The local Gromov-Witten theory of curves

J. Bryan and R. Pandharipande

March 12, 2008

Abstract

We study the equivariant Gromov-Witten theory of a rank 2 vector bundle N over a nonsingular curve X of genus g:

(i) We define a TQFT using the Gromov-Witten partition functions. The full theory is determined in the TQFT formalism from a few exact calculations. We use a reconstruction result proven jointly with C. Faber and A. Okounkov in the appendix.

(ii) If $N \cong K_X^{1/2} \oplus K_X^{1/2}$ and N is equipped with the anti-diagonal $\mathbb{C}^*$-action, the partition function is

$$\sum_{\rho \vdash d} \left(\frac{d!}{\dim_Q \rho} \right)^{2g-2}$$

where $Q = e^{iu}$, u is the genus parameter, and the sum is over irreducible representations of the symmetric group S_d. The formula is a Q-deformation of the classical Hurwitz formula for counting unramified covers.

(iii) An equivariant version of the Gromov-Witten/Donaldson-Thomas correspondence is formulated and discussed in detail for the case of N.

The theory generalizes the local Calabi-Yau theory of X defined and studied in [2, 4].

Contents

1 Introduction

1.1 Summary

The Gromov-Witten theory of threefolds, particularly Calabi-Yau threefolds, is a very rich subject. The study of *local theories*, Gromov-Witten theories of non-compact targets, has revealed much of the structure. Let X be a complete, nonsingular, irreducible curve of genus g over $\mathbb{C}$, and let

$$N \to X$$

be a rank 2 vector bundle with $\det N \cong K_X$. Then, N is a non-compact Calabi-Yau[1] threefold, and the Gromov-Witten theory, defined and studied in [2, 3, 4, 6, 20], is called the *local Calabi-Yau theory of* X. We study here the local theory of curves without imposing the Calabi-Yau condition $\det N \cong K_X$ on the bundle N.

The study of non Calabi-Yau local theories has several motivations. The calculations of [6, 20, 22] predict a uniform structure for all threefold theories closely related to the Calabi-Yau case. The introduction of non Calabi-Yau bundles N yields a more flexible mathematical framework in which new methods arise. We present a complete solution of the local theory of curves via recursions. The result requires a nonsingularity statement proven in the Appendix with C. Faber and A. Okounkov.

The precise local-to-global formalism for a Calabli-Yau threefold Y is not yet understood. The relationship between the local theory of curves and the global Gromov-Witten theory Y of is discussed in [3].

Our solution of the local theory of curves is a starting point for several lines of inquiry:

The Gromov-Witten/Donaldson-Thomas correspondence of [14, 15] may be naturally studied in the context of local theories. In Section 9, we present an equivariant version of the correspondence that is applicable to local theories.

The local theory of the *trivial* rank 2 bundle over $\mathbb{P}^1$ has recently been connected to the quantum cohomologies of the orbifold $(\mathbb{C}^2)/S_n$ and the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ of points in $\mathbb{C}^2$, see [1, 19]. A brief discussion appears in Section 10.

1.2 Results

Let N be a rank 2 bundle on X with a direct sum decomposition,

$$N = L_1 \oplus L_2.$$

Let k_i denote the degree of L_i on X. We call the pair (k_1, k_2) the *level* of the theory. The torus,

$$T = \mathbb{C}^* \times \mathbb{C}^*,$$

acts on N by scaling the line bundles L_1 and L_2. We study here the Gromov-Witten residue invariants of the total space N with respect to the T-action.

[1] We call any quasi-projective threefold with trivial canonical bundle Calabi-Yau.

The Gromov-Witten residue invariants of N, defined in Section 2.2, take values in the localized equivariant cohomology ring of T. The invariants are rational functions in the classes s_1 and s_2 corresponding to the generators of the equivariant cohomology. The basic objects of study in our paper are the *partition functions*

$$Z_d(g \,|\, k_1, k_2) \in \mathbb{Q}(s_1, s_2)((u)),$$

which are generating functions for the degree d residue invariants of N.

The residue invariants specialize to the local invariants of X in a Calabi-Yau threefold defined in [2, 4] if the level satisfies

$$k_1 + k_2 = 2g - 2$$

and the variables are equated,

$$s_1 = s_2.$$

Equating the variables is equivalent to considering the the residue theory of N with respect to the diagonal action of a 1-dimensional torus.

For the Gromov-Witten residue invariants of N, we develop a gluing theory in Section 6 following [2]. The interpretation of the local theory as TQFT is discussed in Section 4. In Sections 5 - 6 and the Appendix (with C. Faber and A. Okounkov), the gluing relations together with a few basic integrals are proven to determine all the full local theory of curves. The level freedom of the theory plays an essential role. We provide explicit formulas in Sections 7 and 8.

A parallel equivariant Donaldson-Thomas residue theory can be defined for the threefold N. We conjecture a Gromov-Witten/Donaldson-Thomas correspondence for equivariant residues in the framework of [14, 15], see Section 9. An important consequence of our theory is Theorem 6.4: after suitable normalization, $Z_d(g \,|\, k_1, k_2)$ is a *rational* function of the variables s_1, s_2, and

$$q = -e^{iu}.$$

The result verifies a predictions of the GW/DT correspondence (see Conjecture 2R of Section 9).

The residue invariants of N are of special interest when the variable reduction,

$$s_1 + s_2 = 0,$$

is taken. The reduction is equivalent to considering the residue theory of N with respect to the *anti-diagonal* action of a 1-dimensional torus. In Theorem 7.1, we obtain a general closed formula for the partition function in the anti-diagonal case.

If we additionally specialize to the Calabi-Yau case, our formula is particularly attractive. The residue partition function here is simply a Q-deformation of the classical formula for unramified covers (see Corollary 7.2):

$$Z_d(g \,|\, k, 2g-2-k) = (-1)^{d(g-1-k)} \sum_{\rho} \left(\frac{d!}{\dim_Q \rho} \right)^{2g-2} Q^{-c_\rho (g-1-k)}$$

where $Q = e^{iu}$ and the sum is over partitions. With the anti-diagonal action, N is *equivariantly* Calabi-Yau.

The above formula suggests a connection of our theory to 2D Yang-Mills theory on X, see, for example, [16]. The genus 1 case was studied in [23].

The anti-diagonal action is exactly *opposite* to the original motivations of the project. It would be very interesting to find connections between the anti-diagonal case and the original questions of the Gromov-Witten theory of curves in Calabi-Yau threefolds.

1.3 Acknowledgments

The authors thank G. Farkas, T. Graber, S. Katz, C. Teleman, M. Thaddeus, C. Vafa, and R. Vakil for valuable discussions. We thank J. Kock for the use of his cobordism LaTeX macros. A first draft of the Appendix was completed during a visit by C. Faber to Princeton in the summer of 2004.

J. B. was partially supported by the NSERC, the Clay Institute, and the Aspen Institute. R. P. was partially supported by the Packard foundation and the NSF.

2 The residue theory

2.1 Gromov-Witten residue invariants

Let Y be a nonsingular, *quasi-projective*, algebraic threefold. Let $\overline{M}^{\bullet}_h(Y, \beta)$ denote the moduli space of stable maps

$$[f : C \to Y]$$

of genus h and degree $\beta \in H_2(Y, \mathbb{Z})$. The superscript $\bullet$ indicates the possibility of disconnected domains C. We require f to be nonconstant on each connected component of C. The genus, $h(C)$, is defined by

$$h(C) = 1 - \chi(\mathcal{O}_C)$$

and may be negative.

Let Y be equipped with an action by an algebraic torus T. We will define Gromov-Witten residue invariants under the following assumption.

Assumption 1. *The T-fixed point set $\overline{M}^\bullet_h(Y, \beta)^T$ is compact.*

We motivate the definition of the residue invariants of Y as follows. We would like to define the reduced Gromov-Witten partition function $\mathsf{Z}'_{GW}(Y)_\beta$ as a generating function of the integrals of the identity class over the moduli spaces of maps,

$$\mathsf{Z}'_{GW}(Y)_\beta \text{ `` = '' } \sum_{h \in \mathbb{Z}} u^{2h-2} \int_{[\overline{M}^\bullet_h(Y,\beta)]^{vir}} 1. \tag{1}$$

However, the integral on the right might not be well-defined if X is not compact.

If Y has trivial canonical bundle and $\overline{M}^\bullet_h(Y, \beta)$ is compact, then the integral (1) is well-defined. The resulting series $\mathsf{Z}'_{GW}(Y)_\beta$ is then the usual reduced partition function for the degree β disconnected Gromov-Witten invariants[2] of Y. We can use the virtual localization formula to express $\mathsf{Z}'_{GW}(Y)_\beta$ as a residue integral over T-fixed point locus.

More generally, under Assumption 1, the series $\mathsf{Z}'_{GW}(Y)_\beta$ can be *defined* via localization.

Definition 2.1. *The reduced partition function for the degree β residue Gromov-Witten invariants of Y is defined by:*

$$\mathsf{Z}'_{GW}(Y)_\beta = \sum_{h \in \mathbb{Z}} u^{2h-2} \int_{[\overline{M}^\bullet_h(Y,\beta)^T]^{vir}} \frac{1}{e(\mathrm{Norm}^{vir})}. \tag{2}$$

[2]We follow the notation of [14, 15] for the reduced partition function. The prime indicates the removal of the degree 0 contributions. In [14, 15], the moduli space $\overline{M}^\bullet_h(Y, \beta)$ is denoted by $\overline{M}'_h(Y, \beta)$. However, to maintain notational consistency with [2], we will *not* adopt the latter convention.

The T-fixed part of the perfect obstruction theory for $\overline{M}^{\bullet}_h(Y,\beta)$ induces a perfect obstruction theory for $\overline{M}^{\bullet}_h(Y,\beta)^T$ and hence a virtual class [9]. The equivariant virtual normal bundle of the embedding,

$$\overline{M}^{\bullet}_h(Y,\beta)^T \subset \overline{M}^{\bullet}_h(Y,\beta),$$

is Norm^{vir} with equivariant Euler class $e(\text{Norm}^{vir})$. The integral in (2) denotes equivariant push-forward.

Let r be the rank of T, and let $s_1,\ldots,s_r$ be generators for the equivariant cohomology of T,

$$H^*_T(\text{pt}) \cong \mathbb{Q}[s_1,\ldots,s_r].$$

By Definition 2.1, $\mathsf{Z}'_{GW}(Y)_\beta$ is a Laurent series in u with coefficients given by rational functions of the variables $s_1,\ldots,s_r$ of homogeneous degree equal to minus the virtual dimension of $\overline{M}^{\bullet}_h(Y,\beta)$.

2.2 Gromov-Witten residue invariants of N

Let X be a nonsingular, irreducible, projective curve of genus g. Let

$$N = L_1 \oplus L_2$$

be a a rank 2 bundle on X. The residue invariants of the threefold N with respect to the 2-dimensional scaling torus action can be written in terms of integrals over the moduli space of maps to X.

A stable map to N which is T-invariant must factor through the zero section. Hence,

$$\overline{M}^{\bullet}_h(N,d[X])^T \cong \overline{M}^{\bullet}_h(X,d).$$

Moreover, the T-fixed part of the perfect obstruction theory of $\overline{M}^{\bullet}_h(N,d[X])$, restricted to $\overline{M}^{\bullet}_h(N,d[X])^T$, is exactly the usual perfect obstruction theory for $\overline{M}^{\bullet}_h(X,d)$. Hence,

$$[\overline{M}^{\bullet}_h(N,d[X])^T]^{vir} \cong [\overline{M}^{\bullet}_h(X,d)]^{vir}.$$

The virtual normal bundle of $\overline{M}^{\bullet}_h(N,d[X])^T \subset \overline{M}^{\bullet}_h(N,d[X])$, considered as an element of K-theory on $\overline{M}^{\bullet}_h(X,d)$, is given by

$$\text{Norm}^{vir} = R^{\bullet}\pi_* f^*(L_1 \oplus L_2)$$

where

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & X \\ {\scriptstyle\pi}\big\downarrow & & \\ \overline{M}^{\bullet}_h(X,d) & & \end{array}$$

is the universal diagram for $\overline{M}^{\bullet}_h(X,d)$.

The reduced Gromov-Witten partition function of the residue invariants may be written in the following form via equivariant integration:

$$\mathsf{Z}'_{GW}(N)_{d[X]} = \sum_{h\in\mathbb{Z}} u^{2h-2} \int_{[\overline{M}^{\bullet}_h(X,d)]^{vir}} e(-R^{\bullet}\pi_* f^*(L_1 \oplus L_2)).$$

We will be primarily interested in a partition function with a shifted exponent,

$$Z_d(g\,|\,k_1,k_2) = u^{d(2-2g+k_1+k_2)}\ \mathsf{Z}'_{GW}(N)_{d[X]}.$$

The shift can be interpreted geometrically as

$$\int_{d[X]} c_1(T_N) = d(2-2g+k_1+k_2),$$

where T_N is the tangent bundle of the threefold N.

The explicit dependence on the equivariant parameters s_1 and s_2 may be written as follows. Let b_1 and b_2 be non-negative integers satisfying

$$b_1 + b_2 = 2h - 2 + d(2-2g)$$

where $2h-2+d(2-2g)$ is the virtual dimension of $\overline{M}^{\bullet}_h(X,d)$. Let

$$Z_d^{b_1,b_2}(g\,|\,k_1,k_2) = \int_{[\overline{M}^{\bullet}_h(X,d)]^{vir}} c_{b_1}(-R^{\bullet}\pi_* f^* L_1) c_{b_2}(-R^{\bullet}\pi_* f^* L_2),$$

where $\int$ here denotes *ordinary* integration. The equivariant Euler class $e(-R^{\bullet}\pi_* f^*(L_1\oplus L_2))$ is easily expressed in terms of the equivariant parameters and the *ordinary* Chern classes of $-R^{\bullet}\pi_* f^*(L_1)$ and $-R^{\bullet}\pi_* f^*(L_2)$,

$$Z_d(g\,|\,k_1,k_2) = \\ u^{d(k_1+k_2)} s_1^{d(g-1-k_1)} s_2^{d(g-1-k_2)} \sum_{b_1,b_2=0}^{\infty} u^{b_1+b_2} s_1^{\frac{b_2-b_1}{2}} s_2^{\frac{b_1-b_2}{2}} Z_d^{b_1,b_2}(g\,|\,k_1,k_2).$$

Since b_1+b_2 is even, the exponents of s_1 and s_2 are integers. We see $Z_d(g\,|\,k_1,k_2)$ is a Laurent series in u with coefficients given by rational functions of s_1 and s_2 of homogeneous degree $d(2g-2-k_1-k_2)$.

2.3 Specialization to the local Calabi-Yau theory.

In [2], we defined the (reduced) partition function for the local Calabi-Yau invariants of a genus g curve,

$$Z_d(g) = \sum_{b=0}^{\infty} t^b \int_{[\overline{M}^{\bullet}_h(X,d)]^{vir}} c_b(-R^{\bullet}\pi_* f^*(\mathcal{O} \oplus K_X))$$

where

$$b = 2h - 2 + d(2 - 2g).$$

We can recover $Z_d(g)$ from $Z_d(g \,|\, k_1, k_2)$ by setting

$$k_1 + k_2 = 2g - 2$$

and specializing $s_1 = s_2$,

$$Z_d(g) = t^{d(2-2g)} \; Z_d(g \,|\, k_1, k_2)|_{k_1+k_2=2g-2,\ s_1=s_2,\ u=t}.$$

The prefactor $t^{d(2-2g)}$ relates different generating conventions.

3 Gluing formulas

3.1 Notation and conventions for partitions

We first review the partition notation used in the paper. By definition, a partition λ is a finite sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots)$$

where

$$|\lambda| = \sum_i \lambda_i = d.$$

We use the notation $\lambda \vdash d$ to indicate that λ is a partition of d.

The number of parts of λ is called the *length* of λ and is denoted $l(\lambda)$. Let $m_i(\lambda)$ be the number times that i occurs in the partition λ. We may write a partition in the format:

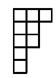
$$\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \cdots) \, .$$

The combinatorial factor,

$$\mathfrak{z}(\lambda) = \prod_{i=1}^{\infty} m_i(\lambda)! i^{m_i(\lambda)},$$

arises frequently.

A partition λ is uniquely determined by the associated Ferrar's diagram which is the collection of d boxes located at (i, j) where $1 \leq j \leq \lambda_i$. For example

$$(3, 2, 2, 1, 1) = (1^2 2^2 3) =$$  .

The *conjugate partition* λ' is obtained by reflecting the Ferrar's diagram of λ about the $i = j$ line.

In Section 7, we will require the following standard quantities. Given a box in the Ferrar's diagram, $\square \in \lambda$, define the *content* $c(\square)$ to be $i - j$, and the *hook length* $h(\square)$ to be $\lambda_i + \lambda'_j - i - j + 1$. The total content

$$c_\lambda = \sum_{\square \in \lambda} c(\square)$$

and the total hooklength

$$\sum_{\square \in \lambda} h(\square)$$

satisfy the following identities (page 11 of [13]):

$$\sum_{\square \in \lambda} h(\square) = n(\lambda) + n(\lambda') + d, \qquad c_\lambda = n(\lambda') - n(\lambda), \tag{3}$$

where

$$n(\lambda) = \sum_{i=1}^{l(\lambda)} (i-1)\lambda_i.$$

3.2 Relative invariants

To formulate our gluing laws for the residue theory of rank 2 bundles on X, we require relative versions of the residue invariants.

Let $(X, x_1, \ldots, x_r)$ be a non-singular curve of genus g with r distinct marked points. Let $\overline{M}^\bullet_h(X, \lambda^1, \ldots, \lambda^r)$ denote the moduli space of (possibly disconnected) relative stable maps to X with prescribed ramification λ^i at x_i. Here $\lambda^1, \ldots, \lambda^r$ are r partitions of d. We use the conventions of Definition 3.1

in [2] for $\overline{M}^\bullet_h(X, \lambda^1, \ldots, \lambda^r)$. We do *not* mark the prescribed ramification points on the domain, and we require the maps to be nonconstant on all connected components.

We define the relative reduced partition function via equivariant integration over relative maps spaces:

$$\mathsf{Z}'_{GW}(N)_{d[X],\lambda^1,\ldots,\lambda^r} \;=\; \sum_{h\in\mathbb{Z}} u^{2h-2} \int_{[\overline{M}^\bullet_h(X,\lambda^1,\ldots,\lambda^r)]^{vir}} e(-R^\bullet\pi_* f^*(L_1 \,\oplus\, L_2)).$$

Again, we will be primarily interested in a shifted generating function,

$$Z_d(g\,|\,k_1,k_2)_{\lambda^1,\ldots,\lambda^r} = u^{d(2-2g+k_1+k_2-r)+\sum_{i=1}^r l(\lambda^i)}\ \mathsf{Z}'_{GW}(N)_{d[X],\lambda^1,\ldots,\lambda^r}.$$

The exponent of u in the partition function $Z_d(g\,|\,k_1,k_2)$ of the non-relative theory is

$$2h-2+\int_{d[X]} c_1(T_N).$$

In the relative theory, the $2h-2$ term in the exponent is replaced with $2h-2+\sum l(\lambda^i)$, the negative Euler characteristic of the *punctured* domain. The class $c_1(T_N)$ is replaced with the dual of log canonical class of N with respect to the relative divisors. The outcome is the modified exponent of u in the partition function $Z_d(g\,|\,k_1,k_2)_{\lambda^1,\ldots,\lambda^r}$.

As before, we can make the dependence on s_1 and s_2 explicit. Let

$$b_1+b_2 = 2h-2+d(2-2g)-\delta,$$

where

$$\delta = \sum_{i=1}^r (d-l(\lambda^i)).$$

Here, b_1+b_2 equals the virtual dimension of $\overline{M}^\bullet_h(X,\lambda^1\ldots\lambda^r)$. Let

$$Z_d^{b_1,b_2}(g\,|\,k_1,k_2)_{\lambda^1,\ldots,\lambda^r} = \int_{[\overline{M}^\bullet_h(X,\lambda^1,\ldots,\lambda^r)]^{vir}} c_{b_1}(-R^\bullet\pi_* f^* L_1)c_{b_2}(-R^\bullet\pi_* f^* L_2).$$

Then, we have

$$\begin{aligned} Z_d(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r} &= u^{d(k_1+k_2)} s_1^{d(g-1-k_1)} s_2^{d(g-1-k_2)} \\ &\quad\cdot \sum_{b_1,b_2=0}^{\infty} u^{b_1+b_2} s_1^{\frac{b_2-b_1+\delta}{2}} s_2^{\frac{b_1-b_2+\delta}{2}} Z_d^{b_1,b_2}(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r}. \quad (4)\end{aligned}$$

Since the parity of $b_1 + b_2$ is the same as δ, the exponents of s_1 and s_2 are integers.

We see $Z_d(g \,|\, k_1, k_2)_{\lambda^1,\ldots,\lambda^r}$ is a Laurent series in u with coefficients given by rational functions in s_1 and s_2 of homogeneous degree

$$d(2g - 2 - k_1 - k_2) + \delta.$$

In [2], the combinatorial factor $\mathfrak{z}(\lambda)$ is used to raise the indices for the relative invariants. For the residue invariants, an additional additional factor $(s_1 s_2)^{l(\lambda)}$ must be included. We define:

$$Z_d(g \,|\, k_1, k_2)^{\nu^1 \ldots \nu^t}_{\mu^1 \ldots \mu^s} = \left(\prod_{i=1}^{t} \mathfrak{z}(\nu^i)(s_1 s_2)^{l(\nu^i)} \right) Z_d(g \,|\, k_1, k_2)_{\mu^1 \ldots \mu^s, \nu^1 \ldots \nu^t}. \quad (5)$$

3.3 Gluing formulas

The gluing formulas are determined by the following result.

Theorem 3.1. *For any sets $\{\mu^1, \ldots, \mu^s\}$ and $\{\nu^1, \ldots, \nu^t\}$ of partitions of d, and integers satisfying $g = g' + g''$, $k_1 = k_1' + k_1''$, and $k_2 = k_2' + k_2''$ we have*

$$Z_d(g \,|\, k_1, k_2)^{\nu^1 \ldots \nu^t}_{\mu^1 \ldots \mu^s} = \sum_{\lambda \vdash d} Z_d(g' \,|\, k_1', k_2')^{\lambda}_{\mu^1 \ldots \mu^s} Z_d(g'' \,|\, k_1'', k_2'')^{\nu^1 \ldots \nu^t}_{\lambda}$$

and

$$Z_d(g + 1 \,|\, k_1, k_2)_{\mu^1 \ldots \mu^s} = \sum_{\lambda \vdash d} Z_d(g \,|\, k_1, k_2)^{\lambda}_{\mu^1 \ldots \mu^s \lambda}.$$

Proof. The proof follows the derivation of the gluing formulas in [2]. The only difference is the modified metric term

$$\mathfrak{z}(\lambda)(s_1 s_2)^{l(\lambda)}.$$

The first factor, $\mathfrak{z}(\lambda)$, is obtained from degeneration formula for the virtual class [11] as in [18].

The second factor, $(s_1 s_2)^{l(\lambda)}$, arises from normalization sequences associated to the fractured domains. Let

$$f : C \to X$$

be an element of $\overline{M}^{\bullet}_h(X, \mu^1, \ldots, \mu^s, \nu^1, \ldots, \nu^t)$. Consider a reducible degeneration of the target,

$$X = X' \cup X'',$$

over which the line bundles L_1 and L_2 extend with degree splittings

$$k_1 = k_1' + k_1'',$$

$$k_2 = k_2' + k_2''.$$

In a degeneration of type λ, the domain curve degenerates,

$$C = C' \cup C'',$$

into components lying over X' and X'' and satisfying

$$|C' \cap C''| = l(\lambda).$$

For each line bundle L_i, we have a normalization sequence,

$$0 \to f^*(L_i)|_C \to f^*(L_i)|_{C'} \oplus f^*(L_i)|_{C''} \to f^*(L_i)|_{C' \cap C''} \to 0. \qquad (6)$$

The last term yields a trivial bundle of rank $l(\lambda)$ with scalar torus action over the moduli space of maps of degenerations of type λ. The factor $(s_1 s_2)^{l(\lambda)}$ is obtained from the higher direct images of the normalization sequences (6). The analysis for irreducible degenerations of X is identical.

The exponent of u in the series $Z_d(g \,|\, k_1, k_2)^{\nu^1 \ldots \nu^t}_{\mu^1 \ldots \mu^s}$ of relative invariants has been precisely chosen to respect the gluing rules. □

4 TQFT formulation of gluing laws

4.1 Overview

The gluing structure of the residue theory of rank 2 bundles on curves is most concisely formulated as a functor of tensor categories,

$$\mathbf{Z}_d(-) : 2\mathbf{Cob}^{L_1, L_2} \to R\mathbf{mod}.$$

Our discussion follows Sections 2 and 4 of [2] and draws from Chapter 1 of [10]. Modifications of the categories have to be made to accommodate the more complicated objects studied here.

4.2 2Cob and $2\mathbf{Cob}^{L_1,L_2}$

We first define the category $2\mathbf{Cob}$ of 2-cobordisms. The objects of $2\mathbf{Cob}$ are compact oriented 1-manifolds, or equivalently, finite unions of oriented circles. Let Y_1 and Y_2 be objects of the category. A morphism,

$$Y_1 \to Y_2,$$

is an equivalence class of oriented cobordisms W from Y_1 to Y_2. Two cobordisms are equivalent if they are diffeomorphic by a boundary preserving oriented diffeomorphism. Composition of morphisms is obtained by concatenation the corresponding cobordisms. The tensor structure on the category is given by disjoint union.

The category $2\mathbf{Cob}^{L_1,L_2}$ is defined to have the same objects as $2\mathbf{Cob}$. A morphism in $2\mathbf{Cob}^{L_1,L_2}$,

$$Y_1 \to Y_2,$$

is an equivalence class of triples (W, L_1, L_2) where W is an oriented cobordism from Y_1 to Y_2 and L_1, L_2 are complex line bundles on W, trivialized on ∂W. The triples (W, L_1, L_2) and (W', L_1', L_2') are equivalent if there exists a boundary preserving oriented diffeomorphism,

$$f : W \to W',$$

and bundle isomorphisms

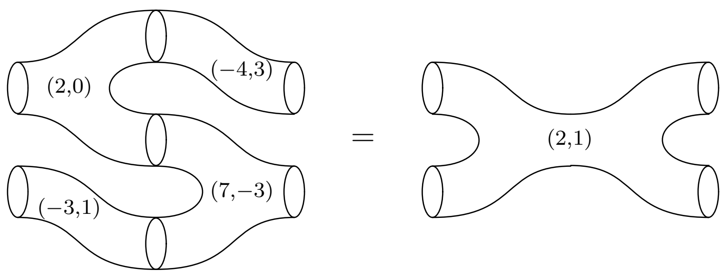
$$L_i \cong f^* L_i'.$$

Composition is given by concatenation of the cobordisms and gluing of the bundles along the concatenation using the trivializations.

The isomorphism class of L_i is determined by the Euler class

$$e(L_i) \in H^2(W, \partial W),$$

which assigns an integer to each component of W. For a connected cobordism W, we refer to the pair of integers (k_1, k_2), determined by the Euler classes of L_1 and L_2, as the *level*. Under concatenation, the levels simply add. For example:



The empty manifold is a distinguished object in $2\mathbf{Cob}$ and $2\mathbf{Cob}^{L_1,L_2}$. A morphism in $2\mathbf{Cob}^{L_1,L_2}$ from the empty manifold to itself is given by a compact, oriented, closed 2-manifold X together with a pair of complex line bundles $L_1 \oplus L_2 \to X$.

The full subcategory of $2\mathbf{Cob}^{L_1,L_2}$ obtained by restricting to level $(0,0)$ line bundles is clearly isomorphic to the category $2\mathbf{Cob}$.

More generally, we obtain an embedding $2\mathbf{Cob} \subset 2\mathbf{Cob}^{L_1,L_2}$ for any fixed integers (a,b) by requiring the level of any connected cobordism to be $(a\chi, b\chi)$ where χ is the Euler characteristic of the cobordism.

If $a+b=-1$, such an embedding is termed *Calabi-Yau* since the threefold

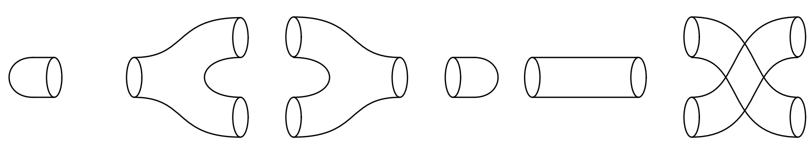
$$L_1 \oplus L_2 \to X$$

has numerically trivial canonical class if

$$\deg(L_1) + \deg(L_2) = -\chi.$$

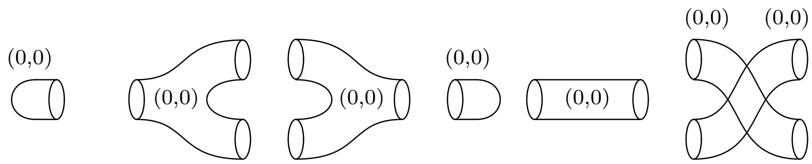
4.3 Generators for $2\mathbf{Cob}$ and $2\mathbf{Cob}^{L_1,L_2}$

The category $2\mathbf{Cob}$ is generated by the morphisms



In other words, any morphism (cobordism) can be obtained by taking compositions and tensor products (concatenations and disjoint unions) of the above list (Proposition 1.4.13 of [10]).

The category $2\mathbf{Cob}^{L_1,L_2}$ is then clearly generated by the morphisms



along with the morphisms

Let R be a commutative ring with unit, and let $R\mathbf{mod}$ be the tensor category of R modules. By a well-known result (see Theorem 3.3.2 of [10]), a 1+1 dimensional R-valued TQFT, which is by definition a tensor functor

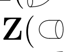
$$\mathbf{Z} : 2\mathbf{Cob} \to R\mathbf{mod}, \tag{7}$$

is equivalent to a commutative Frobenius algebra over R.

Given a tensor functor (7), the underlying R-module of the Frobenius algebra is given by

$$H = \mathbf{Z}(S^1)$$

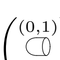
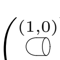
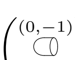
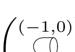
and the Frobenius algebra structure is determined as follows:

multiplication	$\mathbf{Z}($$)$:	$H \otimes H \to H$
unit	$\mathbf{Z}($$)$:	$R \to H$
comultiplication	$\mathbf{Z}($$)$:	$H \to H \otimes H$
counit	$\mathbf{Z}($$)$:	$H \to R.$

Let $\mathbf{Z}$ be a tensor functor on the larger category $2\mathbf{Cob}^{L_1,L_2}$,

$$\mathbf{Z} : 2\mathbf{Cob}^{L_1,L_2} \to R\mathbf{mod}.$$

Since the functor $\mathbf{Z}$ is determined by the values on the generators of $2\mathbf{Cob}^{L_1,L_2}$, the functor $\mathbf{Z}$ is determined by the level $(0,0)$ Frobenius algebra together with the elements

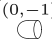
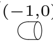
$\mathbf{Z}\Big($$\Big)$, $\mathbf{Z}\Big($$\Big)$, $\mathbf{Z}\Big($$\Big)$, $\mathbf{Z}\Big($$\Big)$,

Since the later two elements are the inverses in the Frobenius algebra of the first two, we obtain half of the following Theorem.

Theorem 4.1. *A tensor functor*

$$\mathbf{Z} : 2\mathbf{Cob}^{L_1,L_2} \to R\mathbf{mod}$$

is uniquely determined by a commutative Frobenius algebra over R for the level $(0,0)$ theory and two distinguished, invertible elements

$\mathbf{Z}\Big($ $^{(0,-1)}$  $\Big)$, $\mathbf{Z}\Big($ $^{(-1,0)}$  $\Big)$.

Proof. Uniqueness was proved above. The existence result will not be used in the paper. We leave the details to the reader. ☐

4.4 The functor $\mathbf{Z}_d(-)$

Let R be the ring of Laurent series in u whose coefficients are rational functions in s_1 and s_2,

$$R = \mathbb{Q}(s_1, s_2)((u)).$$

The collection of partition functions $Z_d(g \,|\, k_1, k_2)_{\lambda^1 \ldots \lambda^r}$ gives rise to a functor

$$\mathbf{Z}_d(-) : 2\mathbf{Cob}^{L_1,L_2} \to R\mathbf{mod}$$

as follows. Define

$$\mathbf{Z}_d(S^1) = H = \bigoplus_{\lambda \vdash d} R e_\lambda$$

to be the free R-module with basis $\{e_\lambda\}_{\lambda \vdash d}$ labelled by partitions of d, and let

$$\mathbf{Z}_d\left(S^1 \coprod \cdots \coprod S^1\right) = H \otimes \cdots \otimes H.$$

Let $W_s^t(g \,|\, k_1, k_2)$ be the connected genus g cobordism from a disjoint union of s circles to a disjoint union of t circles, equipped with lines bundles L_1 and L_2 of level (k_1, k_2). We define the R-module homomorphism

$$\mathbf{Z}_d\left(W_s^t(g \,|\, k_1, k_2)\right) : H^{\otimes s} \to H^{\otimes t}$$

by

$$e_{\eta^1} \otimes \cdots \otimes e_{\eta^s} \mapsto \sum_{\mu^1 \ldots \mu^t \vdash d} Z_d(g \,|\, k_1, k_2)^{\mu^1 \ldots \mu^t}_{\eta^1 \ldots \eta^s} e_{\mu^1} \otimes \cdots \otimes e_{\mu^t}.$$

We extend the definition of $\mathbf{Z}_d(-)$ to disconnected cobordisms by tensor product:

$$\mathbf{Z}_d\left(W[1] \coprod \cdots \coprod W[n]\right) = \mathbf{Z}_d\left(W[1]\right) \otimes \cdots \otimes \mathbf{Z}_d(W[n]).$$

Theorem 4.2. $\mathbf{Z}_d(-) : 2\mathbf{Cob}^{L_1,L_2} \to R\mathbf{mod}$ *is a well-defined functor.*

PROOF: Following the proof of Proposition 4.1 of [2], the gluing laws imply the following compatibility:

$$\mathbf{Z}_d\Big((W, L_1, L_2) \circ (W', L_1', L_2')\Big) = \mathbf{Z}_d(W, L_1, L_2) \circ \mathbf{Z}_d(W', L_1', L_2').$$

We must also prove $\mathbf{Z}_d(-)$ takes identity morphisms to identity morphisms. Since $W_1^1(0;0,0)$ is the identity morphism from S^1 to itself in $2\mathbf{Cob}^{L_1,L_2}$, we require

$$Z_d(0\,|\,0,0)_\mu^\nu = \delta_\mu^\nu. \tag{8}$$

Equation (8) will be proved in Lemma 6.1. □

5 Semisimplicity in level $(0,0)$

5.1 Rings of definition

The partition functions for the level $(0,0)$ relative invariants lie in the ring of power series in u,

$$Z_d(g\,|\,0,0)_{\lambda^1\ldots\lambda^r} \in \mathbb{Q}(s_1, s_2)[[u]],$$

since, by equation (4), no negative powers of u appear. The level $(0,0)$ relative invariants therefore determine a commutative Frobenius algebra over the ring

$$R = \mathbb{Q}(s_1, s_2)[[u]].$$

We will require formal square roots of s_1 and s_2. Let $\tilde{R}$ be the complete local ring of power series in u whose coefficients are rational functions in $s_1^{\frac{1}{2}}$ and $s_2^{\frac{1}{2}}$,

$$\tilde{R} = \mathbb{Q}(s_1^{\frac{1}{2}}, s_2^{\frac{1}{2}})[[u]].$$

5.2 Semisimplicity

A Frobenius algebra A is *semisimple* if A isomorphic to a direct sum of 1-dimensional Frobenius algebras.

Proposition 5.1. *The Frobenius algebra determined by the level* $(0,0)$ *sector of* $\mathbf{Z}_d(-)$ *is semisimple over* $\tilde{R}$.

PROOF: $\tilde{R}$ is a complete local ring with maximal ideal m generated by u. Let F be the Frobenius algebra determined by the level $(0,0)$ theory. The underlying $\tilde{R}$-module of the Frobenius algebra F,

$$H = \oplus_{\lambda \vdash d} \tilde{R} e_\lambda,$$

is freely generated. By Proposition 2.2 of [2], F is semisimple if and only if F/mF is semisimple over $\tilde{R}/m\tilde{R}$.

The structure constants of the multiplication in F/mF are given by the $u=0$ specialization of the invariants $Z_d(0\,|\,0,0)^{\gamma}_{\alpha\beta}$. By (4), after the $u=0$ specialization, only the

$$b_1 = b_2 = 0$$

terms remain. The latter are the expected dimension 0 terms with domain genus

$$2h-2 = d - l(\alpha) - l(\beta) - l(\gamma).$$

In the expected dimension 0 case, the moduli space $\overline{M}^{\bullet}_h(\mathbb{P}^1;\alpha,\beta,\gamma)$ is nonsingular of *actual* dimension 0. We conclude:

$$\begin{aligned} Z_d(0\,|\,0,0)^{\gamma}_{\alpha\beta}|_{u=0} &= \mathfrak{z}(\gamma)(s_1 s_2)^{l(\gamma)}\ Z_d(0;0,0)_{\alpha\beta\gamma}|_{u=0} \\ &= \mathfrak{z}(\gamma)(s_1 s_2)^{\frac{1}{2}(d-l(\alpha)-l(\beta)+l(\gamma))} \int_{[\overline{M}^{\bullet}_h(\mathbb{P}^1;\alpha,\beta,\gamma)]} 1 \\ &= \mathfrak{z}(\gamma)(s_1 s_2)^{\frac{1}{2}(d-l(\alpha)-l(\beta)+l(\gamma))}\ \mathbf{H}^{\mathbb{P}^1}_d(\alpha,\beta,\gamma), \end{aligned}$$

where $\mathbf{H}^{\mathbb{P}^1}_d(\alpha,\beta,\gamma)$ is the Hurwitz number of degree d covers of $\mathbb{P}^1$ with prescribed ramification α, β, and γ at 0, 1, and ∞.

Modulo factors of s_1 and s_2, the quotient F/mF is the Frobenius algebra associated to the TQFT studied by Dijkgraaf-Witten and Freed-Quinn [5, 8]. The latter Frobenius algebra is isomorphic to $\mathbb{Q}[S_d]^{S_d}$, the center of the group algebra of the symmetric group, and well-known to be semisimple.

We derive below an explicit idempotent basis for F/mF analogous to the well known idempotent basis for $\mathbb{Q}[S_d]^{S_d}$. It is here where we need the formal square roots of s_1 and s_2.

Let ρ be an irreducible representation of S_d. The conjugacy classes of S_d are indexed by partitions λ of size d. Let $\chi_\rho(\lambda)$ denote the trace of ρ on the conjugacy class λ. The Hurwitz numbers are determined by the following formula:

$$\mathbf{H}^{\mathbb{P}^1}_d(\alpha,\beta,\gamma) = \sum_\rho \frac{d!}{\dim\rho}\frac{\chi_\rho(\alpha)}{\mathfrak{z}(\alpha)}\frac{\chi_\rho(\beta)}{\mathfrak{z}(\beta)}\frac{\chi_\rho(\gamma)}{\mathfrak{z}(\gamma)},$$

see, for example, [18] equation 0.8. The above sum is over all irreducible representations ρ of S_d. The structure constants for multiplication in F/mF are

$$Z_d(0\,|\,0,0)^{\gamma}_{\alpha\beta}|_{u=0} = (s_1 s_2)^{\frac{1}{2}(d-l(\alpha)-l(\beta)+l(\gamma))} \sum_{\rho} \frac{d!}{\dim\rho} \frac{\chi_\rho(\alpha)}{\mathfrak{z}(\alpha)} \frac{\chi_\rho(\beta)}{\mathfrak{z}(\beta)} \chi_\rho(\gamma). \quad (9)$$

We define a new basis $\{v^0_\rho\}$ for F/mF by

$$v^0_\rho = \frac{\dim\rho}{d!} \sum_{\alpha} \left(s_1^{\frac{1}{2}} s_2^{\frac{1}{2}}\right)^{l(\alpha)-d} \chi_\rho(\alpha) e_\alpha. \quad (10)$$

The elements $\{v^0_\rho\}$ form an idempotent basis:

$$v^0_\rho \cdot v^0_{\rho'} = \delta^{\rho''}_{\rho\rho'} v^0_{\rho''}.$$

By Proposition 2.2 of [2], there exists a unique idempotent basis $\{v_\rho\}$ of F, such that $v_\rho = v^0_\rho \mod m$. □

Remark 1. *In general, $v_\rho \neq v^0_\rho$ but for the anti-diagonal specialization*

$$s_1 = -s_2,$$

the equality $v_\rho = v^0_\rho$ holds (see Section 7).

5.3 Structure

Semisimplicity leads to a basic structure result.

Theorem 5.2. *There exist universal series, $\lambda_\rho, \eta_\rho \in \tilde{R}$, labelled by partitions ρ for which*

$$Z_d(g\,|\,k_1,k_2) = \sum_{\rho\vdash d} \lambda_\rho^{g-1} \eta_\rho^{-k_1} \overline{\eta}_\rho^{-k_2}.$$

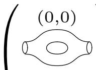
Here, $\overline{\eta}_\rho$ is obtained from η_ρ be interchanging s_1 with s_2.

Proof: Let $\{v_\rho\}$ be an idempotent basis for the level $(0,0)$ Frobenius algebra of $\mathbf{Z}_d(-)$.

Define λ_ρ to be the inverse of the counit evaluated on v_ρ:

$$\lambda_\rho^{-1} = \mathbf{Z}_d\left(\overset{(0,0)}{\bigcirc}\right)(v_\rho).$$

Equivalently, λ_ρ is the eigenvalue for the eigenvector v_ρ under the *genus adding operator* G:

$G = \mathbf{Z}_d$ (

) $: H \to H.$

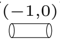
Let η_ρ (respectively $\overline{\eta}_\rho$) be the coefficient of v_ρ in the element

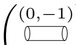
$\eta = \mathbf{Z}_d$ (

) $\in H$, (respectively $\overline{\eta} = \mathbf{Z}_d$ (

) $\in H$).

Equivalently, η_ρ (respectively $\overline{\eta}_\rho$) is the eigenvalue for the eigenvector v_ρ under the *left annihilation operator* (respectively *right annihilation operator*):

$A = \mathbf{Z}_d$ (

) (respectively $\overline{A} = \mathbf{Z}_d$ (

)).

The gluing rules imply:

$$Z_d(g \,|\, k_1, k_2) = \operatorname{tr}(G^{g-1} A^{-k_1} \overline{A}^{-k_2}).$$

The operators G, A, and $\overline{A}$ are simultaneously diagonalized by the basis $\{v_\rho\}$, so the Theorem is equivalent to the above formula. □

6 Computing the theory

6.1 Overview

The *full local theory of curves* is the set of all series

$$Z_d(g \,|\, k_1, k_2)_{\lambda^1 \dots \lambda^r}. \tag{11}$$

The functors $\mathbf{Z}_d(-)$ contain the data of the full local theory. By Theorem 4.1, the full local theory is determined by the following *basic* series:

$$Z_d(0 \,|\, 0,0)_{\lambda\mu\nu}\ ,\ Z_d(0 \,|\, 0,0)_{\lambda\mu}\ ,\ Z_d(0 \,|\, 0,0)_{\lambda}\ , \tag{12}$$
$$Z_d(0 \,|\, -1,0)_{\lambda}\ ,\ Z_d(0 \,|\, 0,-1)_{\lambda}.$$

We present a recursive method for calculating the full local theory of curves using the TQFT formalism. Four of the basics series,

$$Z_d(0 \,|\, 0,0)_{\lambda\mu}\ ,\ Z_d(0 \,|\, 0,0)_{\lambda}\ ,\ Z_d(0 \,|\, -1,0)_{\lambda}\ ,\ Z_d(0 \,|\, 0,-1)_{\lambda}\ ,$$

are determined by closed formulas for all d. The first two are easily obtained by dimension considerations (Lemmas 6.1 and 6.2). The last two have been determined in [2] in case the equivariant parameters s_i are set to 1. The insertion of the equivariant parameters is straightforward (Lemma 6.3).

The level $(0,0)$ pair of pants series

$$Z_d(0\,|\,0,0)_{\lambda\mu\nu}$$

are much more subtle. The main result of the Appendix (with C. Faber and A. Okounkov) is the determination of all degree d level $(0,0)$ pair of pant series from the *single* series

$$Z_d(0\,|\,0,0)_{(d),(d),(1^{d-2}2)} \tag{13}$$

using the TQFT associativity relations, level $(0,0)$ series of *lower degree*, and Hurwitz numbers of covering genus 0. A closed formula for (13) is derived in Section 6.4.3. The outcome is a computation of the full local theory of curves via recursions in degree (Theorem 6.6).

6.2 The level $(0,0)$ tube and cap

We complete the proof of Theorem 4.2 by calculating the series $Z_d(0\,|\,0,0)_\alpha^\beta$.

Lemma 6.1. *The invariants of the level* $(0,0)$ *tube are given by:*

$$Z_d(0\,|\,0,0)_{\alpha\beta} = \begin{cases} \frac{1}{\mathfrak{z}(\alpha)(s_1s_2)^{l(\alpha)}} & \text{if } \alpha=\beta \\ 0 & \text{if } \alpha\neq\beta. \end{cases}$$

Consequently, we have

$$Z_d(0\,|\,0,0)_\alpha^\beta = \delta_\alpha^\beta$$

as was required for $\mathbf{Z}_d(-)$ *to be a functor.*

Proof: The virtual dimension of the moduli space $\overline{M}_h(\mathbb{P}^1,\alpha,\beta)$ with *connected* domains is

$$2h-2+l(\alpha)+l(\beta).$$

Let $\mathbb{E}^\vee$ be the rank h dual Hodge bundle on $\overline{M}_h(\mathbb{P}^1,\alpha,\beta)$. Since the line bundles L_i may be taken to be trivial,

$$c(-R^\bullet\pi_*f^*(L_i)) = c(\mathbb{E}^\vee),$$

where the equality is of ordinary (non-equivariant) Chern classes. The integral

$$\int_{[\overline{M}_h(\mathbb{P}^1,\alpha,\beta)]^{vir}} c_{b_1}(-R^\bullet\pi_* f^*(L_1))c_{b_2}(-R^\bullet\pi_* f^*(L_2)) = \int_{[\overline{M}_h(\mathbb{P}^1,\alpha,\beta)]^{vir}} c_{b_1}(\mathbb{E}^\vee)c_{b_2}(\mathbb{E}^\vee) \quad (14)$$

is zero if

$$2h - 2 + l(\alpha) + l(\beta) > 2h.$$

The only possible non-zero integrals are for $l(\alpha) = l(\beta) = 1$. Moreover, since

$$c_h(\mathbb{E}^\vee)^2 = 0,$$

by Mumford's relation, the integral (14) is zero unless $h = 0$.

Therefore, the only connected stable map which contributes to the integral (14) is the unique degree d map

$$f_d : \mathbb{P}^1 \to \mathbb{P}^1$$

totally ramified over 0 and ∞. The only disconnected maps which contribute are disjoint unions of genus 0 totally ramified maps of lower degree. Given a partition $\alpha \vdash d$, let

$$f_\alpha : \bigsqcup_{l(\alpha)} \mathbb{P}^1 \to \mathbb{P}^1$$

be the map determined by f_{α_i} on the ith component. The map f_α has ramification profile α over both 0 and ∞. The map is isolated in moduli and has an automorphism group of order $\mathfrak{z}(\alpha)$. Thus

$$Z_d^{b_1,b_2}(0\,|\,0,0) = \begin{cases} \frac{1}{\mathfrak{z}(\alpha)} & \text{if } b_1 = b_2 = 0 \text{ and } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

The Lemma then follows directly from equation (4). □

The level $(0,0)$ cap is has a simple form obtained by a similar argument.

Lemma 6.2. *The invariants of the level* $(0,0)$ *cap are given by*

$$Z_d(0\,|\,0,0)_\lambda = \begin{cases} \frac{1}{d!(s_1 s_2)^d} & \text{if } \lambda = (1^d) \\ 0 & \text{if } \lambda \neq (1^d). \end{cases}$$

PROOF: The (connected domain) moduli space $\overline{M}_h(\mathbb{P}^1, \lambda)$ has virtual dimension

$$2h - 2 + d + l(\lambda).$$

Hence,

$$\int_{[\overline{M}_h(\mathbb{P}^1,\lambda)]^{vir}} c_{b_1}(\mathbb{E}^\vee) c_{b_2}(\mathbb{E}^\vee) = 0 \tag{15}$$

if

$$2h - 2 + d + l(\lambda) > 2h.$$

In order for (15) to be non-zero, we must have $d = l(\lambda) = 1$. The virtual dimension is then $2h$ which implies $h = 0$ by Mumford's relation.

The only connected stable map for which the integral (15) is non-zero is the isomorphism

$$f : \mathbb{P}^1 \to \mathbb{P}^1.$$

The Lemma is the obtained from (4) by accounting for disconnected covers. □

6.3 The Calabi-Yau cap

Lemma 6.3. *The invariants of the level* $(-1, 0)$ *cap are given by*

$$Z_d(0 \,|\, -1, 0)_\lambda = (-1)^d (-s_2)^{-l(\lambda)} \frac{1}{\mathfrak{z}(\lambda)} \prod_{i=1}^{l(\lambda)} \left(2 \sin \frac{\lambda_i u}{2} \right)^{-1}$$

PROOF: The calculation has already been done by localization in the proof of Theorem 5.1 in [2] in case $s_1 = s_2 = 1$. We must insert the equivariant parameters. The relevant *connected* integrals are

$$\int_{[\overline{M}_h(\mathbb{P}^1,\lambda)]^{vir}} c_{b_1}(-R^\bullet \pi_* f^* \mathcal{O}(-1)) c_{b_2}(-R^\bullet \pi_* f^* \mathcal{O}).$$

The virtual dimension of the moduli space $\overline{M}_h(\mathbb{P}^1, \lambda)$ is

$$2h - 2 + d + l(\lambda).$$

The object $-R^\bullet \pi_* f^* \mathcal{O}(-1)$ is represented by a bundle of rank $h - 1 + d$. Similarly, $-R^\bullet \pi_* f^* \mathcal{O}$ is represented by a bundle of rank h (minus a trivial factor). Consequently, the integral is zero unless $b_1 = h - 1 + d$, $b_2 = h$, and $\lambda = (d)$. From equation (4), we find the insertion of the equivariant parameters yields a factor of s_2^{-1}.

Since the disconnected invariant is a product of $l(\lambda)$ connected integrals, the invariant has the factor $s_2^{-l(\lambda)}$. The series $Z_d(0\,|\,0,-1)_\lambda$ is obtained from $Z_d(0\,|\,-1,0)_\lambda$ by exchanging s_1 and s_2. □

6.4 The level $(0,0)$ pair of pants

6.4.1 Normalization

We will study the local theory of curves here with a slightly different normalization. Recall

$$\delta = \sum_{i=1}^{r}(d - l(\lambda^i)).$$

Let

$$\begin{aligned} \widehat{Z}_d(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r} &= (-iu)^{d(2-2g+k_1+k_2)-\delta}\ \mathsf{Z}'_{GW}(N)_{d[X],\lambda^1,\ldots,\lambda^r} \\ &= \ (-i)^{d(2-2g+k_1+k_2)-\delta}\ Z_d(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r}. \end{aligned} \tag{16}$$

With the altered metric,

$$\widehat{Z}_d(g\,|\,k_1,k_2)^{\nu^1\ldots\nu^t}_{\mu^1\ldots\mu^s} = \left(\prod_{i=1}^{t}\mathfrak{z}(\nu^i)(-s_1s_2)^{l(\nu^i)}\right)\widehat{Z}_d(g;k_1,k_2)_{\mu^1\ldots\mu^s\nu^1\ldots\nu^t},$$

the partition functions (16) satisfy the same gluing rules as partition functions $Z_d(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r}$. Moreover, a tensor functor,

$$\widehat{\mathbf{Z}} : 2\mathbf{Cob}^{L_1,L_2} \to R\mathbf{mod}.$$

is defined just as before.

The reason for the altered normalization is the following result proven in the Appendix.

Theorem 6.4. *The product*

$$e^{\frac{idu}{2}(2-2g+k_1+k_2)}\ \widehat{Z}_d(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r},$$

is a rational function *of s_1, s_2, and $q = -e^{iu}$ with $\mathbb{Q}$-coefficients.*

The Theorem is closely related to the GW/DT correspondence discussed in Subsection 9.5. The Calabi-Yau cap provides a good example:

$$e^{\frac{idu}{2}}\ \widehat{Z}_d(0\,|\,-1,0)_\lambda \;=\; e^{\frac{idu}{2}}(-i)^{l(\lambda)}(-1)^d(-s_2)^{-l(\lambda)}\frac{1}{\mathfrak{z}(\lambda)}\prod_{j=1}^{l(\lambda)}\left(2\sin\frac{\lambda_j u}{2}\right)^{-1}$$

$$= (-1)^{d-l(\lambda)}\frac{1}{\mathfrak{z}(\lambda)}\frac{1}{s_2^{l(\lambda)}}\prod_{j=1}^{l(\lambda)}\frac{1}{1-(-q)^{-\lambda_j}}.$$

6.4.2 The degree 1 case

The level $(0,0)$ tube and cap in degree 1 are:

$$\widehat{Z}_1(0\,|\,0,0)_{\square,\square} = -\frac{1}{s_1s_2},\ \ \widehat{Z}_1(0\,|\,0,0)_{\square} = -\frac{1}{s_1s_2}.$$

By the gluing formula,

$$\widehat{Z}_1(0\,|\,0,0)_{\square,\square,\square}\ (-s_1s_2)\ \widehat{Z}_1(0\,|\,0,0)_{\square} = \widehat{Z}_1(0\,|\,0,0)_{\square,\square}.$$

We conclude,

$$\widehat{Z}_1(0\,|\,0,0)_{\square,\square,\square} = -\frac{1}{s_1s_2}.$$

Hence, all the basic series in degree 1 are known.

6.4.3 The series $\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(1^{d-2}2)}$

The degree $d\geq 2$ series $\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(1^{d-2}2)}$ plays a special role in the level $(0,0)$ theory.

Theorem 6.5. *For $d\geq 2$,*

$$\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(1^{d-2}2)} \;=\; -\frac{i}{2}\frac{s_1+s_2}{s_1s_2}\left(d\cot\left(\frac{du}{2}\right)-\cot\left(\frac{u}{2}\right)\right)$$

Proof: We abbreviate the partition $(1^{d-2}2)$ by (2). After adjusting equation (4) for the new normalization, we find

$$\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(2)} =$$

$$-\frac{i}{(s_1s_2)^{\frac{1}{2}}}\sum_{b_1,b_2=0}^{\infty}u^{b_1+b_2}\left(\frac{s_1}{s_2}\right)^{\frac{b_2-b_1}{2}}\int_{[\overline{M}^\bullet_h(\mathbb{P}^1,(d),(d),(2))]^{vir}}c_{b_1}(\mathbb{E}^\vee)c_{b_2}(\mathbb{E}^\vee).$$

The domains of the maps in the moduli space $\overline{M}^\bullet_h(\mathbb{P}^1,(d),(d),(2))$ are necessarily connected since there exists a point of total ramification. Since virtual dimension is

$$\text{virdim}\, \overline{M}_h(\mathbb{P}^1,(d)(d)(2)) = 2h-1,$$

the only values of (b_1,b_2) which contribute to $\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(2)}$ are $(h,h-1)$ and $(h-1,h)$. We obtain:

$$\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(2)} = -i\frac{s_1+s_2}{s_1 s_2}\sum_{h=1}^{\infty} u^{2h-1}\int_{[\overline{M}_h(\mathbb{P}^1,(d)(d)(2))]^{vir}} \rho^*(-\lambda_h\lambda_{h-1}).$$

Here, λ_k is the k^{th} Chern class of the Hodge bundle on $\overline{M}_{h,2}$, and

$$\rho : \overline{M}_h(\mathbb{P}^1,(d),(d),(2)) \to \overline{M}_{h,2}$$

is the natural map which takes a relative stable map to the domain marked by the two totally ramified points.

Let $H_d \subset M_{h,2}$ be the locus of curves admitting a degree d map to $\mathbb{P}^1$ which is totally ramified at the marked points. Equivalently, H_d is the locus of curves

$$[C,x_1,x_2]$$

for which $\mathcal{O}(x_1-x_2)$ is a nonzero d-torsion point in $\text{Pic}^0(C)$. Let

$$\overline{H}_d \subset \overline{M}_{h,2}$$

be the closure of H_d.

Consider the locus of maps with nonsingular domains,

$$M_h(\mathbb{P}^1,(d),(d),(2)) \subset \overline{M}_h(\mathbb{P}^1,(d),(d),(2)),$$

and let

$$\partial\overline{M}_h(\mathbb{P}^1,(d),(d),(2))$$

denote the complement. Let

$$\partial\overline{M}_{h,1} \subset \overline{M}_{h,1}$$

denote the nodal locus. Let

$$\epsilon : \overline{M}_{h,2} \to \overline{M}_{h,1}$$

be the map forgetting the first point. An elementary argument yields

$$\rho\left(\partial\overline{M}_h(\mathbb{P}^1,(d),(d),(2))\right) \subset \epsilon^{-1}(\partial\overline{M}_{h,1}).$$

The restriction of the virtual class to $M_h(\mathbb{P}^1,(d),(d),(2))$ is well-known to equal the ordinary fundamental class of the moduli space, see [21]. Since

$$\rho : M_h(\mathbb{P}^1,(d),(d),(2)) \to H_d$$

is a proper cover of degree $2h$, we conclude

$$\rho_*[\overline{M}_h(\mathbb{P}^1,(d),(d),(2))]^{vir} = 2h[\overline{H}_d] + B \tag{17}$$

where B is a cycle supported on $\epsilon^{-1}(\partial\overline{M}_{h,1})$.

Since $\lambda_h\lambda_{h-1}$ vanishes on cycles supported on the boundary of $\overline{M}_{h,1}$, we find

$$\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(2)} = i\frac{s_1+s_2}{s_1 s_2}\sum_{h=1}^{\infty} u^{2h-1}c_h(d),$$

where

$$c_h(d) = 2h\int_{[\overline{H}_d]}\lambda_h\lambda_{h-1}.$$

The cycle $[H_d]$ can be described as follows. Let

$$\begin{array}{c}\mathcal{P}\mathrm{ic}^0\\ s\Big\uparrow\Big\downarrow\pi\\ M_{h,2}\end{array}$$

be the universal Picard bundle with section

$$s : [C,x_1,x_2] \mapsto \mathcal{O}(x_1-x_2).$$

Let $P_d\subset\mathcal{P}\mathrm{ic}^0$ be the locus of *nonzero* d-torsion points. Then, by our previous characterization of H_d,

$$[H_d] = \pi_*\left(s_*[M_{h,2}]\cap P_d\right) \in A_*(M_{h,2}).$$

By a result of Looijenga using the Fourier-Mukai transform, locus of d-torsion points of *any* family of abelian varieties is a multiple of the zero section in the Chow ring [12]. Hence,

$$[P_d] = \frac{d^{2h}-1}{2^{2h}-1}[P_2]$$

and

$$[H_d] = \frac{d^{2h}-1}{2^{2h}-1}[H_2].$$

We conclude

$$c_h(d) = \frac{d^{2h}-1}{2^{2h}-1}c_h(2).$$

Consider the $d=2$ case. In genus 1, the class

$$[\overline{H}_2] \in A_*(\overline{M}_{1,2})$$

pushes forward to $3[\overline{M}_{1,1}]$ under the map ϵ. For genus $h>1$, let $\overline{H} \subset \overline{M}_h$ denote the hyperelliptic locus. There are

$$(2h+2)(2h+1)$$

ways of marking two of the Weierstrass points on each curve in H. Consequently, the class

$$[\overline{H}_2] \in A_*(\overline{M}_{h,2})$$

pushes forward to $(2h+2)(2h+1)[\overline{H}]$ under the forgetful map

$$\overline{M}_{h,2} \rightarrow \overline{M}_h.$$

We find

$$\begin{aligned}
\widehat{Z}_2(0\,|\,0,0)_{(2),(2),(2)} &= i\frac{s_1+s_2}{s_1s_2}\left(6u\int_{\overline{M}_{1,1}}\lambda_1 + \sum_{h=2}^{\infty}\frac{(2h+2)!}{(2h-1)!}u^{2h-1}\int_{\overline{H}}\lambda_h\lambda_{h-1}\right) \\
&= i\frac{s_1+s_2}{s_1s_2}\left(\frac{u^4}{96} + \sum_{h=2}^{\infty}u^{2h+2}\int_{\overline{H}}\lambda_h\lambda_{h-1}\right)''' \\
&= i\frac{s_1+s_2}{s_1s_2}\left(u^2H(u)\right)'''
\end{aligned}$$

where $H(u)$ is defined in [7] on page 222. By Corollary 2 of [7],

$$(u^2H(u))'' = -\log\left(\cos\left(\frac{u}{2}\right)\right),$$

and thus

$$\widehat{Z}_2(0\,|\,0,0)_{(2),(2),(2)} = \frac{i}{2}\frac{s_1+s_2}{s_1s_2}\tan\left(\frac{u}{2}\right).$$

We conclude

$$\sum_{h=1}^{\infty} c_h(2)u^{2h-1} = \frac{1}{2}\tan\left(\frac{u}{2}\right).$$

The function $\cot\left(\frac{u}{2}\right)$ is an odd series in u with a simple pole at $u=0$. We define b_h by

$$\cot\left(\frac{u}{2}\right) = \sum_{h=0} b_h u^{2h-1}.$$

The identity

$$\frac{1}{2}\tan\left(\frac{u}{2}\right) = \frac{1}{2}\cot\left(\frac{u}{2}\right) - \cot\left(2\frac{u}{2}\right)$$

yields

$$c_h(2) = (\frac{1}{2} - 2^{2h-1})b_h.$$

Hence,

$$c_h(d) = \frac{1}{2}(1-d^{2h})b_h.$$

We obtain

$$\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(2)} = -\frac{i}{2}\frac{s_1+s_2}{s_1 s_2}\left(d\cot\left(\frac{du}{2}\right) - \cot\left(\frac{u}{2}\right)\right)$$

which concludes the proof. □

We may write the series as a rational function,

$$\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(1^{d-2}2)} = \frac{1}{2}\frac{s_1+s_2}{s_1 s_2}\left(d\frac{(-q)^d+1}{(-q)^d-1} - \frac{(-q)+1}{(-q)-1}\right). \qquad (18)$$

6.5 Reconstruction for the level $(0,0)$ pair of pants

The main result proven in the Appendix (with C. Faber and A. Okounkov) is the following.

Theorem 6.6. *Let $d \geq 2$. The set of degree d, level $(0,0)$ pair of pant series*

$$\widehat{Z}_d(0\,|\,0,0)_{\lambda\mu\nu}$$

can be uniquely reconstructed from

$$\widehat{Z}_d(0\,|\,0,0)_{(d),(d),(1^{d-2}2)}$$

via the TQFT associativity relations, lower degree series of level $(0,0)$, and Hurwitz numbers of covering genus 0.

The proof yields an effective method of computing the level $(0,0)$ pair of pants series via recursions in degree. Since all the basic series (12) can be computed, the full local theory of curves is effectively determined. Theorem 6.4 is obtained in the Appendix as a Corollary of Theorem 6.6.

7 The anti-diagonal $\mathbb{C}^*$-action

7.1 Overview

We study a particularly nice special case of the local theory of curves. Consider the action of the anti-diagonal subgroup

$$\mathbb{C}^* \subset \mathbb{C}^* \times \mathbb{C}^*$$

on $N = L_1 \oplus L_2$. The anti-diagonal actions corresponds to the limit

$$s_1 = -s_2$$

in equivariant cohomology. The induced $\mathbb{C}^*$-action on K_N is trivial. Explicit formulas can be found since the level $(0,0)$ Frobenius algebra can be explicitly diagonalized in the anti-diagonal case.

We define Q-dimension of ρ, an irreducible representation of the symmetric group, as follows:

$$\frac{\dim_Q \rho}{d!} = \prod_{\square \in \rho} i \left(Q^{\frac{h(\square)}{2}} - Q^{\frac{-h(\square)}{2}} \right)^{-1},$$

see [17]. Under the substitution $Q = e^{iu}$, the Q-dimension can be expressed:

$$\frac{\dim_Q \rho}{d!} = \prod_{\square \in \rho} \left(2 \sin \frac{h(\square) u}{2} \right)^{-1}.$$

By the hook length formula for $\dim \rho$, the leading term in u of the above expression is $\frac{\dim \rho}{d!}$.

The main result here is a closed formula for the (absolute) local theory of curves with the anti-diagonal action.

Theorem 7.1. *Under the restrictions $s_1 = s$ and $s_2 = -s$,*

$$Z_d(g\,|\,k_1, k_2) = \\ (-1)^{d(g-1-k_2)} s^{d(2g-2-k_1-k_2)} \sum_\rho \left(\frac{d!}{\dim \rho}\right)^{2g-2} \left(\frac{\dim \rho}{\dim_Q \rho}\right)^{k_1+k_2} Q^{\frac{1}{2}c_\rho(k_1-k_2)}$$

where $Q = e^{iu}$ and c_ρ is the total content of ρ (see Section 3.1).

7.2 Corollaries

If $k_1 + k_2 = 2g - 2$, the threefold $N = L_1 \oplus L_2$ is Calabi-Yau. As previously remarked, the $s_1 = -s_2$ limit corresponds to the trivial $\mathbb{C}^*$-action on the canonical bundle. In other words, N is *equivariantly* Calabi-Yau.

Corollary 7.2. *In the equivariantly Calabi-Yau case,*

$$Z_d(g\,|\,k, 2g-2-k) = (-1)^{d(g-1-k)} \sum_\rho \left(\frac{d!}{\dim_Q \rho}\right)^{2g-2} Q^{-c_\rho(g-1-k)}.$$

In particular, for the balanced splitting,

$$k_1 = k_2 = g - 1,$$

the partition function is the Q-deformation of the classical formula for unramified covers.

Corollary 7.3. *In the balanced equivariantly Calabi-Yau case,*

$$Z_d(g\,|\,g-1, g-1) = \sum_\rho \left(\frac{d!}{\dim_Q \rho}\right)^{2g-2}.$$

Another special Calabi-Yau case is when the base curve X is elliptic. We obtain a formula recently derived by Vafa using string theoretic methods (page 8 of [23]).

Corollary 7.4. *Let $L \to E$ be a degree k line bundle on an elliptic curve E. The partition function for the Calabi-Yau action on $L \oplus L^{-1}$ is*

$$Z_d(1\,|\,k, -k) = (-1)^{dk} \sum_\rho Q^{kc_\rho}.$$

7.3 Proof of Theorem 7.1

To derive the formula of Theorem 7.1, we first explicitly diagonalize the level $(0,0)$ Frobenius algebra for the anti-diagonal action.

Lemma 7.5. *For the anti-diagonal action, the level* $(0,0)$ *series have no nonzero terms of positive degree in* u.

Proof of Lemma. Let $\mathbb{C}_s$ denote the line bundle over a point with the $\mathbb{C}^*$-action satisfying

$$c_1(\mathbb{C}_s) = s \in H^*_{\mathbb{C}^*}(\mathrm{pt}).$$

Then the dual line bundle is $\mathbb{C}_s^\vee = \mathbb{C}_{-s}$. The level $(0,0)$ partition functions are built from the following integrals:

$$\int_{[\overline{M}^\bullet_h(X,\lambda^1,\ldots,\lambda^r)]^{vir}} e\left(-R^\bullet\pi_*(\mathcal{O}\otimes\mathbb{C}_s)\right) e(-R^\bullet\pi_*(\mathcal{O}\otimes\mathbb{C}_{-s})).$$

For any vector bundle E, the equivariant Euler class $e(E\otimes\mathbb{C}_s)$ is a polynomial in s whose coefficients are the (ordinary) Chern classes of E. The above integrand is a pure weight factor times

$$e(\mathbb{E}^\vee\otimes\mathbb{C}_s)e(\mathbb{E}^\vee\otimes\mathbb{C}_{-s}) = (-1)^h e\left((\mathbb{E}^\vee\oplus\mathbb{E})\otimes\mathbb{C}_s\right).$$

Since the Chern classes of $\mathbb{E}^\vee\oplus\mathbb{E}$ all vanish by Mumford's relation, the last expression is pure weight. The only non-zero integrals occur when

$$b_1 = b_2 = 0$$

in equation (4). Only the constant terms in u are non-zero. In particular, $Z_d(0\,|\,0,0)^\gamma_{\alpha\beta}$ is given by the $s_1=-s_2$ limit of equation (9). □

The structure constants for the level $(0,0)$ Frobenius algebra are given by:

$$Z_d(0\,|\,0,0)^\gamma_{\alpha\beta} = \left(-s^2\right)^{\frac{1}{2}(d-l(\alpha)-l(\beta)+l(\gamma))}\sum_\rho\left(\frac{d!}{\dim\rho}\right)\frac{\chi_\rho(\alpha)\chi_\rho(\beta)}{\mathfrak{z}(\alpha)\mathfrak{z}(\beta)}\chi_\rho(\gamma),$$

where $s=s_1=-s_2$. As a consequence of the Lemma, multiplication in the level $(0,0)$ Frobenius algebra is diagonalized by the basis v^0_ρ constructed in the proof of Proposition 5.1.

In order to diagonalize the level $(0,0)$ Frobenius algebra, we had to enlarge the coefficient ring to $\tilde{R}$ to include the formal square roots $s_1^{\frac{1}{2}}$ and $s_2^{\frac{1}{2}}$. The specialization

$$s_1^{\frac{1}{2}} = s^{\frac{1}{2}}, \quad s_2^{\frac{1}{2}} = is^{\frac{1}{2}}$$

is compatible with

$$s_1 = s, \quad s_2 = -s.$$

By Lemma 7.5, the idempotent basis $v_\rho^0 = v_\rho$ given by equation (10) is:

$$v_\rho = \frac{\dim\rho}{d!}\sum_\alpha (is)^{l(\alpha)-d}\chi_\rho(\alpha)e_\alpha. \tag{19}$$

To apply Theorem 5.2, we must compute λ_ρ, η_ρ, and $\overline{\eta}_\rho$. We compute λ_ρ as follows:

$$\begin{aligned}
\lambda_\rho^{-1} &= \mathbf{Z}_d\left(\overset{(0,0)}{\bigcirc}\right)(v_\rho) \\
&= \frac{\dim\rho}{d!}\sum_\alpha (is)^{l(\alpha)-d}\chi_\rho(\alpha)Z_d(0\,|\,0,0)_\alpha \\
&= \frac{\dim\rho}{d!}(is)^{l(1^d)-d}\chi_\rho(1^d)\frac{1}{d!(-s^2)^d} \\
&= \left(\frac{\dim\rho}{d!}\right)^2 (is)^{-2d}.
\end{aligned}$$

Hence,

$$\lambda_\rho = (is)^{2d}\left(\frac{d!}{\dim\rho}\right)^2.$$

In order to compute η_ρ, we must express η in terms of the basis $\{v_\rho\}$.

$$\begin{aligned}
\eta &= \mathbf{Z}_d\left(\overset{(-1,0)}{\bigcirc}\right) \\
&= \sum_\alpha Z_d(0\,|\,-1,0)^\alpha e_\alpha \\
&= \sum_\alpha (-1)^d s^{l(\alpha)}\left(\prod_{i=1}^{l(\alpha)}\frac{-1}{2\sin\frac{\alpha_i u}{2}}\right)e_\alpha \\
&= \sum_\alpha (-1)^d (is)^{l(\alpha)}Q^{d/2}\left(\prod_{i=1}^{l(\alpha)}\frac{1}{1-Q^{\alpha_i}}\right)e_\alpha
\end{aligned}$$

where $Q = e^{iu}$ as before. The expression

$$\prod_{i=1}^{l(\alpha)} \frac{1}{1 - Q^{\alpha_i}}$$

arises in the theory of symmetric functions. The power sum symmetric functions are defined by:

$$p_k(x_1, x_2, x_3, \dots) = x_1^k + x_2^k + x_3^k + \dots$$
$$p_\alpha = \prod_{i=1}^{l(\alpha)} p_{\alpha_i}.$$

For the specialization

$$x_1 = 1,\ x_2 = Q,\ x_3 = Q^2, \dots,$$

we obtain

$$p_k(Q) = (1 - Q^k)^{-1}.$$

Hence,

$$\eta = \sum_\alpha (-1)^d (is)^{l(\alpha)} Q^{d/2} p_\alpha(Q)\ e_\alpha$$

and similarly

$$\overline{\eta} = \sum_\alpha (-1)^d (is)^{l(\alpha)} Q^{d/2} (-1)^{l(\alpha)} p_\alpha(Q)\ e_\alpha.$$

Inversion of (19) yields the following formula:

$$e_\alpha = (is)^{d - l(\alpha)} \sum_\rho \frac{d!}{\dim \rho} \frac{\chi_\rho(\alpha)}{\mathfrak{z}(\alpha)} v_\rho \tag{20}$$

After substituting (20) in the expression for η, we find

$$\eta = \sum_\rho v_\rho \left[(-is)^d Q^{d/2} \frac{d!}{\dim \rho} \left(\sum_\alpha \frac{\chi_\rho(\alpha) p_\alpha(Q)}{\mathfrak{z}(\alpha)} \right) \right],$$
$$\overline{\eta} = \sum_\rho v_\rho \left[(+is)^d Q^{d/2} \frac{d!}{\dim \rho} \omega \left(\sum_\alpha \frac{\chi_\rho(\alpha) p_\alpha(Q)}{\mathfrak{z}(\alpha)} \right) \right].$$

Here, ω is the involution on the ring of symmetric functions defined by

$$(-1)^{l(\alpha)} p_\alpha = (-1)^d \omega(p_\alpha).$$

The sum over α in the latter expressions for η and $\overline{\eta}$ is equal to the Schur function $s_\rho(Q)$, see [13] page 114. We have

$$\omega(s_\rho) = s_{\rho'}$$

where ρ' is the dual representation (or conjugate partition), [13] page 42. Thus, we obtain

$$\eta_\rho = (-is)^d Q^{d/2} \frac{d!}{\dim\rho} s_\rho(Q)$$
$$\overline{\eta}_\rho = (+is)^d Q^{d/2} \frac{d!}{\dim\rho} s_{\rho'}(Q).$$

The Schur functions are easily expressed in terms of the Q-dimension. From [13] page 45,

$$\begin{aligned} s_\rho &= Q^{n(\rho)} \prod_{\square\in\rho} \frac{1}{1-Q^{h(\square)}} \\ &= Q^{n(\rho)-\frac{1}{2}(n(\rho)+n(\rho')+d)}(-1)^d \prod_{\square\in\rho} \left(Q^{h(\square)/2} - Q^{-h(\square)/2}\right)^{-1} \\ &= Q^{-\frac{1}{2}(d+c_\rho)} i^d \, \frac{\dim_Q\rho}{d!}. \end{aligned}$$

We have used (3) in the above formulas. We conclude

$$\eta_\rho = (+s)^d \, Q^{-\frac{c_\rho}{2}} \, \frac{\dim_Q\rho}{\dim\rho},$$
$$\overline{\eta}_\rho = (-s)^d \, Q^{+\frac{c_\rho}{2}} \, \frac{\dim_Q\rho}{\dim\rho}.$$

Theorem 7.1 then follows directly from Theorem 5.2. $\square$

8 Low degree calculations

Here we carry out, for low degrees, our general algorithm determining the partition function $Z_d(g\,|\,k_1,k_2)$.

We abbreviate the level $(0,0)$ pair of pants by

$$Z_d(0\,|\,0,0)_{\lambda\mu\nu} = Z_{\lambda\mu\nu}$$

and the Calabi-Yau cap by

$$Z_d(0\,|\,-1,0)_\lambda = C_\lambda.$$

We apply the usual convention (5) for raising indices to $Z_{\lambda\mu\nu}$ and C_λ.

From the proof of Theorem 5.2, the partition function is

$$Z_d(g\,|\,k_1,k_2) = \operatorname{tr}\left(G^{g-1}A^{-k_1}\overline{A}^{-k_2}\right).$$

The genus adding operator G and the right annihilation operator A can be computed in terms of $Z_{\lambda\mu\nu}$ and C_λ by the gluing formula.

$$G^{\mu}_{\nu} = \sum_{\lambda,\epsilon\vdash d} Z^{\mu}_{\lambda\epsilon} Z^{\lambda\epsilon}_{\nu}, \tag{21}$$
$$A^{\mu}_{\nu} = \sum_{\lambda\vdash d} C^{\lambda} Z^{\mu}_{\lambda\nu}.$$

We obtain $\overline{A}$ from A by switching s_1 and s_2.

$Z_{\lambda\mu\nu}$ is determined recursively by Theorem 6.6 and C_λ is given explicitly by Lemma 6.3. We list their values for $d \le 3$ below. The function $Z_{(3)(3)(3)}$ is the only entry whose determination requires the use of the linear system solved in the Appendix.

$C_{(1)} = s_2^{-1}\left(2\sin\frac{u}{2}\right)^{-1}$, $\quad C_{(1,1)} = s_2^{-2}\frac{1}{2}\left(2\sin\frac{u}{2}\right)^{-2}$, $\quad C_{(2)} = -s_2^{-1}\frac{1}{2}\left(2\sin u\right)^{-1}$,

$C_{(1,1,1)} = s_2^{-3}\frac{1}{6}\left(2\sin\frac{u}{2}\right)^{-3}$, $\quad C_{(2,1)} = -s_2^{2}\frac{1}{2}\left(2\sin\frac{u}{2}\right)^{-1}\left(2\sin u\right)^{-1}$, $\quad C_{(3)} = s_2^{-1}\frac{1}{3}\left(2\sin\frac{3u}{2}\right)^{-1}$,

$Z_{(1)(1)(1)} = (s_1s_2)^{-1}$, $\quad Z_{(1,1)(1,1)(1,1)} = \frac{1}{2}(s_1s_2)^{-2}$, $\quad Z_{(1,1)(1,1)(2)} = 0$,

$Z_{(1,1)(2)(2)} = \frac{1}{2}(s_1s_2)^{-1}$, $\quad Z_{(2)(2)(2)} = -\frac{1}{2}\frac{s_1+s_2}{s_1s_2}\tan\frac{u}{2}$, $\quad Z_{(1,1,1)(1,1,1)(1,1,1)} = \frac{1}{6}(s_1s_2)^{-3}$,

$Z_{(1,1,1)(1,1,1)(2,1)} = 0$, $\quad Z_{(1,1,1)(1,1,1)(3)} = 0$, $\quad Z_{(1,1,1)(2,1)(2,1)} = \frac{1}{2}(s_1s_2)^{-2}$,

$Z_{(1,1,1)(2,1)(3)} = 0$, $\quad Z_{(1,1,1)(3)(3)} = \frac{1}{3}(s_1s_2)^{-1}$, $\quad Z_{(2,1)(2,1)(2,1)} = -\frac{1}{2}\frac{s_1+s_2}{(s_1s_2)^2}\tan\frac{u}{2}$

$Z_{(2,1)(2,1)(3)} = (s_1s_2)^{-1}$, $\quad Z_{(2,1)(3)(3)} = \frac{1}{2}\frac{s_1+s_2}{s_1s_2}\left(3\cot\frac{3u}{2} - \cot\frac{u}{2}\right)$,

$$Z_{(3)(3)(3)} = \frac{1}{3} - \frac{(s_1+s_2)^2\left(2\sin\frac{u}{2}\right)^2\left(2\left(2\sin\frac{u}{2}\right)^2-9\right)}{3s_1s_2\left(\left(2\sin\frac{u}{2}\right)^2-3\right)^2}.$$

For $d=1$ we have

$$G_{\square}^{\square} = Z_{\square\square}^{\square} Z_{\square}^{\square\square} = s_1 s_2$$
$$A_{\square}^{\square} = C_{\square} Z_{\square}^{\square\square} = s_1 \left(2\sin\frac{u}{2}\right)^{-1}$$

and so

$$Z_1(g\,|\,k_1,k_2) = (s_1 s_2)^{g-1} s_1^{-k_1} s_2^{-k_2} \left(2\sin\frac{u}{2}\right)^{k_1+k_2}.$$

For $d=2$ we compute the entries of G and A via (21) to obtain:

$$G = \begin{pmatrix} 4(s_1s_2)^2 & -2(s_1s_2)^2(s_1+s_2)\tan\frac{u}{2} \\ -2(s_1s_2)(s_1+s_2)\tan\frac{u}{2} & 4(s_1s_2)^2+2(s_1s_2)(s_1+s_2)^2\tan^2\frac{u}{2} \end{pmatrix},$$

$$A = \begin{pmatrix} s_1^2\left(2\sin\frac{u}{2}\right)^{-2} & -s_1^2 s_2 (2\sin u)^{-1} \\ -s_1(2\sin u)^{-1} & s_1(s_1+s_2)\left(2\cos\frac{u}{2}\right)^{-2}+s_1^2\left(2\sin\frac{u}{2}\right)^{-2} \end{pmatrix}.$$

The matrices G, A, and $\overline{A}$ mutually commute and so we can simultaneously diagonalize them to obtain:

$$Z_2(g\,|\,k_1,k_2) = \lambda_+^{g-1}\eta_+^{-k_1}\overline{\eta}_+^{-k_2} + \lambda_-^{g-1}\eta_-^{-k_1}\overline{\eta}_-^{-k_2} \tag{22}$$

where

$$\lambda_\pm = \frac{s_1s_2}{(1-q)^2}\left(-\Theta \pm (1+q)(s_1+s_2)\sqrt{\Theta}\right)$$
$$\eta_\pm = \frac{qs_1}{2(1-q^2)^2}\left((s_1-s_2)(1+q)^2 - 8s_1q \pm (1+q)\sqrt{\Theta}\right)$$
$$\overline{\eta}_\pm = \frac{qs_2}{2(1-q^2)^2}\left((s_2-s_1)(1+q)^2 - 8s_2q \pm (1+q)\sqrt{\Theta}\right)$$
$$\Theta = (s_1-s_2)^2(1+q)^2 + 16qs_1s_2$$
$$q = -e^{iu}.$$

For the specialization

$$s_1 = s_2 = s,$$

the above equations simplify to

$$\lambda_\pm = \frac{4s^4}{1 \mp \sin\frac{u}{2}}$$

$$\eta_\pm = \overline{\eta}_\pm = \frac{s^2}{4\sin^2\frac{u}{2}\left(1 \mp \sin\frac{u}{2}\right)},$$

and so we find

$$\begin{aligned} Z_2(g\,|\,k_1,k_2)|_{s_1=s_2=s} =& s^{2(2g-2-k_1-k_2)} 4^{g-1}\left(2\sin\frac{u}{2}\right)^{2(k_1+k_2)} \\ & \cdot \left\{\left(1+\sin\frac{u}{2}\right)^{k_1+k_2-1+g} + \left(1-\sin\frac{u}{2}\right)^{k_1+k_2-1+g}\right\}. \end{aligned}$$

In particular, the local $d = 2$ Calabi-Yau partition function (see Subsection 2.3) is given by:

$$Z_2(g) = \left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^{4g-4} \left\{\left(4 - 4\sin\frac{t}{2}\right)^{g-1} + \left(4 + 4\sin\frac{t}{2}\right)^{g-1}\right\},$$

as was announced in [2]. The partition function $Z_2(g)$ is easily seen to satisfy the BPS integrality of Gopakumar-Vafa.

By expanding $\lambda_\pm$, $\eta_\pm$, and $\overline{\eta}_\pm$ as power series in q, we see the leading terms in (22) are given by:

$$\begin{aligned} Z_2(g\,|\,k_1,k_2) = & \quad 2^{g-1-2k_2}(s_2-s_1)^{g-1+k_1-k_2} s_1^{2g-2-3k_1} s_2^{g-1-k_2}\; q^{-2k_1-k_2} + \dots \\ & + 2^{g-1-2k_1}(s_1-s_2)^{g-1+k_2-k_1} s_2^{2g-2-3k_2} s_1^{g-1-k_1}\; q^{-2k_2-k_1} + \dots \end{aligned}$$

By the equivariant GW/DT correspondence (see Section 9), the coefficient of the lowest order term in q should be the Donaldson-Thomas invariant for subschemes of minimal Euler characteristic. For $Z_2(g\,|\,k_1,k_2)$ the verification is straightforward: the coefficients of the above two terms are exactly the Donaldson-Thomas residue invariants of the two $\mathbb{C}^* \times \mathbb{C}^*$ invariant pure double structure subschemes on $L_1 \oplus L_2$.

For $d = 3$, to avoid excessively complicated formulas, we restrict ourselves to the local Calabi-Yau case,

$$Z_3(g\,|\,k_1,k_2)_{k_1+k_2=2g-2,\ s_1=s_2} = \mathrm{tr}\left(\left(A^{-2}G\right)^{g-1}\right)_{s_1=s_2}.$$

After computing and simplifying, we find that the matrix $A^{-2}G$, evaluated at

$$s = s_1 = s_2$$

is given by:

$$\frac{1}{9}\begin{pmatrix} w^6(162+351w^2+72w^4-w^6) & 3sw^7v(117+48w^2-w^4) & 2s^2w^6(w^2-3)(w^4-36w^2-27) \\ s^{-1}w^7v(117+48w^2-w^4) & 3w^6(w^2-4)(w^4-30w^2-27) & 2sw^7v(w^2-21)(w^2-3) \\ s^{-2}w^6(w^2-3)(w^4-36w^2-27) & 3s^{-1}w^7v(w^2-21)(w^2-3) & -w^6(2w^2-27)(w^2-3)^2 \end{pmatrix},$$

where $w = 2\sin(t/2)$ and $v = 2\cos(t/2)$. Taking traces of the first few powers of the above matrix and converting to the variable $-q = e^{it}$, we calculate

$$Z_3(g\,|\,g-1, g-1)_{s_1=s_2}.$$

For $g = 2$,

$$Z_3(2\,|\,1,1) = \frac{(1+q)^6}{q^5}(q^4 + 54q^3 + 187q^2 + 54q + 1),$$

For $g = 3$,

$$Z_3(3\,|\,2,2) = \frac{(1+q)^{12}}{3q^{10}}(3q^8+316q^7+9438q^6+63792q^5+117353q^4+63792q^3+9438q^2+316q+3).$$

We note $Z_3(3\,|\,2,2)$ does *not* have integer coefficients. Because of the existence of infinitessimal deformations of triple covers [3], the precise local to global formalism is not yet clear. The fractions in $Z_3(3\,|\,2,2)$ therefore do *not* contradict the conjectured global integral structure of Calabi-Yau threefolds.

Empirically, for $g > 0$, the local Calabi-Yau partition functions

$$Z_d(g\,|\,g-1, g-1)_{s_1=s_2}$$

appear to be Laurent *polynomials* in q with *positive* coefficients.

9 The GW/DT correspondence for residues

9.1 Overview

A Gromov-Witten/Donaldson-Thomas correspondence parallel to [14, 15] is conjectured here for equivariant residues in both the absolute and relative cases. The computation of the full local Gromov-Witten theory of curves together with the GW/DT correspondence predicts the full local Donaldson-Thomas theory of curves.

9.2 Residue Invariants in Donaldson-Thomas theory

Let Y be a nonsingular, *quasi-projective*, algebraic threefold. Let $I_n(Y, \beta)$ denote the moduli space of of ideal sheaves

$$0 \to I_Z \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0$$

of subschemes Z of degree $\beta = [Z] \in H_2(Y, \mathbb{Z})$ and Euler characteristic $n = \chi(\mathcal{O}_Z)$. Though Y may not be compact, we require Z to have *proper* support.

Let Y be equipped with an action by an algebraic torus T. We will define Donaldson-Thomas residue invariants under the following assumption.

Assumption 2. *The T-fixed point set $I_n(Y, \beta)^T$ is compact.*

The definition of the Donaldson-Thomas residue invariants of Y follows the strategy of the Gromov-Witten case. We define $\mathsf{Z}_{DT}(Y)_\beta$ formally by:

$$\mathsf{Z}_{DT}(Y)_\beta \text{ “ = ” } \sum_{n \in \mathbb{Z}} q^n \int_{[I_n(Y,\beta)]^{vir}} 1. \tag{23}$$

The variable q indexes the Euler number n. Under Assumption 2, the integral on the right of (23) is well-defined by the virtual localization formula as an equivariant residue.

Definition 9.1. *The partition function for the degree β Donaldson-Thomas residue invariants of Y is defined by:*

$$\mathsf{Z}_{DT}(Y)_\beta = \sum_{n \in \mathbb{Z}} q^n \int_{[I_n(Y,\beta)^T]^{vir}} \frac{1}{e(\mathrm{Norm}^{vir})}. \tag{24}$$

The T-fixed part of the perfect obstruction theory for $I_n(Y, \beta)$ induces a perfect obstruction theory for $I_n(Y, \beta)^T$ and hence a virtual class [9, 14]. The equivariant virtual normal bundle of the embedding,

$$I_n(Y, \beta)^T \subset I_n(Y, \beta),$$

is Norm^{vir} with equivariant Euler class $e(\mathrm{Norm}^{vir})$. The integral in (24) denotes equivariant push-forward.

As defined, $\mathsf{Z}_{DT}(Y)_\beta$ is *unprimed* since the degree 0 contributions have not yet been removed. In Gromov-Witten theory, the degree 0 contributions are removed geometrically by forbidding such components in the moduli problem. Since a geometrical method of removing the degree 0 contribution from Donaldson-Thomas theory does not appear to be available, a formal method is followed.

Definition 9.2. *The reduced partition function* $\mathsf{Z}'_{DT}(Y)_\beta$ *for the degree* β *Donaldson-Thomas residue invariants of* Y *is defined by:*

$$\mathsf{Z}'_{DT}(Y)_\beta = \frac{\mathsf{Z}_{DT}(Y)_\beta}{\mathsf{Z}_{DT}(Y)_0}.$$

Let r be the rank of T, and let $s_1, \ldots, s_r$ be generators of the equivariant cohomology of T. By definition, $\mathsf{Z}'_{DT}(Y)_\beta$ is a Laurent series in q with coefficients given by rational functions of $s_1, \ldots, s_r$ of homogeneous degree equal to minus the virtual dimension of $I_n(Y, \beta)$.

9.3 Conjectures for the absolute theory

The equivariant degree 0 series is conjecturally determined in terms of the McMahon function,

$$M(q) = \prod_{n \geq 1} \frac{1}{(1-q^n)^n},$$

the generating series for 3-dimensional partitions.

Conjecture 1. The degree 0 Donaldson-Thomas partition function is determined by:

$$\mathsf{Z}_{DT}(Y)_0 = M(-q)^{\int_Y c_3(T_Y \otimes K_Y)},$$

where the integral in the exponent is defined via localization on Y,

$$\int_Y c_3(T_Y \otimes K_Y) = \int_{Y^T} \frac{c_3(T_Y \otimes K_Y)}{e(N_{Y^T/Y})} \in \mathbb{Q}(s_1, \ldots, s_r).$$

The subvariety Y^T is compact as a consequence of Assumption 2. By Theorem 1 of [15], Conjecture 1 holds for toric Y.

Conjecture 2. The reduced series $\mathsf{Z}'_{DT}(Y)_\beta$ is a rational function of the equivariant parameters s_i and q.

The GW/DT correspondence for absolute residue invariants can now be stated.

Conjecture 3. After the change of variables $e^{iu} = -q$,

$$(-iu)^{\int_\beta c_1(T_Y)}\, \mathsf{Z}'_{GW}(Y)_\beta = (-q)^{-\frac{1}{2}\int_\beta c_1(T_Y)}\, \mathsf{Z}'_{DT}(Y)_\beta.$$

Conjectures 1-3 are equivariant versions of the conjectures of [14, 15]. In [15], a GW/DT correspondence for primary and (certain) descendent field insertions is presented. The equivariant correspondence with insertions remains to be studied.

9.4 The relative conjectures

A Gromov-Witten/Donaldson-Thomas residue correspondence for relative theories may also be defined. Let

$$S \subset Y$$

be a nonsingular, T-invariant, divisor. Let $\beta \in H_2(Y, \mathbb{Z})$ be a curve class satisfying

$$\int_\beta [S] \geq 0.$$

Let η be a partition of $\int_\beta [S]$ weighted by the equivariant cohomology of S,

$$H_T^*(S, \mathbb{Q}).$$

We follow here the notation of [15]. The reduced Gromov-Witten partition function,

$$\mathsf{Z}'_{GW}(Y/S)_{\beta,\eta} = \sum_{h \in \mathbb{Z}} u^{2h-2} \int_{[\overline{M}^\bullet_h(Y/S,\beta,\eta)^T]^{vir}} \frac{1}{e(\mathrm{Norm}^{vir})},$$

is well-defined if $\overline{M}^\bullet_h(Y/S, \beta, \eta)^T$ is compact. The weighted partition η specifies the relative conditions imposed on the moduli space of maps.

We refer the reader to [15] for a discussion of relative Donaldson-Thomas theory. The relative Donaldson-Thomas partition function,

$$\mathsf{Z}_{DT}(Y/S)_{\beta,\eta} = \sum_{n \in \mathbb{Z}} q^n \int_{[I_n(Y/S,\beta,\eta)^T]^{vir}} \frac{1}{e(\mathrm{Norm}^{vir})},$$

is well-defined if $I_n(Y/S, \beta, \eta)^T$ is compact. Let

$$\mathsf{Z}'_{DT}(Y/S)_{\beta,\eta} = \frac{\mathsf{Z}_{DT}(Y/S)_{\beta,\eta}}{\mathsf{Z}_{DT}(Y/S)_0}.$$

denote the reduced relative partition function.

The weighted partition η in Donaldson-Thomas theory specifies the relative conditions imposed on the moduli space of ideal sheaves. The partition η determines an element of the Nakajima basis of the T-equivariant cohomology of the Hilbert scheme of points of S.

Conjecture 1R. The degree 0 relative Donaldson-Thomas partition function is determined by:

$$\mathsf{Z}_{DT}(Y/S)_0 = M(-q)^{\int_Y c_3(T_Y[-S]\otimes K_Y[S])},$$

where T_Y is the sheaf of tangent fields with logarithmic zeros and K_Y is the logarithmic canonical bundle.

Conjecture 2R. The reduced series $\mathsf{Z}'_{DT}(Y/S)_{\beta,\eta}$ is a rational function of the equivariant parameters s_i and q.

Conjecture 3R. After the change of variables $e^{iu} = -q$,

$$(-iu)^{\int_\beta c_1(T_Y)+l(\eta)-|\eta|}\ \mathsf{Z}'_{GW}(Y)_\beta = (-q)^{-\frac{1}{2}\int_\beta c_1(T_Y)}\ \mathsf{Z}'_{DT}(Y)_\beta,$$

where $|\eta| = \int_\beta [S]$.

Conjectures 1R-3R are equivariant versions of the relative conjectures of [15] without insertions.

9.5 The local theory of curves

Let X be a nonsingular curve of genus g. Let N be a rank 2 bundle on a X with a direct sum decomposition,

$$N = L_1 \oplus L_2.$$

Let k_i denote the degree of L_i on X.

We have studied the Gromov-Witten residue theory of N relative to the T-invariant divisor

$$S = \bigcup_{p\in D} N_p \ \subset N,$$

where $D \subset X$ is a finite set of points. Since

$$H_T^*(S) = \bigoplus_{p\in D} H_T^*(p),$$

η is simply a list of partitions indexed by D.

We have defined the $\widehat{Z}$-partition functions in relative Gromov-Witten in Section 6.4.1. The $\widehat{Z}$-partition functions may also be defined in Donaldson-Thomas theory by:

$$\widehat{Z}_d^{DT}(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r} = (-q)^{-\frac{d}{2}(2-2g+k_1+k_2)}\ \mathsf{Z}'_{DT}(N)_{d[X],\lambda^1,\ldots,\lambda^r}.$$

Conjecture 3R implies the two definitions coincide after the variable change $e^{iu}=-q$,

$$\widehat{Z}_d^{GW}(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r} = \widehat{Z}_d^{DT}(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r}. \tag{25}$$

Hence, the superscripts GW and DT will usually be omitted.

The full local Donaldson-Thomas theory of curves can be obtain from the Gromov-Witten side using the correspondence. The GW/DT correspondence (25) and Conjecture 2R together predict

$$e^{\frac{idu}{2}(2-2g+k_1+k_2)}\ \widehat{Z}(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r}$$

to be rational function of s_1, s_2, and q — precisely the statement of Theorem 6.4.

9.6 An example

We conclude with an example of the GW/DT correspondence for the local theory of curves. By (18),

$$\begin{aligned}\widehat{Z}_2(0\,|\,0,0)_{(2),(2),(2)} &= \frac{1}{2}\frac{s_1+s_2}{s_1s_2}\left(2\frac{(-q)^2+1}{(-q)^2-1} - \frac{(-q)+1}{(-q)-1}\right).\\ &= \frac{1}{2}\frac{s_1+s_2}{s_1s_2}\left(-1-2q+\ldots\right).\end{aligned}$$

Hence, on the Donaldson-Thomas side, we expect

$$\mathsf{Z}'_{DT}(0\,|\,0,0)_{2[X],(2),(2),(2)} = \frac{1}{2}\frac{s_1+s_2}{s_1s_2}\left(-q^2-2q^3+\ldots\right). \tag{26}$$

We will verify the first coefficient of the Donaldson-Thomas series (26). Let N be trivial rank 2 bundle over $\mathbb{P}^1$,

$$N = \mathbb{C}^2\times\mathbb{P}^1.$$

Let $S \subset N$ be the relative divisor determined by the points $p_1, p_2, p_3 \in \mathbb{P}^1$. The moduli spaces $I_n(N/S, 2[\mathbb{P}^1])$ are empty for $n < 2$. Hence, the terms of $\mathsf{Z}'_{DT}(0\,|\,0,0)_{2[X],(2),(2),(2)}$ of order less than q^2 vanish.

The moduli space $I_2(N/S, 2[\mathbb{P}^1])$ is isomorphic to the Hilbert scheme of 2 points of $\mathbb{C}^2$,

$$I_2(N/S, 2[\mathbb{P}^1]) \cong \mathrm{Hilb}(\mathbb{C}^2, 2). \tag{27}$$

Since $I_2(N/S, 2[\mathbb{P}^1])$ is nonsingular of expected dimension 4, the virtual class reduces to the ordinary fundamental class of the moduli space.

Under the isomorphism (27), the T-fixed locus of the moduli space consists of the two fixed double points at the origin of $\mathbb{C}^2$,

$$I_2(N/S, 2[\mathbb{P}^1])^T = \{f_1, f_2\}.$$

The equivariant Euler classes of the normal bundles,

$$e(\mathrm{Norm}_{f_1}) = s_1 s_2 (s_1 - s_2)(2s_2),$$

$$e(\mathrm{Norm}_{f_2}) = s_1 s_2 (s_2 - s_1)(2s_1),$$

are simply the tangent weights to the Hilbert scheme of points.

We now consider the relative conditions on the moduli space of ideal sheaves $I_2(N/S, 2[\mathbb{P}^1])$. Let

$$\epsilon : I_2(N/S, 2[\mathbb{P}^1]) \to \prod_{i=1}^{3} \mathrm{Hilb}(\mathbb{C}^2_{p_i}, 2)$$

be the intersection maps to the Hilbert scheme of points of S, see[15]. Let

$$H_i \subset \mathrm{Hilb}(\mathbb{C}^2_{p_i}, 2)$$

denote the locus of double points. The Nakajima basis element corresponding to the relative condition (2) over p_i is $[H_i]/2$ where H_i denote the double point locus.

Let H denote the double point locus in $\mathrm{Hilb}(\mathbb{C}^2, 2)$. By definition, the q^2 coefficient of $\mathsf{Z}'_{DT}(0\,|\,0,0)_{2[X],(2),(2),(2)}$ is:

$$\int_{I_2(N/S,2[\mathbb{P}^1])^T} \frac{\epsilon^*(\frac{[H_1]}{2} \cup \frac{[H_2]}{2} \cup \frac{[H_3]}{2})}{e(\mathrm{Norm})} =$$

$$\frac{[H]^3_{f_1}}{8 s_1 s_2 (s_1 - s_2)(2s_2)} + \frac{[H]^3_{f_2}}{8 s_1 s_2 (s_2 - s_1)(2s_1)}.$$

Using the restriction formulas

$$[H]_{f_1} = 2s_2, \quad [H]_{f_2} = 2s_1,$$

we conclude the q^2 coefficient equals

$$-\frac{1}{2}\frac{s_1+s_2}{s_1 s_2}$$

in agreement with (26).

10 Further directions

10.1 The Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$

The level $(0,0)$ theory of $\mathbb{P}^1$ is proven in [19] to *coincide* with the 3-point functions of the equivariant Gromov-Witten theory of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ of n points in $\mathbb{C}^2$:

Let the 2-dimensional torus T act on $\mathbb{C}^2$ by scaling the factors. Consider the induced T-action on $\mathrm{Hilb}^n(\mathbb{C}^2)$. The T-equivariant cohomology of $\mathrm{Hilb}^n(\mathbb{C}^2)$,

$$H_T^*(\mathrm{Hilb}^n(\mathbb{C}^2), \mathbb{Q}),$$

has a canonical Nakajima basis,

$$\{ \ |\mu\rangle \ \}_{|\mu|=n}$$

indexed by partitions of n. The degree of a curve in $\mathrm{Hilb}^n(\mathbb{C}^2)$ is determined by intersection with the divisor

$$D = -|2, 1^{n-2}\rangle.$$

Define the series $\langle \lambda, \mu, \nu \rangle_{0,3,d}^{\mathrm{Hilb}_n}$ of 3-pointed, genus 0, T-equivariant Gromov-Witten invariants by a sum over curve degrees:

$$\langle \lambda, \mu, \nu \rangle^{\mathrm{Hilb}^n(\mathbb{C}^2)} = \sum_{d \geq 0} q^d \langle \lambda, \mu, \nu \rangle_{0,3,d}^{\mathrm{Hilb}^n(\mathbb{C}^2)}.$$

The equality,

$$\widehat{Z}(0\,|\,0,0)_{\lambda\mu\nu} = (-1)^n \langle \lambda, \mu, \nu \rangle^{\mathrm{Hilb}^n(\mathbb{C}^2)}, \tag{28}$$

after the variable change $e^{iu} = -q$, is proven in [19].

10.2 The orbifold $(\mathbb{C}^2)^n/S_n$

The 3-pointed, genus 0, T-equivariant Gromov-Witten invariants of the orbifold $(\mathbb{C}^2)^n/S_n$ are easily related to $\widehat{Z}(0\,|\,0,0)_{\lambda\mu\nu}$, see [1].

The Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ is a crepant resolution of the (singular) quotient variety $(\mathbb{C}^2)^n/S_n$. The equivalence (28) may be viewed as relating the T-equivariant quantum cohomology of the quotient *orbifold* $(\mathbb{C}^2)^n/S_n$ and the T-equivariant quantum cohomology of the resolution $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Mathematical conjectures relating the quantum cohomologies of orbifolds and their crepant resolutions in the non-equivariant case have been pursued by Ruan (motivated by the physical predictions of Zaslow and Vafa). Equality (28) suggests the correspondence also holds in the equivariant context.

A Appendix: Reconstruction Result

By J. Bryan, C. Faber, A. Okounkov, and R. Pandharipande

A.1 Overview

Let λ^i be partitions of $d > 0$. The associativity relation of the TQFT yields the following identity:

$$\sum_{|\mu|=d} \widehat{Z}(0\,|\,0,0)_{\lambda^1\lambda^2\mu}\widehat{Z}(0\,|\,0,0)^{\mu}_{\lambda^3\lambda^4} = \sum_{|\mu|=d} \widehat{Z}(0\,|\,0,0)_{\lambda^1\lambda^3\mu}\widehat{Z}(0\,|\,0,0)^{\mu}_{\lambda^2\lambda^4}.$$

Theorem 6.6 can be proven inductively in d by explicitly calculating the lowest degree terms of

$$\widehat{Z}(0\,|\,0,0)_{\lambda\mu\nu}$$

and establishing the nonsingularity of the full system of associativity equations.

We present here a simpler proof using a closed formula for the series

$$\widehat{Z}(0\,|\,0,0)_{\lambda,(1^{d-2}2),\nu}$$

obtained from Theorem 6.5 and the semisimplicity of the Frobenius algebra associated to the level $(0,0)$ theory.

A.2 Fock space

By definition, the Fock space $\mathcal{F}$ is freely generated over $\mathbb{Q}$ by commuting creation operators

$$\alpha_{-k}, \quad k \in \mathbb{Z}_{>0},$$

acting on the vacuum vector $v_\emptyset$. The annihilation operators

$$\alpha_k, \quad k \in \mathbb{Z}_{>0},$$

kill the vacuum

$$\alpha_k \cdot v_\emptyset = 0, \quad k > 0\,,$$

and satisfy the commutation relations

$$[\alpha_k, \alpha_l] = k\,\delta_{k+l,0}\,.$$

A natural basis of $\mathcal{F}$ is given by the vectors

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod_{i=1}^{l(\mu)} \alpha_{-\mu_i}\, v_\emptyset \,. \tag{29}$$

indexed by partitions μ. After extending scalars to $\mathbb{Q}(s_1, s_2)$, we define the following *nonstandard* inner product on $\mathcal{F}$:

$$\langle \mu | \nu \rangle = \frac{1}{(-s_1 s_2)^{l(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)} \,. \tag{30}$$

A.3 The matrix M_2

Define the linear transformation M_2 on $\mathcal{F}$ by

$$\langle \mu | \mathsf{M}_2 | \nu \rangle = \widehat{Z}(0\,|\,0,0)_{\mu,(1^{|\mu|-2}2),\nu}\ \delta_{|\mu|,|\nu|},$$

after an extending scalars to $\mathbb{Q}(s_1, s_2)[[u]]$. The matrix M_2 can be written in closed form:

$$-\,\mathsf{M}_2 = \frac{s_1+s_2}{2} \sum_{k>0} \left(k \frac{(-q)^k+1}{(-q)^k-1} - \frac{(-q)+1}{(-q)-1} \right) \alpha_{-k}\,\alpha_k + \frac{1}{2} \sum_{k,l>0} \Big[s_1 s_2\, \alpha_{k+l}\, \alpha_{-k}\, \alpha_{-l} - \alpha_{-k-l}\, \alpha_k\, \alpha_l \Big] , \tag{31}$$

where $-q = e^{iu}$.

Formula (31) can be obtained as follows. Using dimension counts similar to those in subsection 6.2, the disconnected invariants $\widehat{Z}(0\,|\,0,0)_{\mu,(1^{|\mu|-2}2),\nu}$ are easily reduced to connected invariants of one of two possible types. First, there are the (necessarily connected) invariants $\widehat{Z}(0\,|\,0,0)_{(d)(d)(1^{d-2}2)}$ (computed in Theorem 6.5), and second there are domain genus 0 Hurwitz numbers. The combinatorics of writing disconnected invariants in terms of connected invariants is most efficiently handled with the Fock space formalism and yields Equation (31).

The first summand of Equation (31) gives the diagonal terms of the matrix M_2. The second summand gives the off diagonal terms with the $s_1 s_2$ term of the summand appearing below the diagonal and the other term appearing above.

Lemma A.1. *The eigenvalues of* M_2 *are distinct.*

The eigenvalues are symmetric functions in s_1 and s_2. In the $s_1 s_2 = 0$ limit, M_2 is *upper-triangular*. Hence it suffices to show that the diagonal entries are distinct. By Equation (31), the diagonal entry at a partition μ is

$$-\frac{s_1+s_2}{2}\sum_{k>0} k m_k(\mu) F_k \tag{32}$$

where

$$F_k = k\frac{(-q)^k+1}{(-q)^k-1} - \frac{(-q)+1}{(-q)-1}.$$

The rational functions $\{F_k\}_{k>1}$ are easily seen to be linearly independent over $\mathbb{Q}$ (by, for example, studying the poles of F_k), and hence the diagonal entries are distinct. □

A.4 Proof of Theorem 6.6

Let $d > 0$. We abbreviate the partition $(1^{d-2}2)$ by (2), and we abbreviate a list $(2), \ldots, (2)$ of r copies of (2) by $(2)^r$. The gluing formula yields the equation

$$\langle \mu \,|\, \mathsf{M}_2^r \,|\, \nu \rangle = \widehat{Z}(0\,|\,0,0)_{\mu,(2)^r,\nu}$$

for partitions μ, ν of d.

A second application of the gluing formula yields the following computation:

$$\begin{aligned}
\widehat{Z}(0\,|\,0,0)_{\mu,(2)^r,\nu} &= \sum_{|\gamma|=d} \widehat{Z}(0\,|\,0,0)_{\mu\gamma\nu} \widehat{Z}(0\,|\,0,0)^{\gamma}_{(2)^r} \\
&= \sum_{|\gamma|=d} \widehat{Z}(0\,|\,0,0)_{\mu\gamma\nu}\ \mathfrak{z}(\gamma)(-s_1 s_2)^{l(\gamma)}\ \left\langle \gamma \,|\, \mathsf{M}_2^r \,|(1^d)\right\rangle.
\end{aligned}$$

The second equality uses the level $(0,0)$ cap calculation of Lemma 6.2.

Taken together, the above results provide a system of linear equations for the degree d, level $(0,0)$ pair of pants integrals,

$$\langle \mu \,|\, \mathsf{M}_2^r \,|\, \nu \rangle = \sum_{|\gamma|=d} \widehat{Z}(0\,|\,0,0)_{\mu\gamma\nu}\ \mathfrak{z}(\gamma)(-s_1 s_2)^{l(\gamma)}\ \left\langle \gamma \,|\, \mathsf{M}_2^r \,|(1^d)\right\rangle. \tag{33}$$

The linear equations have coefficients in the field $\mathbb{Q}(s_1, s_2, q)$. The proof of the Theorem is concluded by demonstrating the nonsingularity of the system (33).

Let $\mathcal{F}_d \subset \mathcal{F}$ be the subspace spanned by the vectors $|\mu\rangle$ satisfying $|\mu| = d$. The transformation M_2 preserves $\mathcal{F}_d$.

The eigenvectors for M_2 restricted to $\mathcal{F}_d$ are the idempotent basis of the semisimple Frobenius algebra associated to the degree d, level $(0,0)$ theory. The identity vector $|(1^d)\rangle$ of the Frobenius algebra has the coefficient 1 in each component of the idempotent basis. Hence, the set of vectors

$$\{\ \mathsf{M}_2^r\,|(1^d)\rangle\ \}_{r\geq 0}$$

have coefficients given by powers of the eigenvalues of M_2 restricted to $\mathcal{F}_d$. These eigenvalues are distinct by Lemma A.1, thus the above set of vectors spans $\mathcal{F}_d$ and the linear system (33) is nonsingular. □

The above proof of Theorem 6.6 was motivated by the study of the quantum cohomology of $\mathrm{Hilb}^n(\mathbb{C}^2)$ in [19]. The formula (31) for M_2 was discovered there.

A.5 Proof of Theorem 6.4

Since $-q = e^{-iu}$, we may disregard all integral terms in the exponent of the prefactor

$$e^{\frac{idu}{2}(2-2g+k_1+k_2)}.$$

Consider the product

$$e^{\frac{idu}{2}(k_1+k_2)}\ \widehat{Z}_d(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r}.$$

The series $\widehat{Z}_d(g\,|\,k_1,k_2)_{\lambda^1\ldots\lambda^r}$ can be calculated by gluing in terms of the caps

$$\widehat{Z}_d(0\,|\,\pm 1,0)_\lambda,\quad \widehat{Z}_d(0\,|\,0,\pm 1)_\lambda \tag{34}$$

and the pair of pant series

$$\widehat{Z}_d(0\,|\,0,0)_{\lambda\mu\nu}.$$

By the proof of Theorem 6.6, the pair of pant series lie in $\mathbb{Q}(s_1,s_2,q)$. By the calculation of Section 6.4.1,

$$e^{-\frac{idu}{2}}\widehat{Z}_d(0\,|\,-1,0)_\lambda,\ e^{-\frac{idu}{2}}\widehat{Z}_d(0\,|\,0,-1)_\lambda \in \mathbb{Q}(s_1,s_2,q).$$

Since the opposite caps are inverses in the Frobenius algebra, we conclude

$$e^{\frac{idu}{2}}\widehat{Z}_d(0\,|\,1,0)_\lambda,\ e^{\frac{idu}{2}}\widehat{Z}_d(0\,|\,0,1)_\lambda \in \mathbb{Q}(s_1,s_2,q).$$

The Theorem is proven by distributing a factor of $e^{\pm\frac{idu}{2}}$ to each cap of type (34) in the gluing formula. □

References

[1] Jim Bryan and Tom Graber. In preparation.

[2] Jim Bryan and Rahul Pandharipande. Curves in Calabi-Yau 3-folds and Topological Quantum Field Theory. *Duke Math Journal*. To appear. Preprint version: math.AG/0306316.

[3] Jim Bryan and Rahul Pandharipande. On the rigidity of stable maps to Calabi-Yau threefolds. To appear in the proceedings of the BIRS workshop on the interaction of finite type and Gromov-Witten invariants. Preprint version: math.AG/0405204.

[4] Jim Bryan and Rahul Pandharipande. BPS states of curves in Calabi-Yau 3-folds. *Geom. Topol.*, 5:287–318 (electronic), 2001. arXiv: math.AG/0009025.

[5] Robbert Dijkgraaf and Edward Witten. Topological gauge theories and group cohomology. *Comm. Math. Phys.*, 129(2):393–429, 1990.

[6] C. Faber and R. Pandharipande. Hodge integrals and Gromov-Witten theory. *Invent. Math.*, 139(1):173–199, 2000.

[7] C. Faber and R. Pandharipande. Logarithmic series and Hodge integrals in the tautological ring. *Michigan Math. J.*, 48:215–252, 2000. With an appendix by Don Zagier, Dedicated to William Fulton on the occasion of his 60th birthday.

[8] Daniel S. Freed and Frank Quinn. Chern-Simons theory with finite gauge group. *Comm. Math. Phys.*, 156(3):435–472, 1993.

[9] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

[10] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

[11] Jun Li. A degeneration formula of GW-invariants. *J. Differential Geom.*, 60(2):199–293, 2002.

[12] Eduard Looijenga. On the tautological ring of M_g. *Invent. Math.*, 121(2):411–419, 1995.

[13] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[14] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory, I. arXiv:math.AG/0312059.

[15] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory, II. arXiv:math.AG/0406092.

[16] Gregory Moore. Two-dimensional Yang-Mills theory and topological field theory. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1292–1303, Basel, 1995. Birkhäuser. arXiv:hep-th/9409044.

[17] A Okounkov and R Pandharipande. Hodge integrals and invariants of the unknot. *Geom. Topol.*, 8:675–699, 2004. arXiv:math.AG/0307209.

[18] Andrei Okounkov and Rahul Pandharipande. Gromov-Witten theory, Hurwitz theory, and completed cycles. arXiv:math.AG/0204305.

[19] Andrei Okounkov and Rahul Pandharipande. Quantum cohomology of the Hilbert scheme of points in the plane. In preparation.

[20] R. Pandharipande. Hodge integrals and degenerate contributions. *Comm. Math. Phys.*, 208(2):489–506, 1999.

[21] R. Pandharipande. The Toda equations and the Gromov-Witten theory of the Riemann sphere. *Lett. Math. Phys.*, 53(1):59–74, 2000.

[22] R. Pandharipande. Three questions in Gromov-Witten theory. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 503–512, Beijing, 2002. Higher Ed. Press.

[23] Cumrum Vafa. Two Dimensional Yang-Mills, Black Holes and Topological Strings. arXiv:hep-th/0406058.