proof: By using the standard rescaling argument we may construct for every rigid singular point a nonconstant J-holomorphic curve $v : \mathbf{C} \to M^{\pm}$(*or* $\mathbf{R} \times \widetilde{M}$). In case $u(\mathbf{C}) \bigcap M_0^{\pm} \neq \emptyset$ we use the argument as in Lemma 4.5. Now we assume that $u(\mathbf{C})$ lies in the cylindrical part. By Lemma 5.3 $\widetilde{E}(v) \neq 0$. Then we have $E(v) \geq \hbar_0$ by lemma 5.5. Then the assertion follows from (5.16). □

We consider a nonremovable singular point of u, say 0. We construct bubbles in a similar way as in Section 4 with small changes: Suppose that

$$\lim_{s\to-\infty} \widetilde{u}(s,t) = x(kt),$$

for some k-periodic orbit x. Since $\mathbf{k}$ is bounded by the cardinality of the intersection of A and Z, there are only finitely many components we have to consider. As in Subsection 5.1, we can make $\mathcal{A}(\gamma_0, \gamma)$ a single-valued function in a neighborhood of S_k. Then there is $\hbar' > 0$ such that either $\mathcal{A}(\gamma_0, x_k) = \mathcal{A}(\gamma_0', x_k')$ or $|\mathcal{A}(\gamma_0, x_k) - \mathcal{A}(\gamma_0', x_k')| > \hbar'$. Put

$$\hbar_0 = \min\{\hbar, \hbar'\}.$$

As in Section 4 we may assume that u_i is defined in the disk $D(\varepsilon)$ such that the center of mass of the measure $|\Pi du_i|^2$ is on the z-axis. For every i there exists $\delta_i > 0$ such that

$$E_\phi(u_i; D_{y_i}(\delta_i)) = m_0 - \frac{1}{2}\hbar_0.$$

Then $\delta_i \to 0$. Put $z = \delta_i w = \delta_i e^{r+2\pi i t}$. Define the J-holomorphic curve v_i by

$$v_i(r,t) = \left(a_i(\log\delta_i + r, t) - a_i(\log\delta_i, t_0), \widetilde{u}_i(\log\delta_i + r, t)\right).$$

Then Lemma 4.9 remains valid. It is easy to check that Lemma 4.11 and Lemma 4.15 are also hold. The same procedure of constructing bubbles and lemmas apply to double points.

We define the notion of relative stable holomorphic curves as in Definition 4.16. $\overline{\mathcal{M}}_A(M^+, g, m, \mathbf{k})$ has an obvious stratification indexed by the combinatorial type of the domain with the following data:

(1) the topological type of the domain as an abstract Riemann surface with marked points and puncture points;

(2) sets of periodic orbits corresponding to each puncture point;

(3) an integral 2-dimensional class A_i.

Suppose that $\mathcal{D}^{J,A,\mathbf{k}}_{g,m+\nu}$ is the set of indices.

Lemma 5.7 $\mathcal{D}^{J,A,\mathbf{k}}_{g,m+\nu}$ *is a finite set.*

Proof: By Lemma 5.6 and a standard argument one can show that there are finitely many combinatorial types of the domain as an abstract nodal surface and integral 2-dimensional classes A_i. The class A determines the bound on $\mathbf{k}$. Since for a J-holomorphic curve in

$$\mathcal{M}_{A_i}(\mathbf{R}\times\widetilde{M}, g, m, \mathbf{k}^+, \mathbf{k}^-)$$

the difference

$$\sum_j k_j^+ - \sum_j k_j^-$$

is determined by the class A_i, we conclude that the set of periods is finite. □

By using the above lemmas one immediately obtain

Theorem 5.8 $\overline{\mathcal{M}}_A(M^\pm, g, m)$ *is compact.*

Theorem 5.9 $\overline{\mathcal{M}}_A(M^\pm, g, m, \mathbf{k})$ *is compact.*

The definition of stable J-holomorphic curves in M_∞ is the same as in Subsection 4.3. Denote by $\overline{\mathcal{M}}_A(M_\infty, g, m)$ the moduli space of stable J-holomorphic curves in M_∞. By using the same method as in Section 4, one can prove the following convergence theorem:

Theorem 5.10 *Let $\Gamma_r \in \overline{\mathcal{M}}_A(M_r, g, m)$ be a sequence. Then there is a subsequence, which "weakly converges" to a stable J-holomorphic curve in $\overline{\mathcal{M}}_A(M_\infty, g, m)$.*

5.3 The Fredholm Index

Let $(\overset{\circ}{\Sigma}, p_1, ..., p_\nu)$ be a Riemann surface with ν ends and $\pi : E \to \overset{\circ}{\Sigma}$ be a $2n$-dimensional symplectic vector bundle with symplectic form ω. We assume that over each end p_i, using the cylindrical coordinates (s,t) near p_i, $\pi^{-1}(s,t)$ converges to a bundle $\pi : E_{(i,\infty)} \to S^1$ as $s \to \infty$. We choose a complex structure J and a Riemannian metric G on E, both of which are compatible with ω. There is a connection $\bigtriangledown$ on E which satisfies

$$d < \zeta, \eta > = < \bigtriangledown\zeta, \eta > + < \zeta, \bigtriangledown\eta > \tag{5.28}$$

$$\bigtriangledown(J\zeta) = J \bigtriangledown \zeta \qquad\qquad \forall \zeta, \eta \in E. \tag{5.29}$$

We introduce a Riemannian metric g on $\overset{\circ}{\Sigma}$ which is compatible with the complex structure j of $\overset{\circ}{\Sigma}$ and agrees with the standard tubular metric over each end. Using the metrics G and g we define $W^p_1(E)$ and $L^p(T^{0,1} \overset{\circ}{\Sigma} \otimes E)$ for $p > 2$.

Let $D : W^p_1(E) \to L^p(T^{0,1} \overset{\circ}{\Sigma} \otimes E)$ be a $\bar\partial$-operator. We choose a unitary trivialization

$$\overset{\circ}{\Sigma} \times \mathbf{R}^{2n} \to E$$

$$(p, \zeta) \mapsto \phi(p)\zeta,$$

near each pucture p_i, where $\phi = (\phi_1, \cdots, \phi_{2n})$ is a local unitary frame field on $\overset{\circ}{\Sigma}$ such that

$$J\phi = \phi J_0; \qquad \phi^*\omega = \omega_0; \tag{5.30}$$

$$< \phi\zeta, \phi\zeta' > = \zeta^T \zeta' \qquad \forall \zeta, \zeta' \in \mathbf{R}^{2n}. \tag{5.31}$$

In the coordinates $\zeta = \phi^{-1}\eta, \forall \eta \in W^p_1(E)$, the $\bar\partial$-operator D over a tube $(0,\infty) \times S^1$ is of the form

$$D\zeta = \frac{\partial\zeta}{\partial s} + J_0\frac{\partial\zeta}{\partial t} + S\zeta \tag{5.32}$$

where $S(s,t)$ is a $2n \times 2n$-matrix. We assume that

$$S(s,t) = S(s,t)^T = S(s,t+1). \tag{5.33}$$

$$S(s,t) \to S_\infty(t) \tag{5.34}$$

as $s \to \infty$ uniformly in t. Let

$$L_\infty = J_0 \frac{\partial}{\partial t} + S_\infty : W_1^p(S^1; \mathbf{R}^{2n}) \to L^p(S^1; \mathbf{R}^{2n}).$$

For any $\zeta(s,0) \in E_{(s,0)}$ let $\zeta(s,t)$ be the solution of the equation

$$J_0 \bigtriangledown_t \zeta + S(s,t)\zeta = 0 \tag{5.35}$$

with initial value $\zeta(s,0)$. This induces a linear map $\sigma(s,t) : E_{(s,0)} \to E_{(s,t)}, \zeta(s,0) \mapsto \zeta(s,t)$. For a given unitary trivialization ϕ, we define $\psi(s,t)$ by

$$\psi(s,t) = \phi^{-1}(t)\sigma(s,t)\phi(0). \tag{5.36}$$

Then $\psi(s,t)$ satisfies

$$\bigtriangledown_t \psi(s,t) = J_0 S(s,t)\psi(s,t), \qquad \psi(s,0) = I. \tag{5.37}$$

Let $s \to \infty$, we obtain

$$\psi(s,t) \to \psi_\infty(t) \tag{5.38}$$

$$\bigtriangledown_t \psi_\infty(t) = J_0 S_\infty(t)\psi_\infty(t) \tag{5.39}$$

$$\psi_\infty(0) = I. \tag{5.40}$$

It is easy to see that

$$\psi_\infty(t) \in S_p(2n, \mathbf{R}).$$

We call an end nondegenerate if $\psi_\infty(t) \in \mathcal{SP}^*$, where $\mathcal{SP}^*$=$\{\psi : [0,1] \to S_p(2n,\mathbf{R}) | \psi(0) = I, \det(I - \psi(1)) \neq 0\}$. Recall that $\psi_\infty \in \mathcal{SP}^*$ if and only if $0 \notin \sigma(L_\infty)$, the spectrum of L_∞. For a nondegenerate end we can define the Maslov index $\mu(\psi_\infty)$ (see[SZ]). If all ends are nondegenerate then D is a Fredholm operator. The proof is standard. So we can define in this case the Fredholm index

$$indD = \dim kerD - \dim cokerD. \tag{5.41}$$

Choose a global unitary trivialization

$$\overset{\circ}{\Sigma} \times \mathbf{R}^{2n} \to E$$

$$(p, \zeta) \mapsto \phi(p)\zeta.$$

The Maslov class $\mu(E, P)$ is defined to be

$$\mu(E, P) = \sum_{1}^{\nu} \mu(\psi_{\infty,i}). \tag{5.42}$$

The Maslov class $\mu(E, P)$ is independent of choices of trivializations. The following Theorems are well-known:

Theorem 5.11 *Let $D : W_1^p(E) \to L^p(T^{0,1} \overset{\circ}{\Sigma} \otimes E)$ be a $\bar{\partial}$-operator. Suppose that over each end*

$$S(s,t) = S(s,t)^T$$

$$S(s,t) \to S_\infty(t)$$

as $s \to \infty$ uniformly in t. If all ends are nondegenerate, then D is a Fredholm operator and

$$indD = n(2 - 2g - \nu) - \mu(E, P). \tag{5.43}$$

where g is the genus of Σ, and ν is the number of ends.

The index invariant $indD$ has a simple addition property. Let $(\overset{\circ}{\Sigma}_1, p_1, ..., p_\nu)$ and $(\overset{\circ}{\Sigma}_2, q_1, ..., q_\nu)$ be Riemann surfaces with ends. Let $\Sigma = \Sigma_1 \# \Sigma_2$ along the punctures. Given two symplectic vector bundles $E_1 \to \overset{\circ}{\Sigma}_1$ and $E_2 \to \overset{\circ}{\Sigma}_2$ with $E_{1,(i,\infty)} = E_{2,(i,\infty)}$. The composition $E = E_1 \# E_2$ is a vector bundle over Σ. Let D_1 on $W_1^p(E_1)$ and D_2 on $W_1^p(E_2)$ be two $\bar{\partial}$ -operators such that over each end $L_{1,(i,\infty)} = L_{2,(i,\infty)}$. Let $D = D_1 \# D_2$ denote the induced operator on E.

Theorem 5.12 *$indD = indD_1 + indD_2$.*

Now we turn to finite energy J-holomorphic curves in N, where N is one of $\{M^+, M^-, \mathbf{R} \times \widetilde{M}\}$. We consider the case $N = M^\pm$, for $N = \mathbf{R} \times \widetilde{M}$ the situation is identical. Let $(\Sigma; y_1, ..., y_m, p_1, ..., p_\nu) \in \mathcal{M}_{g,m+\nu}$, $\overset{\circ}{\Sigma} = \Sigma - \{p_1, ..., p_\nu\}$. Let $u : \overset{\circ}{\Sigma} \to N$ be a finite energy J-holomorphic curve. Suppose that $u(z)$ converges to a k_i-periodic orbit x_{k_i} with $k_i \in \mathbf{Z}$ as z tends to p_i. We introduce the holomorphic cylindrical coordinates (s, t) on Σ near each p_i and a Darboux coordinate (a, θ, w) on $\mathbf{R} \times \widetilde{M}$ near $x_i(t)$ and write

$$u(s,t) = (a(s,t), \widetilde{u}(s,t)).$$

Then

$$\widetilde{u}(s,t) \to x_i(k_i t) \quad as \;\; s \to \infty$$

We choose a local frame field $e_1 = \frac{\partial}{\partial a}, e_2 = X_H$ and $e_3, ..., e_{2n+2} \in \xi$. Over the tube the linearized operator

$$D_u = D\bar{\partial}_J(u) : C^\infty(\Sigma; u^*TN) \to \Omega^{0,1}(u^*TN)$$

takes the following form

$$D_u = \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S.$$

Denote by $\bigtriangledown$ the Levi-Civita connection with respect to the metric $\langle\ ,\ \rangle$. Then

$$S_{ij} = \left\langle e_i, \bigtriangledown_s e_j + J \bigtriangledown_t e_j + (\bigtriangledown_{e_j} J)\frac{\partial u}{\partial t} \right\rangle. \tag{5.45}$$

Recall that for a J-holomorphic curve u, D_u is independent of the choice of connection. Note that $S_{1j} = S_{j1} = S_{2j} = S_{j2} = 0$. Since

$$\frac{\partial \widetilde{u}}{\partial s} \to 0$$

$$\frac{\partial u}{\partial t} \to X_H$$

exponentially and uniformly in t as $s \to \pm\infty$ we have

$$S_{ij}(s,t) \to 0 \quad as \quad s \to \pm\infty \tag{5.46}$$

uniformly in t. Therefore the operator $L_s = J_0 \frac{d}{dt} + S$ converges to $L_\infty = J_0 \frac{d}{dt}$. Obviously, the operator D_u is not a Fredholm operator because over each end the operator $L_\infty = J\frac{D}{dt}$ has zero eigenvalue. The $\ker L_\infty$ consists of constant vectors. To recover a Fredholm theory we use weighted function spaces. We choose a weight α for each end . Fix a positive function W on $\overset{\circ}{\Sigma}$ which has order equal to $e^{\alpha|s|}$ on each end, where α is a small constant such that $0 < \alpha < \delta$ and over each end $L_\infty - \alpha = J_0 \frac{d}{dt} - \alpha$ is invertible. We will write the weight function simply as $e^{\alpha|s|}$. For any section $h \in C^\infty(\Sigma; u^*TN)$ and section $\eta \in \Omega^{0,1}(u^*TN)$ we define the norms

$$\|h\|_{1,p,\alpha} = \left(\int_\Sigma (|h|^p + |\nabla h|^p) d\mu\right)^{1/p} + \left(\int_\Sigma e^{2\alpha|s|}(|h|^2 + |\nabla h|^2) d\mu\right)^{1/2} \tag{5.47}$$

$$\|\eta\|_{p,\alpha} = \left(\int_\Sigma |\eta|^p d\mu\right)^{1/p} + \left(\int_\Sigma e^{2\alpha|s|}|\eta|^2 d\mu\right)^{1/2} \tag{5.48}$$

for $p \geq 2$, where all norms and covariant derivatives are taken with respect to the metric $\langle\ \ \rangle$ on u^*TN defined in (5.17) and the metric on Σ. Denote

$$\mathcal{C}(\Sigma; u^*TN) = \{h \in C^\infty(\Sigma; u^*TN); \|h\|_{1,p,\alpha} < \infty\}, \tag{5.49}$$

$$\mathcal{C}(u^*TN \otimes \wedge^{0,1}) = \{\eta \in \Omega^{0,1}(u^*TN); \|\eta\|_{p,\alpha} < \infty\}. \tag{5.50}$$

Denote by $W^{1,p,\alpha}(\Sigma; u^*TN)$ and $L^{p,\alpha}(u^*TN \otimes \wedge^{0,1})$ the completions of $\mathcal{C}(\Sigma; u^*TN)$ and $\mathcal{C}(u^*TN \otimes \wedge^{0,1})$ with respect to the norms (5.47) and (5.48) respectively. Let $h_{i0} = (b_{i0}, \widetilde{h}_{i0}) \in \ker L_{i\infty}$. Put

$$L_\infty = (L_{1\infty}, ..., L_{\nu\infty})$$

$$h_0 = (h_{10}, ..., h_{\nu 0}).$$

We fix a cutoff function ϱ:

$$\varrho(s) = \begin{cases} 1, & if\ |s| \geq L, \\ 0, & if\ |s| \leq \frac{L}{2} \end{cases}$$

where L is a large positive number. We can consider h_0 as a vector field defined in the Darboux coordinate neighborhood we introduced previously. We put

$$\hat{h}_0 = \varrho h_0.$$

Then $\hat{h}_0$ is a section in $C^\infty(\Sigma; u^*TN)$ supported in the tube $\{(s,t)||s| \geq \frac{L}{2}, t \in S^1\}$. Put

$$\mathcal{W}^{1,p,\alpha} = \{h + \hat{h}_0 | h \in W^{1,p,\alpha}, h_0 \in ker L_\infty\}.$$

The operator

$$D_u : \mathcal{W}^{1,p,\alpha} \to L^{p,\alpha}$$

is a Fredholm operator so long as α does not lie in the spectrum of the operator $L_{i\infty}$ for all $i = 1, \cdots, \nu,$. We thus have a Fredholm index $ind(D_u, \alpha)$.

Remark 5.13 The index $ind(D_u, \alpha)$ does not change if α is varied in such a way that α savoids the spectrum of $L_{\infty i}$. Conversely, the index will change if α is moved across an eigenvalue. We will choose α slightly larger than zero such that at each ends it does not across the first positive eigenvalue.

We next consider the addition formula for operator D_u. Let $u = (u^+, u^-) : (\Sigma^+, \Sigma^-) \to (M^+, M^-)$ be J-holomorphic curves such that u^+ and u^- have ν ends and they converge to the same periodic orbits at each end. Note that according to our convention $\Sigma^\pm$ may not be connected (see the end of Section 4). In this case $Ind(D_{u^\pm}, \alpha)$ denotes the sum of indices of its components. Suppose that $\Sigma = \Sigma^+ \wedge \Sigma^-$ has genus g and $[u(\Sigma)] = A$. Denote by j (resp. $j^\pm$) the complex structure on Σ

(resp. $\Sigma^{\pm}$), and by $Ind(D_u,j)$ (resp. $Ind(D_{u^{\pm}+,j^{\pm}},\alpha)$) the corresponding index. We have the index addition formula of Bott-type:

$$Ind(D_{u^+,j^+},\alpha)+Ind(D_{u^-,j^-},\alpha)-2(n+1)\nu=2C_1(A)+(n+1)(2-2g), \tag{5.51}$$

where we used the fact that at every end $\dim\ker L_\infty=2(n+1)$. The proof of the index addition formula (5.51) is basically a similar gluing argument as in an unpublished book of Donaldson. We will give the proof in the appendix. Considering the variation of the complex structures on the Riemann surfaces we have from (5.51) that

Theorem 5.14

$$Ind(D_{u^+},\alpha)+Ind(D_{u^-},\alpha)=2n\nu+2C_1(A)+(n+1)(2-2g)+6g-6. \tag{5.52}$$

Prof Let us cosider a simple case, the general case is identical. Suppose that Σ^+ is a connected smooth Riemann surface of genus g^+, and Σ^- consists of two components: Σ_1^- of genus g_1^- and Σ_2^- of genus g_2^-. Suppose that Σ^+ intersect with Σ_1^- at ν_1 points, intersect with Σ_2^- at ν_2 points. Σ^+ and Σ^- form a Riemann surface Σ of genus

$$g=g^++g_1^-+g_2^-+\nu_1+\nu_2-2.$$

We add $6g-6$ to both side of (5.51). Note that

$$Ind(D_{u^+},\alpha)=Ind(D_{u^+,j^+},\alpha)+6g^+-6+2(\nu_1+\nu_2),$$

$$Ind(D_{u^-},\alpha)=Ind(D_{u_1^-,j_1^-},\alpha)+6g_1^--6+2\nu_1+Ind(D_{u_2^-,j_2^-},\alpha)+6g_2^--6+2\nu_2.$$

Then (5.52) follows from the above equalities.□

Remark 5.15: *Let u be a J-holomorphic curve from $(\overset{\circ}{\Sigma};y_1,...,y_m,p_1,...,p_\nu)$ into $M^{\pm}$ such that each end converges to a periodic orbit. By using the removable singularity theorem we get a J-holomorphic curve $\bar{u}$ from Σ into $\overline{M}^{\pm}$. Therefore, we have a natural identification of finite energy pseudo-holomorphic curves in $M^{\pm}$ and closed pseudo-holomorphic curves in the closed symplectic manifolds $\overline{M}^{\pm}$. Moreover, the operator D_u is identified with the operator $D_{\bar{u}}$ in a natural way. Under this identification, the condition that u converges to a k-multiple periodic orbit at a marked point y is naturally interpreted as $\tilde{u}$ being tangent to Z at y with order k. Since $\ker L_\infty$ consists of constant vectors, we can identify the vector fields in $\mathcal{W}^{1,p,\alpha}_{\pm}$ along u with the vector fields in $\mathcal{W}^{1,p,\alpha}_{\pm}$*

along $\bar{u}$, the space $L^{p,\alpha}_{\pm}$ along u is also identified with the space $L^{p,\alpha}_{\pm}$ along $\bar{u}$. In case of closed manifolds the definitions of $L^{p,\alpha}_{\pm}$ and $\mathcal{W}^{1,p,\alpha}_{\pm}$ are the same as that of [Liu]).

Thus we have

Proposition 5.16

$$Ind(D_u,\alpha) = IndD_{\bar{u}}. \tag{5.53}$$

6 A gluing theorem

Consider the symplectic cutting. In this section we prove a gluing theorem for two solutions, with ends converging to the same periodic orbits, of the stabilization equation.

6.1 Stabilization equation

We use the notations in Subsection 5.3. We are going to construct a local virtual neighborhood around (Σ, u) and a stabilization equation $\mathcal{S}_e = 0$. We first consider the case $N = M^{\pm}$.

When $2g + m + \nu \geq 3$, Σ is stable and $\mathcal{M}_{g,m+\nu}$ is a V-manifold. Hence, the automorphism group Aut_Σ of Σ is finite. Furthermore, there is an Aut_Σ-equivariant holomorphic fiber bundle

$$\pi_\Sigma : U_\Sigma \to O_\Sigma \tag{6.1}$$

such that O_Σ/Aut_Σ is a neighborhood of Σ in $\mathcal{M}_{g,m+\nu}$ and fiber $\pi_\Sigma^{-1}(b) = b$. We may choose a local smooth trivialization

$$\phi_\Sigma : U_\Sigma \to O_\Sigma \times \Sigma \tag{6.2}$$

such that O_Σ is the set of complex structures on $(\Sigma, \mathbf{y}, \mathbf{p})$. Let $stb_u \subset Aut_\Sigma$ be the subgroup preserving u. Then the metric $u^*\langle\ \rangle$ on u^*TN is obviousely stb_u-invariant. By averaging we may assume that both the weight function W and the cutoff function ϱ are stb_u-invariant. The group stb_u acts on $\mathcal{C}(\Sigma; u^*TN)$ and $\mathcal{C}(u^*TN \otimes \wedge^{0,1})$ in a natural way. Then a neighborhood $\mathcal{U}_{\Sigma,u}$ of u can be described as

$$O_\Sigma \times \{\exp_u(h + \hat{h}_0); h \in \mathcal{C}(\Sigma; u^*TN), h_0 \in \ker L_\infty, \|h\|_{1,p,\alpha} + |h_0| < \epsilon\}/stb_u. \tag{6.3}$$

$\mathcal{U}_{\Sigma,u}$ has a natural Frechet V-manifold structure. Since only a finite group is involved, it is obviously Hausdorff.

For the case $g = 0, m + \nu \leq 2, A \neq 0$, Σ is no longer stable and the automorphism group Aut_Σ is infinite. Here, we fix our marked points at 0 or 0, 1. First of all, stb_u is finite for any $u \in Map_A(N, 0, m+\nu)$ with $A \neq 0$. Locally

$$\mathcal{U}_{\Sigma,u} = Map_A(N, 0, m+\nu)/Aut_\Sigma. \tag{6.4}$$

By using the same argument as in [R5] one can shows that $\mathcal{U}_{\Sigma,u}$ is Hausdorff. Furthermore, one can construct a slice W_u of the action Aut_Σ such that W_u/stb_u is a neighborhood of (Σ, u). We outline the construction of the slice W_u. We marked extra points e_i such that $du(e_i)$ is of maximal rank, and (Σ, e_i) has three marked points. For simplicity, we assume that we only need one extra marked point e_1 to stabilize Σ. The construction of the case with two extra marked points is the same. For the extra marked point e_1, $du(e_1)$ is a 2-dimensional vector space. Clearly,

$$Q_{(u,e_1)} = \oplus_{\tau \in stb_u} du(\tau(e_1)) \subset (T_{u(e_1)}N)^{|stb_u|}$$

is stb_u-invariant. Now we want to construct a 2-dimensional subspace $E_{e_1} \subset Q_{(u,e_1)}$ which is the orbit of action Aut_Σ. In this case, a neighborhood of id in Aut_Σ can be identified with a neighborhood of e_1 by the relation $\tau_x(e_1) = x$ for $x \in D^2(e_1)$. $\frac{d}{dx}\tau_x(f)(y)|_{x=e_1} = du(y)(v(y))$, where $v = \frac{d}{dx}\tau_x|_{x=e_1}$ is a holomorphic vector field. By our identification, v is determined by its value $v(e_1) \in T_{e_1}S^2$. Given any $v \in T_{e_1}S^2$, we use $v_{e_1} \in T_{id}Aut_\Sigma$ to denote its extension. Therefore, v determines $v_{e_1}(\tau(e_1))$. Put

$$E_{e_1} = \{\oplus_{\tau \in stb_u} du(\tau(e_1))(v_{e_1}(\tau(e_1))); v \in T_{e_1}S^2\} \tag{6.5}$$

It is easy to check that E_{e_1} is indeed stb_u-invariant. We can identify E_{e_1} with $T_{e_1}S^2$ by

$$v \to \oplus_{\tau \in stb_u} du(\tau(e_1))(D\tau Ad_\tau(v)), \tag{6.6}$$

Hence, E_{e_1} is 2-dimensional. Given any $h \in \mathcal{C}(\Sigma; u^*TN)$, we say that $h \perp E_{e_1}$ *if* $\oplus_{\tau \in stb_u} h(\tau(e_1))$ *is orthogonal to* E_{e_1}. The slice W_u can be constructed as

$$W_u = \exp_u\{h \in \mathcal{C}(\Sigma; u^*TN); \|h\|_{1,p,\alpha} < \epsilon, \|w\|_{C^1(D_{\delta_0}(g(e_1)))} < \epsilon \text{ for } g \in stb_u, h \perp E_{e_1}\}, \tag{6.7}$$

where δ_0 is a small fixed constant. It is shown in [R5] that W_u has the following properties:

(1) W_u is invariant under stb_u;

(2) If $h\tau \in W_u$ for $h \in W_u$, then $\tau \in stb_u$;

(3) There is a neighborhood U of $id \in Aut$ such that the multiplication

$$F : U \times W_u \to Map_A(N, 0, m + \nu)$$

is a homeomorphism onto a neighborhood of u.

W_u/stb_u has obviously a Frechet V-manifold structure. We can write again

$$\mathcal{U}_{\Sigma,u} = O_\Sigma \times W_u/stb_u$$

with $O_\Sigma = point$.

Next we consider the case $N = \mathbf{R} \times \widetilde{M}$. Let $(\Sigma, \mathbf{y}, \mathbf{p}, \mathbf{q}) \in \mathcal{M}_{g^c, m+\nu+l}$ be a Riemann surface of genus g^c with m marked points and $\nu + l$ ends. Let $u : \overset{\circ}{\Sigma} \to N$ be a finite energy J-holomorphic curve. Suppose that

$$\widetilde{u}(z) \to k_i - periodic\ \ orbit\ \ x_i,\ \ a(z) \to -\infty\ \ as\ \ z \to p_i$$

$$\widetilde{u}(z) \to k'_j - periodic\ \ orbit\ \ x'_j,\ \ a(z) \to \infty\ \ as\ \ z \to q_j.$$

We call p_i *a negative* end, and q_j *a positive* end. We must mod the group G_0 generated by the S^1-action and the translation along $\mathbf{R}$. We fix a point $y_0 \in \overset{\circ}{\Sigma}$ different from the marked points. Fix a local coordinate system a, θ, w on $\mathbf{R} \times \widetilde{M}$ such that $u(y_0) = (0, 0, 0)$. If $(\Sigma, \mathbf{y}, \mathbf{p}, \mathbf{q})$ is stable, then

(6.8)
$$\mathcal{U}_{\Sigma,u} = O_\Sigma \times \{\exp_u(h + \hat{h}_0); h \in \mathcal{C}(\Sigma; u^*TN), h_0 \in \ker L_\infty,$$
$$h(y_0) = (0, 0, *), \|h\|_{1,p,\alpha} + |h_0| < \epsilon\}/stb_u.$$

If $(\Sigma, \mathbf{y}, \mathbf{p}, \mathbf{q})$ is unstable, then

(6.9)
$$\mathcal{U}_{\Sigma,u} = pt \times \{\exp_u(h + \hat{h}_0); h \in \mathcal{C}(\Sigma; u^*TN), h_0 \in \ker L_\infty,$$
$$h \perp E_{e_i^u}, h(y_0) = (0, 0, *), \|h\|_{1,p,\alpha} + |h_0| < \epsilon\}/stb_u.$$

$\mathcal{U}_{\Sigma,u}$ is obviously a Frechet V-manifold.

Now we construct the stabilization equation $\mathcal{S}_e = 0$. We have an infinite dimensional vector bundle $\mathcal{F} \to \mathcal{U}_{\Sigma,u}$ whose fiber at (j, v) is the space

$$\mathcal{F}_{j,v} = \mathcal{C}(v^*TN \otimes \wedge^{0,1}).$$

The $\bar{\partial}$-operator $\bar{\partial}$ defines a section of this vector bundle. $Coker D_u \bar{\partial}_J$ is a finite dimensional subspace of $\mathcal{C}(u^*TN \otimes \wedge^{0,1})$ invariant under stb_u. We choose a stb_u-invariant cut-off function vanishing in a small neighborhood of the ends. Then we multiple it to the element of $CokerD_u\bar{\partial}_J$ and denote the resulting finite dimensional space by F_u. By the construction, F_u is stb_u-invariant. When the neighborhood is small enough, F_u will have the same dimension as $CokerD_u$ and

$$D_u + Id : \mathcal{C}(\Sigma; u^*TN) \to \mathcal{C}(u^*TN \otimes \wedge^{0,1}) \tag{6.10}$$

is surjective. We extend each element η of F_u to a smooth section of $\mathcal{C}(\exp_u(h+\hat{h_0})^*TN \otimes \wedge^{0,1})$ by parallel transportation along $exp_u t(h+\hat{h_0})$. Then η can be considered as a smooth section of the bundle $\mathcal{F} \to \mathcal{U}_{\Sigma,u}$. Thus the identity map of F_u is extended to a stb_u-invariant bundle map from $\mathcal{U}_{\Sigma,u} \times F_u$ to $\mathcal{F}$. Our stabilization equation will be of the form

$$\mathcal{S}_e = \bar{\partial}_u + \eta_u : \mathcal{U}_{\Sigma,u} \times F_u \to \mathcal{F}. \tag{6.11}$$

In the following we will denote by $F_{u^\pm}, F_{u^c}, S_e^\pm, S_e^c$, etc the spaces, sections, ... corresponding to $M^\pm$ and $\mathbf{R} \times \widetilde{M}$. Put

$$\mathcal{M}^\pm_{\mathcal{S}_e}(A^\pm, k_1, ..., k_\nu) = (\mathcal{S}_e^\pm)^{-1}(0)/stb_{u^\pm}, \tag{6.12}$$

$$\mathcal{M}^c_{\mathcal{S}_e}(A^c, k_1, ..., k_\nu; k_1', ..., k_l') = (\mathcal{S}_e^c)^{-1}(0)/stb_{u^c}. \tag{6.13}$$

6.2 Pre-gluing

For simplicity we shall glue two smooth Riemann surfaces with one end. The general case can be reduced to this case. We first glue curves in M^+ and M^-. Denote

$$\overline{\mathcal{M}}(A^+, A^-, g^+, g^-, m^+, m^-, k) = \tag{6.14}$$

$$\left\{(\Gamma^+, \Gamma^-) \in \overline{\mathcal{M}}_{A^+}(M^+, g^+, m^+, k) \times \overline{\mathcal{M}}_{A^-}(M^-, g^-, m^-, k) \mid P^+(\Gamma^+) = P^-(\Gamma^-)\right\}.$$

Let $(\Sigma, u) \in \overline{\mathcal{M}}(A^+, A^-, g^+, g^-, m^+, m^-, k)$. Suppose that $\Sigma = \Sigma^+ \wedge \Sigma^-$ with intersection point p, where $\Sigma^\pm$ are smooth Riemann surfaces with $m^\pm$ marked points and of genus $g^\pm$. We have

$$g = g^+ + g^-$$

$$m = m^+ + m^-.$$

Let $j^\pm$ be the complex structure of $\Sigma^\pm$, which is the standard complex structure i near p or marked points. We write

$$\Sigma^+ - \{p\} = \overset{\circ}{\Sigma^+} = \Sigma_0^+ \bigcup \{[0,\infty) \times S^1\},$$

$$\Sigma^- - \{p\} = \overset{\circ}{\Sigma^-} = \Sigma_0^- \bigcup \{(-\infty, 0] \times S^1\}.$$

We fix a Kahler metric on each component such that over the tube it is the standard tube metric. Let $u = (u^+, u^-)$, where $u^\pm : \Sigma^\pm \to M^\pm$ are stable holomorphic curves such that p is a nonremovable singular point. Moreover, $u^+(z^+)$ and $u^-(z^-)$ converge to the same k-periodic orbit $x(t)$ with $k > 0$ as $z^\pm \to p$. To be more precisely, we introduce the holomorphic cylindrical coordinates $(s^\pm, t^\pm)$ on $\Sigma^\pm$ near p and a Darboux coordinate (θ, w) on $\widetilde{M}$ near $x(t)$ and write

$$u^\pm(s^\pm, t^\pm) = \left(a^\pm(s^\pm, t^\pm), \widetilde{u}^\pm(s^\pm, t^\pm)\right).$$

Then

$$\widetilde{u}^+(s^+, t^+) \to x(kt^+) \quad as \;\; s^+ \to \infty$$

$$\widetilde{u}^-(s^-, t^-) \to x(kt^-) \quad as \;\; s^- \to -\infty.$$

It was shown in Section 5 that there are constants $\ell^\pm, \theta_0^\pm, \delta > 0$ such that for all $r = (r_1, r_2) \in \mathbf{Z^2}$ the following hold :

$$|\partial^r[a^\pm(s^\pm, t^\pm) - ks^\pm - \ell^\pm]| \le c_r e^{-\delta|s^\pm|}, \tag{6.15}$$

$$|\partial^r[\theta(s^\pm, t^\pm) - kt^\pm - \theta_0^\pm]| \le c_r e^{-\delta|s^\pm|}, \tag{6.16}$$

$$|\partial^r w^\pm(s^\pm, t^\pm)| \le c_r e^{-\delta|s^\pm|}, \tag{6.17}$$

where c_r are constants depending on $u = (u^+, u^-)$, δ is a constant independent of $u = (u^+, u^-)$. Since u^+ , u^- converge to the same k-periodic orbit $(kt, 0, 0)$, we have

$$\theta^- = \theta^+ \;\; mod\, 1$$

,

$$t^+ = t^- + \frac{\theta_0^- - \theta_0^+ + n}{k} \tag{6.18}$$

for some constant $n \in Z_k$, where $Z_k = \{0, 1, ..., k-1\}$.

We have constructed a local virtual neighborhood and a stabilization equation around $(\Sigma^\pm, u^\pm)$ in the previous subsection. Put

$$\mathcal{U}_{u^+,u^-} = \left\{(a, b) \in \mathcal{U}_{\Sigma^+,u^+} \times \mathcal{U}_{\Sigma^-,u^-} | P^+(a) = P^-(b)\right\}, \tag{6.19}$$

(6.20) $$\widetilde{\mathcal{U}}_{u^+,u^-} = \left\{(a,b,n) \mid (a,b) \in \mathcal{U}_{u^+,u^-}, n \in Z_k\right\}.$$

The group $stb_{u^+} \times stb_{u^-}$ acts on $\mathcal{U}_{u^+,u^-}$ and $\widetilde{\mathcal{U}}_{u^+,u^-}$ by

$$(\tau^+, \tau^-)(a,b) = (\tau^+ a, \tau^- b)$$

$$(\tau^+, \tau^-)(a,b,n) = (\tau^+ a, \tau^- b, n).$$

It is obvious that both $\mathcal{U}_{u^+,u^-}$ and $\widetilde{\mathcal{U}}_{u^+,u^-}$ are Hausdorff Frechet V-manifolds. The infinite dimensional vector bundles $\mathcal{F} \to \mathcal{U}_{\Sigma^+,u^+}$ and $\mathcal{F} \to \mathcal{U}_{\Sigma^-,u^-}$ are glued as a bundle $\mathcal{F} \to \widetilde{\mathcal{U}}_{u^+,u^-}$. The $\bar\partial$-operator $\bar\partial$ is extended over $\widetilde{\mathcal{U}}_{u^+,u^-}$ in a natural way. Put

(6.21) $$\mathcal{W}^{1,p,\alpha}_{\pm} = \{h^{\pm} + \hat h_0 | h^{\pm} \in W^{1,p,\alpha}_{\pm}, h_0 \in \ker L_\infty\}.$$

$\mathcal{W}^{1,p,\alpha}_{\pm}$ is a Banach space. By construction, the derivative

$$D\mathcal{S}^{\pm}_e = D_{u^\pm} + Id : O_{\Sigma^\pm} \times \mathcal{W}^{1,p,\alpha}_{\pm} \times F^{\pm} \to L^{p,\alpha}_{\pm}$$

is a surjective map. Denote

(6.22) $$D\mathcal{S}_{e,u^+,u^-} = (D\mathcal{S}^+_e, D\mathcal{S}^-_e),$$

(6.23) $$\mathcal{W}^{1,p,\alpha}_{u^+,u^-} = \{(h^+ + \hat h^+_0, h^- + \hat h^-_0) \in \mathcal{W}^{1,p,\alpha}_{+} \times \mathcal{W}^{1,p,\alpha}_{-} | h^+_0 = h^-_0\}.$$

Note that the elements of $F_{u^\pm}$ are $L^{p,\alpha}$ sections of the bundle $(u^\pm)^* TM^\pm \to \Sigma^\pm$, with support away from singular points. We can consider $F = F^+ \times F^-$ as a finite dimensional subbundle of the bundle $\mathcal{F} \to \widetilde{\mathcal{U}}_{u^+,u^-}$. In general

$$D\mathcal{S}_{e,u^+,u^-} : O_{\Sigma^+} \times O_{\Sigma^-} \times \mathcal{W}^{1,p,\alpha}_{u^+,u^-} \times F \to L^{p,\alpha}_{+} \times L^{p,\alpha}_{-}$$

is not surjective. But we can choose another finite dimensional space $F^{\pm} \in L^{p,\alpha}_{\pm}$ such that the above map is surjective. Let

(6.24) $$\mathcal{M}_{\mathcal{S}_e}(A^+, A^-) = \left\{(a,b) \in \mathcal{M}^+_{\mathcal{S}_e}(A^+, k) \times \mathcal{M}^-_{\mathcal{S}_e}(A^-, k) | P^+(a) = P^-(b)\right\}.$$

(6.25) $$\widetilde{\mathcal{M}}_{\mathcal{S}_e}(A^+, A^-) = \left\{(a,b,n) \mid (a,b) \in \mathcal{M}_{\mathcal{S}_e}(A^+, A^-), n \in Z_k\right\}.$$

Then both $\mathcal{M}_{\mathcal{S}_e}(A^+, A^-)$ and $\widetilde{\mathcal{M}}_{\mathcal{S}_e}(A^+, A^-)$ are smooth manifolds of finite dimension. There is a family of complex structures on $\Sigma^\pm$, parametrized by a neighborhood of 0 in $\frac{\mathbf{C}^{3g^\pm - 3 + m^\pm}}{Aut(\Sigma^\pm, m^\pm)}$. Let

$O_{\pm,\epsilon,deform} = \{\xi \in \mathbf{C}^{3g^{\pm}-3+m^{\pm}} ||\xi| < \epsilon\}$. We change the complex structure in a compact set $K_{\pm,deform}$ of $\Sigma^{\pm}$ away from the singular or marked points. Let $K_{\pm,obstr}$ be a compact subset of $\Sigma^{\pm}$ away from singular points such that each element of $F_{u^{\pm}}$ support in $K_{\pm,obstr}$.

We glue M^+ and M^- with parameter r to get M_r as in Section 4. Now we construct a Riemann surface $\Sigma_{\xi,r} = \Sigma^+ \#_r \Sigma^-$ as follows. We cut off the part of $\Sigma^{\pm}$ with cylindrical coordinate $|s^{\pm}| > \frac{3r}{k}$ and glue the remainders along the collars of length $\frac{2r}{k}$ of the cylinders. We have the following gluing formulas:

$$a^+ = a^- + 4r$$

$$\theta^- = \theta^+ \ \ mod\, 1$$

$$s^+ = s^- + \frac{4r + \ell^- - \ell^+}{k}$$

$$t^+ = t^- + \frac{\theta_0^- - \theta_0^+ + n}{k}$$

for some $n \in Z_k$. Without loss of generality we assume in the following that $\ell^+ = \ell^-$, $\theta_0^- = \theta_0^+$. Put $O_\epsilon = O_{+,\epsilon,deform} \times O_{-,\epsilon,deform}$. For any element $(\xi_+, \xi_-) \in O_\epsilon$ and $n \in Z_k$ we put $\xi = (\xi_+, \xi_-, n)$. We get a curve $\Sigma_{\xi,r}$. Now we perturb the map (u^+, u^-) to get a map u_r from $\Sigma_{\xi,r}$ to M_r. We set

$$u_r = \begin{cases} u^+ \quad on \;\; \Sigma_0^+ \bigcup \{(s^+,t^+) | 0 \le s^+ \le \frac{r}{k}, t^+ \in S^1\} \\ \\ (ks^+, x(kt^+)) \quad on \;\; \{(s^+,t^+) | \frac{3r}{2k} \le s^+ \le \frac{5r}{2k}, t^+ \in S^1\} \\ \\ u^- \quad on \;\; \Sigma_0^- \bigcup \{(s^-,t^-) | 0 \ge s^- \ge -\frac{r}{k}, t^- \in S^1\} \\ \end{cases}$$

To define the map u_r in the remaining part we fix a smooth cutoff function $\beta : \mathbf{R} \to [0,1]$ such that

$$\beta(s) = \begin{cases} 1 & if \;\; s \ge 1 \\ 0 & if \;\; s \le 0 \end{cases}$$

and $|\beta'(s)| \le 2$. We assume that r is large enough such that $u^{\pm}$ maps the tube $\{(s^{\pm}, t^{\pm}) ||s^{\pm}| \ge \frac{r}{k}, t^{\pm} \in S^1\}$ into a domain with Darboux coordinates (a, θ, w) on $\mathbf{R} \times \widetilde{M}$ near x(t). We write in this coordinate system

$$u^{\pm} = (a^{\pm}, \widetilde{u}^{\pm}) = (a^{\pm}, \theta^{\pm}, w^{\pm}) \,,$$

$$u_r = (a_r, \widetilde{u}_r) = (a_r, \theta_r, w_r)$$

and define

$$a_r^+ = ks^+ + \left(\beta(3 - \frac{2ks^+}{r})(a^+(s^+,t^+) - ks^+) + \beta(\frac{2ks^+}{r} - 5)(a^-(s^-,t^+) - ks^-)\right)$$

$$\widetilde{u}_r = x(kt^+) + \left(\beta(3 - \frac{2ks^+}{r})(\widetilde{u}^+(s^+,t^+) - x(kt^+)) + \beta(\frac{2ks^+}{r} - 5)(\widetilde{u}^-(s^-,t^+) - x(kt^+))\right).$$

For any $\eta \in \Omega^{0,1}(\Sigma_r; u_r^* TM_r)$, let $\eta_{(r,\pm)}$ be its restriction to the part $\Sigma_0^\pm \bigcup \{(s^\pm, t^\pm) |\ |s^\pm| < \frac{3r}{k}$. We define

$$\|\eta\|_{p,\alpha,r} = \|\eta_{(r,+)}\|_{p,\alpha} + \|\eta_{(r,-)}\|_{p,\alpha}.$$

For any $h \in C^\infty(\Sigma_r; u_r^* TM_r)$ let

$$h_0 = \int_{S^1} h(\frac{2r}{k}, t)dt.$$

We decomposite h uniquely into $h = h' + \hat{h}_0$, where $\hat{h}_0$ is defined from $h_0 \in \ker L_\infty$ as in Section 5. Then we define

$$\|h\|_{1,p,\alpha,r} = \|h'_{(r,+)}\|_{1,p,\alpha} + \|h'_{(r,-)}\|_{1,p,\alpha} + |h_0|.$$

We denote the resulting completed spaces by $L_r^{p,\alpha}$ and $W_r^{1,p,\alpha}$ respectively.

We choose a large number R_0 and let $\frac{r}{k} \geq R_0$. Put $\mathcal{N}_{R_0,r} = \{(s^+,t) | R_0 \leq s^+ \leq \frac{2r}{k}, t \in S^1\} \bigcup \{(s^-,t) | -R_0 \geq s^- \geq -\frac{2r}{k}, t \in S^1\}$. We may assume that both $K_{\pm,deform}$ and $K_{\pm,obstr}$ are away from $\mathcal{N}_{R_0,r}$. Since the elements of $F_{u^\pm}$ are $L^{p,\alpha}$ sections of the bundle $(u^\pm)^* TM^\pm \to \Sigma^\pm$, which have support away from singular points, we may consider the elements of $F := F_{u^+} \times F_{u^-}$ as sections of $u_r^* TM_r \otimes \wedge^{0,1}$. We have a stabilization equation $\mathcal{S}_{e,\xi,r}(u,\eta) = 0$. By using the exponential decay of $u^\pm$ one can easily prove the following lemma:

Lemma 6.1

$$\|\mathcal{S}_{e,\xi,r}(u_r,\eta)\|_{p,\alpha,r} \leq C|\xi| < C\epsilon \quad on \ \ \Sigma_{\xi,r} - \mathcal{N}_{R_0,r}; \tag{6.26}$$

$$\|\bar{\partial}_{\Sigma_{\xi,r}}(u_r)\|_{p,\alpha,r} \leq Ce^{-(\delta-\alpha)R_0} \quad on \ \ \mathcal{N}_{R_0,r}. \tag{6.27}$$

The constants C in the above estimates are independent of r.

Thus we have a family of approximate solutions of the stabilization equation.

6.3 An estimate for the right inverse

Consider the linearized operator

$$D\mathcal{S}_{e,\xi,r} = D_{u_r} + Id : O_{\Sigma_r} \times W_r^{1,p,\alpha} \times F \to L_r^{p,\alpha}.$$

We shall prove that $D\mathcal{S}_{e,\xi,r}$ is surjective and there are uniform right inverse $Q_{\xi,r}$ for r large enough and ϵ small enough. The proof is almost a copy of McDuff- Salamon's in [MS]. By our construction,

$$D\mathcal{S}_{e,u^+,u^-} : O_{\Sigma^+} \times O_{\Sigma^-} \times \mathcal{W}^{1,p,\alpha}_{u^+,u^-} \times F \to L^{p,\alpha}_+ \times L^{p,\alpha}_-$$

is surjective and there is a right inverse Q_{u^+,u^-}. For any element $\xi = (\xi_+, \xi_-) \in O_\epsilon$ we first construct an approximate right inverse $Q'_{\xi,r}$ such that

$$\|Q'_{\xi,r}\| \leq C \tag{6.28}$$

and

$$\|D\mathcal{S}_{e,\xi,r}Q'_{\xi,r} - Id\| \leq \frac{1}{2}. \tag{6.29}$$

Then the operator $D\mathcal{S}_{e,\xi,r}Q'_{\xi,r}$ is invertible and a right inverse $Q_{\xi,r}$ is given by

$$Q_{\xi,r} = Q'_{\xi,r}(D\mathcal{S}_{e,\xi,r}Q'_{\xi,r})^{-1}. \tag{6.30}$$

Given $\eta \in L^{p,\alpha}_r$ we define the pair $(\eta^+, \eta^-) \in L^{p,\alpha}_+ \times L^{p,\alpha}_-$ by

$$\eta^+(s^+,t^+) = \begin{cases} \eta & on \;\; \Sigma^+_0 \bigcup \{(s^+,t^+) | 0 \leq s^+ \leq \frac{2r}{k}, t^+ \in S^1\} \\ \\ 0 & if \;\; s^+ \geq \frac{2r}{k} \end{cases}$$

$$\eta^-(s^-,t) = \begin{cases} \eta & on \;\; \Sigma^-_0 \bigcup \{(s^-,t^-) | 0 \geq s^- \geq -\frac{2r}{k}, t^- \in S^1\} \\ \\ 0 & if \;\; s^- \leq -\frac{2r}{k} \end{cases}$$

Then $\Big((\xi'_+, \xi'_-), (h^+ + \hat{h}_0, h^- + \hat{h}_0); (\zeta^+, \zeta^-)\Big)$ is defined in terms of the right inverse Q_{u^+,u^-} by

$$\Big((\xi'_+, \xi'_-), (h^+ + \hat{h}_0, h^- + \hat{h}_0); (\zeta^+, \zeta^-)\Big) = Q_{u^+,u^-}(\eta^+, \eta^-).$$

We define $h = Q'_{\xi,r}\eta$ by

$$h = \begin{cases} h^+ + \hat{h}_0 & on \;\; \Sigma^+_0 \bigcup \{(s^+,t^+) | 0 \leq s^+ \leq \frac{3r}{2k}, t^+ \in S^1\} \\ \\ h^+ + \hat{h}_0 + \beta\left(\frac{2ks^+}{r} - 3\right)h^- & on \;\; \{(s^+,t^+) | \frac{3r}{2k} \leq s^+ \leq \frac{2r}{k}\}, t^+ \in S^1 \\ \\ h^- + \hat{h}_0 + \beta\left(\frac{-2ks^-}{r} - 3\right)h^+ & on \;\; \{(s^-,t^-) | \frac{-3r}{2k} \geq s^- \geq \frac{-2r}{k}\} \\ \\ h^- + \hat{h}_0 & on \;\; \Sigma_0 \bigcup \{(s^-,t^-) | 0 \geq s^- \geq -\frac{3r}{2k}, t^- \in S^1\} \end{cases}$$

Lemma 6.2 . *The operator $Q'_{\xi,r}$ satisfies the estimates (6.28) and (6.29).*

Proof: The first inequality is easy. We prove the second inequality. We must prove that

$$\|D\mathcal{S}_{e,\xi,r}h - \eta\|_{p,\alpha,r} \leq \frac{1}{2}\|\eta\|_{p,\alpha,r}.$$

By our construction we have

$$D\mathcal{S}_e^{\pm}(h^{\pm} + \hat{h}_0) = \eta^{\pm} \tag{6.31}$$

$$\|D\mathcal{S}_{e,\xi,r} - DS_e^{\pm}\| \leq C|\xi| \tag{6.32}$$

on $\Sigma_0^{\pm} \bigcup\{(s^{\pm}, t^{\pm}) | |s^{\pm}| \leq \frac{3r}{2k}, t^{\pm} \in S^1\}$. The constant C is independent of r. Now we estimate the remaining part. In view of the symmetry of our formula it suffices to estimate the left hand side in the collar

$$\{(s^+, t^+)|\frac{3r}{2k} \leq s^+ \leq \frac{2r}{k}, t^+ \in S^1\}$$

Note that in this region the elements of F vanish, and

$$D_{u^+}(h^+ + \hat{h}_0) = \eta^+$$

$$D_{u^-}(h^- + \hat{h}_0) = \eta^- = 0$$

Since in this region $u_r = (ks^+, x(kt^+))$, and the almost complex structure is the standard complex structure J_0 along the periodic orbit $x(t)$, we see that $D_{u_r}h = \frac{\partial h}{\partial s^+} + J_0\frac{\partial h}{\partial t^+}$, and $D_{u_r}\hat{h}_0 = 0$. Therefore we have in this region

$$D_{u_r}h - \eta = (D_{u_r} - D_{u^+})(h^+ + \hat{h}_0) + \bar{\partial}\beta h^- + \beta(D_{u_r} - D_{u^-})(h^- + \hat{h}_0). \tag{6.33}$$

The cutoff function satisfies the estimate

$$|\bar{\partial}\beta\left(\frac{-2ks^-}{r}\right)| \leq \frac{C}{r} \leq \frac{C}{R_0}. \tag{6.34}$$

On the other hand, by the exponential decay of $u^{\pm}$ we have

$$\|(D_{u_r} - D_{u^{\pm}})(h^{\pm} + \hat{h}_0)\|_{p,\alpha} \leq Ce^{-\delta R_0}\|h\|_{p,\alpha}. \tag{6.35}$$

¿From the above estimates we finally obain

$$\|D\mathcal{S}_{e,\xi,r}h - \eta\|_{p,\alpha,r} \leq C\left(\epsilon + \frac{1}{R_0} + e^{-\delta R_0}\right)\|h\|_{p,\alpha} \leq \frac{1}{2}\|\eta\|_{p,\alpha,r}. \tag{6.36}$$

The last inequality follws from the right inverse Q_{u^+,u^-} and choosing ϵ small enough and R_0 large enough. The lemma has been proved. □

¿From lemma 6.2 we get

Proposition 6.3 *There are constants C, ϵ, R_0 independent of r such that for any $|\xi| < \epsilon, r \geq R_0$ there is a right inverse $Q_{\xi,r}$ of $D\mathcal{S}_{e,\xi,r}$ such that*

$$D\mathcal{S}_{e,\xi,r} Q_{\xi,r} = Id$$

$$\|Q_{\xi,r}\| \leq C.$$

Remark 6.4 *We proved Proposition 6.3 in a special case,i.e., Σ^+ and Σ^- are smooth Riemann surfaces with one end . It can be extended to two general stable curves with ν ends.*

6.4 Gluing

Denote

$$\Gamma^{\pm} = (u^{\pm}, \Sigma^{\pm}; y_{1^{\pm}}, ..., y_{m^{\pm}}, p),$$

$$\mathcal{M}^r_{\mathcal{S}_e}(A) = \mathcal{S}_r^{-1}(0)/stb_{u_r},$$

$$\mathcal{M}^{(r)}_{\mathcal{S}_e} = \bigcup_{r=0}^{\infty} \mathcal{M}^r_{\mathcal{S}_e}(A) \times \{r\}.$$

We glue u^+ and u^- to get u_r , which represents a homology class denoted by $A^+ + A^-$.

Theorem 6.5 *There is a neighborhood O of (Γ^+, Γ^-) in $\mathcal{M}_{\mathcal{S}_e}(A^+, A^-)$ and $R_0 > 0$ such that*

$$f_r|_O : \mathcal{M}_{\mathcal{S}_e}(A^+, A^-) \times Z_k \to \mathcal{M}^r_{\mathcal{S}_e}(A^+ + A^-)$$

for $r > R_0$ is a family of orientation preserving local diffeomorphisms. Moreover, $(R_0, \infty) \times Z_k \times (O/(stb_{u^+} \times stb_{u^-}))$ is a local orbifold chart of $\mathcal{M}^{(r)}_{\mathcal{S}_e}$.

Proof: For any $n \in Z_k$, any $(\xi^+, \xi^-) \in O_{\Sigma^+} \times O_{\Sigma^-}$ and any r we glue Σ^+ and Σ^- to get a Riemann surface $\Sigma_{\xi,r}$ as in subsection 6.2. To simplify notations we use simply $D\mathcal{S}_r$, Q_r to denote $D\mathcal{S}_{e,\xi,r}$, $Q_{\xi,r}$. Consider the Taylor expansion

$$\mathcal{S}_e(\exp_{u_r}(h) = \mathcal{S}_e(u_r) + D\mathcal{S}_r(h) + N_r(h).$$

The nonlinear term $N_r(h)$ satisfies the following estimates (see [Liu])

$$\|N_r(h)\|_{p,\alpha} \leq C\|h\|_\infty \|h\|_{1,p,\alpha}$$

$$\|N_r(h) - N_r(k)\|_{p,\alpha} \leq C\left(\|h\|_{1,\alpha} + \|k\|_{1,\alpha}\right)\|h-k\|_{1,\alpha}$$

¿From the above inequalities and the uniform estimates (6.28),(6.29) for right inverse we get

$$\|Q_rN_r(h) - Q_rN_r(k)\| \leq C\left(\|h\|_{1,\alpha} + \|k\|_{1,\alpha}\right)\|h-k\|_{1,\alpha}$$

$$\|Q_r\left(\mathcal{S}_e(u_r)\right)\|_{1,p,\alpha} \leq C\|\mathcal{S}_e(u_r)\|_{p,\alpha} < C\left(\epsilon + e^{-(\delta-\alpha)R_0}\right).$$

We choose ϵ small and R_0 large enough. Then we can use the Implicite Function Theorem to get an injective gluing map

$$f_r : O \times Z_k \to \mathcal{M}^r_{\mathcal{S}_e}(A^+ + A^-),$$

where O is a neighborhood of (Γ^+, Γ^-) in $\mathcal{M}_{\mathcal{S}_e}(A^+, A^-)$. Recall that $\mathcal{M}^r_{\mathcal{S}_e}(A^+ + A^-)$ is a smooth manifold of dimension

$$IndDS_e = IndD_{u_r} + 6g - 6 + 2m + \dim F^+ + \dim F^-.$$

By our choice of F, $\mathcal{M}_{\mathcal{S}_e}(A^+, A^-)$ is a smooth manifold. We calculate its dimension. Using the index addition formula for Bott-type we have

$$Ind(D_{u^+}, \alpha^-) + Ind(D_{u^-}, \alpha^-) - \dim\ker L_\infty = IndD_{u_r}.$$

Hence

$$\begin{aligned} & \dim \mathcal{M}_{S_e}(A^+, A^-) = Ind(D_{u^+}, \alpha^-) + 6g^+ - 6 + 2 + 2m^+ \\ + \quad & Ind(D_{u^-}, \alpha^-) + 6g^- - 6 + 2 + 2m^- + \dim F^+ + \dim F^- - (\dim\ker L_\infty - 2) \\ = \quad & IndD_{u_r} + 6g - 6 + 2m + \dim F^+ + \dim F^- \\ = \quad & \dim \mathcal{M}^r_{\mathcal{S}_e}(A^+ + A^-). \end{aligned}$$

It follows that f_r is a local diffeomorphism. By a standard argument one can show that f_r preserves the orientation. The first claim follows.

Consider the map

$$f : (R_0, \infty) \times O \times Z_k \to \mathcal{M}^{(r)}_{\mathcal{S}_e}$$

given by

$$f(r, a, b, n) = (r, f_r(a, b, n)).$$

It is easy to see that f is a one-to-one map. By using the Implicite Function Theorem with parameters we conclude that f is a diffeomorphism onto its image. The group $stb_{u^+} \times stb_{u^-}$ acts on $\mathcal{M}^r_{\mathcal{S}_e}(A)$ in a natural way. It follows that $(R_0, \infty) \times Z_k \times O/(stb_{u^+} \times stb_{u^-})$ is a local orbifold chart. $\Box$

Next we glue curves in M^+ and $\mathbf{R} \times \widetilde{M}$. Note that the group G_0 acts on $\mathbf{R} \times \widetilde{M}$. Put

$$\mathcal{M}_{\mathcal{S}_e}(A^+, A^c) = \left\{(a,b) \in \mathcal{M}^+_{\mathcal{S}_e}(A^+, k) \times \mathcal{M}^c_{\mathcal{S}_e}(A^c, k, k'_1, ..., k'_l))|P^+(a) = P^-(b)\right\}/(stb_{u^+} \times stb_{u^c}).$$

Denote

$$S^2 = \mathbf{C} \cup \{\infty\},$$

$$N_{R_0} = \{(\theta, r) \in S^2 | r \geq R_0\}.$$

For any $(\theta, r) \in N_{R_0}$ we glue M^+ and $\mathbf{R} \times \widetilde{M}$ with the following gluing formula

$$a^+ = a^c + 4r$$

$$\theta^+ = \theta^c + \theta.$$

We glue u^+ and u^c in a similar way as before to get u^+_r, which represents a homology class denoted by $A^+ + A^c$. By using a similar argument as in Theorem 6.5, we may obtain

Theorem 6.6 *There is a neighborhood O of (Γ^+, Γ^c) in $\mathcal{M}_{\mathcal{S}_e}(A^+, A^c)$ and $R_0 > 0$ large enough, such that there is an an orientation preserving local diffeomorphism:*

$$f : \mathcal{M}_{\mathcal{S}_e}(A^+, A^-) \times Z_k \times N_{R_0} \to \mathcal{M}^+_{\mathcal{S}_e}(A^+ + A^c, k'_1, ..., k'_l).$$

7 Relative invariants

Let (V, ω) be a compact symplectic manifold, and B a compact, real codimension two symplectic submanifold of V. Then the restriction of ω to the normal bundle ν is a symplectic form, i.e., we may consider ν to be a symplectic vector bundle. We choose a compatible complex structure J_ν on ν such that

$$g_{J_\nu}(h, k) = \omega(x)(h, J_\nu(x)k),$$

for all $x \in B,\ h, k \in \nu_x$, defines a smooth fibrewise metric for ν. Then we can consider ν as a complex vector bundle with fibre $\mathbf{C}$ and with a Hermitian metric. We may identify a neighborhood of B in V with a neighborhood of the 0-section of ν. Consider the local Hamiltonian function

$$H(x, z) = |z|^2 - 1,$$

for $x \in B$ and $z \in \nu_x$. The S^1-action is given by

$$e^{2\pi it}(x, z) = \left(x, e^{2\pi it}z\right).$$

Let $H^{-1}(0) = \widetilde{M}$. We can write locally

$$H^{-1}(I) = I \times \widetilde{M}.$$

Then by performing symplectic cutting we can easily see that $V = \overline{M}^{+}$ for some $\overline{M}^{+}$ such that B corresponds to the symplectic reduction Z. We choose an almost complex structure $\widetilde{J}$ on Z and choose an almost complex structure J as in Subsection 5.1. Then $J|_{\mathbf{C}} = J_\nu$. In this section, we define *relative GW-invariants* for the pair $(V, B) = (\overline{M}^{+}, B)$. Furthermore, we prove a gluing formula representing the GW-invariants of M in terms of the relative GW-invariants of $(\overline{M}^{\pm}, Z)$.

7.1 Virtual neighborhoods

We identify B with Z, $\overline{M}^{\pm} - B$ with $M^{\pm}$. To define the relative GW-invariants we use the virtual neighborhood technique developed in [R5]. The construction of virtual neighborhoods in the present case is basically the same as in [R5] with minor changes.

Consider the moduli space $\overline{\mathcal{M}}_A(M^+, g, m, \mathbf{k})$. By Theorem 5.9, $\overline{\mathcal{M}}_A(M^+, g, m, \mathbf{k})$ is compact. Note that we need only consider a neighborhood of $\overline{\mathcal{M}}_A(M^+, g, m, \mathbf{k})$ in the configuration space $\overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k})$ of C^∞-stable (holomorphic or not) maps. For a smooth Riemann surface Σ and a stable J-holomorphic map u, we may assume that our configuration space is $\mathcal{U}_{\Sigma,u}$. From the discussion in Subsection 6.1 we immediately obtain

Lemma 7.1 *For any smooth marked Riemann surface both* $\mathcal{B}_A(M^+, g, m, \mathbf{k})$ *and* $\mathcal{B}_A(\mathbf{R}\times\widetilde{M}, g, m, \mathbf{k}, \mathbf{k}')$ *are Hausdorff Frechet V-manifolds for any* $2g + l \geq 3$ *or* $g = 0, l \leq 2, A \neq 0$, *where* l *is the number of marked points and puncture points.*

For any $D \in \mathcal{D}^{J,A,\mathbf{k}}_{g,m+\nu}$, let $\mathcal{B}_D \subset \overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k})$ be the set of stable maps whose domain and the corresponding fundamental class of each component and periodic orbits have type D. Then, $\mathcal{B}_D$ is a stratum of $\overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k})$.

Lemma 7.2 $\mathcal{B}_D$ *is a Hausdorff Frechet V-manifold.*

Proof: For simplicity, let's consider a special case. Let $D = (\Sigma_1; \mathbf{y}_1, p) \wedge (\Sigma_2; \mathbf{y}_2, p)$ joining at p. If p is a removable singular point for (u_1, u_2), the argument is the same as in [R5]. If p is an nonremovable singular point, it follows from the discussion in Subsection 6.2. For general D we repeat this argument. □

For each removable singular point p we associate a neighborhood $N(\epsilon)$ of 0 in $\mathbf{C}$. For a nonremovable singular point p we associate $Z_k \times N_{R_0,\epsilon}$. The resulting manifold is denoted by $\widetilde{\mathcal{B}}_D$. We define attaching maps from $\widetilde{\mathcal{B}}_D$ into $\overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k})$ as in [R5] and in Subsection 6.2, which defines the topology on $\overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k})$. By the same argument one can prove

Theorem 7.3 $\overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k})$ *is a Hausdorff stratified Frechet V-manifold with finitely many strata.*

For a smooth Riemann surface Σ and for $D = \Sigma_1 \wedge \Sigma$ we can define $\mathcal{F}_D$ in the same way as in [R5] and in Subsection 6.1. This is easily extended to general strata. Then we can define $\overline{\mathcal{F}}_A(M^+, g, m, \mathbf{k})$ such that

$$\overline{\mathcal{F}}_A(M^+, g, m, \mathbf{k})|\mathcal{B}_D = \mathcal{F}_D.$$

The operator $\bar{\partial}$ is also extended as a section of the bundle

$$\overline{\mathcal{F}}_A(M^+, g, m, \mathbf{k}) \to \overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k}).$$

Thus we have a compact V-triple $(\mathcal{B}, \mathcal{F}, \bar{\partial})$.

To construct a global stabilization equation we need a result of Siebert.

Definition 7.4 *We say that M is a fine V-manifold if any local V-bundle is dominated by a global oriented V-bundle. Namely, Let $U_\alpha \times_{\rho_\alpha} E/G_\alpha$ be a local V-bundle, where $\rho_\alpha : G_\alpha \to GL(E)$ is a representation. There is a global oriented V-bundle $E \to M$ such that $U_\alpha \times_{\rho_\alpha} E_\alpha/G_\alpha$ is a subbundle of E_{U_α/G_α}.*

By remark 5.15, every component of the element $f \in \overline{\mathcal{B}}_A(\overline{M}^+, g, m, \mathbf{k})$ can be identified with a map into a closed symplectic manifold (either $\overline{M}^+$ or the quotient of $\tilde{M} \times \mathbf{R}$ by the S^1 action on the boundary). The later can be identified with the closure of the normal bundle of B in X. From the introduction, we can choose ω such that $\omega, \omega^{\pm}$ have integral periods. Over the closure of normal bundle of B, we use the pull-back of integral symplectic form over Z. In this case, Siebert's construction (see [R5], Appendix) applies and $\overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k})$ is fine.

For a smooth Riemann surface Σ and for $\Sigma = \Sigma_1 \wedge \Sigma_2$ we have constructed in Subsection 6.1 a stabilization equation

$$\mathcal{S}_e = \bar{\partial}_u + \eta_u : \mathcal{U}_{\Sigma,u} \times F_u \to \mathcal{F}$$

in a neighborhood $\mathcal{U}_{\Sigma,u}$ of (Σ, u). This construction is extended to any stratum D. By the compactness of $\overline{\mathcal{M}}_A(M^+, g, m, \mathbf{k})$ we may choose finitely many such neighborhoods $\mathcal{U}_{u_i}$ such that

$$\overline{\mathcal{M}}_A(M^+, g, m, \mathbf{k}) \subset \bigcup \mathcal{U}_{u_i}.$$

For each i, using the result of Siebert, there exists an oriented global finite dimensional V-bundle $\mathcal{E}_i$ over $\mathcal{U} = \bigcup_i \mathcal{U}_{u_i}$ such that $\mathcal{U}_{u_i} \times F_{u_i}$ is a subbundle of $\mathcal{E}_i$. We put

$$\mathcal{E} = \oplus_i \mathcal{E}_i. \tag{7.1}$$

By multiplying a cut-off function and by averaging over a finite group, we may assume that each η_{u_i} is stb_{u_i}-invariant and supported in $\mathcal{U}_{u_i}$. Put

$$\eta = \sum_i \eta_{u_i}. \tag{7.2}$$

Our stabilization operator is the bundle map

$$\mathcal{S}_e = \bar{\partial} + \eta : \mathcal{E} \to \mathcal{F}. \tag{7.3}$$

By the gluing theorem, $U_{\mathcal{S}_e} = (\mathcal{S}_e)^{-1}(0)$ is a smooth V-manifold. Following [R5], we can assume that $U_{\mathcal{S}_e}$ is compatible with the stratification of $\overline{\mathcal{B}}_A$ by defining η_i inductively over the strata of $\overline{\mathcal{B}}_A$. Namely, we define η_i on lower strata first. Then, we extend it over $\overline{\mathcal{B}}_A$ such that η_i is supported in a neighborhood of lower strata. Then, we define η_{i+1} supported away from lower strata. Once η_i is defined in such a fashion. $U_{\mathcal{S}_e}$ has the property that if $\mathcal{B}_{D'} \subset \overline{\mathcal{B}}_D$ is a lower stratum,

$$U_{\mathcal{S}_e} \cap \mathcal{E}|_{\mathcal{B}_{D'}} \subset U_{\mathcal{S}_e} \cap \mathcal{E}|_{\overline{\mathcal{B}}_D} \tag{7.3A}$$

is a submanifold of codimension at least 2. There are two maps, the inclusion map

$$S : U_{\mathcal{S}_e} \to \mathcal{E} \tag{7.4}$$

and the projection

$$\pi : U_{\mathcal{S}_e} \to \mathcal{U}. \tag{7.5}$$

S can be viewed as a section of the bundle $E = \pi^*\mathcal{E}$, and $S^{-1}(0) = \overline{\mathcal{M}}_A(M^+, g, m, \mathbf{k})$.

Lemma 7.5 *S is a proper map.*

The proof is the same as the proof of the compactness theorem for $\overline{\mathcal{M}}_A(M^+, g, m, \mathbf{k})$. By the same method as in [R5] one can prove that $U_{\mathcal{S}_e}$ is an orientable C^1 V-manifold. Thus we have constructed a virtual neighborhood $(U_{\mathcal{S}_e}, E, S)$.

Using the virtual neighborhood we can define the relative GW-invariants. Recall that we have two natural maps

$$\Xi_{g,m} : \overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k}) \to (M^+)^m \tag{7.6}$$

defined by evaluating at marked points and

$$P : \overline{\mathcal{B}}_A(M^+, g, m, \mathbf{k}) \to \prod_i^{\nu} S_{k_i}$$

defined by projecting to its periodic orbits. To define the relative GW-invariants, choose r-form Θ on E supported in a neighborhood of the zero section, where r is the dimension of the fiber, such that

$$\int_{E_x} i^*\Theta = 1 \tag{7.7}$$

for any $x \in U_{\mathcal{S}_e}$, where i is the inclusion map $E_x \to E$. We call Θ a Thom form. The relative GW-invariant can be defined as

$$\Psi^{(V,B)}_{(A,g,m,\mathbf{k})}(\alpha_1, ..., \alpha_m; \beta_1, ..., \beta_\nu) = \int_{U_{\mathcal{S}_e}} \Xi^*_{g,m} \prod_1^m \alpha_i \wedge P^* \prod_1^\nu \beta_i \wedge S^*\Theta. \tag{7.8}$$

for $\alpha_i \in H^*(M^+, \mathbf{R})$ and $\beta_j \in H^*(S_{k_i}, \mathbf{R})$ represented by differential form. Here we omitted the notation π^* for forms α and (V, B) to simplify notations. Clearly, $\Psi = 0$ if $\sum \deg(\alpha_i) + \sum \deg(\beta_i) \neq ind$. By the same argument as in [R5] one can easily show that

Theorem 7.6

(i). $\Psi^{(M,B)}_{(A,g,m,\mathbf{k})}(\alpha_1, ..., \alpha_m; \beta_1, ..., \beta_\nu)$ *is well-defined, multi-linear and skew symmetric.*

(ii). $\Psi^{(M,B)}_{(A,g,m,\mathbf{k})}(\alpha_1, ..., \alpha_m; \beta_1, ..., \beta_\nu)$ *is independent of the choice of forms* α_i, β_j *representing the cohomology classes* $[\beta_j], [\alpha_i]$*, and the choice of virtual neighborhoods.*

(iii). $\Psi^{(M,B)}_{(A,g,m,\mathbf{k})}(\alpha_1, ..., \alpha_m; \beta_1, ..., \beta_\nu)$ *is independent of the choice of* $\widetilde{J}$ *and* J_ν*.*

7.2 A gluing formula

In this subsection we prove a general gluing formula relating GW-invariants of a closed symplectic manifold in terms of relative GW-invariants of its symplectic cutting.

First, we make a remark about an error in the draft concerning the homology class of the general gluing formula. The gluing theorem in the previous section shows that one can glue two pseudo-holomorphic curves (f_+, f_-) in M^+, M^- with the same end point to a pseudo-holomorphic curve f in M. Suppose that the homology classes of f_+, f_-, f are A^+, A^-, A. Then, we carelessly wrote $A = A^+ + A^-$. R. Fintushel and E. Ionel pointed to us that in general the homology class of f depends on the pseudo-holomorphic curve representatives f_+, f_- instead of homology classes A^+, A^-. One can also understand it as follows. Recall that there is a map

$$\pi : M \to \overline{M}^+ \cup_Z \overline{M}^-.$$

π induces a homomorphism

$$\pi_* : H_2(M, \mathbf{Z}) \to H_2(\overline{M}^+ \cup_Z \overline{M}^-, \mathbf{Z}).$$

Using Mayer-Vietoris sequence for $(\overline{M}^+, \overline{M}^-, \overline{M}^+ \cup_Z \overline{M}^-)$, (f_+, f_-) defines a homology class $[f^+ + f^-] \in H_2(\overline{M}^+ \cup_Z \overline{M}^-, \mathbf{Z})$. The existence of glued map f implies $[f^+ + f^-] = \pi_*([f])$. If (f'_+, f'_-) is another representative and glued to f',

$$\pi_*([f']) = [f'_+ + f'_-] = [f_+ + f_-] = \pi_*([f]).$$

When $\ker \pi_* \neq 0$, $[f], [f']$ could be different from a vanishing cycle in $\ker \pi_*$. For the application of our gluing formula to the main theorems, there is no vanishing cycle (Remark 2.A, 2.B, section 2). Hence, this problem does not arise.

However, the original statement of our general gluing formula is incorrect. Instead, our argument yields following modified statement. Let $[A] = A + \ker \pi_*$. Then, we define

$$\Psi_{([A], \cdots)} = \sum_{B \in [A]} \Psi_{(B, \cdots)}.$$

Note that for $B, B' \in [A]$ $\omega(B) = \omega(B')$ by (1.15A). By the compactness theorem, there are only finite many such B to be represented by stable J-holomorphic maps. Hence the summation of right hand side is finite. To abuse the notation, we use $[A] = A^+ + A^-$ to represent the set of homology classes of glued maps. Then we replace $\Psi_{(A, \cdots)}$ by $\Psi_{([A], \cdots)}$ in all the statements in this section and original proof is still valid.

The proof of our gluing formula is similar to the proof of composition law of GW-invariants and has two steps. The first step is to define an invariant for M_∞ and prove that it is the same as the invariant of M_r. Then, we write the invariant of M_∞ in terms of relative invariants of $M^\pm$.

We first construct a virtual neighborhood for M_∞. The moduli space $\overline{\mathcal{M}}_{[A]}(M_\infty, g, m)$ consists of the components indexed by the following data:

(1) The combinatorial type of $(\Sigma^{\pm}, u^{\pm})$: $\{A_i^{\pm}, g_i^{\pm}, m_i^{\pm}, (k_1^{\pm}, ..., k_{\nu}^{\pm})\}, i = 1, ..., l^{\pm}$;

(2) A map $\rho : \{p_1^+, ..., p_{\nu}^+\} \to \{p_1^-, ..., p_{\nu}^-\}$, where $(p_1^{\pm}, ..., p_{\nu}^{\pm})$ denote the puncture points of $\Sigma^{\pm}$.

Suppose that $\mathcal{C}^{J,A}_{g,m,\mathbf{k}}$ is the set of indices. Let $C \in \mathcal{C}^{J,A}_{g,m,\mathbf{k}}$. Denote by $\mathcal{M}_C$ the set of stable maps corresponding to C. The following lemma is obvious.

Lemma 7.7 *$\mathcal{C}^{J,A}_{g,m,\mathbf{k}}$ is a finite set.*

To construct a virtual neighborhood for M_{∞} we consider a simple case: both Σ^+ and Σ^- have a connected component and an end. Suppose that the combinatorial type is

$$C = \{A^+, g^+, m^+, k; A^-, g^-, m^-, k\}.$$

For any J-holomorphic curve $(\Sigma, u) \in \overline{\mathcal{M}}(A^+, A^-, g^+, g^-, m^+, m^-, k)$, by using the same method as in Subsection 6.2, we construct a local virtual neighborhood and a local stabilization equation in $\mathcal{U}_{\Sigma,u}$. Then we use the same method as in the above subsection to construct a virtual neighborhood for this component. In general we can use the method of the above subsection to construct a virtual neighborhood (U, E, S) for $\overline{\mathcal{M}}_{[A]}(M_{\infty}, g, m)$ starting inductively from the lowest stratum. But U is usually not a smooth manifold. To see this, we observe that the configuration space $\overline{\mathcal{B}}_{[A]}(M_{\infty}, g, k)$ can be identified as

$$\cup_C \overline{\mathcal{B}}_C(M_{\infty}, g, k).$$

$\overline{\mathcal{B}}_C(M_{\infty}, g, k)$,$\overline{\mathcal{B}}_{C'}(M_{\infty}, g, k)$ may intersect each other at lower stratas. Hence, $U = \cup_C U_C$, where U_C is a virtual neighborhood of $\mathcal{M}_C$. Note that $\mathcal{M}_{\mathcal{C}}$ and $\mathcal{M}_{C'}$ may intersect each other, where the intersection corresponds to the stable map with some component in $\mathbf{R} \times \widetilde{M}$. Then, U_C may intersect each other. By our construction, U has the same stratification structure as that of $\overline{\mathcal{M}}_{[A]}(M_{\infty}, g, m)$. Hence, for $C \neq C'$, $U_C \bigcap U_{C'}$ is a stratum of $U_C, U_{C'}$ codimension at least 2 (7.3A).

A differential form over U is a set of differential forms over each U_C which agree over the intersections. The integration theory can be extended to such a space in an obvious fashion. Namely, we take the sum of integrals over each U_C, and then substract the integral of overlaping submanifolds and so on. Choose a Thom Θ form of E. It restricts to a Thom form over each U_C which agrees at the intersection. The pull-back of form K, α_i have the same property. Then, we can define GW-invariants $\Psi_{(M_{\infty},[A],g,m)}$ using the same integral formula. We can also define

GW-invariants Ψ_C for each component C. It is easy to see that

$$\Psi_{(M_\infty,[A],g,m)} = \sum_{\mathcal{C}^{J,[A]}_{g,m,\mathbf{k}}} \Psi_C. \tag{7.9}$$

Remark 7.8 *It is easy to see that*

(i) *For $C = \{A^+, g^+, m^+\}$, we have*

$$\Psi_C(\alpha^+) = \Psi^{(\overline{M}^+,Z)}_{(A^+,g^+,m^+)}(\alpha^+); \tag{7.10}$$

(ii) *For $C = \{A^-, g^-, m^-\}$, we have*

$$\Psi_C(\alpha^-) = \Psi^{(\overline{M}^-,Z)}_{(A^-,g^-,m^-)}(\alpha^-). \tag{7.11}$$

Theorem 7.9 *For any r, $0 < r < \infty$, we have*

$$\Psi_{(M_\infty,[A],g,m)} = \Psi_{(M_r,[A],g,m)}. \tag{7.12}$$

Proof: Choose $(\mathcal{E}_{(r)}, \eta)$ of $\mathcal{F}_{(r)} \to \mathcal{U}_{(r)}$ such that

$$\delta(\mathcal{S}^r + \eta) \tag{7.13}$$

is surjective to $\mathcal{F}_{\mathcal{U}_{(r)}}$ where $\mathcal{M}_{\mathcal{S}_{(t)}} \subset \mathcal{U}_{(t)} \subset \mathcal{B}_{(t)}$. Repeating the previous argument, we construct a virtual neighborhood $(U_{(t)}, E_{(t)}, S_{(t)})$. Then, it is easy to check that (U_0, E_0, S_0) is a virtual neighborhood of $\mathcal{S}_0$ defined by $(\mathcal{E}_0, \eta(0))$ and $(U_\infty, E_\infty, S_\infty)$ is a virtual neighborhood of $\mathcal{S}_\infty$ defined by $(\mathcal{E}_\infty, \eta(\infty))$. In general U_∞ is not a smooth boundary of $U_{(t)}$. In order to apply the Stokes Theorem we need a clear description of neighborhoods of each lower strata of U_∞ in $U_{(t)}$. This is basically a gluing argument. We consider a simple case , the argument for general case is similar. Suppose that $U_C \bigcap U_{C'} = \mathcal{M}^{(+,0,-)}$, where

$$\mathcal{M}^{(+,0,-)} = \{(a,b,d) \in \mathcal{M}^+_{\mathcal{S}_e}(A^+,1) \times \mathcal{M}^c_{\mathcal{S}_e}(A^c,1) \times \mathcal{M}^-_{\mathcal{S}_e}(A^-,1)$$

$$|P^+(a) = P^-(b), P^+(b) = P^-(d)\}.$$

Suppose that $(\Gamma^+, \Gamma_0, \Gamma^-) \in \mathcal{M}^{(+,0,-)}$, where

$$\Gamma^\pm = ((u^\pm, \Sigma^\pm, p^\pm), \eta^\pm)\,,$$

$$\Gamma_0 = ((u_0, \Sigma_0, q^+, q^-), \eta_0)\,.$$

Let $O_{(+,0.-)}$ be a neighborhood of $(\Gamma^+, \Gamma_0, \Gamma^-)$ in $\mathcal{M}^{(+,0,-)}$. For any (θ, r_1, r_2) we glue $M^+, \mathbf{R}\times\widetilde{M}$ and M^- to get $M_{(\theta,r_1,r_2)}$ with the following gluing formulas

$$a^+ = a_0 + 4r_1, \quad a_0 = a^- + 4r_2$$

$$\theta^+ = \theta^- = \theta_0 + \theta \;\; mod\, 1.$$

Given complex structure $\xi = (\xi^+, \xi_0, \xi^-)$ we construct $\Sigma_{(\xi,r_1,r_2)} = \Sigma^+\#_{r_1}\Sigma_0\#_{r_2}\Sigma^-$ and construct a pre-gluing map $u_{(\theta,r_1,r_2)} : \Sigma_{(\xi,r_1,r_2)} \to M_{(\theta,r_1,r_2)}$ in a similar way as in Section 6 (Recall that we perturbe $u^\pm$ and u_0 only in the gluing domain). Then we use a similar way as in Section 6 to prove that a neighborhood of $(\Gamma^+, \Gamma_0, \Gamma^-)$ in $U_{(t)}$ can be descripted as $O_{(+,0.-)} \times N$ where $N : S^1 \times (R_1, \infty] \times (R_2, \infty] \to \mathbf{R}^4$ is a local hypersurface given by

$$y = (\frac{1}{r_1}cos\theta, \frac{1}{r_1}sin\theta, \frac{1}{r_2}cos\theta, \frac{1}{r_2}sin\theta).$$

Both

$$O_{(+,0.-)} \times N(S^1 \times \{\infty\} \times (R_2, \infty])$$

and

$$O_{(+,0.-)} \times N(S^1 \times (R_1, \infty] \times \{\infty\})$$

are its boundary, which can be identified with a open set of U_C and $U_{C'}$ respectively by Theorem 6.6.

We show that the Stokes theorem still applies to such a space. Without loss of generality we consider a domain D, $(0,0,0,0) \in D \subset N$. The boundary ∂D consists of three parts:

1. γ_1: the boundary of D in the hypersurface N, we assume it to be smooth;
2. $\gamma_2 = D \cap N(S^1 \times \{\infty\} \times (R_2, \infty])$;
3. $\gamma_3 = D \cap N(S^1 \times (R_1, \infty] \times \{\infty\})$.

We draw a small ϵ-ball $B(\epsilon)$ around $(0,0,0,0)$ in $\mathbf{R}^4$, then $D - B(\epsilon)$ is a polyhedron (union of simplexes) in Euclidean space, for which the Stokes Theorem applies. We use Stokes Theorem and then let $\epsilon \to 0$. Note that the volume elements of N and γ_1, γ_2 are respectively $(s_1+s_2)ds_1\wedge ds_2\wedge d\theta$, $s_1 dr_1 \wedge ds_2 \wedge d\theta$ and $s_2 ds_1 \wedge ds_2 \wedge d\theta$, where we denoted $s_i = \frac{1}{r_i}$. If the coefficients of a differential forms α and $d\alpha$ with respect to the coordinates (s_1, s_2, θ) are bounded by a constant , we can obtain

$$\int_D d\alpha = \int_{\partial D} \alpha.$$

This argument works obviousely for $O_{(+,0.-)} \times N$.

Next we show that for any differential form $\alpha \in H^*(M_{(\theta,r_1,r_2)}, R)$ the coefficients of $\Xi^*_{g,m}\alpha$ are bounded by a constant independent of r_1, r_2, and for any Thom form Θ, $S^*\Theta$ has also bounded coefficients with respect to $(z^+, z_0, z^-, s_1, s_2, \theta)$, where we use the coordinates (z^+, z_0, z^-) for $O_{(+,0,-)}$ and $s_i = 1/r_i$ for N. For any $(z^+, z_0, z^-, s_1, s_2, \theta)$ denote by $(u_{(z^+,z_0,z^-,s_1,s_2,\theta)}, \eta_{(z^+,z_0,z^-,s_1,s_2,\theta)})$ the corresponding solution of stabilization equation. Let $x \in \Sigma_{(\xi,r_1,r_2)}$ be a marked point. Without loss of generality we may assume that x is a fixed point in Σ^+ and x is out of the gluing domain. Then for large r_1, r_2 , $u_{(z^+,z_0,z^-,s_1,s_2,\theta)}(x)$ lies in a small neighborhood of $u^+(x)$. Since there is a uniformly bounded right inverse (see Section 6), by using the Implicity Function Theorem , we may conclude that the norm of derivatives of η with respect to $(z^+, z_0, z^-, s_1, s_2, \theta)$ are unifomly bounded. At the same time, the W^p_1-norm of derivatives of $u_{(z^+,z_0,z^-,s_1,s_2,\theta)}$ with respect to $(z^+, z_0, z^-, s_1, s_2, \theta)$ are also uniformly bounded . Then we use Sobolev emmbedding Theorem to show its derivatives at x are uniformly bounded. Then we can use Stokes Theorem. Theorem 7.9 is proved. $\Box$

We derive a gluing formula for the component $C = \{A^+, g^+, m^+, k; A^-, g^-, m^-, k\}$. For any component C we can use repeately this formula. Choose a homology basis $\{\beta_b\}$ of $H^*(S_k, \mathbf{R})$. Let (δ_{ab}) be its intersection matrix.

Theorem 7.10 *Suppose that* $\alpha_i^+|_Z = \alpha_i^-|_Z$ *and hence* $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{M}^+ \cup_Z \overline{M}^-, \mathbf{R})$. *Let* $\alpha_i = \pi^*(\alpha_i^+ \cup_Z \alpha_i^-)$ *(1.15A).*

For $C = \{A^+, g^+, m^+, k; A^-, g^-, m^-, k\}$, *we have the gluing formula*

(7.14)

$$\Psi_C(\alpha_1, ..., \alpha_{m^++m^-}) = k \sum \delta^{ab} \Psi^{(\overline{M}^+,Z)}_{(A^+,g^+,m^+,k)}(\alpha_1^+, ..., \alpha^+_{m^+}, \beta_a) \Psi^{(\overline{M}^-,Z)}_{(A^-,g^-,m^-,k)}(\alpha^-_{m^++1}, ..., \alpha^-_{m^++m^-}, \beta_b).$$

Proof: We denote $U_C = U|_C$, $U_{(A^\pm,k)} = (S_e^\pm)^{-1}(0)/stb_{u^\pm}$, and

$$U_{(A^+,A^-)} = \left\{(a,b) \in U_{(A^+,k)} \times U_{(A^-,k)} | P^+(a) = P^-(b)\right\}.$$

There is a natural map of degree k

$$Q : U_C \to U_{(A^+,A^-)}$$

defined by

$$Q(a,b,n) = (a,b),$$

and a map

$$P : U_{(A^+,k)} \times U_{(A^-,k)} \to S_k \times S_k$$

defined by

$$P(a,b)=(P^+(a),P^-(b)).$$

Note that

$$\mathcal{M}_{S_e}(A^+,A^-)=P^{-1}(\Delta),$$

where $\Delta\subset S_k\times S_k$ is the diagonal. The Poincare dual Δ^* of Δ is

$$\Delta^*=\Sigma\delta^{ab}\beta_a\wedge\beta_b.$$

Choose Thom form $\Theta=\Theta^+\wedge\Theta^-$, where $\Theta^\pm$ are Thom forms in $F^\pm$ supported in a neighborhood of the zero section . Then

$$\Psi_C(\alpha_1,...,\alpha_{m^++m^-})=\int_{U_C}\prod_1^{m^+}\alpha_i\wedge\prod_{m^+}^{m^++m^-}\alpha_j\wedge S^*\Theta$$

$$=k\int_{U_{(A^+,A^-)}}\prod_1^{m^+}\alpha_i\wedge\prod_{m^+}^{m^++m^-}\alpha_j\wedge S^*\Theta$$

$$=k\int_{U_{(A^+,k)}\times U_{(A^-,k)}}\sum\delta^{ab}\prod_1^{m^+}\alpha_i^+\wedge S^*\Theta^+\wedge\beta_a\wedge\prod_{m^+}^{m^++m^-}\alpha_j^-\wedge S^*\Theta^-\wedge\beta_b$$

$$=k\sum\delta^{ab}\Psi^{(\overline{M}^+,Z)}_{(A^+,g^+,m^+,k)}(\alpha_1^+,...,\alpha^+_{m^+},\beta_a)\Psi^{(\overline{M}^-,Z)}_{(A^-,g^-,m^-,k)}(\alpha^-_{m^++1},...,\alpha^-_{m^++m^-},\beta_b).\quad\square$$

8 Proofs of the Main Theorems

Proof of Theorem A

Let M be a 3-fold. After a small contraction we get M_s. By using Wilson's argument we may assume that all singularities of M_s are ordinary double points. Without loss of generality we assume that M_s has only one singularity. By blowing up the singularity we get M_b with exceptional loci $P^1\times P^1$. Then we contract one ruling to get M. If we contract other ruling we get the flop M_f. Note that both M and M_f have an exceptional smooth rational curve with normal bundle $\mathbf{O}(-1)+\mathbf{O}(-1)$.

We perform the symplectic cutting for M. Choose the symplectic form

$$\omega+dz_1\wedge d\bar z_1+dz_2\wedge d\bar z_2$$

on $\mathbf{O}(-1)+\mathbf{O}(-1)$. Conside the Hamiltonian function $H(x,z_1,z_2)=|z_1|^2+|z_2|^2-\epsilon$ with the S^1-action given by

$$e^{2\pi it}(x,z_1,z_2)=\left(x,e^{2\pi it}z_1,e^{2\pi it}z_2\right).$$

We stretch M along the hypersurface $\widetilde{M} = H^{-1}(0)$. We have $\overline{M}^- = M_b$,and from the example 2 in Section 2,

$$\overline{M}^+ = P(\mathbf{O}(-1) + \mathbf{O}(-1) + \mathbf{O}).$$

The same procedure applies to M_f. We have

$$\overline{M}^+ = \overline{M}_f^+ = P(\mathbf{O}(-1) + \mathbf{O}(-1) + \mathbf{O}), \tag{8.1}$$

$$\overline{M}^- = \overline{M}_f^- = M_b. \tag{8.2}$$

By remark 2.A, 2.B, there is no vanishing 2-cycles in our cases. Now we use the gluing formula 7.14 to prove our assertion. First, we assume that $deg(\alpha_i) \geq 4$ and is Poincare dual to a point or a 2-dimensional homology class Σ Then, we can choose the Pseudo-submanifold representative of Σ such that Σ is in M^-. Hence, (hence α_i) is supported in M^-. We will show that the contribution of J-holomorphic curves which go through the middle to the GW-invariant is zero. For simplicity we consider a special component:

$$C = \{A^+, g^+, m^+, k; A^-, g^-, m^-, k\}.$$

The general case can be treated in the same way.

Following argument depends only on an index calculation. By the appendix, the index has an additive property. If a relative stable map has more than one component, we can always construct a pre-gluing map and use the index of pre-gluing map. Hence, we can assume that the stable map under consideration has only two components (u^+, u^-), where $u^\pm$ is a J-map in $M^\pm$.

Since M is a 3-fold, using the addition formula (5.52) for the index we have

$$Ind(D_{u^-}, \alpha^-) + Ind(D_{u^+}, \alpha^-) - 4 = Ind(D_u). \tag{8.3}$$

By (5.53),

$$Ind(D_{u^+}, \alpha^-) = Ind(D_{\bar{u}^+}). \tag{8.4}$$

Note that $\overline{M}^+ = P(\mathbf{O}(-1) + \mathbf{O}(-1) + \mathbf{O})$. Next we claim that

$$Ind(D_{\bar{u}^+}) \geq 6. \tag{8.5}$$

Note that $\bar{u}^+$ can be identified as a holomorphic curve h in $P(\mathbf{O}(-1) + \mathbf{O}(-1) + \mathbf{O})$ over $\mathbf{P}^1$ by remark 5.15. Then,

$$Ind(D_h) = 2(C_1([h]) - k + 1), \tag{8.5A}$$

where $C_1(M^+)$ is represented precisely by $3Z$. A simple index calculation shows that $C_1([h]) = 3k$. Hence, if $k > 0$

$$Ind(D_{\bar{u}^+}) \geq 6.$$

¿From (8.3)-(8.5) we have

$$Ind(D_{u^-}, \alpha^-) \leq -2. \tag{8.6}$$

Hence $Ind(D_{u^-}, \alpha^-) < Ind(D_u)$. However, the representatives of α_i is supported in M^-. It is clear that $\alpha_i^+ = 0$ and only nontrivial terms in the gluing formula are

$$k \sum \delta^{ab} \Psi^{(\bar{M}^+, Z)}_{(A^+, g^+, m^+, k)}(\beta_a) \Psi^{(\bar{M}^-, Z)}_{(A^-, g^-, m^-, k)}(\{\alpha_i^-\}, \beta_b).$$

However, $\sum_i deg(\alpha_i) = Ind(D_u) > Ind(D_{u^-})$. Then for any β_b,

$$\Psi^{(\overline{M}^-, Z)}_{(A^-, g^-, m^-, k)}(\{\alpha_i^-\}, \beta_b) = 0.$$

If $u^{\pm}$ has more than one end, said ν ends, $Ind(D_{u^-})$ increases faster than 4ν. It will force $Ind(D_{u^-})$ to become even more negative. It follows that $\Psi_C = 0$ except $C = \{A^+, g^+, m^+\}$ or $C = \{A^-, g^-, m^-\}$. Then the assertion follows from Remark 7.8 and (8.1),(8.2). When $deg(\alpha_i) = 2$, we can use formula (1.1) to eliminate α_i. The same formula follows from previous case. □

Only corollary A.2 needs a proof. Others are immediate consequences of Theorem A and Corollary A.2.

Proof of Corollary A.2

For simplicity, we assume that all the exceptional curves are in the same homology class $[\Gamma]$. Let $n_\Gamma = n_{\hat{\Gamma}}$ be the number of such curves, $[\Gamma]$ and $[\hat{\Gamma}]$ be the homology classes of the exceptional curves respectively in M and M_f. Then for any $0 < k \in \mathbf{Z}$, by the formula for multiple cover maps, $\Phi^M_{(k[\Gamma], 0)} = \Phi^{M_f}_{(k[\hat{\Gamma}], 0)} = \frac{n_\Gamma}{k^3}$ (see [V]). Then the total 3-point function can be written in the form

$$\Psi^M_\omega(\beta_1, \beta_2, \beta_3) = \beta_1 \wedge \beta_2 \wedge \beta_3 + \sum_{A \neq k[\Gamma]} \Psi^M_{(A,0,3)}(\beta_1, \beta_2, \beta_3) q^A$$

$$+ \sum_{n[\Gamma], n \neq 0} \Psi^M_{(A,0,3)}(\beta_1, \beta_2, \beta_3) q^A. \tag{8.5}$$

For the last term, it is zero except that $deg(\beta_i) = 2$ by the dimension reason. If $deg(\beta_i) = 2$,

$$\sum_{n[\Gamma], n \neq 0} \Psi^M_{(A,0,3)}(\beta_1, \beta_2, \beta_3) q^{n[\Gamma]} = \frac{q^{[\Gamma]}}{1 - q^{[\Gamma]}} \beta_1([\Gamma]) \beta_2([\Gamma]) \beta_3([\Gamma]) n_\Gamma.$$

For the first term, it is also zero except that $(deg(\beta_1), deg(\beta_2), deg(\beta_3)) = (2,2,2), (4,2,0), (6,0,0)$. There is a similar expression for the 3-point function of M_f. We see that, by Theorem A, only the first and last terms are different. Suppose that $\beta_i = \varphi^* \alpha_i$. Let's first consider the case that $(deg(\beta_1), deg(\beta_2), deg(\beta_3)) = (4,2,0), (6,0,0)$. In these cases, the last term are zero. We claim that the first terms are the same as well. This is obvious for the case of $(6,0,0)$. For the case $(4,2,0)$, we can assume that $\alpha_3 = 1$. By the definition, the map $H^4(M_f, \mathbf{R}) \to H^4(M, \mathbf{R})$ is Poincare dual to the inverse of the map $H_2(M, \mathbf{Z}) \to H_2(M_f, \mathbf{Z})$. By our construction, the α_1, β_1 are Poincare dual to the same 2-manifold Σ disjoint from exceptional loci. Hence,

$$\varphi^* \alpha_2 \wedge \varphi^* \alpha_1 = \varphi^* \alpha_2(\Sigma) = \alpha_2(\Sigma) = \alpha_2 \wedge \alpha_1.$$

For the case $(2,2,2)$, both the first term and last term are not zero. Now we use Morrison's argument. The assertion follows from the following lemma. □

Lemma 8.1 ([Mo1]) For $q^{[\Gamma]} \neq 1$ we have

$$B_1 B_2 B_3 + \frac{q^{[\Gamma]}}{1 - q^{[\Gamma]}} (B_1 \Gamma)(B_2 \Gamma)(B_3 \Gamma) n_\Gamma = \hat{B}_1 \hat{B}_2 \hat{B}_3 + \frac{q^{[\hat{\Gamma}]}}{1 - q^{[\hat{\Gamma}]}} (\hat{B}_1 \hat{\Gamma})(\hat{B}_2 \hat{\Gamma})(\hat{B}_3 \hat{\Gamma}) n_{\hat{\Gamma}}.$$

Proof of Theorem B

The proof is similar to that of Theorem A. We take our extremal transition model in Section 1

$$M_e = M_s^+ \cup M_{t_0}^-.$$

Choose ϵ' slightly large than ϵ. Let $\widetilde{M} = M_e \cap S(\epsilon')$. We perform the symplectic cutting. One part $\overline{M_e}^-$ is M_b. Another part $\overline{M_e}^+$ is, by example 3 in Section 2, the projective compactification of $\{F = 0\}$ at infinity gluing with $\{F = t_0\}$, which is symplectic deformation equivalent to $Y = \{F - t_0 z_5^2 = 0\}$. Again, we first argue the case that $deg(\alpha_i) \geq 4$, where we choose support of α_i inside M^-. By a simple index calculation, if h is a holomorphic curve of Y tangent to the infinity divisor with order k, its index is $2(3k - k + 1) \geq 6$ if $k > 0$. By the same argument as in Theorem A we conclude that we need only consider those J-holomorphic curves which don't go through $\widetilde{M}$. Theorem B follows easily. Then, we use formula (1.1) to reduce the case $deg(\alpha_i) = 2$ to previous case. □

Only Corollary B.2 needs a proof. Others are immediate consequences of Theorem B and Corollary B.2.

Proof of Corollary B.2

The proof is a generalization of Tian's argument [Ti]. The surjective map

$$\varphi : H_2(M, \mathbf{R}) \to H_2(M_e, \mathbf{R}) \tag{8.7}$$

induces an injective map

$$\varphi^* : H^2(M_e, \mathbf{R}) \to H^2(M, \mathbf{R}).$$

By the definition, the map on H^4 is Poincare dual to a right inverse of (8.7). We claim that the ordinary cup product remains to be the same after transition. Assume that $\beta_i = \varphi^*\alpha_i$. Again, we need to consider the case that $(deg(\beta_1), deg(\beta_2), deg(\beta_3)) = (2,2,2), (4,2,0), (6,0,0)$. The case $(6,0,0)$ is obvious. The proof of case $(4,2,0)$ is similar to that of Corollary of Theorem A. β_1 is Poincare dual to $A_1 \in H_2(M, \mathbf{R})$ such that α_1 is Poincare dual to $\varphi_*(A_1)$. Hence,

$$\varphi^*\alpha_2 \wedge \varphi^*\alpha_1 = \varphi^*\alpha_2(A_1) = \alpha_2(\varphi_*(A_1)) = \alpha_2 \wedge \alpha_1.$$

For the case $(2,2,2)$, clearly $\varphi^*(\beta_i)(\Gamma) = 0$. Without loss of generality, assume that there is a 4-manifold representing $\varphi^*(\beta_i)$ which is disjoint from Γ. Hence, it can be viewed as a submanifold of M_e. Clearly, the same 4-manifold represents β_i. Hence,

$$\beta_1 \wedge \beta_2 \wedge \beta_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3.$$

Therefore,

$$\Psi^M_{\varphi^* w}(\varphi^*(\beta_1), \varphi^*(\beta_2), \varphi^*(\beta_3)) = \varphi^*(\beta_1) \wedge \varphi^*(\beta_2) \wedge \varphi^*(\beta_3)$$

$$+ \sum_{A \neq k[\Gamma]} \sum_m \frac{1}{m!} \Psi^M_{(A,0,m+3)}(\varphi^*(\beta_1), \varphi^*(\beta_2), \varphi^*(\beta_3), \varphi^* w, \cdots, \varphi^* w) q^A$$

$$+ \sum_{k[\Gamma], k \neq 0} \sum_m \frac{1}{m!} \Psi^M_{(A,0,m+3)}(\varphi^*(\beta_1), \varphi^*(\beta_2), \varphi^*(\beta_3), w, \cdots, w) q^{k[\Gamma]}. \tag{8.8}$$

By the previous argument, the first term is the same as $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$. The last term is always zero. Now we change the formal variable by $q^A \to q^{\varphi_*(A)}$ and apply the Theorem B. Then we prove

$$\Psi^M_{\varphi^* w}(\varphi^*(\alpha_1), \varphi^*(\alpha_2), \varphi^*(\alpha_3)) = \Psi^{M_e}_w(\alpha_1, \alpha_2, \alpha_3).$$

9 Appendix

In this appendix we prove the index addition formula of Bott-type (5.51). The proof is basically a similar gluing argument as in an unpublished book of Donaldson. For simplicity we shall consider a simple case, for general case the proof is the same. We use the notations in our paper. Suppose that $\Sigma = \Sigma^+ \wedge \Sigma^-$ with a intersection point p, where $\Sigma^{\pm}$ are smooth Riemann surfaces. We fix complex structure $j^{\pm}$ on $\Sigma^{\pm}$. Let $u = (u^+, u^-) : (\Sigma^+, \Sigma^-) \to (M^+, M^-)$ be J-holomorphic curves such that u^+ and u^- converge to the same k-periodic orbit x at the double point p. We introduce the following norms:

$$\|h\|_{L^{1,2,\alpha}} = \left(\int_{\Sigma^{\pm}} e^{2\alpha|s^{\pm}|}(|h|^2 + |\nabla h|^2)d\mu\right)^{1/2} \quad \forall\ h \in C^{\infty}(\Sigma^{\pm}; (u^{\pm})^*TM^{\pm}) \tag{9.1}$$

$$\|\eta\|_{L^{2,\alpha}} = \left(\int_{\Sigma^{\pm}} e^{2\alpha|s^{\pm}|}|\eta|^2 d\mu\right)^{1/2} \quad \forall\ \eta \in \Omega^{0,1}((u^{\pm})^*TM^{\pm}) \tag{9.2}$$

$$\|(h^+, h^-, h_0)\|_{L^{2,\alpha}} = \|h^+\|_{L^{2,\alpha}} + \|h^-\|_{L^{2,\alpha}} + |h_0| \ \ \forall\ h^{\pm} \in C^{\infty}(\Sigma^{\pm}; (u^{\pm})^*TM^{\pm}), h_0 \in kerL_{\infty}. \tag{9.3}$$

Obviously, $\|h\|_{L^{1,2,\alpha}} < \|h\|_{1,p,\alpha}$, $\|\eta\|_{L^{2,\alpha}} < \|\eta\|_{p,\alpha}$ (recall the definition of the norms $\| \ \|_{1,p,\alpha}$ and $\| \ \|_{p,\alpha}$ in our paper). Conversely, let $h \in kerD_{u^{\pm}}$ with $\|h\|_{L^{1,2,\alpha}} < \infty$. By using standard elliptic estimates, it is easy to show that $\|h\|_{1,p,\alpha} < \infty$. This fact also holds for $cokerD_{u^{\pm}}$. Hence in discussing index we will use the norms (9.1) and (9.2) instead of $\| \ \|_{1,p,\alpha}$ and $\| \ \|_{p,\alpha}$, which we used in our paper. Denote by $L^{1,2,\alpha}_{\pm}$ and $L^{2,\alpha}_{\pm}$ the Banach spaces corresponding to the norms (9.1) and (9.2) respectively. Put

$$\mathcal{L}^{1,2,\alpha}_{\pm} = \{h + \hat{h}_0 | h \in L^{1,2,\alpha}_{\pm}, h_0 \in kerL_{\infty}\}.$$

Denote by $Ind'(D_{u^{\pm},j^{\pm}}, \alpha)$ the index of the operator $D_{u^{\pm}} : L^{1,2,\alpha}_{\pm} \to L^{2,\alpha}_{\pm}$, and by $Ind(D_{u^{\pm},j^{\pm}}, \alpha)$ the index of the operator $D_{u^{\pm}} : \mathcal{L}^{1,2,\alpha}_{\pm} \to L^{2,\alpha}_{\pm}$. It is easy to show that

$$Ind(D_{u^{\pm},j^{\pm}}, \alpha) = Ind'(D_{u^{\pm},j^{\pm}}, \alpha) + dimkerL_{\infty}.$$

We glue M^+ and M^- with parameter r to get M_r, glue Σ^+ and Σ^- with parameter r to get the Riemann surface Σ_r and construct a map $u_r : \Sigma_r \to M_r$ as in the Section 6. For any $\eta \in \Omega^{0,1}(\Sigma_r; u_r^*TM_r)$, let $\eta_{(r,\pm)}$ be its restriction to the part $\Sigma_0^{\pm} \bigcup \{(s^{\pm}, t^{\pm}) | \ |s^{\pm}| < \frac{3r}{k}$. We define

$$\|\eta\|_{2,\alpha,r} = \|\eta_{(r,+)}\|_{L^{2,\alpha}} + \|\eta_{(r,-)}\|_{L^{2,\alpha}}.$$

For any $h \in C^{\infty}(\Sigma_r; u_r^*TM_r)$ let

$$h_0 = \int_{S^1} h(\frac{2r}{k}, t)dt.$$

We decomposite h uniquely into $h = h' + \hat{h}_0$, where $\hat{h}_0$ is defined from $h_0 \in \ker L_{\infty}$ as in Section 5. Then we define

$$\|h\|_{L^{1,2,\alpha,r}} = \|h'_{(r,+)}\|_{L^{1,2,\alpha}} + \|h'_{(r,-)}\|_{L^{1,2,\alpha}} + |h_0|. \tag{9.4}$$

Theorem 9.1

$$Ind(D_{u^+,j^+},\alpha) + Ind(D_{u^-,j^-},\alpha) - 2(n+1) = Ind(D_{u_r}). \tag{9.5}$$

Proof The proof is divided into 2 steps.

Step 1.

Suppose that $D_{u^\pm} : L^{1,2,\alpha}_\pm \to L^{2,\alpha}_\pm$ is surjective, so admit bounded right inverses

$$Q^\pm : L^{2,\alpha}_\pm \to L^{1,2,\alpha}_\pm$$

with $\|Q^\pm(\eta)\|_{L^{1,2,\alpha}} \le C\|\eta\|_{L^{2,\alpha}}$ and $D_{u^\pm}Q^\pm = I$. For large r we construct a injection

$$I : kerD_{u_r} \to kerD_{u^+} \times kerD_{u^-} \times kerL_\infty.$$

We fix cut-off functions λ^+ and λ^- on Σ_r such that

$$(\lambda^+)^2 e^{2\alpha|s^+|} + (\lambda^-)^2 e^{2\alpha|s^-|} = e^{2\alpha|s^+|} + e^{2\alpha|s^-|}$$

with $\lambda^\pm$ supported in $|s^\pm| < \frac{3r}{k}$ and such that $|\nabla\lambda^\pm| < \frac{const.}{r}$. For any $h \in kerD_{u_r}$ we put $I(h) = (h^+, h^-, h_0)$, where

$$h^\pm = \lambda^\pm h' - Q^\pm D_{u^\pm}(\lambda^\pm h').$$

We have

$$\begin{aligned} \|h^\pm - \lambda^\pm h'\|_{L^{2,\alpha}} &= \|Q^\pm D_{u^\pm}(\lambda^\pm h')\|_{L^{2,\alpha}} \\ &\le C\|D_{u^\pm}(\lambda^\pm h')\|_{L^{2,\alpha}} \le \frac{C_1}{r}\|h\|_{L^{2,\alpha,r}}. \end{aligned} \tag{9.6}$$

By (9.6) and the definition of the norm on Σ_r we have for any $h \in kerD_{u_r}$

$$\|\lambda^+ h'\|^2_{L^{2,\alpha}} + \|\lambda^- h'\|^2_{L^{2,\alpha}} + |h_0|^2 = \|h\|^2_{L^{2,\alpha,r}},$$

$$|\|I(h)\|_{L^{2,\alpha}} - \|h\|_{L^{2,\alpha,r}}| = |\|I(h)\|_{L^{2,\alpha}} - \|(\lambda^+ h', \lambda^- h', h_0)\|_{L^{2,\alpha}}| \le \frac{C_2}{r}\|h\|_{L^{2,\alpha,r}},$$

so I is injective once $r > C_2$. It follows that

$$Ind'(D_{u^+,j^+},\alpha) + Ind'(D_{u^-,j^-},\alpha) + 2(n+1) \ge Ind(D_{u_r}). \tag{9.7}$$

Remark 9.2 *If $D_{u^\pm} : L^{1,2,\alpha}_\pm \to L^{2,\alpha}_\pm$ is not surjective, the inequality (9.7) remains hold. In fact, we can choose a finite dimensional subspace $F^\pm$ of $L^{2,\alpha}_\pm$, which elements are supported away from* *the intersect point, such that* $S^{\pm} = D_{u^{\pm}} + I : L^{1,2,\alpha}_{\pm} + F^{\pm} \to L^{2,\alpha}_{\pm}$ *is surjective. Then we consider the operator* $S^{\pm}$ *instead of* $D_{u^{\pm}}$*.*

Step 2

Put

$$\mathcal{L}^{1,2,\alpha}_{u^+,u^-} = \{(h^+ + \hat{h}^+_0, h^- + \hat{h}^-_0) \in \mathcal{L}^{1,2,\alpha}_{+} \times \mathcal{L}^{1,p,\alpha}_{-} | h^+_0 = h^-_0\} \tag{9.8}$$

$$D_{u^+,u^-} = (D_{u^+}, D_{u^-}). \tag{9.9}$$

By the same reason as Remark 9.2 we may assume that

$$D_{u^+,u^-} : \mathcal{L}^{1,2,\alpha}_{u^+,u^-} \to L^{2,\alpha}_{+} \times L^{2,\alpha}_{-}$$

is surjective. By using the same argument as in Subsection 6.2 we can show that the operator D_{u_r} is also surjective for large r, so there is a right inverse Q_r. Now we construct a injection $I' : kerD_{u^+} \times kerD_{u^-} \times kerL_{\infty} \to kerD_{u_r}$. Choose cut-off functions $\beta^{\pm}$ such that

$$\beta^{\pm}(s^{\pm}) = \begin{cases} 1 & if \;\; |s^{\pm}| \leq \frac{r}{2k} \\ 0 & if \;\; |s^{\pm}| \geq \frac{3r}{2k} \end{cases}$$

and $|\beta'(s^{\pm})| \leq \frac{2}{r}$. For element $(h^+, h^-, h_0) \in kerD_{u^+} \times kerD_{u^-} \times kerL_{\infty}$ we set

$$I'(h^+, h^-, h_0) = g - Q_r D_{u_r} g$$

where

$$g = \beta^+ h^+ + \beta^- h^- + \hat{h_0}.$$

As in the first step we have

$$\|Q_r D_{u_r} g\|_{L^{2,\alpha,r}} \leq \frac{C}{r} \|(h^+, h^-, h_0)\|_{L^{2,\alpha}}. \tag{9.10}$$

It remains to show that $\|g\|_{L^{2,\alpha,r}}$ is close to $\|(h^+, h^-, h_0)\|_{L^{2,\alpha}}$ for large r. Let $f_i, i = 1, ..., d$, be an orthonormal basis for the $kerD_{u^+}$. Then $F = \sum |f_i|^2$ is a integrable function. For any $\varepsilon > 0$ we can choose r_0 such that

$$\int_{(r_0/k,\infty)\times S^1} F e^{2\alpha|s^+|} \leq \varepsilon.$$

Then for $r > r_0$

$$\int_{(r/k,\infty)\times S^1} |h^+|^2 e^{2\alpha|s^+|} \leq d\varepsilon \|h^+\|^2_{L^{2,\alpha}}.$$

Similar inequality holds over Σ^-. Note that $\beta^{\pm}=1$ for $|s^{\pm}|\leq \frac{r}{k}$, we have

$$\|(\beta^+h^+ + \beta^-h^- + \hat{h_0})\|_{L^{2,\alpha,r}} \geq (1-2d\varepsilon)\|(h^+,h^-,h_0)\|_{L^{2,\alpha}} \tag{9.11}$$

It follows from (9.10) and (9.11) that I' is injective. So

$$Ind'(D_{u^+,j^+},\alpha) + Ind'(D_{u^-,j^-},\alpha) + 2(n+1) \leq Ind(D_{u_r}). \tag{9.12}$$

The Theorem 9.1 follows from (9.7) and (9.12). □

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