Modules for algebraic groups with finitely many orbits on subspaces

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[1]

Introduction

Let G be a connected linear algebraic group over an algebraically closed field K of characteristic $p \geq 0$. In this paper we determine all finite-dimensional irreducible rational KG-modules V such that G has only a finite number of orbits on the set of vectors in V. We shall call such a module a *finite orbit module* for G. When $K = \mathbb{C}$, the finite orbit modules were classified by Kac in [Ka, Theorem 2]. Note that the problem is essentially equivalent to that of determining all V such that G has a finite number of orbits on $P_1(V)$, the set of all 1-dimensional subspaces of V, since in such a situation, the group K^*G will have finitely many orbits on V.

This is part of the larger problem of classifying pairs of closed subgroups G, H of a reductive algebraic group S such that there are only finitely many (G, H)-double cosets in S. (The case above has $S = GL(V)$, G an irreducible subgroup and H the stabilizer in S of a 1-space.) Our methods also yield a classification of irreducible KG-modules V such that G has finitely many orbits on the set of k-dimensional subspaces of V for some k (i.e. there are finitely many (G, H)-double cosets in $SL(V)$, where H is the stabilizer of a k-space). Moreover, we classify all pairs X, Y of maximal closed subgroups of $SL(V)$ such that there are finitely many (X, Y)-double cosets. Further results on double cosets can be found in [Br1, Br2].

If V is a finite orbit module for G, then G has an open dense orbit on V. Those KG-modules on which a connected algebraic group G has an open dense orbit are called *prehomogeneous spaces*. Prehomogeneous spaces over algebraically closed fields of characteristic zero were classified by Sato and Kimura in [SK]; this was extended to arbitrary characteristic by Chen in [Ch1,Ch2]. Thus a list of candidates for finite orbit modules can be obtained from these papers. However, this is not very useful for our proof; as we shall see, obtaining a list of candidates for V is not hard – but given a candidate for V, deciding whether or not G has finitely many orbits on V can be far from trivial. For example, if $G = A_1A_{2n}$ and $V = V(\lambda_1) \otimes V(\lambda_2)$, then V is a prehomogenous space for all n (see [SK]), but there are finitely many orbits only for $n \leq 3$. Despite this, a curious corollary of our results is that in the case where G

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is simple, every prehomogeneous irreducible KG-module is also a finite orbit module for G (see Corollary 1 below).

One way to obtain finite orbit modules for algebraic groups is the following. Let H be a simple algebraic group over K and let $P = QG$ be a proper parabolic subgroup of H with unipotent radical Q and Levi subgroup G. Let

$$1 = Q_0 < Q_1 < \ldots < Q_r = Q$$

be a G-invariant composition series for Q. By [ABS], each factor Q_{i+1}/Q_i has the structure of a rational irreducible KG-module. The following theorem is an immediate consequence of [Ri, Theorem E] and [ABS, Theorem 1].

Richardson's Theorem *The Levi subgroup G has only finitely many orbits on each of the modules Q_{i+1}/Q_i.*

Richardson's Theorem naturally gives rise to many examples of finite orbit modules Q_{i+1}/Q_i. We shall call such examples *internal Chevalley modules* for G. A list of all internal Chevalley modules can be written down using [ABS, Theorem 2], and we do so in Table B below. In the table, we use the ordering of fundamental dominant weights given in [Bo, p.250]; in the G-column we write just the semisimple part of the Levi subgroup; and in the column headed "V", for G simple we write just λ to indicate that $V = V_G(\lambda)$, the rational irreducible KG-module with high weight λ, and for G non-simple we write $\lambda \otimes \mu \otimes \ldots$ to indicate that V is the tensor product of irreducible modules for the factors of G with high weights $\lambda, \mu, \ldots$. In each row of the final column we give a simple algebraic group H containing a parabolic subgroup QG which gives rise to the internal Chevalley module of that row. Finally, each irreducible module for each simple factor of G is specified only up to graph and field twists (by a field or graph twist of a module V, we mean a module V^α obtained from V by twisting the action of the group by a field or graph automorphism α (i.e. replacing the action $v \to vg$ by $v \to vg^\alpha$)).

We now come to the classification of irreducible finite orbit modules. Observe that if V is an irreducible faithful KG-module then G must be reductive. Also $G = G'Z$ where $Z = Z(G)$ acts on V as scalars; thus if V is a finite orbit module then the semisimple group G' has finitely many orbits on $P_1(V)$, the set of 1-spaces in V. Conversely, given such a semisimple group G', V is a finite orbit module for the group K^*G' (where K^* is the group of all scalars). Thus the problem will be solved if we determine all connected semisimple groups having finitely many orbits on $P_1(V)$ for some rational irreducible module V.

Theorem 1 *Let G be a connected semisimple algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Suppose that V is a faithful rational irreducible finite-dimensional KG-module such that G has finitely many orbits on $P_1(V)$. Then one of the following holds, where we give V up to field or graph twists:*

(i) *V is an internal Chevalley module for G (listed in Table B);*

(ii) *G and V are as in Table A below.*

Conversely, if G and V are as in (i) or (ii), then G has finitely many orbits on $P_1(V)$. (The reference given in the last column of Table A shows where a proof of this fact can be found in the paper.)

Table A

G	V	$\dim V$	reference
$A_n\,(p \neq 0)$	$\lambda_1 + p^i\lambda_1,\ \lambda_1 + p^i\lambda_n\,(i > 0)$	$(n+1)^2$	2.6
$A_2\,(p = 3)$	$\lambda_1 + \lambda_2$	7	2.5
$A_3\,(p = 3)$	$\lambda_1 + \lambda_2$	16	2.7
$B_n\,(n = 4, 5)$	λ_n	2^n	2.9, 2.11
$C_3\,(p = 3)$	λ_2	13	2.12
G_2	λ_1	$7 - \delta_{p,2}$	2.5
$F_4\,(p = 3)$	λ_4	25	2.11
$A_nB_3\,(n = 1, 2)$	$\lambda_1 \otimes \lambda_3$	$8(n+1)$	3.5
$A_nC_3\,(n = 1, 2, p = 2)$	$\lambda_1 \otimes \lambda_3$	$8(n+1)$	3.5
$A_1C_n\,(p \neq 2)$	$2\lambda_1 \otimes \lambda_1$	$6n$	4.4
$A_1G_2\,(p \neq 2)$	$\lambda_1 \otimes \lambda_1$	14	3.4
$A_nG_2\,(p = 2)$	$\lambda_1 \otimes \lambda_1$	$6(n+1)$	3.4
$A_1A_2A_n\,(n \geq 5)$	$\lambda_1 \otimes \lambda_1 \otimes \lambda_1$	$6(n+1)$	4.5

Table B : internal Chevalley modules

G	V	simple group H containing QG
A_n, B_n, C_n, D_n	λ_1	$A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}$ (resp.)
A_n	λ_2	D_{n+1}
$A_n\,(p \neq 2)$	$2\lambda_1$	C_{n+1}
$A_n\,(n = 5, 6, 7)$	λ_3	E_{n+1}
$A_1\,(p \neq 2, 3)$	$3\lambda_1$	G_2
B_3, C_3	λ_3	F_4
$D_n\,(n = 5, 6, 7)$	λ_{n-1}, λ_n	E_{n+1}
E_6	λ_1	E_7
E_7	λ_7	E_8
$A_mA_n, A_mB_n, A_mC_n, A_mD_n$	$\lambda_1 \otimes \lambda_1$	$A_{m+n+1}, B_{m+n+1}, C_{m+n+1}, D_{m+n+1}$ (resp.)
$A_1A_n\,(p \neq 2)$	$2\lambda_1 \otimes \lambda_1$	B_{n+2}
A_3A_n	$\lambda_2 \otimes \lambda_1$	D_{n+4}
$A_1A_1A_n$	$\lambda_1 \otimes \lambda_1 \otimes \lambda_1$	D_{n+3}
$A_1A_n\,(n = 4, 5, 6)$	$\lambda_1 \otimes \lambda_2$	E_{n+2}
$A_nA_4\,(n = 2, 3)$	$\lambda_1 \otimes \lambda_2$	E_{n+5}
$A_1A_2\,(p \neq 2)$	$\lambda_1 \otimes 2\lambda_1$	F_4
$A_nD_5\,(n = 1, 2)$	$\lambda_1 \otimes \lambda_4$	E_{n+6}
A_1E_6	$\lambda_1 \otimes \lambda_1$	E_8
$A_1A_2A_n\,(n = 2, 3, 4)$	$\lambda_1 \otimes \lambda_1 \otimes \lambda_1$	E_{n+4}

Remarks 1. Kac [Ka, Theorem 2] obtains the list of finite orbit modules over $\mathbb{C}$. His list of course includes all internal Chevalley modules, and also the examples B_n, G_2, A_nB_3, A_1C_n, A_1G_2 and $A_1A_2A_n$ in Table A. The other examples in Table A occur only in finite characteristics.

2. Our proofs that the modules in Table B are finite orbit modules sometimes give more information, such as the actual number of orbits, orbit representatives and point stabilizers.

The next result could be verified by comparing the lists of irreducible prehomogeneous spaces for simple algebraic groups given in [SK, Ch1, Ch2] with Theorem 1. For completeness, however, we include a proof at the end of §2.

Corollary 1 *Let G be a simple algebraic group over K, and let V be a rational irreducible KG-module. Then G has finitely many orbits on $P_1(V)$ if and only if G has a dense orbit on $P_1(V)$.*

As remarked above, the conclusion of Corollary 1 is false if G is only assumed to be semisimple.

We now consider the corresponding problem of classifying irreducible modules with finitely many orbits on k-spaces for some $k > 1$. Let G be an irreducible closed connected subgroup of $GL(V)$. Let $P_k(V)$ denote the variety of k-dimensional subspaces of V. Note that if G has finitely many orbits on $P_k(V)$, then G has finitely many orbits on $P_{n-k}(V)$ as well (because we can identify $P_{n-k}(V)$ with $P_k(V^*)$ and V^* is equivalent to V via an automorphism of G). In conclusion (i) below, by the natural module for a classical group G, we mean the module $V_G(\lambda_1)$.

Theorem 2 *Let G be a connected semisimple algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Suppose that V is a faithful rational irreducible finite-dimensional KG-module such that G has finitely many orbits on $P_k(V)$ for some k with $1 < k \leq (\dim V)/2$. Then one of the following occurs (up to field or graph twists) and conversely in each of the cases listed G has finitely many orbits on $P_k(V)$:*

(i) *G is a classical group, V is the natural module for G and k is arbitrary;*

(ii) *$G = SL_r \otimes SL_s$, $V = V(\lambda_1) \otimes V(\lambda_1)$ and one of: $k = 2$, $r \leq 3$, s arbitrary; or $k = 3$, $r = 2$, $s \geq 3$ arbitrary;*

(iii) *G and V are as in Table C below.*

Table C

G	V	k
$A_2\,(p \neq 2)$	$2\lambda_1$	2
$A_n\,(4 \leq n \leq 6)$	λ_2	2
A_4	λ_2	$3,4$
B_3 or $C_3(p = 2)$	λ_3	$2,3$
D_5	λ_5	$2,3$
E_6	λ_1	2
G_2	λ_1	2
$G_2\,(p = 2)$	λ_1	3

We show in §6 that if there are only finitely many (X, Y)-double cosets in SL_n, where X and Y are proper closed subgroups, then either X or Y is contained in a parabolic subgroup. Moreover, if Y is a maximal parabolic subgroup, we determine all examples with X maximal:

Theorem 3 *Let V be an n-dimensional vector space over K, and let $G = SL(V)$. Suppose that X, Y are maximal closed subgroups of G such that there are only finitely many (X, Y)-double cosets in G. Then either X or Y is a parabolic subgroup. Moreover, if $1 \leq k \leq n/2$ and $Y = P_k$ (the stabilizer of a k-subspace) or P_{n-k}, then one of the following holds:*

(i) *X is parabolic;*

(ii) *X is a classical group and V is the natural module for X;*

(iii) *$X^0 = (GL_{n/r})^r \cap G$ with $r|n$, and one of: $r \leq 3$, k arbitrary; or $r \geq 4, k = 1$;*

(iv) *$X^0 = SL_r \otimes SL_s$ ($rs = n$), and one of: $k = 1$; $k = 2$, $r \leq 3$, s arbitrary; or $k = 3$, $r = 2$, $s \geq 3$ arbitrary;*

(v) *X^0, V, k are as in Table D below.*

Conversely, if $Y = P_k$ and X is as in (i)-(v), *then there are only finitely many (X, Y)-double cosets in G.*

Table D

X^0	V	k
A_n	λ_2	1
$A_n\,(p \neq 2)$	$2\lambda_1$	1
$A_n\,(n = 6, 7)$	λ_3	1
$A_3\,(p = 3)$	$\lambda_1 + \lambda_2$	1
$D_n\,(n = 5, 7)$	λ_n	1
$A_2\,(p \neq 2)$	$2\lambda_1$	2
$A_n\,(n \leq 6)$	λ_2	2
A_4	λ_2	$3, 4$
D_5	λ_5	$2, 3$
E_6	λ_1	$1, 2$

Notice that in conclusion (iii), X^0 is not maximal connected in G, but $N_G(X^0)$ is maximal closed.

The next result is an immediate consequence of Theorems 1 and 2.

Corollary 2 *Let G be a connected semisimple algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Suppose that V is a faithful rational irreducible finite-dimensional KG-module. Then G has finitely many orbits on the set of all subspaces of V if and only if one of the following occurs (up to field and graph twists):*

(i) *G is a classical group and V is the natural module for G;*

(ii) *$G = SL_2 \otimes SL_3$ and $V = V(\lambda_1) \otimes V(\lambda_1)$;*

(iii) *$G = G_2$, $p = 2$ and $V = V(\lambda_1)$.*

A well known result concerning finite linear groups states that if G is a subgroup of $GL_n(q)$ (q a prime power), and $j \leq k \leq n/2$, then the number of orbits of G on $P_j(V_n(q))$ is less than or equal to the number of orbits on $P_k(V_n(q))$ (see Lemma 7.1 in §7). Theorems 1 and 2 imply a weak analogue of this for connected semisimple subgroups G of $GL_n(K)$: if G has finitely many orbits on $P_k(V_n(K))$, then G also has finitely many orbits on $P_j(V_n(K))$. The next result, which we shall prove directly using model theory, shows that the same conclusion is true under a much weaker hypothesis on G:

Theorem 4 *Let V be an n-dimensional vector space over K, and let G be a closed subgroup of $GL(V)$. Let $j \leq k \leq n/2$. If G has finitely many orbits on $P_k(V)$, then G also has finitely many orbits on $P_j(V)$.*

The paper has seven further sections. In the first we give some preliminary results. One of these (Proposition 1.2) provides a list of candidates for finite orbit modules for simple groups. Section 2 of the paper is concerned with determining which modules in this list actually are finite orbit modules. In the third section, we classify simple groups having finitely many orbits on k-spaces for $k > 1$. The following two sections concern the non-simple groups; the case $k = 1$ for non-simple groups depends upon the result for $k > 1$ for simple groups. Section 6 deals with the proof of Theorem 3. In the final section we prove Theorem 4.

1 Preliminaries

We begin with a result which shows that the finite orbit property of a G-module is independent of the algebraically closed field of characteristic p over which we work.

Let $G = G(K)$ be a connected reductive algebraic group over the algebraically closed field K. If k is an algebraically closed subfield of K, denote by $G(k)$ the group of k-rational points of $G(K)$. Similarly, if $V = V(K)$ is an affine K-variety defined over k, let $V(k)$ be the set of k-rational points of V.

Proposition 1.1 *Suppose that $G(K)$ acts algebraically on the affine variety $V(K)$, and the action is defined over the algebraically closed subfield k. Then $G(K)$ has finitely many orbits on $V(K)$ if and only if $G(k)$ has finitely many orbits on $V(k)$. Moreover, if this holds, then the number of orbits is the same in each case, and each $G(K)$-orbit has a representative in $V(k)$.*

Proof We begin by establishing
(1) *Let $V(K), W(K)$ be affine K-varieties defined over k. If $f : V(k) \to W(k)$ is a surjective morphism, then the morphism $f_0 : V(K) \to W(K)$ which restricts to f on $V(k)$ is also surjective.*

To see this, let A be the automorphism group of K/k. Observe that k is the fixed field of A, and since k is algebraically closed, every orbit of A either has size 1 or is infinite. Note that A acts on $V(K)$ and $W(K)$ with fixed points $V(k)$ and $W(k)$, respectively. View $W(K)$ as a closed subvariety of affine n-space over K.

Suppose that f_0 is not surjective, and choose $w \in W(K)$ not in the image. Since f_0 is defined over k, it commutes with the action of A and so the A-orbit of w misses the image of f_0. Since f_0 is surjective over k, we have $w \notin W(k)$, and hence the A-orbit of w is infinite.

The image of f_0 contains $W(k)$, which is dense in $W(K)$. Therefore the image of f_0 is dense in $W(K)$, and hence contains a dense open subset of $W(K)$. If $W(K)$ has dimension 1, this implies that the image of f_0 is cofinite, contradicting the previous paragraph. This completes the proof when $W(K)$ has dimension 1.

Now assume that $W(K)$ has dimension greater than 1; in particular, $n > 1$. We may certainly assume that K has finite transcendence degree over k (adjoin the coordinates of w to k, and replace K by the algebraic closure of this); by induction then, we may assume that K has transcendence degree 1 over k. Write $w = (w_1, \ldots, w_n)$. We may take it that $w_1 \notin k$. Then each w_i is algebraically dependent on w_1 over k, and thus satisfies an equation $g_i(w_1, w_i) = 0$ with g_i a nonzero polynomial over k. Therefore w lies in the subvariety $C = C(K)$ of $W(K)$ defined by $g_i(y_1, y_i) = 0$ for $2 \leq i \leq n$, where the y_i are the coordinate functions on affine space intersected with $W(K)$. Every A-conjugate of w also lies in C, so C has positive dimension. Then C is defined over k and has dimension 1 (if we project to the first coordinate, the inverse image of any point is finite). Let $U(k)$ be the inverse image of $C(k)$ under f. Then $f : U(k) \to C(k)$ is surjective; applying the dimension 1 case proved above, we deduce that $f_0 : U(K) \to C(K)$ is surjective. But this means that w lies in the image of f_0, a contradiction. This completes the proof of (1).

Next we show

(2) *If $v, w \in V(k)$ are in distinct $G(k)$-orbits, then they are in distinct $G(K)$-orbits.*

To see this, define $X = \{g \in G : vg = w\}$. Then X is a closed subvariety of G, defined over k. If $X(k)$ is empty then the ideal defining $X(k)$ is the whole coordinate ring over k; hence the ideal defining $X(K)$ is the whole coordinate ring over K, which implies that $X(K)$ is empty. This gives (2).

We now complete the proof of the proposition. Assume that $G(k)$ has m orbits on $V(k)$, with representatives $v_1, \ldots, v_m$. By (2), $v_1, \ldots, v_m$ lie in distinct orbits of $G(K)$ on $V(K)$. Now define $f : G(k) \times \{1, \ldots, m\} \to V(k)$ by $f(g, i) = v_i g$ for all $g \in G(k)$ and all i. Then f is a surjective morphism. Hence (1) implies that the v_i form a complete set of orbit representatives for $G(K)$ on $V(K)$.

Conversely, assume that $G(K)$ has m orbits on $V(K)$. Then by (2), $G(k)$ has at most m orbits on $V(k)$, and the previous paragraph applies. This completes the proof. $\square$

The next result provides a list of candidates for the finite orbit modules of a simple algebraic group.

Proposition 1.2 *Let G be a simple algebraic group in characteristic p, and suppose that the nontrivial rational irreducible KG-module $V = V_G(\lambda)$ satisfies* $\dim V \leq \dim G + 1$. *Then, up to graph and field twists, either*

(i) *V is a composition factor of the adjoint module for G, or*

(ii) *G, λ are as follows:*

G	λ
A_n	$\lambda_1,\ 2\lambda_1,\ \lambda_2,\ \lambda_1 + p^i\lambda_1 (i > 0),\ \lambda_1 + p^i\lambda_n (i > 0),$ $3\lambda_1 (n = 1),\ \lambda_1 + \lambda_2 (n = p = 3),\ \lambda_3 (n = 5, 6, 7)$
$B_n,\ C_n$	$\lambda_1,\ \lambda_2,\ \lambda_3 (n = 3),\ \lambda_n (4 \leq n \leq 6, p = 2 \textit{ if } G = C_n)$
D_n	$\lambda_1,\ \lambda_2,\ \lambda_n (n \leq 7)$
G_2	λ_1
F_4	λ_4
E_6	λ_1
E_7	λ_7

Proof This follows from [Li2, 2.2, 2.7 and 2.10] for types A_n, C_n and for exceptional types, and from [Li1, 1.1] for types B_n, D_n. □

Corollary 1.3 *Let G and $V = V_G(\lambda)$ be as in conclusion (i) or (ii) of* 1.2. *Then one of the following holds (V given up to graph and field twists):*

(i) *V is an internal Chevalley module for G;*

(ii) *V is a composition factor of the adjoint module for G;*

(iii) *G and λ are as follows:*

G	λ
A_n	$\lambda_1 + p^i\lambda_1 (i > 0),\ \lambda_1 + p^i\lambda_n (i > 0),\ \lambda_1 + \lambda_2 (n = p = 3)$
C_n	λ_2
$B_n,\ C_n\ (n = 4, 5, 6)$	$\lambda_n\ (p = 2 \textit{ if } G = C_n)$
G_2	λ_1
F_4	λ_4

Proof The composition factors of the adjoint module for G are well known (see [LS3, 1.10] for example). The result follows from 1.2 and the list of internal Chevalley modules given in Table B in the Introduction. □

2 Proof of Theorem 1 for G simple

Suppose that G is a connected simple algebraic group over the algebraically closed field K of characteristic p. In this section we prove Theorem 1, classifying those KG-modules $V = V_G(\lambda)$ such that G has finitely many orbits on $P_1(V)$. Certainly if V is such a module, then $\dim G \geq \dim V - 1$, so G and V are as in 1.3. By Richardson's Theorem, we may assume that V is not an internal Chevalley module for G. Hence by 1.3, we may assume that either V is a composition factor of the adjoint module $L(G)$ or G, λ are as in 1.3(iii).

Most of the results in this section concern actions on $P_1(V)$, but there are also a couple (2.2 and 2.10) which deal with actions on $P_k(V)$ with $k > 1$; these will be used in §3, via 3.2.

We begin with three preliminary results, in which V is an arbitrary rational G-module. Let T be a maximal torus of G, and $W = N_G(T)/T$ the Weyl group.

Lemma 2.1 *Let v, v' be vectors in the zero weight space of V relative to T. Then v and v' are in the same G-orbit if and only if they are in the same W-orbit.*

Proof Let Σ be the root system of G relative to T, Π a basis of fundamental roots, and Σ^+ the set of positive roots determined by Π. Let U be the group generated by the root groups U_α $(\alpha \in \Sigma^+)$, and for $w \in W$, let U_w be the subgroup generated by those root groups U_α such that α is a positive root and $w(\alpha)$ is a negative root.

Suppose $g(v) = v'$ for some $g \in G$. By the Bruhat decomposition we may write $g = unu'$, where $u \in U, n \in N_G(T)$ and $u' \in U_w$ with $w = nT \in W$. Then

$$(nu'n^{-1})(nv) = u^{-1}(v').$$

The right hand side is a sum of v' and weight vectors whose weights are sums of positive roots, while the left hand side is a sum of nv and weight vectors whose weights are sums of negative roots. Hence $nv = v'$, and so v and v' are W-conjugate. □

Corollary 2.2 *Let V_0 be the zero weight space of V relative to T. Suppose one of the following holds:*

(i) *G has finitely many orbits on either $P_1(V)$ or $P_2(V)$;*

(ii) *G has a dense orbit on $P_1(V)$, and this dense orbit has a representative in $P_1(V_0)$.*

Then $\dim V_0 \leq 1$.

Proof First suppose (i) holds, and assume that V_0 has dimension at least 2. By 2.1 and the finiteness of W, there are infinitely many 1-spaces L_i $(i = 1, 2, \ldots)$ in V_0 such that the L_i are in distinct G-orbits. This shows that G has infinitely many orbits on $P_1(V)$. Hence by (i), G has finitely many orbits on $P_2(V)$.

Let v be a weight vector for T of nonzero weight and let M_i be the 2-space spanned by v and L_i. Note that T stabilizes M_i and has 3 orbits on the set of 1-spaces in M_i, for each i. Thus, for fixed i, the set $M_i \cap (G \cdot L_j)$ is empty for all but finitely many j. Consequently there must be infinitely many distinct G-orbits represented by the M_i, contradicting (i).

Now suppose (ii) holds. If Δ is the open dense orbit of G on $P_1(V)$, then $\Delta \cap P_1(V_0)$ is dense in $P_1(V_0)$. This forces $\dim V_0 \leq 1$ by 2.1. □

Lemma 2.3 *Let V_0 be the zero weight space of V relative to T, and let $C = C_G(V_0)^0$. Suppose* $\dim G - \dim C = \dim V - \dim V_0$ *and* $\dim V_0 > 1$. *Then G has no dense orbit on $P_1(V)$.*

Proof Assume G has a dense orbit on $P_1(V)$. If $v \in V_0$ then $C_G(v)$ contains T, so there are only finitely many possibilities for $C_G(v)$. Therefore $\{v \in V_0 : C < C_G(v)^0\} = \bigcup_{C<D=D^0} C_{V_0}(D)\}$ is a finite union of proper closed subsets of V_0, hence is proper closed in V_0. It follows that $\Delta = \{v \in V_0 : C_G(v)^0 = C\}$ is an open dense subset of V_0.

We now claim that the dense orbit of G on $P_1(V)$ contains $\langle v_0 \rangle$ for some $v_0 \in \Delta$. Let $\phi : G \times \Delta \to V$ be the morphism $(g, v) \to g(v)$. Pick $v_0 \in \Delta$. Then by 2.1, $\phi^{-1}(v_0)$

has a component $\{(g, v_0) : g \in C_G(v_0)^0 = C\}$, and hence $\dim \phi^{-1}(v_0) = \dim C$. Therefore, by the hypothesis of the lemma,

$$\dim \mathrm{Im}\phi = \dim G + \dim V_0 - \dim C = \dim V.$$

Thus $G\Delta$ contains an open dense subset of V, and hence the dense orbit of G on $P_1(V)$ has a representative in $P_1(V_0)$, which is a contradiction by 2.2(ii). □

We now use 2.1 – 2.3 to deal with some of the candidates for the finite orbit module $V = V_G(\lambda)$ given by 1.3.

Lemma 2.4 *Suppose that one of the following holds:*

(i) *V is a nontrivial composition factor of the adjoint module $L(G)$, and V is not an internal Chevalley module;*

(ii) *$G = C_n\,(n \geq 3)$, $V = V_G(\lambda_2)$;*

(iii) *$G = F_4$, $V = V_G(\lambda_4)$.*

If G has a dense orbit on $P_1(V)$, then $p = 3$ and $(G, \lambda) = (A_2, \lambda_1 + \lambda_2)$, (G_2, λ_1), (C_3, λ_2) or (F_4, λ_4) (up to graph and field twists).

Proof Suppose G has a dense orbit on $P_1(V)$. Let V_0 be the zero weight space of V relative to T.

First assume (i) holds. The composition factors of $L(G)$ can be found in [LS3, 1.10]. Suppose (G, p) not one of the special pairs $(B_n, 2)$, $(C_n, 2)$, $(G_2, 3)$ or $(F_4, 2)$. Assuming (as we may) that G is simply connected, we have $V = L(G)/Z$, where $Z = Z(L(G))$ has dimension at most 2 (and has dimension 2 if and only if $G = D_n$ with n even and $p = 2$). The zero weight space V_0 is equal to $L(T)/Z$ and has dimension $\mathrm{rank}(G) - \dim Z$. Hence, excluding the case $(G, p) = (A_2, 3)$ in the conclusion, and also the case where $G = A_1$ (in which case V is an internal module), we have $\dim V_0 \geq 2$. Now $C_G(V_0)^0$ centralizes $L(T)$. As $C_G(L(T))^0 = T$ by [LS3, 1.11], we have $C_G(V_0)^0 = T$. But now 2.3 gives a contradiction.

Now assume (i) holds with (G, p) a special pair $(B_n, 2)$, $(C_n, 2)$, $(G_2, 3)$ or $(F_4, 2)$. By [LS3, 1.10], up to graph and field twists we have $V = V_G(\lambda)$, where $\lambda = \lambda_2, \lambda_2, \lambda_1, \lambda_4$, respectively (recall that internal modules are excluded in (i)). The first two cases are dealt with under (ii) below (note that B_n and C_n have the same image group in $SL(V)$), and the F_4 case is handled under (iii) below. Finally, the G_2 case occurs in the conclusion.

Next consider (ii), so $(G, \lambda) = (C_n, \lambda_2)$ with $n \geq 3$. Let M be the usual module $V_G(\lambda_1)$ for G. By [CPS, Table 4.5], $V_G(\lambda_2)$ is a section of $\bigwedge^2 M$, of dimension $\dim(\bigwedge^2 M) - 1 - \delta$, where $\delta = 1$ if $p|n$ and $\delta = 0$ otherwise. Since T has zero weight space on $\bigwedge^2 M$ of dimension n, it follows that either $\dim V_0 \geq 2$ or $(n, p) = (3, 3)$. Exclude the latter case, which occurs in the conclusion. Thus $\dim V_0 \geq 2$. Now T lies in a commuting product A_1^n of fundamental A_1's, and $C_G(V_0)$ contains $\langle A_1^n, N_G(T)\rangle$. ¿From this it is easy to see that $C_G(V_0)^0 = A_1^n$. Now 2.3 gives a contradiction.

Finally, consider case (iii), $(G, \lambda) = (F_4, \lambda_4)$. The case where $p = 3$ occurs in the conclusion, so assume $p \neq 3$. Then $\dim V = 26$, and if we let D be a maximal rank subgroup D_4 of G generated by long root subgroups, then by [LS2, 2.3],

$V \downarrow D$ has composition factors of high weights $\lambda_1, \lambda_3, \lambda_4, 0, 0$. Therefore $\dim V_0 = 2$. Since $V_D(\lambda_i)$ $(i = 1, 3, 4)$ do not have indecomposable extensions by the trivial module, $C_G(V_0)^0$ contains D; and as D is maximal connected in G, we conclude that $C_G(V_0)^0 = D$. Now 2.3 gives a contradiction. □

Notice that each of the exceptions in the previous lemma occurs in Table A of Theorem 1. We establish the finite orbit property for two of these in the next lemma, and postpone this for the other two.

Lemma 2.5 *Suppose $p = 3$ and $(G, \lambda) = (A_2, \lambda_1 + \lambda_2)$ or (G_2, λ_1). Then G has finitely many orbits on $P_1(V)$.*

Proof This is well known for the G_2 case (see [LSS, 1.3,1.4] for example).

Now consider $(G, \lambda) = (A_2, \lambda_1 + \lambda_2)$. Here $V = L/Z$, where L is the space of 3×3 matrices of trace zero over K and Z is the 1-dimensional subspace of scalar matrices (note that $Z < L$ as $p = 3$); G acts on L by conjugation. Every matrix in L is G-conjugate to a matrix in Jordan canonical form; modulo Z, the Jordan forms are $\mathrm{diag}\,(\alpha, -\alpha, 0)$ $(\alpha \in K)$, E_{12} and $E_{12} + E_{23}$, where E_{ij} denotes the matrix with 1 in the ij-entry and 0 elsewhere. Thus G has three orbits on $P_1(V)$ in this case. □

In the rest of this section we deal with the remaining possibilities for the finite orbit module V given by 1.3(iii).

Lemma 2.6 *Suppose that $G = SL_n(K)$, $p \neq 0$ and $V = V_G(\lambda)$ with $\lambda = \lambda_1 + p^i\lambda_1$ or $\lambda_1 + p^i\lambda_{n-1}$ $(i > 0)$. Then G has finitely many orbits on $P_1(V)$.*

Proof Let W be the usual n-dimensional module $V_G(\lambda_1)$. Define the KG-module $W^{(p^i)}$ to be the space W with G-action $w * g = wg^\sigma$ $(w \in W, g \in G)$, where σ is the Frobenius morphism of G which raises matrix entries to the power p^i. Then $V_G(\lambda_1 + p^i\lambda_1) \cong W \otimes W^{(p^i)}$ and $V_G(\lambda_1 + p^i\lambda_{n-1}) \cong W \otimes W^{*(p^i)}$.

We first consider $W \otimes W^{*(p^i)}$. In fact it is convenient to deal with the dual of this, so let $V = W^* \otimes W^{(p^i)}$. We can identify V with $M_n(K)$, the space of $n \times n$ matrices over K, with G-action given by

$$g : A \to g^{-1}Ag^\sigma \quad (A \in M_n(K))$$

where σ is as above. We shall show that $G_1 = GL_n(K)$ has finitely many orbits on V in this action.

To do this, we begin by reinterpreting the question. Write σ also for the p^i-power automorphism of K, and let $L = K[x; \sigma]$ be the twisted polynomial ring consisting of polynomials in x over K, with multiplication determined by $xa = a^\sigma x$ for $a \in K$. This is a (noncommutative) left and right principal ideal domain (see [Ja] for a general reference). Given an $n \times n$ matrix A over K, we can define an L-module $M(A)$ as follows: $M(A)$ is the space of column vectors in K^n, and for $v \in M(A)$, we set $xv = Av^\sigma$.

In fact, any L-module which is n-dimensional over K is of this form. For if M is such a module, choose a K-basis for M. Then x defines a K-semilinear map from M

to itself. Let A denote the matrix of x with respect to this basis and identify M (as a K-space) with the space of column vectors via this basis. Then $xv = Av$ for each v in the basis, whence by the definition of L, it follows that $xv = Av^\sigma$ for v arbitrary. Thus, $M \cong M(A)$.

Suppose $A, B \in M_n(K)$, and f is a K-linear mapping of $M(A)$ into $M(B)$. Define $U \in M_n(K)$, by $Ue_i = f(e_i)$, where $e_1, \ldots, e_n$ is the standard basis for the space of column vectors. Then $f(xv) = f(Av^\sigma) = UAv^\sigma$, while $xf(v) = B(Uv)^\sigma = BU^\sigma v^\sigma$. Thus $\mathrm{Hom}_L(M(A), M(B)) = \{U \in M_n(K) : BU^\sigma = UA\}$. In particular, $M(A) \cong M(B)$ if and only if both A and B lie in $M_n(K)$ and there exists $g \in GL_n(K)$ such that $g^{-1}Ag^\sigma = B$. Thus the number of $GL_n(K)$-orbits on V is equal to the number of isomorphism classes of L-modules which are K-spaces of dimension n.

Let M be an L-module which is a K-space of dimension n. The structure of finitely generated modules over non-commutative principal ideal domains is given in [Ja]. We deduce that $M \cong M_1 \oplus M_2$, where M_1 is a direct sum of modules of the form L/x^iL, and x induces a bijection on M_2.

We know that $M_2 \cong M(C)$ for some $m \times m$ matrix C, where $m = \dim_K M_2$. If C is singular, then $Cw = 0$ for some nonzero $w \in M_2$, and so the submodule $\{v \in M_2 : xv = 0\}$ is nonzero, a contradiction. Therefore C is nonsingular. By Lang's theorem [La], $C = D^{-1}D^\sigma$ for some $D \in GL_m(K)$. Thus $M_2 \cong M(I_m)$; in particular, the isomorphism class of M_2 is determined by its dimension.

The above arguments show that the number of isomorphism classes of L-modules which are K-spaces of dimension n is at most $1 + \sum_{i=1}^n p(i)$, where $p(i)$ is the number of partitions of i (in fact it is equal to this). In particular this number is finite, as required.

Now consider $V = W \otimes W^{(p^i)}$. Here we can identify V with $M_n(K)$, with G-action given by

$$g : A \to g^T A g^\sigma \quad (A \in M_n(K))$$

(where g^T is the transpose of g). By Lang's theorem applied to the morphism $g \to (g^\sigma)^{-T}$ (which has finite unitary fixed point group), in the above action G is transitive on nonsingular matrices. We need to show that there are only finitely many orbits on singular matrices.

In fact, this conclusion follows from [Ga] and [RS, §9]; we sketch the argument. Let $A \in M_n(K)$ and let X be the space of n-dimensional column vectors. Then A determines a σ-sesquilinear form f_A on X defined by $f_A(u, v) = u^T A v^\sigma$ $(u, v \in X)$. We say that (X, f_A) is a σ-sesquilinear module over K. There is an obvious notion of morphisms of σ-sesquilinear modules. It is immediate that A and B are in the same G-orbit if and only if $(X, f_A) \cong (X, f_B)$. Also A is nonsingular if and only if the space (X, f_A) is non-degenerate. Thus by our observation above, the only invariant for a non-degenerate sesquilinear module is its dimension. It follows (see [RS, Section 9]) that a sesquilinear space X can be decomposed into an orthogonal sum of indecomposable sesquilinear spaces. By the previous sentence we may assume that X has no non-degenerate summands. Then $X = X_{II} - X_{III}$ where the Kronecker modules $K(X_{II})$ and $K(X_{III})$ have only indecomposable summands of a certain type (again, see [RS, Section 9]). Moreover, the isomorphism classes of X_{II} and X_{III} are determined by the isomorphism classes of their Kronecker modules. Since the dimensions of the Kronecker modules are bounded by a function of $n = \dim X$, and there are only finitely many possibilities for indecomposable Kronecker modules of a given dimension ([RS, Section 9]), it follows that there are only finitely many isomorphism types possible for X. The conclusion follows. $\Box$

Lemma 2.7 *If $p = 3$, then the group $G = GL_4$ has finitely many orbits on the vectors of the 16-dimensional module $V_G(\lambda_1 + \lambda_2)$.*

Proof This is proved by Cohen and Wales in [CW], following preliminary work of Chen [Ch3] on this module. $\Box$

Lemma 2.8 *If $G = B_6$ and $V = V_G(\lambda_6)$, then G has no dense orbit on $P_1(V)$. In particular, G has infinitely many orbits on $P_1(V)$.*

Proof Suppose G has a dense orbit on $P_1(V)$. We have $G < D < SL(V)$, where $D = D_7$ and $V = V_D(\lambda_6)$. Choose a subgroup B_3B_3 of D; then $V \downarrow B_3B_3 = V_{B_3}(\lambda_3) \otimes V_{B_3}(\lambda_3)$ (see [LS2, 2.7]). A subgroup G_2 of B_3 fixes a unique 1-space in $V_{B_3}(\lambda_3)$. Hence there is a 1-space $\alpha \in P_1(V)$ such that D_α contains G_2G_2.

We claim that $D_\alpha^0 = G_2G_2$. For suppose false. As G_2G_2 lies in no parabolic, D_α^0 is reductive. Since G_2 is maximal in B_3, D_α^0 must contain a subgroup B_3G_2. But B_3G_2 does not fix a 1-space of V, so this is impossible.

Thus $D_\alpha^0 = G_2G_2$, as claimed. Since $\dim D - \dim G_2G_2 = 63 = \dim P_1(V)$, α^D is the dense orbit of D on $P_1(V)$. Therefore, as we are assuming G has a dense orbit also, this orbit lies in α^D, and so there is a dense (G, G_2G_2)-double coset in D. Therefore G_2G_2 has a dense orbit on the coset space $(D : G)$, hence on the set of nonsingular 1-spaces in V_{14}, the natural 14-dimensional module for D.

Assume $p \neq 2$. Then $V_{14} \downarrow G_2G_2 = V_7 \oplus V_7$, a sum of two non-degenerate 7-spaces. The group G_2 has two orbits on $P_1(V_7)$, with connected stabilizers A_2 and P_1 (see [LSS, 1.2,1.3]). Hence the smallest possible dimension of the stabilizer in G_2G_2 of a nonsingular 1-space in V_{14} is $\dim A_2A_2 = 16$. But $\dim G_2G_2 - 16 = 12 < \dim(D_7 : B_6)$. Hence G_2G_2 has no dense orbit on nonsingular 1-spaces in V_{14}, a contradiction.

Finally, assume $p = 2$. Here G fixes a nonsingular vector $v \in V_{14}$. Let $\langle w \rangle$ be a representative of the dense orbit of G_2G_2 on nonsingular 1-spaces in V_{14}. Then $w \notin v^-$; take $(v, w) = 1$.

Let $W = \langle v, w \rangle$, and let H be the stabilizer of W in G_2G_2. We claim that H^0 centralizes W. To see this, observe that since W is non-degenerate, $(H/C_H(W))^0$ is a (possibly trivial) torus. As G_2G_2 fixes v and fixes w modulo v^-, this torus is trivial. The claim follows. Thus H induces a finite group on W.

Suppose $\langle w' \rangle \subseteq W$ is in the G_2G_2-orbit of $\langle w \rangle$. Then $g(w) = \lambda w'$ for some $g \in G_2G_2$, $\lambda \in K^*$. Since g fixes v, we then have $g \in H$. Hence by the previous paragraph, $\langle w \rangle^{G_2G_2} \cap P_1(W)$ is finite (and non-empty, as it contains $\langle w \rangle$). But this is a contradiction, as $\langle w \rangle^{G_2G_2}$ is an open dense orbit, so must intersect $P_1(W)$ in a dense subset. $\Box$

Lemma 2.9 *If $G = B_n$ $(n = 4, 5)$, $V = V_G(\lambda_n)$, a spin module of dimension 2^n, and $p \neq 2$, then G has finitely many orbits on $P_1(V)$.*

Proof The orbits on these spin modules are described for $p \neq 2$ in [Ig, Propositions 5 and 6]; there are three orbits when $n = 4$ and five orbits when $n = 5$. $\square$

By 1.3 together with Lemmas 2.1-2.10, to complete the proof of Theorem 1 for G simple, it remains to show that for the following pairs (G, λ), G has finitely many orbits on $P_1(V_G(\lambda))$:

G :	$B_4\,(p=2)$	$B_5\,(p=2)$	$F_4\,(p=3)$	$C_3\,(p=3)$
λ :	λ_4	λ_5	λ_4	λ_2

We handle all but the C_3 case using a method which involves first determining the orbits of corresponding finite groups; the C_3 case will be deduced from the F_4 case in Lemma 2.12.

The next result is somewhat more general than what we need for the above modules, and will also be useful in the ensuing sections.

Assume that $p > 0$, and as always let K be an algebraically closed field of characteristic p, and V a finite-dimensional vector space over K. For each power q of p, let σ_q be the Frobenius morphism of $SL(V)$, raising all matrix entries to the q^{th} power relative to some fixed basis of V. Assume that G is a closed connected subgroup of $SL(V)$ which is σ_q-stable for some q. For $e \geq 1$, let $G(q^e)$ denote the group of fixed points of σ_{q^e} on G and $V(q^e)$ denote the fixed points of σ_{q^e} on V.

Lemma 2.10 *Under the assumptions of the previous paragraph, the subgroup G of $SL(V)$ has only finitely many orbits on $P_k(V)$ if and only if there exists a constant c such that $G(q^e)$ has at most c orbits on $P_k(V(q^e))$ for all $e \geq 1$. In that case G has at most c orbits on $P_k(V)$.*

Proof Since G is σ_q-invariant, it is defined over $\bar{\mathbb{F}}_q$, the algebraic closure of $\mathbb{F}_q$. Let P_k be the stabilizer in $SL(V)$ of a k-dimensional space, with P_k invariant under σ_q. By 1.1, we may assume that $K = \bar{\mathbb{F}}_q$ (we apply 1.1 to the group $G \times P_k$ acting on $SL(V)$ by $(g_1, g_2)x = g_1 x g_2^{-1}$).

Assume that $G(q^e)$ has at most c orbits on $P_k(V(q^e))$ for all $e \geq 1$. If G has at least $c+1$ orbits on $P_k(V)$, then choose $c+1$ representatives. These are subspaces of $V(q^e)$ for some e (since $K = \bar{\mathbb{F}}_q = \bigcup_{e \geq 1} \mathbb{F}_{q^e}$), whence $G(q^e)$ has at least $c+1$ orbits on $P_k(V(q^e))$, a contradiction. So G has at most c orbits.

Conversely, assume that G has exactly m orbits on $P_k(V)$ (with m finite). Denote these orbits by O_i and let $o_i \in O_i$ be a representative. Let G_i denote the stabilizer in G of o_i. By [SS, I, 2.7], $O_i(q^e)$ breaks up into at most m_i $G(q^e)$ orbits, where $m_i = |G_i : (G_i)^0|$. Thus, we may take $c = \sum m_i$. $\square$

We now return to consideration of the remaining modules listed above for B_4, B_5 and F_4.

Lemma 2.11 *Let (G, λ, p) be $(B_4, \lambda_4, 2)$, $(B_5, \lambda_5, 2)$ or $(F_4, \lambda_4, 3)$. Identify G with a subgroup of $SL(V_G(\lambda))$ which is σ_p-stable. Then for all q, $G(q)$ has the following number of orbits on $P_1(V(q))$:*

G :	B_4	B_5	F_4
no. of orbits:	3	6	5

Consequently (by 2.10), G has finitely many orbits on $P_1(V)$.

Proof For $G = F_4$, this is proved in [CC, (B.1)].

Consider now $G = B_4$. Here $G(q) = B_4(q) < D_5(q) < SL_{16}(q) = X_{\sigma_q}$, and the orbits of $D = D_5(q)$ on $P_1(V(q))$ are given in [Li2, 2.9]; there are just two orbits, with corresponding point-stabilizers D_1, D_2, where D_1 is an A_4-parabolic subgroup of D and $D_2 \cong (\mathbb{F}_q)^8.B_3(q).(q-1)$. If $U = V_{10}(q)$ is the usual $\mathbb{F}_q D$-module, then $G(q)$ is the stabilizer of a nonsingular 1-space in U. Let Ω be the set of all nonsingular 1-spaces in U. Thus, denoting by $\mathrm{orb}(A, S)$ the number of orbits of a group A on a set S, we have

$$\mathrm{orb}(G(q), P_1(V(q))) = \mathrm{orb}(D_1, \Omega) + \mathrm{orb}(D_2, \Omega).$$

It is easy to see that D_1 is transitive on Ω, while D_2 has two orbits (with point-stabilizers $(\mathbb{F}_q)^7.G_2(q).(q-1)$ and $B_3(q)$).

Finally, consider $G = B_5$. We handle this in similar fashion. First, observe that $G(q) = B_5(q) < D_6(q) < SL_{32}(q)$. We argue first that $E = D_6(q)$ has just five orbits on $P_1(V(q))$, with point-stabilizers $E_1, \ldots, E_5$ as follows:

(i) $E_1 = A_5$-parabolic in E, orbit size $|E : E_1| = (q^8-1)(q^5+1)(q^3+1)/(q-1)$;

(ii) $E_2 = A_5(q).2$, orbit size $\frac{1}{2}q^{15}(q^8-1)(q^5+1)(q^3+1)$;

(iii) $E_3 = {}^2A_5(q).2$, orbit size $\frac{1}{2}q^{15}(q^8-1)(q^5-1)(q^3-1)$;

(iv) $E_4 = [q^{17}].(B_3(q) \times A_1(q)).(q-1)$, orbit size $q^3(q^{10}-1)(q^8-1)(q^6-1)/(q^2-1)(q-1)$;

(v) $E_5 = [q^{14}].B_3(q).(q-1)$, orbit size $q^7(q^{10}-1)(q^8-1)(q^6-1)/(q-1)$.

(in (iv) and (v), we use $[q^n]$ simply to denote a group of order q^n). To see this, let $Y = E_7(q)$, with root system as in [Bo]. If α_0 is the longest root, then $P = N_Y(U_{\alpha_0}) = QEH$ is a parabolic subgroup of Y, with $E = D_6(q)$, $|H| = q-1$ and $|Q| = q^{33}$. Moreover, $Q = \langle U_\alpha : \alpha > 0 \text{ involves } \alpha_1 \rangle$, and Q/U_{α_0} has the structure of the spin module $V_E(\lambda_5)$ over $\mathbb{F}_q$ (cf. [ABS, Theorem 2]); so we may take $V(q) = Q/U_{\alpha_0}$.

Observe first that E_1 is the stabilizer of a 1-space containing a maximal vector.

We next obtain the stabilizer E_4. Write $a_1 \ldots a_7$ for the root $\sum a_i \alpha_i$. Let $\alpha = 1123321$, $\beta = 1223221$ (roots for Y), and define $y = u_\alpha(1)u_\beta(1)$. We shall show that E_4 is the stabilizer of the 1-space containing y (modulo U_{α_0}). Let $P_6 = Q_6L_6$ be the D_5A_1-parabolic subgroup of Y obtained by deleting α_6 from the Dynkin diagram. By [AS, p.60], we have $C_Y(y) = U_0L_0$, where $U_0 = Q_6$ and $L_0 \cong B_4(q) \times A_1(q)$ is a subgroup of L_6. We find that $U_0 \cap E = U_1$ has order q^{17} and $L_0 \cap E = B_3(q) \times A_1(q)$. Moreover, conjugation by the element $h_{\alpha_2}(t^{-1})h_{\alpha_6}(t)$ sends $u_\alpha(1)u_\beta(1)$ to $u_\alpha(t)u_\beta(t)$ for $t \in \mathbb{F}_q^*$. Hence the stabilizer of the 1-space $\langle y \rangle$ contains $E_4 = U_1(L_0 \cap E).(q-1)$. Since the stabilizer lies in $U_0L_0.(q-1)$, it is easy to see that it is in fact equal to E_4.

We obtain E_5 similarly as the stabilizer of the 1-space containing $z = u_\gamma(1)u_\delta(1)u_\epsilon(1)$, where $\gamma = 1223210$, $\delta = 1122221$ and $\epsilon = 1123211$.

To obtain stabilizer E_2, consider the action of a Levi subgroup $A = A_5(q)$ of E on $V(q)$. There are two nontrivial composition factors of dimension 15 (with high weights λ_2, λ_3), and a fixed 2-space $F = \langle U_{\alpha_1}, U_{\alpha_0-\alpha_1} \rangle / U_{\alpha_0}$. The normalizer $N_Y(A) = GL_6(q).2$ acts on the $q+1$ points in F with orbits of size $q-1$ and 2; the orbit of size 2 contains points with stabilizer of type E_1, and the orbit of size $q-1$ contains points with stabilizer $SL_6(q).2$.

Finally, stabilizer E_3 is obtained in the same way as E_2 starting with a subgroup ${}^2A_5(q)$ of E.

We have now obtained the stabilizers in (i)-(v) above. Since the corresponding orbit sizes add up to $(q^{32}-1)/(q-1)$, there are no further orbits of E on $P_1(V)$.

As in the previous case, if Ω is the set of nonsingular 1-spaces in the usual 12-dimensional E-module over $\mathbb{F}_q$, we have

$$\text{orb}(G(q), P_1(V(q))) = \sum_i \text{orb}(E_i, \Omega).$$

We find that E_1, E_2, E_3 and E_5 are transitive on Ω (with point-stabilizers A_4-parabolic in $G(q)$, $A_4(q).2$, ${}^2A_4(q).2$ and $[q^{14}].B_2(q).(q-1)$, respectively), while E_4 has two orbits on Ω (with point-stabilizers $[q^{15}].(G_2(q) \times A_1(q)).(q-1)$ and $[q^9].B_3(q).(q-1)$). Therefore $G(q) = B_5(q)$ has 6 orbits on $P_1(V(q))$. □

Lemma 2.12 *If $G = C_3$, $p = 3$ and $V = V_G(\lambda_2)$, then G has finitely many orbits on $P_1(V)$.*

Proof Let $Y = F_4$, and let M be the 25-dimensional Y-module $V_Y(\lambda_4)$. If A_1 denotes a fundamental SL_2 in Y, then $N_Y(A_1) = A_1C_3$, and

$$M \downarrow A_1C_3 = (V_{A_1}(\lambda_1) \otimes V_{C_3}(\lambda_1)) \oplus V_{C_3}(\lambda_2)$$

(see [LS2, 2.3]). Therefore we may take $V = C_M(A_1)$.

We claim that if two 1-spaces in V are Y-conjugate, then they are conjugate under $N_Y(A_1)$. To see this, let α, β be 1-spaces in V, and suppose $\alpha g = \beta$ for some $g \in Y$. Then A_1, A_1^g are fundamental SL_2's of Y, both contained in Y_β. Over finite fields $\mathbb{F}_q$, the stabilizers in $Y(q) = F_4(q)$ of 1-spaces in $M(q)$ are given by [CC, (B.1)]; these stabilizers contain groups $D_4^\epsilon(q), [q^{15}].B_3(q)$ and $[q^{14}].G_2(q)$. Therefore Y_β contains such a subgroup, and it follows that modulo its unipotent radical, Y_β can have only one simple factor, Y_0 say. Write $Q = R_U(Y_\beta)$. Using [LS1, 2.2], we see that the images modulo Q of QA_1 and QA_1^g are fundamental A_1's in the simple group Y_0. Thus QA_1 and QA_1^g are conjugate in Y_β. Moreover, if t is the central involution in A_1, then $C_Y(t)$ normalizes A_1, whence $C_{QA_1}(t) = A_1 \times Q_0$ for some $Q_0 \leq Q$. It follows that A_1 and A_1^g are conjugate in Y_β, say $A_1^g = A_1^y$ with $y \in Y_\beta$. Then $\alpha gy^{-1} = \beta$ with $gy^{-1} \in N_Y(A_1)$, proving the claim.

By 2.11, Y has finitely many orbits on $P_1(M)$. Therefore the result follows by the previous paragraph. □

Observe that Lemmas 2.1-2.12 constitute the proof of Theorem 1 for G simple.

To conclude the section, observe that we have also proved Corollary 1: for if G is simple and has a dense orbit on $P_1(V)$, where V is a faithful irreducible rational G-module, then V is in the conclusion of 1.2. For each candidate given by 1.2, we have shown that either G has finitely many orbits on $P_1(V)$, or G has no dense orbit on $P_1(V)$. Corollary 1 follows.

3 Proof of Theorem 2 for G simple

Let G be a connected simple algebraic group over the algebraically closed field K. Let V be an irreducible rational KG-module, and let $P_k(V)$ denote the variety of k-dimensional subspaces of V. In this section, we classify all such modules such that G has finitely many orbits on $P_k(V)$ for some $k > 1$ (the case $k = 1$ was settled in the previous section).

Let $n = \dim V$. Then G has finitely many orbits on $P_k(V)$ if and only if G has finitely many orbits on $P_k(V^*)$ (because the dual representation is obtained by applying an automorphism of G) if and only if G has finitely many orbits on $P_{n-k}(V)$. Hence we need only classify V up to graph and field twists and we may assume that $k \leq n/2$. Note that $P_k(V)$ is a variety of dimension $k(n-k)$ and so if G has finitely many orbits on $P_k(V)$, then $\dim G \geq k(n-k)$. This eliminates many possibilities.

Proposition 3.1 *Let V be an irreducible rational KG-module of dimension n and suppose that $\dim G \geq k(n-k)$ for some integer k with $2 \leq k \leq n/2$. Then (up to graph and field twists) one of the following holds:*

(i) *G is a classical group and V is the natural module for G;*

(ii) *G, V are as in Table E below.*

Table E

G	V	k
$A_2\,(p \neq 2)$	$2\lambda_1$	2
$A_l\,(l \geq 4)$	λ_2	2
A_4	λ_2	$3, 4$
B_3 or $C_3\,(p = 2)$	λ_3	$2, 3, 4$
B_4 or $C_4\,(p = 2)$	λ_4	2
D_5	λ_5	$2, 3$
D_6	λ_6	2
E_6	λ_1	$2, 3$
E_7	λ_7	2
F_4	λ_4	2
G_2	λ_1	$2, 3$

Proof Clearly G, V are as in the conclusion of 1.2. One checks which of these satisfy the necessary inequality. □

The next proposition gives a relationship between the property of having finitely many orbits on k-spaces and that of having finitely many orbits on 1-spaces, for a tensor product. Write W_n for a K-vector space of dimension n.

Proposition 3.2 *Let a, b be positive integers, let H be a subgroup of $GL(W_b)$, and let $GL_a(K) \times H$ act on $W = W_a \otimes W_b$ in the usual way. Then $GL_a(K) \times H$ has finitely many orbits on W if and only if, for all $i \leq a$, H has finitely many orbits on the set of i-spaces in W_b.*

Proof Let $w_1, \ldots, w_a$ be a basis for W_a. Every vector in W can be written uniquely in the form $\sum_{j=1}^{a} w_j \otimes w'_j$ with $w'_j \in W_b$. There is an element of $GL_a(K) \times H$ sending $\sum_{j=1}^{a} w_j \otimes w'_j$ to $\sum_{j=1}^{a} w_j \otimes w''_j$ if and only if there is an element of H sending the subspace $\langle w'_j : 1 \leq j \leq a \rangle$ of W_b to the subspace $\langle w''_j : 1 \leq j \leq a \rangle$. The result follows. □

We now consider the various possibilities given in Table E.

Lemma 3.3 *Let $G = A_n (n \geq 3)$ and $V = V_{A_n}(\lambda_2)$. Then G has finitely many orbits on $P_2(V)$ if and only if $n \leq 6$.*

Proof By Richardson's Theorem (stated in the Introduction), for $n \leq 6$ the group $A_1 A_n$ has finitely many orbits on $P_1(V(\lambda_1) \otimes V(\lambda_2))$. Hence by 3.2, G has finitely many orbits on $P_2(V)$ for $n \leq 6$.

Set $m = n + 1$ and assume now that $m \geq 8$. Identify V with the space $S_m(K)$ of skew-symmetric $m \times m$ matrices over K, with SL_m-action given by $g : A \to g^T A g$ for $g \in SL_m$, $A \in S_m(K)$. Let J be a nonsingular 2×2 skew-symmetric matrix over K. For $\mathbf{a} = (a_1, a_2, a_3) \in K^3$ with $0, a_1, a_2, a_3$ all distinct, define block-diagonal matrices $v_0, w_{\mathbf{a}} \in S_m(K)$ as follows:

$$v_0 = \mathrm{diag}(J, J, J, 0^{m-6}), \quad w_{\mathbf{a}} = \mathrm{diag}(a_1^{-1} J, a_2^{-1} J, a_3^{-1} J, J, 0^{m-8}),$$

and let $U_{\mathbf{a}} = \langle v_0, w_{\mathbf{a}} \rangle$, a 2-space in $S_m(K)$.

We claim that among the 2-spaces $U_{\mathbf{a}}$ there are infinitely many orbit representatives of SL_m on 2-spaces of $S_m(K)$. To see this, let $a_i, b_i \in K^*$ $(i = 1, 2, 3)$ with a_1, a_2, a_3 distinct and b_1, b_2, b_3 distinct, and suppose $g \in SL_m$ sends $U_{\mathbf{a}} \to U_{\mathbf{b}}$. The matrices of rank 6 in $U_{\mathbf{a}}$ are scalar multiples of v_0 and of $v_0 - a_i w_{\mathbf{a}}$ $(i = 1, 2, 3)$. Hence if we define $a_0 = b_0 = 0$, there is a permutation τ of $\{0, 1, 2, 3\}$ such that $(v_0 - a_i w_{\mathbf{a}})g$ is a scalar multiple of $v_0 - b_{i\tau} w_{\mathbf{b}}$ for $i = 0, 1, 2, 3$. Adjusting g by a scalar, we may take

$$v_0 g = v_0 - b_{0\tau} w_{\mathbf{b}}.$$

We have $(v_0 - a_1 w_{\mathbf{a}})g = \lambda(v_0 - b_{1\tau} w_{\mathbf{b}})$ for some $\lambda \in K^*$, and hence

$$w_{\mathbf{a}} g = a_1^{-1}(1 - \lambda) v_0 + a_1^{-1}(\lambda b_{1\tau} - b_{0\tau}) w_{\mathbf{b}}.$$

The fact that for some $\mu \in K^*$,

$$(v_0 - a_2 w_{\mathbf{a}})g = \mu(v_0 - b_{2\tau} w_{\mathbf{b}})$$

determines λ in terms of $a_1, a_2, b_{0\tau}, b_{1\tau}$ and $b_{2\tau}$. Finally, for some $\gamma \in K^*$ we have

$$(v_0 - a_3 w_{\mathbf{a}})g = \gamma(v_0 - b_{3\tau} w_{\mathbf{b}}),$$

which gives an equation for $b_{3\tau}$ in terms of $a_1, a_2, a_3, b_{0\tau}, b_{1\tau}$ and $b_{2\tau}$. It follows that there are infinitely many orbit representatives among the 2-spaces $U_{\mathbf{a}}$, as claimed. □

Lemma 3.4 *Suppose that $G = G_2$ and $V = V(\lambda_1)$. If $p = 2$ then G has finitely many orbits on $P_k(V)$ for all k; and if $p \neq 2$ then G has finitely many orbits on $P_k(V)$ if and only if $k \leq 2$.*

Proof Suppose first that $p=2$, so that $\dim V = 6$ and $G < Sp(V) = Sp_6$. By [As, 5.1], G is transitive of rank 4 on $P_1(V)$. Hence G has finitely many orbits on $P_1(V)$ and on $P_2(V)$.

Now consider $P_3(V)$. For this case it is convenient to deal with the finite group case $G(q) = G_2(q) < Sp_6(q) = Sp(V(q))$, where q is a power of 2, and then apply 2.10. Let R_1 be the set of 3-spaces in $V(q)$ with 1-dimensional radical, and let R_2 be the set of totally singular 3-spaces in $V(q)$.

Let $W \in R_1$ have radical $\langle v \rangle$. Then the stabilizer in $G(q)$ of $\langle v \rangle$ is a parabolic P. The action of P on the symplectic 4-space $v^-/\langle v \rangle$ induces a parabolic subgroup P_2 of $Sp_4(q)$, and $W/\langle v \rangle$ is a nonsingular 2-space in $v^-/\langle v \rangle$. Thus the number of orbits of $G(q)$ on R_1 is equal to the number of (P_2, N_2)-double cosets in $Sp_4(q)$ (where N_2 is the stabilizer of a nonsingular 2-space). Applying a graph automorphism, this is equal to the number of $(P_1, \Omega_4^+(q))$-double cosets in $Sp_4(q)$, which is just the number of orbits of $\Omega_4^+(q)$ on 1-spaces. This number is 2. Thus $G(q)$ has 2 orbits on R_1.

Now consider R_2. The parabolic P of $G_2(q)$ stabilizes a 3-space in R_2; and so does a subgroup $SL_3(q)$ of $G_2(q)$. We calculate that

$$|G_2(q):P| + |G_2(q):SL_3(q)| = \frac{q^6-1}{q-1} + q^3(q^3+1) = (q+1)(q^2+1)(q^3+1) = |R_2|.$$

Therefore $G(q)$ also has 2 orbits on R_2.

We conclude that for any $q = 2^e$, $G(q)$ has 4 orbits on $P_3(V(q))$. By 2.10 therefore, G has at most 4 orbits on $P_3(V)$. This completes the proof for $p = 2$.

Now suppose $p \neq 2$. Here $\dim V = 7$ and $G < SO(V) = SO_7$. By [LSS, 1.2,1.3], G is transitive on the set of singular 1-spaces and on the set of nonsingular 1-spaces in V, hence has 2 orbits on $P_1(V)$.

Now consider $P_2(V)$. By [As, 5.1], G is transitive of rank 4 on the set of singular 1-spaces in V, hence has finitely many orbits on the set of non-degenerate 2-spaces and on the set of totally singular 2-spaces in V. Let R_3 be the set of 2-spaces remaining. If $W \in R_3$, then W has 1-dimensional radical $\langle v \rangle$. The stabilizer in G of $\langle v \rangle$, acting on the orthogonal space $v^-/\langle v \rangle$, induces a parabolic P_2 of SO_5, and $W/\langle v \rangle$ is a nonsingular 1-space in $v^-/\langle v \rangle$. Hence the number of orbits of G on R_3 is equal to the number of (P_2, N_1)-double cosets in SO_5 (where N_1 is the stabilizer of a nonsingular 1-space). Applying an isomorphism $SO_5 \to Sp_4$, we see that this is equal to the number of (P_1, N_2)-double cosets in Sp_4, which is 3. Thus G has finitely many orbits on $P_2(V)$.

To complete the proof, we show that G has infinitely many orbits on $P_3(V)$. Let f be a nontrivial alternating trilinear form on V preserved by G (see [As]). If f restricts trivially to every non-degenerate 3-space then $f = 0$ (because the set of triples of vectors which span a non-degenerate 3-space is dense). So we may choose a non-degenerate 3-space L with f nontrivial on L. Let e_1, e_2, e_3 be an orthonormal basis for L and normalize f so that $f(e_1,e_2,e_3) = 1$. Note that if e_1', e_2', e_3' is another basis for L, then $f(e_1', e_2', e_3') = d$ where d is the determinant of the linear transformation of L sending e_i to e_i'. In particular, f^2 takes on the same value for any orthonormal basis. Choose a vector $w \in V$ such that w is orthogonal to e_1 and e_2 with $f(e_1, e_2, w) = 0$ (there is a 4-space of such vectors w). We can choose w of norm 1 as well (since a 4-space cannot consist of singular vectors). Set

$w_a = ae_3 + bw$ where $a^2 + b^2 + 2ab(w, e_3) = 1$. (Note that given a, we can solve for b.) Then $f(e_1, e_2, w_a) = a$. It follows that $\langle e_1, e_2, w_a \rangle$ and $\langle e_1, e_2, w_c \rangle$ are not in the same G-orbit unless $a^2 = c^2$. Therefore G has infinitely many orbits on the set of non-degenerate 3-spaces in V, hence on $P_3(V)$. □

Lemma 3.5 *Suppose that $G = B_3$ and $V = V(\lambda_3)$. Then G has finitely many orbits on $P_k(V)$ if and only if $k \leq 3$.*

Proof View B_3 as a subgroup of $D_4 = SO_8(V)$. It follows from [LSS, Theorem B] that G is transitive on singular 1-spaces, on nonsingular 1-spaces, and on one D_4-orbit of totally singular 4-spaces.

We next claim that G has finitely many orbits on totally singular j-spaces for any $j \leq 3$. Let U be such a j-space. Then U is contained in a totally singular 4-space W of either D_4-orbit. Choose W to be in the D_4-orbit of such 4-spaces on which G is transitive. Thus, every G-orbit on totally singular j-spaces is represented by a subspace of W. The stabilizer S of W is an A_2-parabolic subgroup of B_3. In particular, if S_0 is a Levi subgroup of this parabolic, then $W \downarrow S_0$ is the direct sum of a 1-dimensional and a 3-dimensional module (with the latter being the natural A_2-module). It follows that S has finitely many orbits on subspaces of W, whence G has finitely many orbits on totally singular j-spaces.

Now suppose $p \neq 2$. We will show that G has finitely many orbits on j-dimensional subspaces for $j \leq 3$. By the previous paragraph, we need only consider subspaces containing a nonsingular vector v. Since G is transitive on nonsingular 1-spaces, it suffices to show that the stabilizer S of v has finitely many orbits on j-dimensional spaces containing v. This follows from 3.4, since $S = G_2$ and $V/Kv \cong V_{G_2}(\lambda_1)$.

Next assume $p = 2$. By [LSS, 1.2,1.3], G has 2 orbits on $P_1(V)$. As for 2-spaces, we have already established that G has finitely many orbits on totally singular 2-spaces, and G is transitive on non-degenerate 2-spaces by [LSS, Theorem C, Appendix]. Let R be the remaining set of 2-spaces. If $W \in R$, then W contains a nonsingular vector v. By 3.4, $G_v = G_2$ has finitely many orbits on 1-spaces in v^-/Kv. Therefore G has finitely many orbits on R, hence on $P_2(V)$.

Now consider $P_3(V)$ (still assuming $p = 2$). For 3-spaces with radical containing a nonsingular vector v, we may pass to v^-/Kv, which is the 6-dimensional module for $G_v = G_2$, and the result follows from 3.4. So we need only consider 3-spaces with a 1-dimensional radical L which is singular. Then the stabilizer in G of L acting on the orthogonal 6-space L^-/L induces an A_2-parabolic subgroup P_3 of SO_6. Hence the number of orbits of G on such 3-spaces is equal to the number of (P_3, N_2)-double cosets in SO_6. This is equal to the number of $(P_1, (SL_2 \oplus SL_2)T_1)$-double cosets in SL_4, which is finite.

Thus we have shown that G has finitely many orbits on j-dimensional subspaces for $j \leq 3$.

To complete the proof, we show that G has infinitely many orbits on non- degenerate 4-dimensional subspaces. Assume otherwise. Applying a triality automorphism of $D_4 = PSO(V)$ (which fixes the stabilizer of a non-degenerate 4-space), we deduce that the reducible subgroup SO_7 of D_4 has finitely many orbits on non-degenerate

4-spaces. Equivalently, the stabilizer of a non-degenerate 4-space M has finitely many orbits on nonsingular 1-spaces in V. However, if $m \in M$ and $m' \in M^-$ are nonsingular vectors, then the 1-spaces $\langle m + \alpha m' \rangle$ give infinitely many G_M-orbit representatives as α ranges over K, which is a contradiction. This completes the proof. $\square$

Lemma 3.6 *Suppose that $G = D_6$, E_7 or E_6, with $V = V(\lambda_5), V(\lambda_7)$ or $V(\lambda_1)$, respectively. Then G has infinitely many orbits on $P_k(V)$ for $k = 2, 2$ and 3 respectively.*

Proof We begin with the first two cases. Let $W = V(\lambda_1) \otimes V$ for $H = A_1G$. Let X be a simply connected group over K of type E_7 if $G = D_6$, of type E_8 if $G = E_7$. Fix a maximal torus T of X, and let $\Sigma(X)$ be the root system of X relative to T. Pick a fundamental system $\Pi(X) = \{\alpha_1, \ldots, \alpha_l\}$ in $\Sigma(X)$, and label the extended Dynkin diagram of X as in [Bo, p.250]. The subgroup of X obtained by deleting α_1 (respectively α_8) from the extended Dynkin diagram is A_1D_6 (respectively A_1E_7), and we identify H with this subgroup. Let $\{h_\alpha, e_\beta : \alpha \in \Pi(X), \beta \in \Sigma(X)\}$ be the usual Chevalley basis of the Lie algebra $L(X)$. We claim that we may identify W with the subspace of $L(G)$ spanned by all $e_{\pm\beta}$ with $\beta = \sum b_i\alpha_i$ and $b_1 = 1$ (respectively $b_8 = 1$). For H certainly fixes this subspace, since the root elements generating H fix it; moreover, $-\alpha_1$ (respectively $-\alpha_8$) is equal to the weight $\lambda_1 \otimes \lambda_5$ (respectively $\lambda_1 \otimes \lambda_7$) of H when written in terms of fundamental dominant weights of H, so this subspace is H-isomorphic to W, establishing the claim.

With this identification, choose orthogonal roots α, β such that $e_\alpha, e_\beta \in W$. For $a, b \in K$, define

$$x_{a,b} = a(e_\alpha + e_{-\alpha}) + b(e_\beta + e_{-\beta}).$$

When $p \neq 2$, these elements are commuting and semisimple, and lie in a 2-dimensional toral subalgebra of $L(X)$; since fusion in a toral subalgebra is controlled by the action of the Weyl group, it follows that the set of 1-spaces $\langle x_{a,b} \rangle$ $(a, b \in K)$ contains infinitely many H-orbit representatives. When $p = 2$, we apply the above argument, this time using the semisimple parts of the elements $x_{a,b}$. Therefore $H = A_1G$ has infinitely orbits on $P_1(W)$. By 3.2, it follows that G has infinitely many orbits on either $P_1(V)$ or $P_2(V)$. It has finitely many orbits on $P_1(V)$ by Richardson's Theorem (see the Introduction). Hence it has infinitely many on $P_2(V)$.

Now we consider the last case, $H = A_2E_6$; the argument is rather similar to the first two cases. Let $X = E_8$, and identify H with the subgroup of X obtained by deleting α_7 from the extended Dynkin diagram of X. If we define

$$V_1 = \langle e_\beta : \beta = \sum b_i\alpha_i \in \Sigma(X)^+, b_7 = 1 \rangle,$$

$$V_2 = \langle e_{-\beta} : \beta = \sum b_i\alpha_i \in \Sigma(X)^+, b_7 = 2 \rangle,$$

then $\dim V_1 = 54, \dim V_2 = 27$, and we may identify $W := V(\lambda_1) \otimes V$ with $V_1 \oplus V_2$. Now pick roots $\alpha, \beta, \gamma, \delta \in \Sigma(X)^+$, all with α_7 coefficient 1, such that $\langle \alpha, \beta \rangle$ and $\langle \gamma, \delta \rangle$ are both A_2 systems, perpendicular to each other: for example, take

$$\alpha = \alpha_7, \beta = 11222211, \gamma = 12232210, \delta = 00011111.$$

For $a, b \in K$, define

$$x_{a,b} = a(e_\alpha + e_\beta + e_{-\alpha-\beta}) + b(e_\gamma + e_\delta + e_{-\gamma-\delta}).$$

Now the above argument goes through, showing that H has infinitely many orbits on $P_1(W)$. Note that by Richardson's Theorem and 3.2, G has finitely many orbits on $P_k(V)$ for $k \leq 2$. Thus, by 3.2, G has infinitely many orbits on $P_3(V)$. $\square$

Lemma 3.7 *If $G = B_4$ with $V = V(\lambda_4)$, then G has infinitely many orbits on $P_2(V)$.*

Proof There is a subgroup $D \cong D_5$ of $SL(V)$ such that $B_4 < D < SL(V)$. Pick a subgroup A_1B_3 of D (acting on the natural 10-dimensional D-module N as $SO_3 \times SO_7$). Then $V \downarrow A_1B_3 = \lambda_1 \otimes \lambda_3$ (see [LS2, 2.7]). If we choose a subgroup G_2 of the factor B_3, then G_2 fixes a 1-space in $V_{B_3}(\lambda_3)$, and hence the subgroup A_1G_2 of D fixes a 2-space in V; call this 2-space A.

We claim that A_1G_2 is the full connected stabilizer of A in D. For suppose $A_1G_2 < S = (D_A)^0$. As A_1G_2 lies in no parabolic of D, S is reductive. If S is irreducible on N then $S = D_5$, which is clearly impossible; and if S is reducible, then S must be A_1B_3 or B_4 (with $p = 2$ in the latter case) - but both of these are irreducible on V. Hence $(D_A)^0 = A_1G_2$, as claimed.

If there were only finitely many B_4-orbits on the 2-spaces in V, then in particular, the D-orbit containing A would split into finitely many B_4-orbits; equivalently, there would be finitely many (B_4, A_1G_2) double cosets in D. However, we claim that there are infinitely many such double cosets - there are even infinitely many (B_4, A_1B_3) double cosets. To see this, observe that the latter double cosets are in bijective correspondence with the orbits of $SO_3 \times SO_7$ on the set of nonsingular 1-spaces in the natural D-module N. Write $N = V_3 \oplus V_7$ correspondingly, and let v, w be nonsingular vectors in V_3, V_7, respectively. Then there are infinitely many orbit representatives for $SO_3 \times SO_7$ among the 1-spaces $\langle v + cw \rangle$ $(c \in K)$. This completes the proof. $\square$

Lemma 3.8 *If $G = F_4$ and $V = V(\lambda_4)$, then G has infinitely many orbits on $P_2(V)$.*

Proof If $p \neq 3$, then, by the proof of 2.4, the zero weight space V_0 has dimension 2 and $C_G(V_0)^0 = D_4$. Hence 2.2 gives the conclusion.

Now suppose $p = 3$, so that $\dim V = 25$. Pick a subgroup D_4 of F_4 generated by long root groups in a D_4 subsystem, and choose a subgroup G_2 in this D_4. Let $C = C_{F_4}(G_2)^0$, a group of type A_1; then the fixed point space $X = C_V(G_2)$ is 4-dimensional, and $X \downarrow C = 1 \otimes 1^{(3)}$ (see [LS2, 2.5]). View X as an orthogonal 4-space for C.

We now claim that the connected stabilizer in F_4 of any non-degenerate 2-space in X is either G_2 or B_3. For let Y be such a 2-space, and let $S = ((F_4)_Y)^0$, a subgroup containing G_2. If S is reductive then it is G_2, A_1G_2, B_3, D_4 or B_4, and all are determined up to conjugacy by [LS2]. Of these, only G_2 and B_3 can fix a 2-space in V (see [LS2, 2.3, 2.5]). Now suppose S is not reductive. Then it lies in a B_3-parabolic of F_4; let Q be the unipotent radical of this parabolic. Thus $S = UG_2$

or UB_3, where $1 \neq U = R_u(S) \leq Q$. If $U \leq C$ then U fixes 1-dimensional subspaces in Y and $Y^- \cap X$; however, by the representation theory of SL_2, the unipotent radical of a parabolic of C fixes only one 1-space in X, a contradiction. Hence $U \not\leq C$. As G_2 acts irreducibly on Q' ($\cong V(\lambda_1)$), and as $00 \oplus V(\lambda_1)$ on Q/Q', it follows that $Q' \leq U$. Now $Q'G_2 < Q'B_3 < B_4 < F_4$. We have $V \downarrow B_4 = V_1 - V_2$, where $V_1 = V_{B_4}(\lambda_1), V_2 = V_{B_4}(\lambda_4)$ ([LS2, Table 8.7]). Consequently $C_V(Q')$ is the perpendicular sum of a singular 1-space in V_1 and a totally singular 8-space in V_2, hence is totally singular. But the non-degenerate 2-space Y lies in $C_V(Q'G_2)$, which is a contradiction. This proves our claim.

Next, observe that the non-degenerate 2-spaces in X with stabilizer B_3 form a single C-orbit: for any two subgroups B_3 in F_4 containing our subgroup G_2 must be conjugate in the normalizer of this G_2, hence by an element of C.

From the previous two paragraphs, we deduce that the number of orbit representatives for F_4 among the 2-spaces in X is at least the number of orbits of C on the set of non-degenerate 2-spaces in X, minus 1. The non-degenerate 2-spaces form a variety of dimension 4, so this number of orbits is infinite. □

In all the cases in Table E of 3.1 not covered by Lemmas 3.3 – 3.8, it follows by 3.2 and Richardson's Theorem (see the Introduction) that G has finitely many orbits on $P_k(V)$. This completes the proof of Theorem 2 for G simple.

4 Proof of Theorem 1 for G non-simple

Let G be a connected semisimple algebraic group over the algebraically closed field K, and let V be a faithful irreducible rational KG-module. Assume in this section that G is not simple, so $G = G_1 \dots G_r$, a commuting product of simple algebraic groups G_i with $r \geq 2$, and $V = V_1 \otimes \dots \otimes V_r$ with each V_i a nontrivial irreducible KG_i-module.

As in the previous section, we denote by W_n a K-vector space of dimension n. We begin with a result which will be useful in several contexts.

Proposition 4.1 (i) *For $n \geq 2$ there are infinitely many $(Sp_{2n}(K), Sp_{2n}(K))$-double cosets in $SL_{2n}(K)$.*

(ii) *For $n \geq 2$ there are infinitely many $(Sp_{2n}(K), SO_{2n}(K))$-double cosets in $SL_{2n}(K)$.*

Proof (i) Let $G = SL_{2n}(K)$. There is an involutory graph automorphism τ of G such that $G_\tau = Sp_{2n}(K)$. The (G_τ, G_τ)-double cosets in G are in bijective correspondence with the orbits of G_τ on τ^G. Choose a 1-dimensional torus T_1 in G which is inverted by τ (for example, with the usual notation for a Cartan subgroup T and graph automorphism τ, take $T_1 = \{h_{\alpha_1}(c)h_{\alpha_{2n-1}}(c^{-1}) : c \in K^*\}$). Then for $h \in T_1$, we have $\tau\tau^h = h^2$, and this can take infinitely many possible orders. Therefore G_τ has infinitely many orbits on τ^G, proving (i).

(ii) If $p = 2$ then $SO_{2n}(K) \leq Sp_{2n}(K)$, and so (ii) follows from (i). So assume now that $p \neq 2$. There exists an involution $t \in T$ such that if $\delta = \tau t$, then $G_\delta = SO_{2n}(K)$.

As above, for $h \in T_1$, the element $\tau\delta^h$ can take infinitely many different orders. Hence G_τ has infinitely many orbits on δ^G, and (ii) follows. $\Box$

Denote by Cl_n a simple classical algebraic group with natural module W_n of dimension n over K.

The next lemma provides a useful reduction.

Lemma 4.2 *Suppose that $G = G_1 \otimes Cl_n$ acting on $V = W_m \otimes W_n$. Let U be a non-degenerate subspace of W_n (any subspace if $Cl_n = SL_n$), and let S be the stabilizer of U in Cl_n. If G has finitely many orbits on the set of k-subspaces of V, then $G_1 \otimes S$ has finitely many orbits on the set of k-subspaces of $W_m \otimes U$.*

Proof Let $v_1, \ldots, v_m$ be a basis for W_m, and suppose that M and M' are k-dimensional subspaces of $W_m \otimes U$ in the same G-orbit. It suffices to prove that M and M' are in the same $G_1 \otimes S$-orbit. Suppose that $Mg = M'$ with $g = x \otimes y \in G$. Let U' be the largest subspace of U such that $U'y \subseteq U$. By Witt's Lemma (and a much easier argument in the case where $Cl_n = SL_n$), there exists $y' \in S$ such that $wy' = wy$ for all $w \in U'$. Let $v \in M$ and write $v = \sum v_i \otimes w_i$ with $w_i \in U$. Since $vg \in M' \subseteq W_m \otimes U$, it follows that $w_i y \in U$, whence $w_i y = w_i y'$. Therefore $v(x \otimes y) = v(x \otimes y')$, and hence $Mg = M(x \otimes y')$. So M and M' are in the same $G_1 \otimes S$-orbit, as required. $\Box$

In the next two lemmas we consider $G = Cl_m \otimes Cl_n$ with $V = W_m \otimes W_n$; if the first factor is SL_m or the second is SL_n, then $W_m \otimes W_n$ is an internal Chevalley module for G (see Table A in the Introduction), and G has finitely many orbits on $P_1(V)$.

Note that if one of the factors Cl_m or C_n is an orthogonal group SO_{2l+1} in odd dimension, then we take $p \neq 2$, since SO_{2l+1} is reducible on W_{2l+1} when $p = 2$.

Lemma 4.3 *Suppose that $G = Cl_m \otimes Cl_n$ with $Cl_m \neq SL_m, Cl_n \neq SL_n$ and $V = W_m \otimes W_n$. If G has finitely many orbits on $P_1(V)$, then $m = 3$, $n = 2l$ and $G = SO_3 \otimes Sp_{2l}$ (with $p \neq 2$).*

Proof Suppose G has finitely many orbits on $P_1(V)$. When $G = Sp_m \otimes Sp_n$ we have $m, n \geq 4$ by hypothesis, and 4.2 implies that $G_0 = Sp_4 \otimes Sp_4$ has finitely many orbits on $P_1(W_4 \otimes W_4)$. Let $H = GL_4 \times Sp_4$ and identify $W_4 \otimes W_4$ with $M_4(K)$, where $(a, b) \in H$ acts as $A \to a^{-1}Ab$ for $A \in M_4(K)$. The stabilizer of the 1-space $\langle I \rangle$ is $H_{\langle I \rangle} = \{(\lambda b, b) : b \in Sp_4,\ \lambda \in K^*\}$. Now there are finitely many $(G_0, H_{\langle I \rangle})$ double cosets in H. Projecting to the first factor of H, we deduce that there are finitely many $(Sp_4, K^* Sp_4)$ double cosets in GL_4, hence finitely many (Sp_4, Sp_4) double cosets in SL_4. This contradicts 4.1.

If $G = SO_m \otimes SO_n$ then $m, n \geq 3$ (as SO_2 is reducible), so 4.2 implies that $SO_3 \otimes SO_3$ has finitely many orbits on $P_1(W_3 \otimes W_3)$. This is impossible by dimension considerations.

Finally, let $G = SO_m \otimes Sp_n$ with $m \geq 3, n \geq 4$. If $m \geq 4$ then 4.2 implies that $SO_4 \otimes Sp_4$ has finitely many orbits on $P_1(W_4 \otimes W_4)$. Arguing as above, this means that there are finitely many (SO_4, Sp_4) double cosets in SL_4, contrary to 4.1. Therefore $m = 3$ and $G = SO_3 \otimes Sp_n$, as in the conclusion. $\Box$

Lemma 4.4 *$SO_3 \otimes Sp_{2l}(p \neq 2)$ has finitely many orbits on $P_1(W_3 \otimes W_{2l})$.*

Proof Any vector $v \in W_3 \otimes W_{2l}$ lies in a subspace of the form $W_3 \otimes U$ with $\dim U \leq 3$. Since Sp_{2l} has finitely many orbits on subspaces, it suffices to prove that $SO_3 \otimes H$ has finitely many orbits on $P_1(W_3 \otimes U)$ for each U, where H is the group induced on U by the stabilizer of U in Sp_{2l}.

If $\dim U \leq 2$, or if U is totally singular, then H contains $SL(U)$ and the result follows from 3.2. Thus we may assume that $\dim U = 3$, and U has radical of dimension 1, spanned by e_1, say. Extend to a basis e_1, e_2, e_3 of U. Note that $SO_3 \otimes H$ has finitely many orbits on 1-spaces of $(W_3 \otimes U)/(W_3 \otimes \langle e_1 \rangle)$, since the action on this quotient is $SO_3 \otimes SL_2$.

Let $v_1, v_2, v_3 \in W_3$, and write $v = \sum_1^3 v_i \otimes e_i$. By the last remark in the previous paragraph, it suffices to show that there are only finitely many orbits on 1-spaces spanned by vectors of this form, with fixed $v_2 \otimes e_2 + v_3 \otimes e_3$. When the v_i are linearly dependent, there are only finitely many orbits, by the case where $\dim U \leq 2$; hence we may assume that the v_i are linearly independent. By applying an element of the form $1 \otimes h$ with h in the unipotent radical of H, we may add any element in the span of v_2, v_3 to v_1; and for $a \in K^*$, there is an element $g \in H$ sending $e_1 \to ae_1$ and fixing e_2, e_3, whence $1 \otimes g$ multiplies $v_1 \otimes e_1$ by a. The result follows. □

Lemma 4.5 *Suppose that $r \geq 3$ and $G = G_1 \ldots G_r$ has finitely many orbits on $P_1(V_1 \otimes \ldots \otimes V_r)$, where each G_i is simple and each V_i is an irreducible rational G_i-module. Then $G = SL_2 \otimes SL_k \otimes SL_n$ ($k = 2$ or 3), acting in the usual way on $V = W_2 \otimes W_k \otimes W_n$. Conversely, $SL_2 \otimes SL_k \otimes SL_n$ ($k = 2$ or 3) has finitely many orbits on $P_1(W_2 \otimes W_k \otimes W_n)$.*

Proof If $r \geq 4$ then 4.2 forces $SL_2 \times SL_2 \times SL_2 \times SL_2$ to have finitely many orbits on $P_1(W_2 \otimes W_2 \otimes W_2 \otimes W_2)$, which is not the case, by consideration of dimensions. Thus $r = 3$. Let $V = W_a \otimes W_b \otimes W_c$ with $a \leq b \leq c$, so $G \leq SL_a \otimes SL_b \otimes SL_c$. If $a \geq 3$ then 4.2 implies that $SL_3 \times SL_3 \times SL_3$ has finitely many orbits on $P_1(W_3 \otimes W_3 \otimes W_3)$, which is again impossible by dimensions. Therefore $a = 2$.

Suppose $b \geq 4$. Then by 4.2, $SL_2 \times SL_4 \times SL_4$ has finitely many orbits on $P_1(W_2 \otimes W_4 \otimes W_4)$. This implies (by 3.2) that $SL_4 \times SL_4$ has finitely many orbits on 2-spaces in $W_4 \otimes W_4$. We show that this is not the case. We can identify $W_4 \otimes W_4$ with $M_4(K)$, the space of 4×4 matrices over K, so that the action of $(g, h) \in SL_4 \times SL_4$ sends $A \to g^{-1}Ah$ ($A \in M_4(K)$). For $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in K^4$ not a multiple of $(1, 1, 1, 1)$, let $D_\lambda = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and define the 2-space W_λ by

$$W_\lambda = \langle I, D_\lambda \rangle,$$

where I denotes the 4×4 identity matrix. Suppose that $(g, h) \in SL_4 \times SL_4$ sends W_λ to W_μ (where $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$). Then there exist $a, b, c, d \in K$ such that

$$g^{-1}h = aI + bD_\mu, \quad g^{-1}D_\lambda h = cI + dD_\mu.$$

Therefore $g^{-1}D_\lambda g = (cI + dD_\mu)(aI + bD_\mu)^{-1}$, and so the right hand side has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in some order. Thus there is a permutation $\tau \in S_4$ such that for $1 \leq i \leq 4$, $\lambda_{i\tau} = (c + d\mu_i)(a + b\mu_i)^{-1}$, and hence

$$\lambda_{i\tau}a + \lambda_{i\tau}\mu_i b - c - \mu_i d = 0$$

for $1 \leq i \leq 4$. Regard this as a system of 4 linear equations in a, b, c, d. The system has a nonzero solution if and only if a certain determinant, which is a polynomial in λ_i, μ_i, is zero. This means that the W_λ contain infinitely many orbit representatives for $SL_4 \times SL_4$ on 2-spaces of $M_4(K)$, which is a contradiction.

This establishes that $b \leq 3$ and $G \leq SL_2 \otimes SL_b \otimes SL_n$. By 3.2, the third factor of G has finitely many orbits on i-subspaces of W_n for $i \leq 2b$, so by the results of §3 it satisfies the conclusion of Theorem 2 for simple groups. It follows that one of the following holds:

(i) $G = SL_2 \otimes SL_k \otimes SL_n$ ($k = 2$ or 3);

(ii) $G = SL_2 \otimes SL_k \otimes Cl_n$ (where $Cl_n \neq SL_n$ and $k \leq 3$);

(iii) $G = SL_2 \otimes SO_3 \otimes Cl_n$.

(iv) $G = SL_2 \otimes SL_2 \otimes SL_5$ acting on $W_2 \otimes W_2 \otimes V(\lambda_2)$.

Observe that G has infinitely many orbits on $P_1(V)$ in case (iv), since $\dim G < \dim P_1(V)$.

We claim that in cases (ii) and (iii), G has infinitely many orbits on $P_1(V)$. Consider (iii). By 4.3, $Cl_n = SL_n$ or Sp_n. Hence by 4.2, we only need to prove that there are infinitely many orbits when $G = SL_2 \otimes SO_3 \otimes SL_2$. This is clear by consideration of dimensions. Now consider (ii). If $Cl_n = SO_n$, then we may reduce to (iii) (or, if $p = 2$, to $SL_2 \otimes SL_k \otimes SO_4$, which is out by dimension considerations again). If $Cl_n = Sp_n$, then 4.2 shows that we need only consider $G = SL_2 \otimes SL_2 \otimes Sp_4$. However, this is $SO_4 \otimes Sp_4$, which is excluded by 4.3.

For the converse, note that $SL_2 \otimes SL_3 \otimes SL_5$ is a Levi factor acting on an internal Chevalley module (see Richardson's Theorem in the Introduction). Hence by 3.2, $SL_2 \otimes SL_3$ has finitely many orbits on i-spaces of $W_2 \otimes W_3$, for all i. Now another application of 3.2 shows that $SL_2 \otimes SL_3 \otimes SL_n$ has finitely many orbits for all n. By 4.2, $SL_2 \otimes SL_2 \otimes SL_n$ also has finitely many orbits. □

We can now complete the proof of Theorem 1 for G non-simple. Suppose G has finitely many orbits on $P_1(V)$. In view of the previous result, we can suppose that G has just two simple factors, so $G = G_1G_2$ and $V = V_1 \otimes V_2$. By 4.3 and 4.4, we may assume that $G \neq Cl_m \otimes Cl_n$; say $G_1 \neq Cl(V_1)$.

Now by 3.2, G_1 has finitely many orbits on $P_i(V_1)$ for all $i \leq \dim V_2$. Hence by Theorem 2 for simple groups (established in §3), G_1 and V_1 are as in Table C (given in Theorem 2). Let k be the largest integer given in the third column of Table C. If $\dim V_2 > k$ then by 3.2, G_1 has finitely many orbits on $P_{k+1}(V_1)$, which is a contradiction unless $k + 1 > \frac{1}{2} \dim V_1$; this occurs only in the last row of Table C. (In the last row G_1 has finitely many orbits on the set of all subspaces of V_1.) Therefore, except when G_1 is as in the last row, we have $\dim V_2 \leq k$.

If $G_2 = SL(V_2)$, then $G = G_1 \otimes SL(V_2)$, with G_1 as in Table C and $\dim V_2 \leq k$ (except for the last row), and all these groups occur in Table A or B in the conclusion of Theorem 1. Moreover, by 3.2 and Theorem 2 for simple groups, all these groups do have finitely many orbits on $P_1(V)$.

Thus it remains to establish that $G_2 = SL(V_2)$. Suppose this is not the case. Then $k \geq 3$, so from Table C, $(G_1, V_1) = (A_4, V(\lambda_2))$, $(B_3$ or $C_3, V(\lambda_3))$, $(D_5, V(\lambda_5))$

or $(G_2, V(\lambda_1))$; moreover, G_2 lies in $SO(V_2)$ or $Sp(V_2)$. Lemma 4.3 rules out all possibilities except $(A_4, V(\lambda_2))$ and $(D_5, V(\lambda_5))$. In the first case dim $G <$ dim $P_1(V)$. The second case is handled in the following lemma.

Lemma 4.6 *Let $G = D_5 \otimes SO_3 (p \neq 2)$ acting on $V = V(\lambda_5) \otimes W_3$. Then G has infinitely many orbits on $P_1(V)$.*

Proof Suppose false, and write $V_{16} = V_{D_5}(\lambda_5)$. Then by 4.2, $D_5 \otimes SO_2$ has finitely many orbits on $P_1(V_{16} \otimes W_2)$. In other words, $D_5 T_1$ has finitely many orbits on $V_{16} \oplus V_{16}$, where the torus T_1 acts as $\{cI_{16} \oplus c^{-1} I_{16} : c \in K^*\}$. It follows that for any 1-space $\langle v \rangle$ in V_{16}, $(D_5)_{\langle v \rangle}$ has finitely many orbits on $P_1(V_{16})$.

Choose $\langle v \rangle$ in the dense orbit of D_5 on $P_1(V_{16})$, so that $(D_5)^0_{\langle v \rangle} = U_8 B_3 T_1$ (see [Ig, Proposition 2]), lying in a parabolic $U_8 D_4 T_1$ of D_5. Then $C_{V_{16}}(U_8)$ is an 8-dimensional space on which $B_3 T_1$ acts as $V_1 \oplus V_7$ with T_1 inducing scalars, and hence B_3 has finitely many orbits on $P_1(V_1 \oplus V_7)$. However, for suitable vectors $v \in V_1, w \in V_7$, the 1-spaces $\langle v + cw \rangle$ $(c \in K^*)$ give infinitely many orbit representatives for $B_3 T_1$, which is a contradiction. □

The proof of Theorem 1 is now complete.

5 Proof of Theorem 2 for G non-simple

We first consider the case where $G = SL_m \otimes SL_n$ acting on $V = W_m \otimes W_n$ (where as always, W_m, W_n are the natural modules for the factors. Of course, G has finitely many orbits on $P_1(V)$.

Lemma 5.1 *Assume that $m \leq n$ and $1 < k \leq mn/2$. Then $G = SL_m \otimes SL_n$ has finitely many orbits on $P_k(V)$ if and only if one of the following holds:*

(i) *$m = 2$ and $k \leq 3$;*

(ii) *$m = 3$ and $k = 2$.*

Proof By 4.5, for any r, $SL_2 \otimes SL_3 \otimes SL_r$ has finitely many orbits on 1-spaces (of the tensor product of the three natural modules). Application of 3.2 shows that G has finitely many orbits on k-spaces if (i) or (ii) holds.

Now assume that neither (i) nor (ii) holds. Then one of:

(1) $m \geq 4$, $k = 2$;

(2) $m \geq 3$, $k \geq 3$;

(3) $m = 2$, $k \geq 4$.

In case (1) or (3), 4.2 or 3.2 (respectively) implies that $SL_4 \otimes SL_4$ has finitely many orbits on 2-spaces of the tensor product, a possibility ruled out in the proof of 4.5. And in case (2), 4.2 forces $SL_3 \otimes SL_3$ to have finitely many orbits on 3-spaces, which is not so by dimension considerations. □

Lemma 5.2 *Let $G = SL_a \otimes SL_b \otimes SL_c$ with $2 \leq a \leq b \leq c$, and let V be the tensor product of the three natural modules. Then G has infinitely many orbits on $P_k(V)$ for all $2 \leq k \leq abc/2$.*

Proof Assume false, for some k. Comparing the dimension $k(abc - k)$ of $P_k(V)$ with that of G, it follows that $k \leq c$. As usual, 4.2 reduces us to the case $a = b = 2$ and $c = k$. But in this case the dimension of $P_k(V)$ is larger than the dimension of G. □

Lemma 5.3 *Let $G = Cl_m \otimes Cl_n \neq SL_m \otimes SL_n$ with $m \leq n$, acting on $V = W_m \otimes W_n$. Then G has infinitely many orbits on $P_k(V)$ for $2 \leq k \leq mn/2$.*

Proof Assume false for some k. It follows from Lemma 5.1 that $m \leq 3$.

Suppose first that $m = 3$. If $G = SO_3 \otimes SL_n$ has finitely many orbits on $P_k(V)$, then by dimension, $k < n$ and as usual, we may reduce to the case $SO_3 \otimes SL_k$, where again a dimension argument gives a contradiction. If $G = SL_3 \otimes Cl_n$ with $Cl_n < SL_n$, then by dimension, $k \leq n$. By 4.2, if $k \geq 4$ then it suffices to consider $Cl_n = Sp_k$, Sp_{k+1}, SO_k, or SO_{k+1}; and if $k < 4$ then it suffices to consider $Cl_n = Sp_4$, SO_4 or SO_3. All cases give a contradiction by dimension considerations.

Thus $m = 2$ and $Cl_m = SL_2$. By hypothesis, $k \leq n$.

If $Cl_n = Sp_n$ for $n > 4$, then by 4.2, we may reduce to $SL_2 \otimes Sp_{n-2}$ (if $k \leq n-2$, this group still satisfies the hypothesis, while if $k > n - 2$, we replace k with $k' = (2n - 4 - k) \geq 2$). Thus, we find that $SL_2 \otimes Sp_4$ has finitely many orbits on $P_j(V)$ for some $2 \leq j \leq 4$. By dimension considerations, $j = 2$. This implies by 3.2 that $SL_2 \otimes SL_2 \otimes Sp_4 = SO_4 \otimes Sp_4$ has finitely many orbits on 1-spaces, contradicting 4.3.

If $Cl_n = SO_n$, then arguing as above, we reduce to the case where $n = 3$ or 4. Then G has dimension strictly less than $P_k(V)$, a contradiction. □

We now complete the proof of Theorem 2. Assume G has finitely many orbits on $P_k(V)$ with $1 < k \leq (\dim V)/2$. If G is simple, the theorem is already proved in §3, so assume that G is not simple. By 5.2, $G = G_1 G_2$ with G_1, G_2 simple, and $V = V_1 \otimes V_2$ with each V_i an irreducible G_i-module. And by 5.1 we may assume that G is not $SL_m \otimes SL_n$ (with $V = W_m \otimes W_n$). Take $\dim V_1 \leq \dim V_2$. By 5.1 and 5.3, we have $G_1 = SL_2$ or SL_3, with V_1 the natural module, and $k \leq 3$. Also by 5.3, G_2 is not contained in $SO(V_2)$ or $Sp(V_2)$ (and so V_2 is not self dual). It is clear that G_2 has finitely many orbits on $P_k(V_2)$ (by considering subspaces of the form $e \otimes W$ where e is a fixed vector in V_1 and W is a subspace of V_2). We now have a short list of possibilities for G_2 (by Theorem 2 for simple groups). In all cases, one checks easily that $\dim G < \dim P_k(V)$, a contradiction. This completes the proof of Theorem 2.

6 Proof of Theorem 3

We now prove Theorem 3. Suppose X and Y are maximal closed subgroups of SL_n such that there are finitely many (X, Y)-double cosets.

Write $G = SL_n = SL(V)$. Suppose neither X nor Y is a parabolic subgroup. Then X^0 and Y^0 are both reductive.

If X^0 is reducible on V, we claim that $X^0 = (GL_{n/r})^r \cap G$ for some r dividing n. When $V \downarrow X^0$ is not homogeneous, this is clear from Clifford's theorem and the maximality of X. Otherwise, $V \downarrow X^0$ is homogeneous; but then $C_G(X^0)^0 \neq 1$, so $X < N_G(X^0)$, contradicting the maximality of X. This establishes the claim.

If X^0 is irreducible on V, then it is either tensor-decomposable, in which case $X^0 \leq SL_r \otimes SL_s\,(rs = n)$, or tensor-indecomposable, in which case X^0 is simple.

To summarise, either $X^0 = (GL_{n/r})^r \cap G$, or $X^0 \leq SL_r \otimes SL_s$, or X^0 is simple; the same applies to Y^0.

We may assume that $\dim X \geq (n^2-1)/2$. Then X^0 must be simple, and it follows easily using 1.2 that $X = Sp_n$ (and n is even). Then $\dim Y \geq n(n-1)/2 - 1$, from which we deduce (again using 1.2) that $Y^0 = Sp_n, SO_n$ or $(GL_{n/2})^2 \cap G$. The first two possibilities are ruled out by 4.1, and the last by the following lemma.

Lemma 6.1 *For $m \geq 2$, there are infinitely many $(Sp_{2m}, GL_m \times GL_m)$-double cosets in GL_{2m}.*

Proof The action of GL_{2m} on the cosets of Sp_{2m} is equivalent to its action on the set Ω of all invertible skew-symmetric $2m \times 2m$ matrices (the latter action being $g \cdot X = g^T X g$ for $g \in GL_{2m}, X \in \Omega$). Thus it is enough to show that $GL_m \times GL_m$ has infinitely many orbits on Ω.

Let $X \in \Omega$, and write

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

(so X_{11}, X_{22} are skew-symmetric and $X_{21} = -X_{12}^T$). Define $d(X) = \det(X)$ and $e(X) = \det(X_{12})$. If $h = (A, B) \in GL_m \times GL_m$, then $d(h \cdot X) = \det(AB)^2 d(X)$ and $e(h \cdot X) = \det(AB)e(X)$. Therefore e^2/d is a non-constant H-invariant regular function on the variety Ω. Consequently H has no dense orbit on Ω, and the result follows. $\Box$

We have now established that either X or Y is a parabolic subgroup, say Y is. Write $Y = P_k$ or P_{n-k}, where $k \leq n/2$. We may assume that X is not parabolic. The (X, Y)-double cosets in SL_n are in bijective correspondence with the orbits of X on k-spaces (or $(n-k)$-spaces). Thus X has finitely many orbits on $P_k(V)$.

We next handle the case where X^0 is reducible.

Lemma 6.2 *Suppose X^0 is reducible on V. Then $X^0 = (GL_{n/r})^r \cap G$ with $r|n$, and one of: $r \leq 3$, k arbitrary; or $r \geq 4, k = 1$. Conversely, if either of these holds, then X does have finitely many orbits on $P_k(V)$.*

For inductive purposes in the proof, it is convenient to deduce 6.2 from the following more general result.

Lemma 6.3 *Let $V = V_1 \oplus \ldots \oplus V_r$, with each V_i nonzero, and let $H = GL(V_1) \times \ldots \times GL(V_r) \leq GL(V)$. Then H has finitely many orbits on $P_k(V)\,(k \leq n/2)$ if and only if one of: $r \leq 3$, k arbitrary; or $r \geq 4, k = 1$.*

Proof Fix r with $r \geq 4$. We first establish by induction on $\dim V$ that for $1 < k < \dim V - 1$, H has infinitely many orbits on $P_k(V)$. We may assume that $k \leq \dim V/2$.

If all V_i are of dimension 1, then $\dim H/K^* = r-1$, and this is less than $\dim P_k(V)$ as $r \geq 4$, giving the conclusion in this case. Thus we may assume that $\dim V_1 \geq 2$.

Suppose $k > 2$. For $0 \neq v \in V_1$, define

$$\Omega_v = \{W \in P_k(V) : v \in W\}.$$

By induction, $H_{\langle v \rangle}$ has infinitely many orbits on $(k-1)$-spaces in $V/\langle v \rangle$, hence on k-spaces in Ω_v. Also, if $W, W' \in \Omega_v$ are in the same H-orbit, then they are in the same $H_{\langle v \rangle}$-orbit. It follows that H has infinitely many orbits on $P_k(V)$.

Thus we may assume that $k = 2$. Suppose some V_i has dimension 3 or more, say $\dim V_1 \geq 3$. By induction, given a 2-space $X \subset V_1$, H_X has infinitely many orbits on 2-spaces in $X \oplus V_2 \oplus \ldots \oplus V_r$. Therefore H has infinitely many orbits on 2-spaces in V which project to a 2-space of V_1, hence on $P_2(V)$.

We are now down to the case where $k = 2$ and $\dim V_i \leq 2$ for all i. For each i, let π_i be the projection $V \to V_i$, and let X_i be a 1-space in V_i. By the second paragraph, $H_{X_1,\ldots,X_r}$ has infinitely many orbits on the set

$$\{W \in P_2(V) : \pi_i(W) = X_i \text{ for all } i\}$$

Therefore H has infinitely many orbits on the set

$$\{W \in P_2(V) : \dim \pi_i(W) = 1 \text{ for all } i\},$$

hence on $P_2(V)$. This completes the proof of the assertion in the first paragraph.

To complete the proof of the lemma, we must show that if $r \leq 3$ or if $k = 1$, then H has finitely many orbits on $P_k(V)$. This is clear if $k = 1$.

Suppose $r = 2$. As above, by induction H has finitely many orbits on the set of k-spaces W such that $\pi_i(W) \neq V_i$ for some i. If there are any remaining k-spaces W, then $\dim W = \dim V_1 = \dim V_2$; the set of such k-spaces W (projecting onto both V_1 and V_2) form a single orbit of H.

Finally, suppose $r = 3$. This case appears to be more complicated than the others, and rather than argue directly, we apply the theory of quivers as follows. For background, see [Be], for example.

Suppose W and W' are subspaces of $V = V_1 \oplus V_2 \oplus V_3$. Then W gives rise to a representation of the D_4 quiver, by putting W on the central node of the D_4 diagram, and the V_i at the outer nodes, and letting the corresponding maps $W \to V_i$ be the projections.

If the representations of the D_4 quiver corresponding to W and W' are equivalent, then W and W' are in the same H-orbit. However, by a result of Gabriel (see [Be, 4.7.6]), any quiver corresponding to a Dynkin diagram has finite representation type, hence has only finitely many representations of any given dimension type, up to equivalence. Consequently H has finitely many orbits on $P_k(V)$ for each k, as required. □

By 6.2 we can now assume that X^0 is irreducible, and hence is given by Theorems 1 and 2. If X^0 is tensor-decomposable, then by the maximality of X we have $X^0 = SL_r \otimes SL_s$ (note that $X^0 = SL_2 \otimes SL_2 \otimes SL_2$ has normalizer contained in Sp_8). Hence conclusion (iv) of Theorem 3 holds.

It remains to handle the case where X^0 is simple and tensor-indecomposable. If $X = Cl_n$ then (ii) of Theorem 3 holds; so we may assume that the representation of X in SL_n is not self-dual. The possibilities are listed in Table D of Theorem 3.

7 Proof of Theorem 4

We begin with a well known result concerning finite linear groups. For completeness, we sketch a proof.

Lemma 7.1 *Let q be a prime power and let H be a subgroup of $GL_n(q)$. If $j \leq k \leq n/2$, then the number of orbits of H on $P_j(V_n(q))$ is less than or equal to the number of orbits of H on $P_k(V_n(q))$.*

Proof For $i \leq n/2$, let ϕ_i denote the permutation character of $GL_n(q)$ on the set of i-dimensional subspaces. Then, ϕ_i is a sum of i distinct irreducible characters, and $\phi_i = \phi_{i-1} + \chi_i$ where χ_i is irreducible. The number of orbits of H on i-dimensional subspaces is the inner product $(1_H^{GL_n(q)}, \phi_i) \geq (1_H^{GL_n(q)}, \phi_{i-1})$, giving the conclusion. □

We now prove Theorem 4. As usual, K denotes an algebraically closed field of characteristic $p \geq 0$, and V is a finite-dimensional vector space over K. When $p > 0$, q denotes a power of p, and σ_q a Frobenius q-power morphism of $GL(V)$.

The strategy of proof is first to show using model theory that the field K can be assumed to be locally finite; then we use Lang's Theorem [SS, I, 2.7] to reduce to finite groups, in which case we can apply Lemma 7.1.

Regard K as a first order structure in the usual language $(+, ., -, 0, 1)$ for rings. We shall use the following well-known model-theoretic facts. (In fact (4) and (5) hold without the algebraic closure assumption.)

(1) (Corollary A.5.2 of [Ho].) Any two algebraically closed fields of the same characteristic satisfy the same L-sentences.

(2) If an L-sentence τ is true of an algebraically closed field of characteristic 0, then for all but finitely many primes p, it is true of all algebraically closed fields of characteristic p. (This follows easily from (1) by compactness or an ultraproduct argument.)

(3) (2.10 (i) of [DS].) There is a natural number $B = B(n, d)$ (not depending on K) with the following property. Let $f_1, \ldots, f_m \in K[\bar{X}] = K[X_1, \ldots, X_n]$, each of total degree at most d, and let I be the ideal in $K[\bar{X}]$ generated by the f_i. Then I is prime provided, whenever $f, g \in K[\bar{X}]$ have total degree at most B and $fg \in I$, we have $f \in I$ or $g \in I$.

(4) ((I) of [DS].) There is a natural number $C = C(n, d)$ (not depending on the field K) with the following property. Let $f_1, \ldots, f_m, I$ be as in in (4),and $f \in I$. Then there are $g_1, \ldots, g_m \in K[\bar{X}]$ each of total degree at most C such that $f = \sum_{i=1}^m g_i f_i$.

¿From (3) and (4) we easily obtain

(5) Let $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_m)$. Also let $\phi(\bar{x}, \bar{y})$ be an L-formula which is a finite conjunction of formulas each expressing (without quantifiers) that

a certain polynomial with variables from $\bar{x}\bar{y}$ vanishes. Then there is a first order formula $\psi(\bar{y})$ such that for any algebraically closed field F and $\bar{a} \in F^m$, the variety $\{\bar{x} \in F^n : F \models \phi(\bar{x}, \bar{a})\}$ is irreducible if and only if $F \models \psi(\bar{a})$.

Suppose now that G is a counterexample to Theorem 4; thus G is a closed subgroup of $GL(V)$, and G has finitely many orbits on $P_k(V)$ and infinitely many orbits on $P_j(V)$, where $j < k \leq \dim V/2$. Let $\Omega_1, \ldots, \Omega_d$ be the G-orbits on $P_k(V)$, and for each i pick $\omega_i \in \Omega_i$ and define $G_i = G_{\omega_i}$, $e_i = |G_i : G_i^0|$ and $e = e_1 + \ldots + e_d$. We regard $GL(V)$ as a subset of K^n for some n. The domain of $GL(V)$, its group operation, and its action on V are also L-definable (by a single formula which works in all fields). Let $\bar{a}$ be a finite sequence from K over which G, G_i, G_i^0 are defined. There are quantifier free formulas $\phi(\bar{x}, \bar{a})$, $\psi_i(\bar{x}, \bar{a})$ and $\chi_i(\bar{x}, \bar{a})$ whose solution sets in K^n are respectively G, G_i, and G_i^0 (for $i = 1, \ldots, d$). There is also a sentence $\rho(\bar{a})$ which expresses that G is a subgroup of $GL(V)$, that the ω_i form a set of orbit representatives for the action of G on $P_k(V)$, with stabilizers G_i, that $|G_i : G_i^0| = e_i$, that each G_i^0 is connected (this uses (5) above - we express this connectedness in a way which works in all algebraically closed fields) and also that G has at least $e+1$ orbits on $P_j(V)$. The formula $\rho(\bar{y})$ is chosen so that in any algebraically closed field F and for any $\bar{b}$ from F, it expresses that the corresponding group defined over $\bar{b}$ has the corresponding properties.

Suppose first $p > 0$, and let F be the algebraic closure of $\mathbb{F}_p$. Since $K \models \exists\bar{y}\rho(\bar{y})$, by (1) also $F \models \exists\bar{y}\rho(\bar{y})$ so there is $\bar{b}$ from F with $F \models \rho(\bar{b})$. Hence, we may suppose that $K = F$ and $\bar{b} = \bar{a}$. Choose a power q of p large enough so that $\bar{a}$ is fixed by σ_q and at least $e+1$ orbits of $G(F)$ on $P_j(V(F))$ have representatives in $P_j(V(q))$. Then $G(q)$ has at least $e+1$ orbits on $P_j(V(q))$, but, by Lang's Theorem, has at most e orbits on $P_k(V(q))$, contradicting Lemma 7.1.

Finally, suppose that $p = 0$. Since $K \models \exists\bar{y}\rho(\bar{y})$, there is a prime r such that the algebraically closed field F of characteristic r and transcendence degree 0 satisfies $\exists\bar{y}\rho(\bar{y})$ (in fact, any sufficiently large prime will do). Now argue as in the last paragraph.

This completes the proof of Theorem 4.

Remark. The methods used here, combined with the fact that in any algebraically closed field, any formula is equivalent to a quantifier-free formula (Corollary A.5.2 of [Ho]), also give an alternative short proof of Proposition 1.1.

References

[As] M. Aschbacher, "Chevalley groups of type G_2 as the group of a trilinear form", *J. Algebra* **109** (1987), 193–259.

[AS] M. Aschbacher and G.M. Seitz, "Involutions in Chevalley groups over fields of even order", *Nagoya Math. J.* **63** (1976), 1–91.

[ABS] H. Azad, M. Barry and G.M. Seitz, "On the structure of parabolic subgroups", *Comm. in Alg.* **18** (1990), 551-562.

[**Be**] D. Benson, *Representations and cohomology I: basic representation theory of finite groups and associative algebras*, Cambridge University Press, 1991.

[**Bo**] N. Bourbaki, *Groupes et algebres de Lie*, Chapters 4-6, Hermann, Paris, 1968.

[**Br1**] J. Brundan, "Double coset density in exceptional algebraic groups", *J. London Math. Soc.*, to appear.

[**Br2**] J. Brundan, "Double cosets in algebraic groups", Ph.D Thesis, Imperial College, London, 1996.

[**Ch1**] Z. Chen, "A classification of irreducible prehomogeneous vector spaces over an algebraically closed field of characteristic 2, I", *Acta Math. Sinica* **2** (1986), 168-177.

[**Ch2**] Z. Chen, "A classification of irreducible prehomogeneous vector spaces over an algebraically closed field of characteristic p, II", *Chinese Ann. Math. Ser. A* **9** (1988), 10-22.

[**Ch3**] Z. Chen, "A prehomogeneous vector space of characteristic 3", in *Group theory, Beijing 1984* (ed. H.F. Tuan), Lecture Notes in Mathematics 1185, Springer-Verlag, 1986, pp.266-276.

[**CPS**] E. Cline, B. Parshall and L. Scott, "Cohomology of finite groups of Lie type, I", *I.H.E.S. Publ. Math.* **45** (1975), 169-191.

[**CC**] A.M. Cohen and B.N. Cooperstein, "The 2-spaces of the standard $E_6(q)$-module", *Geom. Ded.* **25** (1988), 467-480.

[**CW**] A.M. Cohen and D. Wales, "$GL(4)$-orbits on a 16-dimensional module in characteristic 3", *J. Algebra*, to appear.

[**DS**] L. van den Dries and K. Schmidt, "Bounds in the theory of polynomial rings over fields. A non-standard approach", *Invent. Math.* **76** (1984), 77-91.

[**Ga**] P. Gabriel, "Degenerate bilinear forms", *J. Algebra* **31** (1974), 67-72.

[**Ho**] W.Hodges, *Model theory*, Cambridge University Press, Cambridge, 1993.

[**Ig**] J. Igusa, "A classification of spinors up to dimension twelve", *Amer. J. Math.* **92** (1970), 997-1028.

[**Ja**] N. Jacobson, *The theory of rings*, Amer. Math. Soc., New York, 1943.

[**Ka**] V. Kac, "Some remarks on nilpotent orbits", *J. Algebra* **64** (1980), 190-213.

[**La**] S. Lang, "Algebraic groups over finite fields", *Amer. J. Math.* **78** (1956), 555-563.

[**Li1**] M.W. Liebeck, "On the orders of maximal subgroups of the finite classical groups", *Proc. London Math. Soc.* **50** (1985), 426-446.

[**Li2**] M.W. Liebeck, "The affine permutation groups of rank 3", *Proc. London Math. Soc.* **54** (1987), 477-516.

[**LS1**] M.W. Liebeck and G.M. Seitz, "Subgroups generated by root elements in groups of Lie type", *Annals of Math.* **139** (1994), 293-361.

[**LS2**] M.W. Liebeck and G.M. Seitz, "Reductive subgroups of exceptional algebraic groups", *Mem. Amer. Math. Soc.* **121**, No. 580 (1996), 1-111.

[**LS3**] M.W. Liebeck and G.M. Seitz, "On the subgroup structure of exceptional groups of Lie type", to appear.

[**LSS**] M.W. Liebeck, J. Saxl and G.M. Seitz, "Factorizations of simple algebraic groups", *Trans. Amer. Math. Soc.* **348** (1996), 799-822.

[**Ri**] R. Richardson, "Finiteness theorems for orbits of algebraic groups", *Indag. Math.* **88** (1985), 337-344.

[**RS**] C. Riehm and M.A. Shrader-Frechette, "The equivalence of sesquilinear forms", *J. Algebra* **42** (1976), 495-530.

[**SK**] M. Sato and T. Kimura, " A classification of irreducible prehomogeneous vector spaces and their relative invariants", *Nagoya Math. J.* **65** (1977), 1-155.

[**SS**] T.A. Springer and R. Steinberg, "Conjugacy classes", in: *Seminar on algebraic groups and related topics* (ed. A. Borel et al.), Lecture Notes in Math. 131, Springer, Berlin, 1970, pp. 168-266.

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