Stability and Competitive Equilibrium in Trading Networks *

John William Hatfield† Scott Duke Kominers‡ Alexandru Nichifor§
Michael Ostrovsky¶ Alexander Westkamp‖

June 28, 2011

Abstract

We introduce a model in which agents in a network can trade via bilateral contracts. We find that when continuous transfers are allowed and utilities are quasilinear, the full substitutability of preferences is sufficient to guarantee the existence of stable outcomes for any underlying network structure. Furthermore, the set of stable outcomes is essentially equivalent to the set of competitive equilibria, and all stable outcomes are in the core and are efficient. In contrast, for any domain of preferences strictly larger than that of full substitutability, the existence of stable outcomes and competitive equilibria cannot be guaranteed.

JEL classification: C78; C62; C71; D85; L14

Keywords: Matching; Networks; Stability; Competitive Equilibrium; Core; Efficiency; Submodularity; Lattice; Substitutes

*We thank Peter Cramton, Çağatay Kayı, Paul Milgrom, Al Roth, and Glen Weyl for helpful comments and suggestions. Hatfield and Nichifor appreciate the hospitality of Harvard Business School, which hosted them during parts of this research. Kominers thanks the National Science Foundation and the Yahoo! Key Scientific Challenges Program for support. Nichifor thanks the Netherlands Organisation for Scientific Research (NWO) for support under grant VIDI-452-06-013. Westkamp thanks the German Research Foundation (DFG) for support.

†Graduate School of Business, Stanford University; hatfield@stanford.edu.

‡Department of Economics, Harvard University, and Harvard Business School; skominers@hbs.edu.

§Department of Economics, Maastricht University; a.nichifor@maastrichtuniversity.nl.

¶Graduate School of Business, Stanford University; ostrovsky@gsb.stanford.edu.

‖Economics Department, University of Bonn; awest@uni-bonn.de.

1 Introduction

The analysis of markets with heterogeneous agents and personalized prices has a long tradition in economics, which began with the canonical one-to-one assignment model of Koopmans and Beckmann (1957), Gale (1960), and Shapley and Shubik (1971). In this model, agents on one side of the market are matched to objects (or agents) on the other side, and each "match" generates a pair-specific surplus. Agents' utilities are quasilinear in money, and arbitrary monetary transfers between the two sides are allowed. In this case, the efficient assignment—the one that maximizes the sum of all involved parties' payoffs—can be supported by the price mechanism as a competitive equilibrium outcome. Moreover, several solution concepts (competitive equilibrium, core, and pairwise stability) essentially coincide.

Crawford and Knoer (1981) extend the assignment model to a richer setting, in which heterogeneous firms form matches with heterogeneous workers. One firm can be matched to multiple workers, but each worker can be matched to at most one firm. Crawford and Knoer (1981) assume that preferences are separable across pairs, i.e., the payoff from a particular firm–worker pair is independent of the other matches the firm forms. Crawford and Knoer (1981) do not rely on the linear programming duality theory used in the previous work; instead, they use a modification of the deferred-acceptance algorithm of Gale and Shapley (1962) to prove their results, thus demonstrating a close link between the notions of pairwise stability and competitive equilibrium. Kelso and Crawford (1982) then extend the previous results, showing that the restrictive assumption of the separability of preferences across pairs is inessential: it is enough that firms view workers as *substitutes* for each other.

In this paper, we show that the results from the two-sided models described above continue to hold in a much richer environment, in which a network of heterogeneous agents can trade indivisible goods or services via bilateral contracts. Some agents can be involved in production, buying inputs from other agents, turning them into outputs at some cost, and then selling the outputs. We find that if all agents' preferences satisfy a suitably generalized notion of substitutability, then stable outcomes and competitive equilibria are guaranteed to exist and are efficient. Moreover, in that case, the sets of competitive equilibria and stable outcomes are in a sense equivalent. These results apply to general trading networks and do not require any assumptions on the network structure, such as two-sidedness or acyclicity. The presence of continuously transferable utility is essential for our results: Without transfers, the existence of stable outcomes can only be guaranteed if the underlying trading network does not contain cycles (Ostrovsky, 2008; Hatfield and Kominers, 2010b), and even in acyclic networks, stable outcomes are not guaranteed to be Pareto efficient (Blair, 1988; Westkamp, 2010). Moreover, the substitutability of preferences is also essential: our last main result is a "maximal domain" theorem showing that if any agent's preferences are not substitutable, then substitutable preferences can be found for other agents such that neither competitive equilibria nor stable outcomes exist.

In our model, contracts specify a buyer, a seller, provision of a good or service, and a monetary transfer. An agent may be involved in some contracts as a seller, and in other contracts as a buyer. Agents' preferences are defined by cardinal utility functions over sets of contracts and are quasilinear with respect to the numeraire. To incorporate feasibility constraints, we allow agents' utilities for certain production plans to be unboundedly negative. We say that preferences are *fully substitutable* if contracts are substitutes for each other in a generalized sense, i.e., whenever an agent gains a new purchase opportunity, he becomes both less willing to make other purchases and more willing to make sales, and whenever he gains a new sales opportunity, he becomes both less willing to make other sales and more willing to make purchases. We formally show that this intuitive notion of substitutability, which has appeared in the literature on matching in vertical networks (Ostrovsky, 2008; Hatfield and Kominers, 2010b; Westkamp, 2010), is equivalent to the *gross substitutes and complements* condition of the literature on competitive equilibrium in exchange economies with indivisible objects (Sun and Yang, 2006, 2009). Full substitutability is also equivalent to the submodularity of the indirect utility function (Gul and Stacchetti, 1999; Ausubel and Milgrom, 2002).[1]

Our main results are as follows. We first show that when preferences are fully substitutable, competitive equilibria are guaranteed to exist. Our proof is constructive. Its key idea is to consider an associated two-sided many-to-one matching market, in which "firms" are the agents and "workers" are the possible trades in the original economy. Fully substitutable utilities of the agents in the original economy give rise to substitutable (in the usual Kelso–Crawford sense) preferences of the firms in the associated two-sided market, and the equilibrium outcome in the associated market can be mapped back to a competitive equilibrium of the original economy. While the construction of the associated market is conceptually natural, it involves several additional steps that deal with the potentially unbounded utilities in the original economy and ensure that the equilibrium in the associated economy is "full employment" (which is required to be able to transform it back into an equilibrium of the original economy). Having established the existence of competitive equilibria, we then use standard techniques to demonstrate some of their properties: analogues of the first and second welfare theorems, as well as the lattice structure of the set of competitive equilibrium prices. While these properties are of independent interest, we also use them to prove some

[1] For definitions of the classical notions of substitutability in two-sided settings with and without transfers, see Kelso and Crawford (1982), Roth (1984), Gul and Stacchetti (1999), and Hatfield and Milgrom (2005).

of our subsequent results.

We then turn to our key results establishing the connections between competitive equilibra and stable outcomes. First, we show that even without full substitutability, any competitive equilibrium induces a stable outcome. The proof of this result is similar in spirit to the standard proofs that show that competitive equilibrium outcomes are in the core, but it is more subtle: unlike the core, stability also rules out the possibility that agents may profitably recontract while maintaining some of their prior contractual relationships. Second, we prove the converse: under fully substitutable preferences, any stable outcome corresponds to a competitive equilibrium. These two results establish an essential equivalence between the two solution concepts under full substitutability. While this equivalence is analogous to a similar finding of Kelso and Crawford (1982) for two-sided many-to-one matching markets, it requires a more complex proof. In the setting of Kelso and Crawford (1982), one can construct "missing" prices for unrealized trades simply by considering those trades one by one, because in that setting each worker can be employed by at most one firm. In our setting, that simple procedure would not work, because each agent can be involved in multiple trades. Instead, for a given stable outcome, we consider a new economy consisting of trades that are *not* involved in the stable outcome and modified utilities that assume that the agents have access to trades that *are* involved; show that preferences in this modified economy satisfy full substitutability; use our earlier results to establish the existence of a competitive equilibrium in this modified economy; and then use the prices for all trades in this competitive equilibrium of the modified economy to construct a competitive equilibrium in the original economy.

Thus, fully substitutable preferences are sufficient for the existence of stable outcomes and competitive equilibria and the essential equivalence of these two concepts. Our final main result establishes that full substitutability is also, in a sense, necessary: if any agent's preferences are not fully substitutable, then fully substitutable preferences can be found for other agents such that no stable outcome exists.[2]

[2]In the setting of two-sided many-to-one matching with transfers, Kelso and Crawford (1982) show that substitutability is sufficient for the existence of stable outcomes and competitive equilibria; Gul and Stacchetti (1999) and Hatfield and Kojima (2008) prove corresponding necessity results.

In a setting in which two types of indivisible objects need to be allocated to consumers, Sun and Yang (2006) show that competitive equilibria are guaranteed to exist if consumers view objects of the same type as substitutes and view objects of different types as complements (see also Section 5.3).

Sufficiency and necessity of fully substitutable preferences also obtains in settings of many-to-many matching with and without contracts (Roth (1984), Echenique and Oviedo (2006), Klaus and Walzl (2009), and Hatfield and Kominers (2010a) prove sufficiency results; Hatfield and Kojima (2008) and Hatfield and Kominers (2010a) prove necessity results) and in matching in vertical networks (Ostrovsky (2008) and Hatfield and Kominers (2010b) prove sufficiency; Hatfield and Kominers (2010b) prove necessity). Substitutable preferences are sufficient for the existence of a stable outcome in the setting of many-to-one matching with contracts (Hatfield and Milgrom, 2005), but are not necessary (Hatfield and Kojima, 2008, 2010).

We then analyze the relationship between stability as defined in this paper and several other solution concepts. Generalizing the results of Shapley and Shubik (1971) and Sotomayor (2007), we show that all stable outcomes are in the core (although, unlike in the basic one-to-one assignment model, the converse is not true here). We then consider the notion of strong group stability and show that in our setting, in contrast to the results of Echenique and Oviedo (2006) and Klaus and Walzl (2009) for matching settings without transfers, the set of stable outcomes is in fact equal to the set of strongly group stable outcomes. We also consider chain stability, extending the definition of Ostrovsky (2008). While chain stability is logically weaker than stability, we show that the two concepts are equivalent when agents' preferences are fully substitutable, by presenting an algorithm that shows how to find a chain block for any unstable outcome.[3]

Finally, we show that our model embeds the more common setting in which agents are indifferent over their trading partners. We introduce a condition on utilities formalizing this notion, and show that under this condition, a competitive equilibrium with "anonymous" rather than personalized prices always exists. Our framework also allows for a hybrid case, in which prices are personalized for some goods and are anonymous for others.

The remainder of this paper is organized as follows. In Section 2, we formalize our model. In Section 3, we discuss the notion of full substitutability in detail and prove a result on the equivalence of various alternative definitions. In Section 4, we present our main results. In Section 5, we consider the relationships between competitive equilibria, stable outcomes, and other solution concepts. We conclude in Section 6.

2 Model

There is a finite set I of *agents* in the economy. These agents can participate in bilateral *trades*. Each trade ω is associated with a *seller* $s(\omega) \in I$ and a *buyer* $b(\omega) \in I$, $b(\omega) \neq s(\omega)$. The set of possible trades, denoted Ω, is finite and exogenously given. The set Ω may contain multiple trades that have the same buyer and the same seller. For instance, a worker (seller) may be hired by a firm (buyer) in a variety of different capacities with a variety of job conditions and characteristics, and each possible type of job may be represented by a different trade. One firm may sell multiple units of a good (or several different goods) to another firm, with each unit represented by a separate trade. We also allow Ω to contain trades ω_1 and ω_2 such that $s(\omega_1) = b(\omega_2)$ and $s(\omega_2) = b(\omega_1)$.

It is convenient to think of a trade as representing the nonpecuniary aspects of a transaction between a seller and a buyer (although in principle it could include some "financial"

[3]Hatfield and Kominers (2010b) prove an analogous result for the setting of Ostrovsky (2008).

terms and conditions as well). The purely financial aspect of a trade ω is represented by a price p_ω; the complete vector of prices for all trades in the economy is denoted by $p \in \mathbb{R}^{|\Omega|}$. Formally, a *contract* x is a pair (ω, p_ω), with $\omega \in \Omega$ denoting the trade and $p_\omega \in \mathbb{R}$ denoting the price at which the trade occurs. The set of available contracts is $X \equiv \Omega \times \mathbb{R}$. For any set of contracts Y, we denote by $\tau(Y)$ the set of trades involved in contracts in Y: $\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\}$.

For a contract $x = (\omega, p_\omega)$, we will denote by $s(x) \equiv s(\omega)$ and $b(x) \equiv b(\omega)$ the seller and the buyer associated with the trade ω of contract x. Consider any set of contracts $Y \subseteq X$. We denote by $Y_{\to i}$ the set of "upstream" contracts for i in Y, that is, the set of contracts in Y in which agent i is the buyer: $Y_{\to i} \equiv \{y \in Y : i = b(y)\}$. Similarly, we denote by $Y_{i\to}$ the set of "downstream" contracts for i in Y, that is, the set of contracts in Y in which agent i is the seller: $Y_{i\to} \equiv \{y \in Y : i = s(y)\}$. We denote by Y_i the set of contracts in Y in which agent i is involved as the buyer or the seller: $Y_i \equiv Y_{\to i} \cup Y_{i\to}$. We let $a(Y) \equiv \bigcup_{y\in Y}\{b(y), s(y)\}$ denote the set of agents involved in contracts in Y as buyers or sellers. We use analogous notation to denote the subsets of trades associated with some agent i for sets of trades $\Psi \subseteq \Omega$.

We say that the set of contracts Y is *feasible* if there is no trade ω and prices $p_\omega \neq \hat{p}_\omega$ such that both contracts (ω, p_ω) and $(\omega, \hat{p}_\omega)$ are in Y; i.e., a set of contracts is feasible if each trade is associated with at most one contract in that set. An *outcome* $A \subseteq X$ is a feasible set of contracts.[4] Thus, an outcome specifies which trades get formed and what the associated prices are, but does not specify prices for trades that do not take place. An *arrangement* is a pair $[\Psi; p]$, where $\Psi \subseteq \Omega$ is a set of trades and $p \in \mathbb{R}^{|\Omega|}$ is a vector of prices for all trades in the economy. For any arrangement $[\Psi; p]$ we denote by $\kappa([\Psi; p]) \equiv \cup_{\psi\in\Psi}\{(\psi, p_\psi)\}$, the set of contracts induced by the arrangement. Note that $\kappa([\Psi; p])$ is an outcome, and that $\tau(\kappa([\Psi; p])) = \Psi$.

2.1 Preferences

Each agent i has a valuation function u_i over sets of trades $\Psi \subseteq \Omega_i$; we extend u_i to Ω as follows: $u_i(\Psi) \equiv u_i(\Psi_i)$ for any $\Psi \subseteq \Omega$. The valuation u_i gives rise to a quasilinear utility function U_i over sets of trades and the associated transfers. We formalize this in two different ways. First, for any feasible set of contracts Y, we say that

$$U_i(Y) \equiv u_i(\tau(Y)) + \sum_{(\omega,p_\omega)\in Y_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in Y_{\to i}} p_\omega.$$

[4] In the literature on matching with contracts, the term "allocation" has been used to refer to a set of contracts. Unfortunately, the term "allocation" is also used in the competitive equilibrium literature to denote an assignment of goods, without specifying transfers. For this reason, to avoid confusion, we use the term "outcome" to refer to a feasible set of contracts.

Second, for any arrangement $[\Psi; p]$, we say that

$$U_i([\Psi; p]) \equiv u_i(\Psi) + \sum_{\psi \in \Psi_{i\to}} p_\psi - \sum_{\psi \in \Psi_{\to i}} p_\psi.$$

Note that, by construction, $U_i([\Psi; p]) = U_i(\kappa([\Psi; p]))$.

We allow $u_i(\Psi)$ to take the value $-\infty$ for some sets of trades Ψ in order to incorporate various technological constraints. However, we also assume that for all i, the outside option is finite: $u_i(\varnothing) \in \mathbb{R}$. That is, no agent is "forced" to sign any contracts at extremely unfavorable prices: he always has an outside option of completely withdrawing from the market at some potentially high but finite price.

Many of our results also depend critically on the assumption that preferences are fully substitutable. We present and discuss this assumption in Section 3.

The utility function U_i gives rise to both demand and choice correspondences. The *choice correspondence* of agent i from the set of contracts $Y \subseteq X$ is defined as the collection of the sets of contracts maximizing agent i's utility:

$$C_i(Y) \equiv \underset{Z \subseteq Y_i;\ Z \text{ is feasible}}{\operatorname{argmax}} U_i(Z).$$

The *demand correspondence* of agent i given a price vector $p \in \mathbb{R}^{|\Omega|}$ is defined as the collection of the sets of trades maximizing agent i's utility under prices p:

$$D_i(p) \equiv \underset{\Psi \subseteq \Omega_i}{\operatorname{argmax}} U_i([\Psi; p]).$$

Note that while the demand correspondence always contains at least one (possibly empty) set of trades, the choice correspondence may be empty-valued (e.g., if Y consists of all contracts with prices strictly between 0 and 1). If the set Y is finite, then the choice correspondence is also guaranteed to contain at least one set of contracts.

2.2 Stability and Competitive Equilibrium

The main solution concepts that we study are *stability* and *competitive equilibrium.* Both concepts specify which trades are formed and what the associated transfers are. Competitive equilibria also specify prices for trades that are not formed.

Definition 1. *An outcome A is* stable *if it is*

1. Individually rational: $A_i \in C_i(A)$ *for all* i*;*
2. Unblocked: *There is no feasible nonempty* blocking set $Z \subseteq X$ *such that*
   (a) $Z \cap A = \varnothing$*, and*

(b) for all $i \in a(Z)$, for all of i's choices $Y \in C_i(Z \cup A)$, we have $Z_i \subseteq Y$.

Individual rationality requires that no agent can become strictly better off by dropping some of the contracts that he is involved in. This is a standard requirement in the matching literature. The second condition states that when presented with a stable outcome A, one cannot propose a new set of contracts such that all the agents involved in these new contracts would strictly prefer to form all of them (and possibly drop some of their existing contracts in A) instead of forming only some of them (or none). This requirement is a natural adaptation of the stability condition of Hatfield and Kominers (2010b) to the current setting. We discuss the relationship between our notion of stability and several other stability notions considered in the matching literature, such as the core and chain stability, in Sections 5.1 and 5.2.

Our second solution concept is competitive equilibrium.

Definition 2. *An arrangement $[\Psi; p]$ is a* competitive equilibrium *if for all $i \in I$,*

$$\Psi_i \in D_i(p).$$

This is the standard notion of competitive equilibrium, adapted to the current setting: market-clearing is "built in," because each trade in Ψ carries with it the corresponding buyer and seller, and the condition is simply that each agent is (weakly) optimizing given market prices. Note that this condition implicitly allows for "personalized" prices: e.g., identical goods may be sold by a seller to two different buyers at two different prices. In many settings, sellers may not care who they sell their goods to, and buyers may not care who they buy from (and care only about the characteristics of a good), and thus it is natural to talk about "anonymous" good-specific prices rather than personalized ones. Indeed, this is how the classical models of competitive equilibrium are usually set up and interpreted. In Section 5.3 we show how to formalize these notions in our framework and discuss the relationship between competitive equilibria with personalized and anonymous prices.

3 Full Substitutability

Kelso and Crawford (1982) introduced the gross substitutes condition (GS) in the context of a two-sided many-to-one matching market between firms and workers. The (GS) condition requires that an increase in the salary of some worker cannot cause a firm to drop other initially employed workers whose salaries did not change. This condition explicitly deals with indifferences, by placing restrictions on the behavior of the multi-valued demand correspondence when workers' salaries change. Kelso and Crawford use (GS) to establish the existence of a core outcome for many-to-one matching models in which prices can be either discrete or continuous. Substitutability conditions similar to (GS) are also sufficient for the existence of stable outcomes in several other two-sided settings (Roth and Sotomayor (1990) describe these results).

Ausubel and Milgrom (2002) offered a convenient alternative definition of (GS) for a setting with continuous prices, in which demand is not guaranteed to be single-valued: goods are (gross) substitutes if the demand for each one is nondecreasing in the prices of others when attention is restricted only to the vectors of prices at which demand is single-valued. Additionally, Ausubel and Milgrom (2002) showed that (GS) is equivalent to submodularity of the indirect utility function.

In a vertical network setting with discrete contract sets, Ostrovsky (2008) introduced a combination of two related assumptions: same-side substitutability (SSS) and cross-side complementarity (CSC). These assumptions impose two constraints: First, when an agent's opportunity set on one side of the market expands, that agent does not choose any options previously rejected from that side of the market. Second, when an agent's opportunity set on one side of the market expands, that agent does not reject any options previously chosen from the other side of the market. Hatfield and Kominers (2010b) show that (SSS) and (CSC) are together equivalent to the assumption of quasisubmodularity of the indirect utility function—an adaptation of submodularity to the discrete setting.

Independently, Sun and Yang (2006) introduced the gross substitutability and complementarity (GSC) condition for a setting with continuous transfers in which indivisible objects are allocated to consumers. This condition requires that objects can be divided into two groups such that objects in the same group are substitutes and objects in different groups are complements. The Sun and Yang (2006) definition explicitly deals with indifferences and multi-valued demand functions. Sun and Yang (2009) show that (GSC) is equivalent to the condition that the indirect utility function is submodular.

In this section, we introduce the notion of *full substitutability*[5] for the current setting: When presented with additional contractual options to purchase, an agent both rejects any previously rejected purchase options and continues to choose any previously chosen sale options. Analogously, when presented with additional contractual options to sell, an agent both rejects any previously rejected sale options and continues to choose any previously chosen purchase options.

We introduce several alternative definitions, adapting the ones mentioned above to our setting. For convenience, in this section, we use the approach of Ausubel and Milgrom (2002) and restrict attention to sets and vectors of prices for which choices and demands are

[5] Since preferences are quasilinear in our settting, there is no distinction between gross and net substitutes. Therefore, we drop the "gross" specification.

single-valued. In Appendix A, we introduce additional definitions that explicitly deal with indifferences and multi-valued correspondences, and prove that those definitions, as well as the ones in this section, are all equivalent. We also show that in quasilinear settings, full substitutability implies the Laws of Aggregate Supply and Demand (Hatfield and Kominers, 2010b), extending an analogous result for the (GS) condition (Hatfield and Milgrom, 2005). While these results are of independent interest, we also use them in the proofs of our subsequent results.

3.1 Definitions of Full Substitutability

First, we define full substitutability in the language of sets and choices, adapting and merging (SSS) and (CSC).

Definition 3. *Agent i's preferences are* choice-language fully substitutable *(CFS) if:*

1. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{\to i} - Y^*_{\to i}) \subseteq (Z_{\to i} - Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;*
2. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{i\to} - Y^*_{i\to}) \subseteq (Z_{i\to} - Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.*

In other words, the choice correspondence C_i is fully substitutable if, when attention is restricted to sets for which C_i is single-valued, when the set of options available to i on one side expands, i rejects a larger set of contracts on that side (SSS), and selects a larger set of contracts on the other side (CSC).

Our second definition uses the language of prices and demands, and goes back to the gross substitutes and complements condition (GSC).

Definition 4. *Agent i's preferences are* demand-language fully substitutable *(DFS) if:*

1. *for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;*
2. *for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$, and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.*

The demand correspondence D_i is fully substitutable if, when attention is restricted to prices for which demands are single-valued, a decrease in the price of some inputs for agent

i leads to the decrease in his demand for other inputs and to an increase in his supply of outputs, and an increase in the price of some outputs leads to the decrease in his supply of other outputs and an increase in his demand for inputs.

Our third definition is essentially a reformulation of Definition 4, using a convenient vector notation due to Hatfield and Kominers (2010b). For each agent i, for any set of trades $\Psi \subseteq \Omega_i$, define the *(generalized) indicator function* $e(\Psi) \in \{-1, 0, 1\}^{|\Omega_i|}$ to be the vector with component $e_\omega(\Psi) = 1$ for each upstream trade $\omega \in \Psi_{\to i}$, $e_\omega(\Psi) = -1$ for each downstream trade $\omega \in \Psi_{i\to}$, and $e_\omega(\Psi) = 0$ for each trade $\omega \notin \Psi$. The interpretation of $e(\Psi)$ is that an agent buys a strictly positive amount of a good if he is the buyer in a trade in Ψ, and "buys" a strictly negative amount if he is the seller of such a trade.

Definition 5. *Agent i's preferences are* indicator-language fully substitutable *(IFS) if for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $e_\omega(\Psi) \leq e_\omega(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.*

This definition clarifies the reason for the term "full substitutability": an agent is more willing to "demand" a trade (i.e., keep an object that he could potentially sell, or buy an object that he does not initially own) if prices of other trades increase.

Our final definition concerns neither the choice correspondences nor the demand correspondences, but agents' indirect utility functions. It is a generalization of the earlier submodularity assumption on the indirect utility of an agent (Ausubel and Milgrom, 2002). For price vectors $p, p' \in \mathbb{R}^{|\Omega|}$, let the *join* of p and p', denoted $p \vee p'$, be defined as the pointwise maximum of p and p'; let the *meet* of p and p', denoted $p \wedge p'$, be defined as the pointwise minimum of p and p'.

Definition 6. *The* indirect utility function *of agent i,*

$$V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} U_i([\Psi; p]),$$

is submodular *if, for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$, $V_i(p \wedge p') + V_i(p \vee p') \leq V_i(p) + V_i(p')$.*

3.2 Equivalence of the Definitions

The main result of this section shows that the three definitions of full substitutability above are equivalent, and are also equivalent to the assumption of submodularity of the indirect utility function.[6] Subsequently, we will typically use the term *full substitutability* in place of the notations (CFS), (DFS), and (IFS).

[6] One can also give an equivalent definition of full substitutability in terms of $M^\natural$-concavity, analogous to the equivalent definitions of (GS) of Reijnierse et al. (2002) and Fujishige and Yang (2003).

Theorem 1. *(CFS), (DFS), and (IFS) are all equivalent, and hold if and only if the indirect utility function is submodular.*

4 Main Results

We now present our three main contributions. First, we show that when preferences are fully substitutable, competitive equilibria are guaranteed to exist and satisfy a number of interesting properties, similar to those in two-sided settings. We then show that under full substitutability, the set of competitive equilibria essentially coincides with the set of stable outcomes. Finally, we show that if preferences are not fully substitutable, stable outcomes and competitive equilibria need not exist. The proofs of all results in this and the subsequent sections are presented in Appendix B.

4.1 The Existence and Properties of Competitive Equilibria

Theorem 2. *Suppose agents' preferences are fully substitutable. Then there exists a competitive equilibrium.*

The main idea in the proof of Theorem 2 is to associate to the original market a two-sided many-to-one matching market with transfers, in which each agent corresponds to a "firm" and each trade corresponds to a "worker." The valuation of firm i for hiring a set of workers $\Psi \subseteq \Omega_i$ in the associated two-sided market is given by

$$v_i(\Psi) \equiv u_i\left(\Psi_{\to i} \cup (\Omega - \Psi)_{i\to}\right). \tag{1}$$

Intuitively, we think of the firm as employing all of the workers associated with trades that the firm buys and with trades that the firm *does not* sell. We show that v_i satisfies the gross substitutes condition (GS) of Kelso and Crawford (1982) as long as u_i is fully substitutable.[7] Workers strongly prefer to work rather than being unemployed, and their utilities are monotonically increasing in wages. Also, every worker ω has a strong preference for being employed by $s(\omega)$ and $b(\omega)$ rather than some other firm $i \in I - \{s(\omega), b(\omega)\}$. With these definitions, we have constructed a two-sided market of the type studied by Kelso and Crawford (1982). In this market, a competitive equilibrium is guaranteed to exist, and in every equilibrium, every worker ω is matched to $s(\omega)$ or $b(\omega)$.

We then transform this competitive equilibrium back into a set of trades and prices for the original economy as follows: Trade ω is included in the set of executed trades in the

[7]This construction is analogous to the one used in Sun and Yang (2006) to transform an exchange economy with two types of goods, which are substitutable within each type and complementary across types, into an economy in which preferences satisfy the (GS) condition of Kelso and Crawford (1982).

original economy if the worker ω was hired by $b(\omega)$ in the associated market and is not included if ω was hired by $s(\omega)$. We use the wages in the associated market as prices in the original market. We thus obtain a competitive equilibrium of the original economy: Given the prices generated, a trade ω is demanded by its buyer if and only if it is also demanded by its seller (i.e., *not* demanded by the seller in the associated market).

This construction also provides an algorithm for finding a competitive equilibrium. For instance, once we have transformed the original economy into an associated market, we can use an ascending auction for workers to find the minimal-price competitive equilibrium of the associated market; we may then map that competitive equilibrium back to a competitive equilibrium of the original economy.

One technical complication that we need to address in the proof is that the modified valuation function in Equation (1) may in principle be unbounded and take the value $-\infty$ for some sets of trades, violating the assumptions of Kelso and Crawford (1982). To deal with this issue, we further modify the valuation function by bounding it in a way that preserves full substitutability but at the same time ensures that the equilibrium derived from the "bounded" economy remains an equilibrium of the original one. We also need to ensure that the equilibrium in the associated two-sided market exhibits full employment, in order to be able to go back from an equilibrium of the associated economy to an equilibrium of the original one.

We now turn to the properties of competitive equilibria in this economy. While they are similar to those of competitive equilibria in two-sided settings, it is important to verify that they continue to hold in this richer environment. We also rely on some of these properties in the proofs of our subsequent results.

We first note an analogue of the first welfare theorem in our economy.

Theorem 3. *Suppose* $[\Psi; p]$ *is a competitive equilibrium. Then* Ψ *is an* efficient *set of trades, i.e.,* $\sum_{i\in I} u_i(\Psi) \geq \sum_{i\in I} u_i(\Psi')$ *for any* $\Psi' \subseteq \Omega$.

The proof of this result follows from observing that in a competitive equilibrium, each agent is maximizing his utility under prices p, and when the utilities are added up across all agents, prices cancel out, leaving only the sum of agents' valuations.

Our next result can be viewed as a strong version of the second welfare theorem for our setting, providing a converse to Theorem 3: For any efficient set of trades Ψ and any competitive equilibrium price vector p, the arrangement $[\Psi; p]$ is a competitive equilibrium. Generically, the efficient set of trades is unique, and then this statement follows immediately from Theorem 3. We show that it also holds when there are multiple efficient sets of trades.

Theorem 4. *Suppose agents' preferences are fully substitutable. Then for any competitive equilibrium* $[\Xi; p]$ *and efficient set of trades* Ψ, $[\Psi; p]$ *is also a competitive equilibrium.*

The result of Theorem 4 implies that the notion of a *competitive equilibrium price vector* is well-defined. Our next result shows that the set of such vectors is a lattice.

Theorem 5. *Suppose agents' preferences are fully substitutable. Then the set of competitive equilibrium price vectors is a lattice.*

The lattice structure of the set of competitive equilibrium prices is analogous to the lattice structure of the set of stable outcomes for economies without transferable utility. In those models, there is a buyer-optimal and a seller-optimal stable outcome. In our model, the lattice of equilibrium prices may in principle be unbounded. If the lattice is bounded,[8] then there exist lowest-price and highest-price competitive equilibria.

4.2 The Relationship between Competitive Equilibria and Stable Outcomes

We now show how the sets of stable outcomes and competitive equilibria are related. First, we show that for every competitive equilibrium $[\Psi; p]$, the associated outcome $\kappa([\Psi; p])$ is stable.

Theorem 6. *Suppose* $[\Psi; p]$ *is a competitive equilibrium. Then* $\kappa([\Psi; p])$ *is stable.*

If for some competitive equilibrium $[\Psi; p]$ the outcome $\kappa([\Psi; p])$ is not stable, then either it is not individually rational or it is blocked. If it is not individually rational for some agent i, then $\kappa([\Psi; p])_i \notin C_i(\kappa([\Psi; p]))$. Hence, $\Psi_i \notin D_i(p)$, and so $[\Psi; p]$ is not a competitive equilibrium. If $\kappa([\Psi; p])$ admits a blocking set Z, then all the agents with contracts in Z are strictly better off after the deviation. Hence, there exists an agent $i \in a(Z)$ who is strictly better off choosing an element of $C_i(Z \cup \kappa([\Psi; p]))$ given the original price vector p; hence, $\Psi_i \notin D_i(p)$ and so $[\Psi; p]$ is not a competitive equilibrium. Note that this result does not rely on full substitutability.

However, it is not generally true that all stable outcomes correspond to competitive equilibria. Consider the following example.

Example 1. There are two agents, i and j, and two trades, χ and ψ, where $s(\chi) = s(\psi) = i$ and $b(\chi) = b(\psi) = j$. Agents' valuations are:

[8] E.g., if all valuations u_i are bounded.

Ψ	$\varnothing$	$\{\chi\}$	$\{\psi\}$	$\{\chi, \psi\}$
$u_i(\Psi)$	0	-4	-4	-4 .
$u_j(\Psi)$	0	3	3	3

In this case, $\varnothing$ is stable. Since $\varnothing$ is the only efficient set of trades, by Theorem 4 any competitive equilibrium is of the form $[\varnothing; p]$. However, the preferences of agent i imply that $p_\chi + p_\psi \leq 4$, as otherwise i will choose to sell at least one of ψ or χ. Moreover, the preferences of agent j imply that $p_\chi, p_\psi \geq 3$, as otherwise j will buy at least one of ψ or χ. Clearly, all three inequalities cannot jointly hold. Hence, while $\varnothing$ is stable, there is no corresponding competitive equilibrium.

The key issue is that an outcome A only specifies prices for the trades in $\tau(A)$, while a competitive equilibrium must specify prices for all trades (including those trades that do not transact). Hence, it may be possible, as in Example 1, for an outcome A to be stable, but, because of complementarities in preferences, it may be impossible to assign prices to trades outside of $\tau(A)$ in such a way that $\tau(A)_i$ is in fact an optimal set of trades for every agent i. Note that in Example 1, the preferences of agent j are fully substitutable, but those of agent i are not.

If, however, the preferences of all agents are fully substitutable, then for any stable outcome A we can in fact find a supporting set of prices p such that $[\tau(A); p]$ is a competitive equilibrium and the prices of trades that transact are the same as in A.

Theorem 7. *Suppose that agents' preferences are fully substitutable and A is a stable outcome. Then there exists a price vector $p \in \mathbb{R}^{|\Omega|}$ such that $[\tau(A); p]$ is a competitive equilibrium and if $(\omega, \bar{p}_\omega) \in A$, then $p_\omega = \bar{p}_\omega$.*

To construct a competitive equilibrium from a stable outcome A, we need to find appropriate prices for the trades that are not part of the stable outcome, i.e., trades $\omega \in \Omega - \tau(A)$. In the case of two-sided markets, this can be done on a trade-by-trade basis, because it is sufficient to verify that the price assigned to a trade will not make this trade desirable for either its buyer or its seller given the prices of the trades that they actually make. In our setting, this approach does not work, because the willingness of a buyer to make a new purchase may also depend on the prices assigned to the trades in which he is a potential seller. Thus, equilibrium prices for trades in $\Omega - \tau(A)$ are interdependent, and need to be assigned simultaneously in a consistent manner.

We start with the original market and the stable outcome A, and then construct a modified market. In this modified market, the set of available trades is $\Omega - \tau(A)$, and the valuation of each player i for a set of trades $\Psi \subseteq \Omega - \tau(A)$ is equal to the highest value that he can attain by combining the trades in Ψ_i with various subsets of A_i. We first show that the corresponding utility of each player i is fully substitutable, and thus the modified market has a competitive equilibrium. We then show that at least one such equilibrium has to be of the form $[\varnothing; \hat{p}]$ for some vector $\hat{p} \in \mathbb{R}^{|\Omega - \tau(A)|}$—otherwise, we show that in the original economy, there must exist a nonempty set that blocks A (the proof of this statement relies on Theorems 3 and 4, our "first" and "second" welfare theorems). Assigning prices $\hat{p}$ to the trades that are not part of A, we obtain a competitive equilibrium of the original economy.

4.3 The Necessity of Full Substitutability for Existence

We now show that if the preferences of any one agent are not fully substitutable, then stable outcomes need not exist. In fact, in that case we can construct *simple* preferences for other agents such that no stable outcome exists.

Definition 7. $\psi, \omega \in \Omega_i$ *are*

1. Independent *if* $u_i(\{\psi, \omega\} \cup \Phi) - u_i(\{\omega\} \cup \Phi) = u_i(\{\psi\} \cup \Phi) - u_i(\Phi)$ *for all* $\Phi \subseteq \Omega_i - \{\psi, \omega\}$.
2. Incompatible *if* $\psi, \omega \in \Omega_{\to i}$ *or* $\psi, \omega \in \Omega_{i \to}$ *and* $u_i(\{\psi, \omega\} \cup \Phi) - u(\Phi) = -\infty$ *for all* $\Phi \subseteq \Omega_i - \{\psi, \omega\}$.
3. Dependent *if* $\psi \in \Omega_{\to i}$, $\omega \in \Omega_{i \to}$ *and either* $u_i(\{\psi\} \cup \Phi) - u(\Phi) = -\infty$ *or* $u_i(\{\omega\} \cup \Phi) - u(\Phi) = -\infty$ *for all* $\Phi \subseteq \Omega_i - \{\psi, \omega\}$.

Preferences of agent i are simple *if for all $\psi, \omega \in \Omega_i$, ψ and ω are either independent, incompatible, or dependent.*

Two trades ψ and ω are independent for i if the marginal utility i obtains from performing ψ does not affect the marginal utility that i obtains from performing ω. In contrast, the trades ψ and ω are incompatible for i if i is unable to perform ψ and ω simultaneously; for instance, if ψ and ω both denote the transfer of a particular object, but to different individuals, then $u_i(\{\psi, \omega\}) = -\infty$. Finally, the trades ψ and ω are dependent for i if i can perform one of them only while performing the other; for instance, if ψ denotes the transfer from $s(\psi)$ to i of a necessary input of a production process, and ω denotes the transfer of the output of that process from i to $b(\omega)$, then $u_i(\{\omega\}) = -\infty$.

Simple preferences play a role similar to that of *unit-demand preferences*, used in the Gul and Stacchetti (1999) result characterizing the maximal domain for the existence of competitive equilibria in exchange economies. However, in our setting we must allow an individual agent to act as a set of unit-demand consumers, as each contract specifies both the buyer and the seller, and the violation of substitutability may only occur for an agent i when he holds multiple contracts with another agent.

Our necessity result requires sufficient "richness" of the set of trades. Specifically, we require that the set of trades Ω is *exhaustive*, i.e., that for each $i \neq j \in I$ there exist $\omega_i, \omega_j \in \Omega$ such that $b(\omega_i) = s(\omega_j) = i$ and $b(\omega_j) = s(\omega_i) = j$.

Theorem 8. *Suppose that there exist at least four agents and that the set of trades is exhaustive. Then if the preferences of some agent are not fully substitutable, there exist simple preferences for all other agents such that no stable outcome exists.*[9]

To understand the result, consider the following example.

Example 2. Agent i is just a buyer, and has perfectly complementary preferences over the contracts ψ and ω, and is not interested in other contracts, i.e., $u_i(\{\psi, \omega\}) = 1$ and $u_i(\{\psi\}) = u_i(\{\omega\}) = u_i(\varnothing) = 0$.

Suppose that $s(\psi)$ and $s(\omega)$ also have contracts $\hat{\psi}$ and $\hat{\omega}$ (where $s(\hat{\psi}) = s(\psi)$ and $s(\hat{\omega}) = s(\omega)$) with another agent j, and let the valuations of agents $j \neq i$ be given by:

$$u_{s(\psi)}(\{\hat{\psi}\}) = u_{s(\psi)}(\{\psi\}) = u_{s(\psi)}(\varnothing) = 0, \qquad u_{s(\psi)}(\{\psi, \hat{\psi}\}) = -\infty,$$
$$u_{s(\omega)}(\{\hat{\omega}\}) = u_{s(\omega)}(\{\omega\}) = u_{s(\omega)}(\varnothing) = 0, \qquad u_{s(\omega)}(\{\omega, \hat{\omega}\}) = -\infty,$$
$$u_j(\{\hat{\psi}, \hat{\omega}\}) = u_j(\{\hat{\psi}\}) = u_j(\{\hat{\omega}\}) = \frac{3}{4}, \qquad u_j(\varnothing) = 0.$$

Then in any stable outcome $s(\psi)$ will sell at most one of ψ and $\hat{\psi}$, and similarly for $s(\omega)$. It can not be that $\{\psi, \omega\}$ is part of a stable outcome, as their total price is at most 1, meaning at least one of them has a price at most $\frac{1}{2}$; suppose it is ω—we then have that $\{(\hat{\omega}, \frac{5}{8})\}$ is a blocking set. It also can not be the case that $\{(\hat{\psi}, p_{\hat{\psi}})\}$ or $\{(\hat{\omega}, p_{\hat{\omega}})\}$ is stable: in the former case, $p_{\hat{\psi}}$ must be less than $\frac{3}{4}$, in which case $\{(\psi, \frac{7}{8}), (\omega, \frac{1}{16})\}$ is a blocking set. A symmetric construction holds for the latter case.

The proof of Theorem 8 essentially generalizes Example 2.

Since a stable outcome does not necessarily exist when preferences are not fully substitutable, and (for any preferences) all competitive equilibria generate stable outcomes (by Theorem 6), Theorem 8 immediately implies the following corollary.

Corollary 1. *Suppose that there exist at least four agents and that the set of trades is exhaustive. Then, if the preferences of some agent are not fully substitutable, there exist simple preferences for all other agents such that no competitive equilibrium exists.*

[9]The proof of this result also shows that for two-sided markets with transferable utility if any agent's preferences are not fully substitutable, then if there exists at least one other agent on the same side of the market, simple preferences can be constructed such that no stable outcome exists.

5 Relation with Other Solution Concepts

In this section, we describe the relationships between competitive equilibrium, stability, and other solution concepts that have played important roles in the literature.

5.1 The Core and Strong Group Stability

We start by introducing a classical solution concept: the *core*.

Definition 8. *An outcome A is in the* core *if it is* core unblocked*: there does not exist a set of contracts Z such that, for all $i \in a(Z)$, $U_i(Z) > U_i(A)$.*

The definition of the core differs from that of stability in two ways. First, a core block requires all the agents with contracts in the blocking set to drop their contracts with other agents; this is a more stringent restriction than that of stability, which allows agents with contracts in the blocking set to retain previous relationships. Second, a core block does not require that $Z_i \in C_i(Z \cup A)$ for all $i \in a(Z)$; rather, it requires only the less stringent condition that $U_i(Z) > U_i(A)$.

We now introduce *strong group stability*, which is more restrictive than both core and stability.

Definition 9. *An outcome A is* strongly group stable *if it is*

1. *Individually rational;*
2. Strongly unblocked: *There does not exist a nonempty feasible $Z \subseteq X$ such that*
   (a) *$Z \cap A = \varnothing$, and*
   (b) *for all $i \in a(Z)$, there exists a $Y^i \subseteq Z \cup A$ such that $Z \subseteq Y^i$ and $U_i(Y^i) > U_i(A)$.*

Strong group stability is a more stringent notion than both stability and core as, when considering a block Z, agents may retain previously held contracts (as in the defintion of stability, but not in the definition of the core), and the new set of contracts for each agent need only be an improvement, not optimal (as in the definition of the core, but not the definition of stability).

We call this notion strong group stability as it is more stringent than both strong stability (introduced by Hatfield and Kominers, 2010a) and group stability (introduced by Roth and Sotomayor, 1990, and extended to the setting of many-to-many matching by Konishi and Ünver, 2006). Strong group stability is more stringent than strong stability, as strong stability imposes the additional requirement on blocking sets that each Y^i be individually rational. Strong group stability is also more stringent than group stability, as group stability imposes the additional requirement on blocking sets that if $y \in Y^{b(y)}$, then $y \in Y^{s(y)}$, i.e., that the deviating agents agreed on which contracts from the original allocation would be kept after the deviation. Strong stability and group stability are themselves strengthenings of the notion of setwise stability, introduced by Echenique and Oviedo (2006) and Klaus and Walzl (2009), which imposes both of the above requirements on blocking sets.[10]

Given these definitions, the following result is immediate.

Theorem 9. *Any strongly group stable outcome is stable and in the core. Furthermore, any core outcome is efficient.*

Without additional assumptions on preferences, no additional structure need be present. In particular, it may be that both stable and core outcomes exist for a given set of preferences, but that no outcome is both stable and core.

Example 3. There are two agents, i and j, and two trades, ψ and ω, where $s(\psi) = s(\omega) = i$ and $b(\psi) = b(\omega) = j$. Agents' valuations are:

Ψ	$\varnothing$	$\{\psi\}$	$\{\omega\}$	$\{\psi, \omega\}$
$u_i(\Psi)$	0	-2	-2	-6
$u_j(\Psi)$	0	0	0	7

The set of core outcomes is given by $\{\{(\psi, p_\psi), (\omega, p_\omega)\} : 6 \le p_\psi + p_\omega \le 7\}$. However, the unique stable outcome is $\varnothing$: any outcome of the form $\{(\psi, p_\psi)\}$ or $\{(\omega, p_\omega)\}$ is not individually rational, and any outcome $\{(\psi, p_\psi), (\omega, p_\omega)\}$ can only be individually rational if $p_\psi \ge 4$, $p_\omega \ge 4$, and $p_\psi + p_\omega \le 7$, which cannot all hold simultaneously.

Example C.1 of Appendix C demonstrates that even an outcome that is both stable and in the core need not be strongly group stable.

Even when preferences are fully substitutable, the relationship between strong group stability and stability is unclear. For models without continuously transferable utility (see e.g., Sotomayor, 1999; Echenique and Oviedo, 2006; Klaus and Walzl, 2009; Hatfield and Kominers, 2010a; and Westkamp, 2010), strong group stability is strictly more restrictive than stability. However, with continuously transferable utility and fully substitutable preferences, these solution concepts coincide.

Theorem 10. *If preferences are fully substitutable and A is a stable outcome, then A is strongly group stable and in the core. Moreover, for any core outcome A, there exists a stable outcome $\hat{A}$ such that $\tau(A) = \tau(\hat{A})$.*

[10] The notion of setwise stability used in these works is slightly stronger than the definition of setwise stability introduced by Sotomayor (1999); see Klaus and Walzl (2009) for a discussion of the subtle differences between these two definitions.

However, even for fully substitutable preferences, the core may be strictly larger than the set of stable outcomes.

Example 4. Consider again the setting of Example 3, but take preferences to be:

Ψ	$\varnothing$	$\{\psi\}$	$\{\omega\}$	$\{\psi, \omega\}$
$u_i(\Psi)$	0	0	0	-3
$u_j(\Psi)$	0	5	5	9

In this case, $\{(\psi, 2), (\omega, 2)\}$ is a core outcome, but is not individually rational for agent i; he will choose to drop one of the two contracts. We therefore see that the set of imputed utilities of a core outcome may not correspond to the set of imputed utilities for any stable outcome: in this example, the payoff of agent i in any stable outcome is at least 3, while it is only 1 in the core outcome above.

5.2 Chain Stability

The stability concept used in this paper also appears substantively different and noticeably stronger than the chain stability concept used in Ostrovsky (2008): the former (Definition 1) requires robustness to all blocking sets, while the latter requires robustness only to very specific blocking sets—chains of contracts. However, Hatfield and Kominers (2010b) show that in the presence of fully substitutable preferences, in the setting of Ostrovsky (2008) these conditions are in fact equivalent.[11] In this section, we prove an analogous result for the current setting with continuous prices.

First, we need to define the notion of chain stability. It is more general than that in the earlier work, because in the current setting trading cycles are allowed, and the definition needs to accommodate that. If trading cycles are prohibited (i.e., there is an upstream–downstream partial ordering of agents), then the definition is essentially the same as in Ostrovsky (2008), adjusted only for the presence of continuous prices.

Definition 10. *A non-empty set of contracts Z is a* chain *if its elements can be arranged in some order $y^1, \ldots, y^{|Z|}$ such that $s(y^{\ell+1}) = b(y^\ell)$ for all $\ell < |Z|$.*

Note that under this definition, the buyer in contract $y^{|Z|}$ is allowed to be the seller in contract y^1 (in which case the chain becomes a cycle), and also the same agent can be involved in the chain multiple times (because there is no enforced upstream–downstream partial ordering).

[11] Within the Ostrovsky (2008) setting, Westkamp (2010) characterizes the set of supply-chain networks for which chain and group stability (see p. 18) coincide. He also shows that this set of networks is the largest set for which group stable outcomes are guaranteed to exist.

Definition 11. *An outcome A is* chain stable *if it is*

1. Individually rational*;*
2. Not blocked by a chain*: There does not exist a chain $Z \subseteq X$ such that*
   (a) *$Z \cap A = \varnothing$, and*
   (b) *for all agents $i \in a(Z)$, for all $Y \in C_i(Z \cup A)$, we have $Z_i \subseteq Y$.*

This definition is weaker than the defintion of stability (Definition 1), and so it is immediate that for any preferences, any stable outcome is chain stable. The converse, for fully substitutable preferences, is a direct corollary of the following, more general result, which shows that any blocking set of an arbitrary outcome A can be "decomposed" into blocking chains.

Theorem 11. *Suppose agents' preferences are fully substitutable, and consider any outcome A that is blocked by some nonempty set Z. Then for some $\overline{m} \geq 1$, we can partition the set Z into a collection of $\overline{m}$ chains W^m such that $Z = \cup_{m=1}^{\overline{m}} W^m$, A is blocked by W^1, and for any $m \leq \overline{m} - 1$, the set of contracts $A \cup W^1 \cup \cdots \cup W^m$ is blocked by chain W^{m+1}.*

Corollary 2. *If agents' preferences are fully substitutable, then any chain stable outcome A is stable.*

Full substitutability is critical for these results. Without it, chain stability is strictly weaker than stability. For instance, consider Example 2 above. There is no stable outcome in that example, yet $\{(\hat{\omega}, 0)\}$ is chain stable: any set that blocks $\{(\hat{\omega}, 0)\}$ includes two contracts of the form (ψ, p_ψ) and (ω, p_ω), and so the blocking set does not constitute a blocking chain.

5.3 Competitive Equilibria without Personalized Prices

The notion of competitive equilibrium studied in this paper considers trades as the basic unit of analysis; a price vector specifies one price for each trade. E.g., if agent i has one object to sell, a competitive equilibrium price vector generally specifies a different price for each possible buyer, allowing for personalized pricing. This is in contrast to the notions of competitive equilibrium that assign a single price to each object (see, e.g., Gul and Stacchetti, 1999, and Sun and Yang, 2006). We now introduce a condition on utilities under which it is convenient to study non-personalized pricing.

Definition 12. *Consider an arbitrary agent i. The trades in some set $\Psi \subseteq \Omega_i$ are*

1. mutually incompatible for i *if for all $\Xi \subseteq \Omega_i$ such that $|\Xi \cap \Psi| \geq 2$, $u_i(\Xi) = -\infty$.*

2. perfect substitutes for i *if for all* $\Xi \subseteq \Omega_i - \Psi$ *and all* $\omega, \omega' \in \Psi$, $u_i(\Xi \cup \{\omega\}) = u_i(\Xi \cup \{\omega'\})$.

Theorem 12. *Let agents' preferences be fully substitutable. Suppose for agent i, trades in $\Psi \subseteq \Omega_i$ are mutually incompatible and perfect substitutes and let $[\Xi; p]$ be an arbitrary competitive equilibrium.*

(a) If $\Psi \subseteq \Omega_{i\rightarrow}$, let $\overline{p} = \max_{\psi \in \Psi} p_\psi$ and define q by $q_\psi = \overline{p}$ for all $\psi \in \Psi$ and $q_\psi = p_\psi$ for all $\psi \in \Omega - \Psi$. Then, $[\Xi; q]$ is a competitive equilibrium.

(b) If $\Psi \subseteq \Omega_{\rightarrow i}$, let $\underline{p} = \min_{\psi \in \Psi} p_\psi$ and define q by $q_\psi = \underline{p}$ for all $\psi \in \Psi$ and $q_\psi = p_\psi$ for all $\psi \in \Omega - \Psi$. Then, $[\Xi; q]$ is a competitive equilibrium.

Note that since the preferences of agent i are fully substitutable, a trade $\omega \in \Omega_{i\rightarrow}$ cannot be a perfect substitute for a trade $\omega' \in \Omega_{\rightarrow i}$. Hence, the two cases in the theorem are exhaustive.

This result allows us to embed the model of Sun and Yang (2006) as a special case of our model. In their model, a finite set S of indivisible objects needs to be allocated to a finite set I of agents with quasilinear utility. Objects are partitioned into two groups, S_1 and S_2. Agents' preferences satisfy the (GSC) condition: Objects in the same group are substitutes and two objects belonging to different groups are complements. To embed this model into our setting, one can view each object in S_1 as an agent who can sell goods to agents in I, and each object in S_2 as an agent who can buy goods from agents in I. Each agent in $S = S_1 \cup S_2$ only cares about prices and can not buy from/sell to more than one agent in I. For this embedding (GSC) is equivalent to full substitutability and all our results apply immediately. Since trades are mutually incompatible and perfect substitutes for every agent in S, we can apply the procedure from Theorem 12 to any competitive equilibrium (with personalized prices) in order to obtain a competitive equilibrium without personalized prices. In particular, our Theorem 2 implies the existence result in Sun and Yang (2006).[12]

6 Conclusion

We have introduced a general model in which a network of agents can trade via bilateral contracts. In this setting, when continuous transfers are allowed and agents' preferences are

[12]Note that our network setting is clearly more general, as it cannot be embedded in the setting of Sun and Yang (2006). E.g., consider a simple market with three agents i, j, and k, where i can sell trades to both j and k, j can sell trades only to k, k cannot sell trades to anyone, and all agents' preferences are fully substitutable. In this market, it is impossible to separate trades into two groups S_1 and S_2 in such a way that every agent views trades in one group as substitutes and views trades in different groups as complements.

quasilinear, full substitutability of preferences is sufficient and necessary for the guaranteed existence of stable outcomes. Furthermore, full substitutability implies that the set of stable outcomes is equivalent to the set of competitive equilibria, and that all stable outcomes are in the core and efficient.

Viewing these results in light of the previous matching literature leads to two additional observations.

First, stability may be a natural extension of the notion of competitive equilibrium for some economically important settings in which competitive equilibria do not exist. If the underlying network structure of a market does not contain cycles, stable outcomes exist even if there are restrictions on which contracts the agents are allowed to form, as long as agents' preferences are fully substitutable (Ostrovsky, 2008). For instance, a price floor (or ceiling) may prevent markets from clearing and thus lead to the non-existence of competitive equilibria. When studying a market for a single good, the classical supply–demand diagram may be sufficient for reasoning about the effects of the price floor. However, in more complicated cases, such as supply chain networks or two-sided markets with multiple goods, a simple diagram is no longer sufficient. The results of this paper suggest that stability may be an appropriate extension of competitive equilibrium for those cases: When contractual arrangements are not restricted, the notions of stability and competitive equilibrium are equivalent; and when there are contracting restrictions, stability continues to make predictions. Recent evidence suggests that these predictions are experimentally supported in multi-good markets in which competitive equilibria do not exist due to price floors (Hatfield et al., 2011).

Second, contrasting our results for general networks with previous findings presents a puzzle. Typically, in the matching literature, there are strong parallels between the existence and properties of stable outcomes in markets with fully transferable utility and those in which transfers are either not allowed or restricted. (This similarity was first observed by Shapley and Shubik (1971) for the basic one-to-one matching model, and continues to hold for increasingly complex environments, up to the case of vertical networks.) Our results show that this relationship breaks down for networks with cycles (in which agents' preferences are fully substitutable): with continuous transfers, stable outcomes are guaranteed to exist, while without them, the set of stable outcomes may be empty (Hatfield and Kominers, 2010b). It is an open question why the presence of a continuous numeraire can replace the assumption of supply chain structure in ensuring the existence of stable outcomes in trading networks.

Appendix A: Full Substitutability

In this Appendix, we show the equivalence of eight definitions of full substitutability. Four of these definitions were presented in Section 3. Below we introduce four other definitions, all of which deal directly with indifferences in agents' preferences. These definitions are more convenient to work with in some settings, and also correspond directly to those introduced in the prior literature. After proving the equivalence of these definitions, we introduce the Laws of Aggregate Supply and Demand for the current setting and show that quasilinearity and full substitutability imply those laws. This result is subsequently used in the proof of Theorem 11, and is also of independent interest.

A.1 Equivalent Definitions

The first two definitions are analogues of Definition 3, explicitly considering indifferences in preferences. The first one states what happens when an agent's set of options expands, and the second one states what happens when it shrinks.

Definition A.1. *The preferences of agent i are* choice-language expansion fully substitutable *(CEFS) if:*

1. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that $(Y_{\to i} - Y^*_{\to i}) \subseteq (Z_{\to i} - Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;*
2. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that $(Y_{i\to} - Y^*_{i\to}) \subseteq (Z_{i\to} - Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.*

Definition A.2. *The preferences of agent i are* choice-language contraction fully substitutable *(CCFS) if:*

1. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for every $Z^* \in C_i(Z)$ there exists $Y^* \in C_i(Y)$ such that $(Y_{\to i} - Y^*_{\to i}) \subseteq (Z_{\to i} - Z^*_{\to i})$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;*
2. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for every $Z^* \in C_i(Z)$ there exists $Y^* \in C_i(Y)$ such that $(Y_{i\to} - Y^*_{i\to}) \subseteq (Z_{i\to} - Z^*_{i\to})$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.*

Note that we use Y as the "starting set" in (CEFS) and Z as the "starting set" in (CCFS) to make the two notions more easily comparable. Furthermore, note that in Case

1 of (CEFS) and (CCFS), requiring $Y_{\to i} - Y^*_{\to i} \subseteq Z_{\to i} - Z^*_{\to i}$ is equivalent to requiring that $Z^* \cap Y_{\to i} \subseteq Y^*$, and similarly, in Case 2, requiring $Y_{i\to} - Y^*_{i\to} \subseteq Z_{i\to} - Z^*_{i\to}$ is equivalent to requiring that $Z^* \cap Y_{i\to} \subseteq Y^*$.

The next two definitions are analogues of Definition 4.

Definition A.3. *The preferences of agent i are* demand-language expansion fully substitutable *(DEFS) if:*

1. *for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;*
2. *for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.*

Definition A.4. *The preferences of agent i are* demand-language contraction fully substitutable *(DCFS) if:*

1. *for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$ such that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;*
2. *for all price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$ such that $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.*

Note that we use p as the "starting price vector" in (DEFS) and p' as the "starting price vector" in (DCFS). Also, in Case 1 of (DEFS) and (DCFS), requiring $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ is equivalent to requiring that $\{\omega \in (\Omega_{\to i} - \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} - \Psi'$, and similarly, in Case 2, requiring $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ is equivalent to requiring that $\{\omega \in (\Omega_{i\to} - \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{i\to} - \Psi'$.

A.2 The Equivalence Result

Theorem A.1. *(CFS), (DFS), (IFS), (CEFS), (CCFS), (DEFS), and (DCFS) are all equivalent, and hold if and only if the indirect utility function is submodular.*

Proof. It is immediate that (CEFS) and (CCFS) each imply (CFS). Below we show that (CFS) ⇒ (DFS), (DFS) ⇒ (DEFS), (DFS) ⇒ (DCFS), (DEFS) ⇒ (CEFS), and (DCFS) ⇒ (CCFS), thus proving the equivalence of these six definitions. We then show that these definitions are equivalent to (IFS) and the submodularity of the indirect utility function.

(CFS) ⇒ (DFS)

We first show that Case 1 of (DFS) implies Case 1 of (CFS). For any agent i and price vector $p \in \mathbb{R}^{|\Omega|}$, let $X_i(p) \equiv \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{\to i},\ \hat{p}_\omega \geq p_\omega\} \cup \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{i\to},\ \hat{p}_\omega \leq p_\omega\}$, in essence denoting the set of contracts available to agent i under prices p.

Let price vectors $p, p' \in \mathbb{R}^{|\Omega|}$ be such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$. Let $Y = X_i(p)$ and $Z = X_i(p')$. Clearly, $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Furthermore, it is immediate that $\Psi \in D_i(p)$ if and only if $\kappa([\Psi; p]) \in C_i(Y)$, and similarly, $\Psi' \in D_i(p')$ if and only if $\kappa([\Psi'; p']) \in C_i(Z)$. In particular, we have $|C_i(Y)| = |C_i(Z)| = 1$ and can thus apply (CFS) to the sets Y and Z.

Take the unique $\Psi \in D_i(p)$, let $Y^* = \kappa([\Psi, p])$, and note that $Y^* \in C_i(Y)$. By (CFS), the unique $Z^* \in C_i(Z)$ satisfies $Y_{\to i} - Y^*_{\to i} \subseteq Z_{\to i} - Z^*_{\to i}$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$. Let $\Psi' = \tau(Z^*)$ and note that $\Psi' \in D_i(p')$. We show that Ψ' satisfies the conditions in Case 1 of Definition 4.

Note that $Y_{\to i} - Y^*_{\to i} \subseteq Z_{\to i} - Z^*_{\to i}$ implies $\{\omega \in \Omega_{\to i} - \Psi_{\to i} : p_\omega = p'_\omega\} \subseteq \tau(Y_{\to i}) - \tau(Y^*_{\to i}) \subseteq \tau(Z_{\to i}) - \tau(Z^*_{\to i}) \subseteq \Omega_{\to i} - \Psi'_{\to i}$. Furthermore, $Y^*_{i\to} \subseteq Z^*_{i\to}$ and $p_\omega = p'_\omega$ for each $\omega \in \Omega_{i\to}$ imply $\Psi'_{i\to} \subseteq \Psi_{i\to}$.

The proof that Case 2 of (CFS) implies Case 2 of (DFS) is analogous.

(DFS) ⇒ (DEFS), (DFS) ⇒ (DCFS)

We first show that Case 1 of (DFS) implies Case 1 of (DEFS). Take two price vectors p, p' such that $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and fix an arbitrary $\Psi \in D_i(p)$. We need to show that there exists a set $\Psi' \in D_i(p')$ that satisfies the conditions of Case 1 of (DEFS).

As the statement is trivial when $D_i(p') = \{\Xi : \Xi \subset \Omega_i\}$, we assume the contrary. Furthermore, we assume that $D_i(p) \neq \{\Xi : \Xi \subset \Omega_i\}$; the arguments below are easily extended to the case where this assumption is not satisfied. In the following, let $\tilde{\Omega}_{\to i} = \{\omega \in \Omega_{\to i} : p'_\omega < p_\omega\}$. Let $\varepsilon_1 = V_i(p) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(p)} U_i([\Xi; p])$, $\varepsilon_2 = V_i(p') - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(p')} U_i([\Xi; p'])$, and $\varepsilon_3 = \min_{\omega \in \tilde{\Omega}_{\to i}} (p_\omega - p'_\omega)$. Let $\varepsilon = \frac{\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}}{3|\Omega_i|}$. Note that by construction, $\varepsilon > 0$.

We now define a price vector q^1 by

$$q^1_\omega = \begin{cases} p_\omega - \varepsilon & \omega \in \Omega_{i\to} - \Psi \text{ or } \omega \in \Psi_{\to i} \\ p_\omega + \varepsilon & \omega \in \Omega_{\to i} - \Psi \text{ or } \omega \in \Psi_{i\to} \\ 0 & \omega \notin \Omega_i. \end{cases}$$

Clearly, we must have $D_i(q^1) = \{\Psi\}$. Now define q^2 by $q^2_\omega = q^1_\omega$ for all $\omega \in \Omega - \tilde{\Omega}_{\to i}$ and $q^2_\omega = p'_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$. We claim that $D_i(q^2) \subseteq D_i(p')$. To see this, fix an arbitrary

$\Phi \in D_i(p')$ and an arbitrary $\Xi \notin D_i(p')$. Then we must have

$$U([\Phi; q^2]) \geq U([\Phi; p']) - |\Phi|\varepsilon > U([\Xi; p']) + |\Xi|\varepsilon \geq U([\Xi; q^2]),$$

where the first and third inequalities follow directly from the definitions of q^2, and the second inequality follows from $(|\Xi| + |\Phi|)\varepsilon \leq 2|\Omega_i|\varepsilon < U([\Phi; p']) - U([\Xi; p'])$.

We will now show that the condition in Case 1 of Definition 4 is satisfied for any set of trades $\Psi' \in D_i(q^2)$. Take any such Ψ'. Similar to the above, we define $\delta_1 = V_i(q^1) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(q^1)} U_i([\Xi; q^1])$, $\delta_2 = V_i(q^2) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(q^2)} U_i([\Xi; q^2])$, and $\delta_3 = \min_{\omega \in \tilde{\Omega}_{\to i}} (q^1_\omega - p'_\omega)$. Let $\delta = \frac{\min\{\delta_1, \delta_2, \delta_3\}}{3|\Omega_i|}$, and define price vector q^3 as

$$q^3_\omega = \begin{cases} q^2_\omega - \delta & \omega \in \Omega_{i\to} - \Psi' \text{ or } \omega \in \Psi'_{\to i} \\ q^2_\omega + \delta & \omega \in \Omega_{\to i} - \Psi' \text{ or } \omega \in \Psi'_{i\to} \\ 0 & \omega \notin \Omega_i. \end{cases}$$

Clearly, we must have $D_i(q^3) = \{\Psi'\}$. Now define q^4 by $q^4_\omega = q^3_\omega$ for all $\omega \in \Omega - \tilde{\Omega}_{\to i}$ and $q^4_\omega = q^1_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$. Similar to the above, we can show that $D_i(q^4) \subseteq D_i(q^1)$, and therefore $D_i(q^4) = \{\Psi\}$. Since $q^3_\omega < q^4_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$ and $q^3_\omega = q^4_\omega$ for all $\omega \in \Omega - \tilde{\Omega}_{\to i}$, we can now apply Case 1 of (DFS) to conclude that Ψ' satisfies the condition in Case 1 of (DEFS).

The proofs that Case 2 of (DFS) implies Case 2 of (DEFS), and that (DFS) implies (DCFS) are completely analogous.

(DEFS) ⇒ (CEFS), (DCFS) ⇒ (CCFS)

We first prove Case 1 of (CEFS). Take agent i and any sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$. Define *usable* and *unusable* trades in Y as follows. Take trade $\omega \in Y_{i\to}$. If there exists real number r such that (i) $(\omega, r) \in Y$ and (ii) for any $r' > r$, $(\omega, r') \notin Y$, then trade ω is usable in Y; otherwise, it is unusable in Y. Similarly, take trade $\omega \in Y_{\to i}$. If there exists real number r such that (i) $(\omega, r) \in Y$ and (ii) for any $r' < r$, $(\omega, r') \notin Y$, then trade ω is usable in Y; otherwise, it is unusable in Y. Note that an unusable trade cannot be a part of any contract involved in any optimal choice in $C_i(Y)$. The definitions of trades usable and unusable in Z are completely analogous.

We now construct preliminary price vectors q and q' as follows. First, for every trade $\omega \notin \Omega_i$, $q_\omega = q'_\omega = 0$. Second, for every trade ω unusable in Y, $q_\omega = 0$, and for every trade ω unusable in Z, $q'_\omega = 0$. Next, for any trade $\omega \in \Omega_{i\to}$ usable in Y, $q_\omega = \max\{r : (\omega, r) \in Y\}$, and similarly, for any trade $\omega \in \Omega_{i\to}$ usable in Z, $q'_\omega = \max\{r : (\omega, r) \in Z\}$. Finally, for any trade $\omega \in \Omega_{\to i}$ usable in Y, $q_\omega = \min\{r : (\omega, r) \in Y\}$ and for any trade $\omega \in \Omega_{\to i}$ usable in

Z, $q'_\omega = \min\{r : (\omega, r) \in Z\}$.

We now construct price vectors p and p'. First, for any trade $\omega \notin \Omega_i$, $p_\omega = p'_\omega = 0$. Second, for any trade $\omega \in \Omega_i$ that is usable in both Y and Z, let $p_\omega = q_\omega$ and let $p'_\omega = q'_\omega$. Finally, we need to set prices for trades unusable in Y or Z. We already noted that for any trade ω unusable in set Y, it has to be the case that ω is not involved in any contract in any optimal choice in $C_i(Y)$; and likewise, if ω is unusable in Z, then ω is not involved in any contract in any optimal choice in $C_i(Z)$. Thus, in forming prices p and p', we will need to assign to these trades prices that are so large (or small, depending on which side the trade is on) that the corresponding trades are not demanded by agent i.

Let Π be a very large number. For instance, let

$$\Delta_1 = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i(\Omega_1; q) - U_i(\Omega_2; q)|,$$

$$\Delta_2 = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i(\Omega_1; q') - U_i(\Omega_2; q')|,$$

and $\Pi = 1 + \Delta_1 + \Delta_2 + \max_{\omega \in \Omega_i} |q_\omega| + \max_{\omega \in \Omega_i} |q'_\omega|$. For all $\omega \in \Omega_{i\to}$ that are unusable in Y (and thus also in Z), let $p_\omega = p'_\omega = -\Pi$. For all $\omega \in \Omega_{\to i}$ that are unusable in both Y and Z, let $p_\omega = p'_\omega = \Pi$. For all $\omega \in \Omega_{\to i}$ that are unusable in Y but not in Z, let $p_\omega = \Pi$ and $p'_\omega = q'_\omega$. Finally, for all $\omega \in \Omega_{\to i}$ that are unusable in Z but not in Y, let $p_\omega = p'_\omega = q_\omega$. Note that for any such ω, since $Y \subset Z$, $(\omega, q_\omega) \in Z$; also, as ω is unusable in Z, there are no contracts involving ω in any optimal choice in $C_i(Z)$.

Now, $p'_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$ and $p'_\omega \le p_\omega$ for all $\omega \in \Omega_{\to i}$. Take any $Y^* \in C_i(Y)$, and let $\Psi = \tau(Y^*)$. By construction, $\Psi \in D_i(p)$. By (DEFS), there exists $\Psi' \in D_i(p')$ such that $\{\omega \in (\Omega_{\to i} - \Psi_{\to i}) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} - \Psi'_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$. Let $Z^* = \kappa([\Psi', p'])$. Again, by construction, $Z^* \in C_i(Z)$. We now show that this set of contracts satisfies the conditions in Case 1 of (CEFS).

First, take some $y \in Y_{\to i} - Y^*_{\to i}$ and suppose that contrary to what we want to show, $y \in Z^*_{\to i}$. The latter implies that $y = (\omega, p'_\omega)$ for some trade ω, which, in turn, implies that $p_\omega = p'_\omega$ (because $y = (\omega, p'_\omega) \in Y$ and, since $Y \subset Z$, $(\omega, r) \notin Y$ for any $r < p'_\omega$). But then, by construction, $\{\omega \in (\Omega_{\to i} - \Psi_{\to i}) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} - \Psi'_{\to i}$, contradicting $y \in Z^*_{\to i}$. Second, since $Y^*_{i\to} = \{(\omega, p_\omega) : \omega \in \Psi_{i\to}\}$, $Z^*_{i\to} = \{(\omega, p_\omega) : \omega \in \Psi'_{i\to}\}$, and $\Psi_{i\to} \subseteq \Psi'_{i\to}$, it is immediate that $Y^*_{i\to} \subseteq Z^*_{i\to}$. This completes the proof that Case 1 of (DEFS) implies Case 1 of (CEFS).

The proofs that Case 2 of (DEFS) implies Case 2 of (CEFS) and that (DCFS) implies (CCFS) are completely analogous.

(DFS) $\Rightarrow$ (IFS)

Take two price vectors p, p' such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$. Let $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$ be the unique demanded sets. We have to show that $e_\omega(\Psi') \geq e_\omega(\Psi)$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

First, let p^1 be a price vector such that $p^1_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p^1_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$. By (DCFS) there must exist a $\Psi^1 \in D_i(p^1)$ such that $\{\omega \in \Psi_{\to i} : p^1_\omega = p_\omega\} \subseteq \Psi^1$ and $\Psi^1_{i\to} \subseteq \Psi_{i\to}$. This immediately implies $e_\omega(\Psi^1) \geq e_\omega(\Psi)$ for all $\omega \in \Omega_i$ such that $p^1_\omega = p_\omega$. Now by (DEFS) we must have $\{\omega \in \Omega_{\to i} - \Psi^1 : p^1_\omega = p_\omega\} \subseteq \Omega - \Psi'$ and $\Psi^1_{\to i} \subseteq \Psi'_{\to i}$, implying $e_\omega(\Psi') \geq e_\omega(\Psi^1)$ for all $\omega \in \Omega_i$ such that $p'_\omega = p^1_\omega$. Combining this with the above, we obtain the desired statement.

(IFS) $\Rightarrow$ (DFS)

This follows immediately, because the price change conditions in both Cases 1 and 2 of (DFS) are special cases of the price change condition of (IFS).

Submodularity

The proof of the equivalence of the (DFS) condition and the submodularity of the indirect utility function is completely analogous to the proof of Theorem 2 of Sun and Yang (2009). We cannot apply Theorem 2 of Sun and Yang (2009) directly, because in addition to (DFS) ((DCFS) to be precise), Sun and Yang impose monotonicity and boundedness conditions on u_i. The proof of that theorem, however, does not rely on these additional assumptions. More specifically, it only relies on (i) (DCFS), (ii) the equivalence of (DCFS) and (DEFS), which we have shown above to hold in our setting as well, and (iii) the monotonicity of the *indirect* utility function, which always holds. □

A.3 Laws of Aggregate Supply and Demand

An important property of fully substitutable preferences in two-sided quasilinear settings is the Law of Aggregate Demand (Hatfield and Milgrom, 2005). Its analogues for the current setting are the Laws of Aggregate Supply and Demand, introduced by Hatfield and Kominers (2010b). Below we show that in the current network setting with quasilinear utilities and continuous transfers, full substitutability implies that preferences satisfy these laws.

Definition A.5. *Agent i's preferences satisfy the* Law of Aggregate Demand *if for all finite sets of contracts Y and Y' such that $Y_{i\to} = Y'_{i\to}$ and $Y_{\to i} \subseteq Y'_{\to i}$, for any $W \in C_i(Y)$, there exists $W' \in C_i(Y')$ such that $|W'_{\to i}| - |W_{\to i}| \geq |W'_{i\to}| - |W_{i\to}|$.*

Agent i's preferences satisfy the Law of Aggregate Supply *if for all finite sets of contracts Y and Y' such that $Y_{i\to} \subseteq Y'_{i\to}$ and $Y_{\to i} = Y'_{\to i}$, for any $W \in C_i(Y)$, there exists $W' \in C_i(Y')$ such that $|W'_{i\to}| - |W_{i\to}| \geq |W'_{\to i}| - |W_{\to i}|$.*

Theorem A.2. *If the preferences of agent i are fully substitutable and quasilinear in the numeraire, then they satisfy the Laws of Aggregate Supply and Demand.*

Proof. We prove the Law of Aggregate Demand; the proof of the Law of Aggregate Supply is analogous.

Fix a fully substitutable valuation function u_i for agent i. First, take two finite sets of contracts Y and Y' such that $|C_i(Y)| = |C_i(Y')| = 1$, $Y_{i\to} = Y'_{i\to}$, and $Y_{\to i} \subseteq Y'_{\to i}$. Assume that for any $\omega \in \Omega_{i\to}$, $(\omega, r) \in Y_{i\to}$ and $(\omega, r') \in Y_{i\to}$ implies $r = r'$ (this is without loss of generality, because for a given trade in $\Omega_{i\to}$, agent i, as a seller, can only choose a contract with the highest price available for that trade, and thus we can disregard all other contracts involving that trade). Let $W \in C_i(Y)$ and $W' \in C_i(Y')$. Define a modified valuation $\tilde{u}_i$ on $\tau(Y'_i)$ for agent i by setting, for each $\Psi \subseteq \tau(Y'_i)$,

$$\tilde{u}_i(\Psi) = u_i(\Psi_{\to i} \cup (\tau(Y') - \Psi)_{i\to}).$$

Let $\tilde{C}_i$ denote the associated choice correspondence. Using arguments analogous to those in the proof of Theorem 2, one can show that $\tilde{u}_i$ satisfies the gross substitutes condition of Kelso and Crawford (1982). Furthermore, we must have $\tilde{C}_i(Y) = \{W_{\to i} \cup (Y' - W)_{i\to}\}$ and $\tilde{C}_i(Y') = \{W'_{\to i} \cup (Y' - W')_{i\to}\}$. Since we assume quasilinearity, the Law of Aggregate Demand for two-sided markets applies to $\tilde{C}_i$ (Theorem 7 in Hatfield and Milgrom, 2005). Since $Y \subseteq Y'$, this implies $|W'_{\to i} \cup (Y' - W')_{i\to}| \geq |W_{\to i} \cup (Y' - W)_{i\to}|$. The last inequality is equivalent to $|W'_{\to i}| - |W_{\to i}| \geq |W'_{i\to}| - |W_{i\to}|$, which is precisely what we needed to show. The proof that the Law of Aggregate Demand for the case in which choice correspondences are single-valued implies the more general case in which they can be multi-valued is analogous to the proof of the implication (DFS)⇒(DEFS) in Theorem A.1. □

Appendix B: Proofs of Results in Sections 4 and 5

Proof of Theorem 2

The proof consists of four steps: (1) transforming the original valuations into bounded ones, (2) constructing a two-sided many-to-one matching market with transfers, based on the network market with bounded valuations, (3) picking a full-employment competitive equilibrium in the two-sided market, and (4) using that equilibrium to construct a competitive equilibrium in the original market.

Step 1: We first transform a fully substitutable but potentially unbounded from below valuation function u_i into a fully substitutable and bounded valuation function $\hat{u}_i$. For this purpose, we now introduce a very high price Π. Specifically, for each agent i, let $\overline{u}_i$ be the highest possible absolute value of the utility of agent i from a combination of trades, i.e., $\overline{u}_i = \max_{\{\Psi \in \Omega_i : |u_i(\Psi)| < \infty\}} |u_i(\Psi)|$. Then set $\Pi = 2\sum_i \overline{u}_i + 1$. Consider the following *modified economy.* Assume that for every trade, the buyer of that trade can always purchase a perfect substitute for that trade for Π and the seller of that trade can always produce this trade at the cost of Π with no inputs needed. Formally, for each agent i, for a set of trades $\Psi \subseteq \Omega_i$, let

$$\hat{u}_i(\Psi) = \max_{\Psi' \subseteq \Psi} \left[u_i(\Psi') - \Pi \cdot |\Psi - \Psi'| \right].$$

For the economy with valuations $\hat{u}_i$, let $\hat{U}_i$ denote agent i's utility function and let $\hat{D}_i$ denote the modified demand correspondence. Note that by the choice of Π, for any $\Psi \subseteq \Omega_i$, $\overline{u}_i \geq \hat{u}_i(\Psi) \geq \max\{u_i(\varnothing) - \Pi \cdot |\Psi|, u_i(\Psi)\}$, and that $\hat{u}_i(\Psi) = u_i(\Psi)$ whenever $u_i(\Psi) \neq -\infty$. We will use these facts throughout the proof.

The rest of Step 1 consists of proving the following lemma.

Lemma B.1. *Utility function $\hat{U}_i$ is fully substitutable.*

Proof. We will first prove the following auxiliary statement. Take any fully substitutable valuation function u_i. Take any trade $\phi \in \Omega_{i\rightarrow}$. Consider a modified valuation function u_i^ϕ:

$$u_i^\phi(\Psi) = \max[u_i(\Psi), u_i(\Psi - \phi) - \Pi].$$

I.e., this valuation function allows (but does not require) agent i to pay Π instead of forming one particular trade, ϕ. Then valuation function u_i^ϕ is also fully substitutable.

To see this, consider utility U_i^ϕ and demand D_i^ϕ corresponding to valuation u_i^ϕ. We will show that D_i^ϕ satisfies (IFS). Fix two price vectors p and p' such that $p \leq p'$ and $|D_i^\phi(p)| = |D_i^\phi(p')| = 1$. Take $\Psi \in D_i^\phi(p)$ and $\Psi' \in D_i^\phi(p')$. We need to show that for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$, $e_\omega(\Psi) \leq e_\omega(\Psi')$.

Let price vector q coincide with p on all trades other than ϕ, and set $q_\phi = \min\{p_\phi, \Pi\}$. Note that if $p_\phi < \Pi$, then $p = q$ and $D_i^\phi(p) = D_i(p)$. If $p_\phi > \Pi$, then under utility U_i^ϕ, agent i always wants to form trade ϕ at price p_ϕ, and the only decision is whether to "buy it out" or not at the cost Π; i.e., the agent's effective demand is the same as under price vector q. Thus, $D_i^\phi(p) = \{\Xi \cup \{\phi\} : \Xi \in D_i(q)\}$. Finally, if $p_\phi = \Pi$, then $p = q$ and $D_i^\phi(p) = D_i(p) \cup \{\Xi \cup \{\phi\} : \Xi \in D_i(p)\}$. Construct price vector q' corresponding to p' analogously.

Now, if $p_\phi \leq p'_\phi < \Pi$, then $D_i^\phi(p) = D_i(p)$, $D_i^\phi(p') = D_i(p')$, and thus $e_\omega(\Psi) \leq e_\omega(\Psi')$ follows directly from (IFS) for demand D_i.

If $\Pi \leq p_\phi \leq p'_\phi$, then (since we assumed that D_i^ϕ was single-valued at p and p') it has to be the case that D_i is single-valued at the corresponding price vectors q and q'. Let $\Xi \in D_i(q)$ and $\Xi' \in D_i(q')$. Then $\Psi = \Xi \cup \{\phi\}$, $\Psi' = \Xi' \cup \{\phi\}$, and the statement follows from (IFS) for demand D_i, because $q \leq q'$.

Finally, if $p_\phi < \Pi \leq p'_\phi$, then $p = q$, Ψ is the unique element in $D_i(p)$, and Ψ' is equal to $\Xi' \cup \{\phi\}$, where Ξ' is the unique element in $D_i(q')$. Then for $\omega \neq \phi$, the statement follows from (IFS) for demand D_i, because $p \leq q'$. For $\omega = \phi$, the statement does not need to be checked, because $p_\phi < p'_\phi$.

Thus, valuation function u_i^ϕ is fully substitutable. The proof for the case $\phi \in \Omega_{\to i}$ is completely analogous.

To complete the proof of the lemma, it is now enough to note that valuation function $\hat{u}_i(\Psi) = \max_{\Psi' \subseteq \Psi} [u_i(\Psi') - \Pi \cdot |\Psi - \Psi'|]$ can be obtained from the original valuation u_i by allowing agent i to buy out all of his trades, one by one, and since each such transformation preserves substitutability, $\hat{u}_i$ is substitutable as well. □

Step 2: We now transform the modified economy with bounded and fully substitutable valuations $\hat{u}_i$ into an associated two-sided many-to-one matching market with transfers, which will satisfy the assumptions of Kelso and Crawford (1982; subsequently KC). The set of firms in this market is I, and the set of workers is Ω.

Worker ω can be matched to at most one firm. His utility is defined as follows. If he is matched to firm $i \in \{s(\omega), b(\omega)\}$, then his utility is equal to the monetary transfer that he receives from that firm, i.e., his salary $p_{i,\omega}$, which can in principle be negative. If he is matched to any other firm i, his utility is equal to $-\Pi - 1 + p_{i,\omega}$, where Π is as defined in Step 1 and $p_{i,\omega}$ is the salary firm i pays him. If worker ω remains unmatched, his utility is equal to $-2\Pi - 2$.

Firm i can be matched to any set of workers, but only its matches to workers $\omega \in \Omega_i$ have an impact on its valuation. Formally, firm i's valuation from hiring a set of workers $\Psi \subseteq \Omega$ is given by

$$\tilde{u}_i(\Psi) = \hat{u}_i\left(\Psi_{\to i} \cup (\Omega - \Psi)_{i\to}\right) - \hat{u}_i\left(\Omega_{i\to}\right),$$

where the second term in the difference is simply a constant, which ensures that $\tilde{u}_i(\varnothing) = 0$ and thus valuation function $\tilde{u}_i$ satisfies assumption (NFL) of KC. Hiring a set of workers

$\Psi \subseteq \Omega$ when the salary vector is $p \in \mathbb{R}^{|I|\times|\Omega|}$ yields i a utility of

$$\tilde{U}_i([\Psi; p]) \equiv \tilde{u}_i(\Psi_i) - \sum_{\omega \in \Psi} p_{i,\omega}.$$

The associated demand correspondence is denoted by $\tilde{D}_i$.

Assumption (MP) of KC requires that any firm's change in valuation from adding a worker, ω, to any set of other workers is at least as large as the lowest salary worker ω would be willing to accept from the firm when his only alternative is to remain unmatched. This assumption is also satisfied in our market: A worker's utility from remaining unmatched is $-2\Pi - 2$, while his valuation, excluding salary, from matching with any firm is at least $-\Pi - 1$, and so he would strictly prefer to work for any firm for negative salary $-\Pi$ instead of remaining unmatched. At the same time, the change in valuation of any firm i from adding worker ω to a set of workers Ψ is equal to $\tilde{u}_i(\Psi \cup \{\omega\}) - \tilde{u}_i(\Psi) \geq -\bar{u}_i - \bar{u}_i > -\Pi$, and thus every firm i would also always strictly prefer to hire worker ω for the negative salary $-\Pi$.

Finally, we show that i's preferences in this market satisfy the gross substitutes (GS) condition of KC. Take two salary vectors $p, p' \in \mathbb{R}^{|I|\times|\Omega|}$ such that $p \leq p'$ and $|\tilde{D}_i(p)| = |\tilde{D}_i(p')| = 1$. Let $\Psi \in \tilde{D}_i(p)$ and $\Psi' \in \tilde{D}_i(p')$. Denote by $q = (p_{i,\omega})_{\omega\in\Omega}$ and $q' = (p'_{i,\omega})_{\omega\in\Omega}$ the vectors of salaries that i faces under p and p', respectively. Note that $\Psi \in \tilde{D}_i(p)$ if and only if $(\Psi_{\to i} \cup (\Omega - \Psi)_{i\to}) \in \hat{D}_i(q)$ and $\Psi' \in \tilde{D}_i(p')$ if and only if $(\Psi'_{\to i} \cup (\Omega - \Psi')_{i\to}) \in \hat{D}_i(q')$. In particular, $|\hat{D}_i(q)| = |\hat{D}_i(q')| = 1$. Since $q \leq q'$ and demand $\hat{D}_i$ is fully substitutable, (IFS) implies that for any $\omega \in \Psi_{\to i}$ such that $q_\omega = q'_\omega$, we have $\omega \in \Psi'_{\to i}$, and for any $\omega \notin (\Omega_{i\to} - \Psi_{i\to})$ such that $q_\omega = q'_\omega$, we have $\omega \notin (\Omega_{i\to} - \Psi'_{i\to})$. In other words, for every $\omega \in \Psi$ such that $q_\omega = q'_\omega$, we have $\omega \in \Psi'$, and thus the (GS) condition is satisfied for all salary vectors for which demand $\tilde{D}_i$ is single-valued. By the implication (DFS)$\Rightarrow$(DCFS) of Theorem A.1, it follows that (GS) is satisfied for all salary vectors.

Step 3: By the results of KC (Theorem 2 and the discussion in Section 2), there exists a full-employment competitive equilibrium of the two-sided market constructed in Step 2. Take one such equilibrium, and for every ω and i, let $\mu(\omega)$ denote the firm matched to ω in this equilibrium and let $r_{i,\omega}$ denote equilibrium salary of ω at i.

Note that in this equilibrium, it must be the case that every worker ω is matched to either $b(\omega)$ or $s(\omega)$. Indeed, suppose ω is matched to some other firm $i \notin \{b(\omega), s(\omega)\}$. Since by definition, for any $\Psi \subset \Omega$, $\tilde{u}_i(\Psi \cup \{\omega\}) - \tilde{u}_i(\Psi) = 0$, it must be the case that $r_{i,\omega} \leq 0$. Then, for worker ω to weakly prefer to work for i rather than $b(\omega)$, it must be the case that $r_{b(\omega),\omega} \leq -\Pi - 1$. But at that salary, firm $b(\omega)$ strictly prefers to hire ω, contradicting the assumption that ω is not matched to $b(\omega)$ in this equilibrium.

Note also that if $\mu(\omega) = b(\omega)$, then $r_{b(\omega),\omega} \geq r_{s(\omega),\omega}$, and if $\mu(\omega) = s(\omega)$, then $r_{s(\omega),\omega} \geq r_{b(\omega),\omega}$ (otherwise, worker ω would strictly prefer to change his employer). Now, define prices $p_{i,\omega}$ as follows: if $i \neq b(\omega)$ and $i \neq s(\omega)$, then $p_{i,\omega} = r_{i,\omega}$. Otherwise, $p_{i,\omega} = \max\{r_{b(\omega),\omega}, r_{s(\omega),\omega}\}$. Note that matching μ and associated prices $p_{i,\omega}$ also constitute a competitive equilibrium of the two-sided market.

Step 4: We can now construct a competitive equilibrium for the original economy. Let $p^* \in \mathbb{R}^{|\Omega|}$ be defined as $p^*_\omega \equiv p_{\mu(\omega),\omega}$ for each $\omega \in \Omega$, i.e., the salary that ω actually receives in the equilibrium of the two-sided market. Let $\Psi^* \equiv \{\omega \in \Omega : \mu(\omega) = b(\omega)\}$, i.e., the set of trades/workers who in the equilibrium of the two-sided market are matched to their buyers (and thus not matched to their sellers!).

We now claim that $[\Psi^*; p^*]$ is a competitive equilibrium of the network economy with bounded valuations $\hat{u}_i$. Take any set of trades $\Psi \in \Omega_i$. We will show that $\hat{U}_i([\Psi^*; p^*]) \geq \hat{U}_i([\Psi; p^*])$. By construction, for any $\omega \in \Omega_{\to i}$, $\omega \in \Psi^*$ if and only if $i = \mu(\omega)$, and for any $\omega \in \Omega_{i\to}$, $\omega \in \Psi^*$ if and only if $i \neq \mu(\omega)$. Thus, in the equilibrium of the two-sided market, firm i is matched to the set of workers $\Psi^*_{\to i} \cup (\Omega_{i\to} - \Psi^*_{i\to})$, which implies that

$$\tilde{u}_i(\Psi^*_{\to i} \cup (\Omega_{i\to} - \Psi^*_{i\to})) - \sum_{\omega \in \Psi^*_{\to i}} p_{i,\omega} - \sum_{\omega \in (\Omega_{i\to} - \Psi^*_{i\to})} p_{i,\omega}$$
$$\geq \tilde{u}_i(\Psi_{\to i} \cup (\Omega_{i\to} - \Psi_{i\to})) - \sum_{\omega \in \Psi_{\to i}} p_{i,\omega} - \sum_{\omega \in (\Omega_{i\to} - \Psi_{i\to})} p_{i,\omega}. \quad (2)$$

Using the definition of $\tilde{u}_i$ and the fact that for any set $\Phi \subseteq \Omega_{i\to}$, $\sum_{\omega \in (\Omega_{i\to} - \Phi)} p_{i,\omega} = \left(\sum_{\omega \in \Omega_{i\to}} p_{i,\omega}\right) - \left(\sum_{\omega \in \Phi} p_{i,\omega}\right)$, we can rewrite the inequality (2) as

$$\hat{u}_i(\Psi^*_{\to i} \cup \Psi^*_{i\to}) - \sum_{\omega \in \Psi^*_{\to i}} p_{i,\omega} + \sum_{\omega \in \Psi^*_{i\to}} p_{i,\omega} \geq \hat{u}_i(\Psi_{\to i} \cup \Psi_{i\to}) - \sum_{\omega \in \Psi_{\to i}} p_{i,\omega} + \sum_{\omega \in \Psi_{i\to}} p_{i,\omega},$$

which in turn can be rewritten as

$$\hat{U}_i([\Psi^*; p^*]) = \hat{u}_i(\Psi^*_i) - \sum_{\omega \in \Psi^*_{\to i}} p^*_\omega + \sum_{\omega \in \Psi^*_{i\to}} p^*_\omega \geq \hat{u}_i(\Psi) - \sum_{\omega \in \Psi_{\to i}} p^*_\omega + \sum_{\omega \in \Psi_{i\to}} p^*_\omega = \hat{U}([\Psi; p^*]).$$

We now show that $[\Psi^*; p^*]$ is an equilibrium of the original economy with valuations u_i. Suppose to the contrary that there exists an agent i and a set of trades $\Xi \subset \Omega_i$, such that $U_i([\Xi; p^*]) > U_i([\Psi^*; p^*])$. Since $\hat{U}_i([\Xi; p^*]) \leq \hat{U}_i([\Psi^*; p^*])$, and by the construction of $\hat{u}_i$, $\hat{U}_i([\Xi; p^*]) \geq U_i([\Xi; p^*])$, it follows that $\hat{U}_i([\Psi^*; p^*]) > U_i([\Psi^*; p^*])$. This, in turn, implies that for some nonempty set $\Phi \subseteq \Psi_i^*$, we have $\hat{u}_i(\Psi_i^*) = u_i(\Psi_i^* - \Phi) - \Pi \cdot |\Phi| \leq \overline{u}_i - \Pi$. This implies that $\sum_{j \in I} \hat{u}_j(\Psi^*) = \hat{u}_i(\Psi^*) + \sum_{j \neq i} \hat{u}_j(\Psi^*) \leq \overline{u}_i - \Pi + \sum_{j \neq i} \overline{u}_j = \sum_{j \in I} \overline{u}_j - \Pi = -\sum_{j \in I} \overline{u}_i - 1 < \sum_{j \in I} u_j(\varnothing)$, contradicting Theorem 3. (The proof of Theorem 3 is entirely self-contained.)

Proof of Theorem 3

If $[\Psi; p]$ is a competitive equilibrium, then for any $\Xi \subseteq \Omega$, we have

$$u_i(\Psi) + \sum_{\omega \in \Psi_{i \rightarrow}} p_\omega - \sum_{\omega \in \Psi_{\rightarrow i}} p_\omega = U_i([\Psi; p]) \geq U_i([\Xi; p]) = u_i(\Xi) + \sum_{\omega \in \Xi_{i \rightarrow}} p_\omega - \sum_{\omega \in \Xi_{\rightarrow i}} p_\omega$$

for every $i \in I$. By summing these inequalities over all $i \in I$, we get

$$\sum_{i \in I} u_i(\Psi) \geq \sum_{i \in I} u_i(\Xi).$$

Proof of Theorem 4

We use an approach analogous to the one Gul and Stacchetti (1999) use to prove their Lemma 6. Suppose $[\Xi; p]$ is a competitive equilibrium and $\Psi \subseteq \Omega$ is an efficient set of trades. Since Ψ is efficient, we have

$$\begin{aligned} \sum_{i \in I} \left[u_i(\Psi) + \sum_{\omega \in \Psi_{i \rightarrow}} p_\omega - \sum_{\omega \in \Psi_{\rightarrow i}} p_\omega \right] &= \sum_{i \in I} U_i([\Psi; p]) \\ &\geq \sum_{i \in I} U_i([\Xi; p]) = \sum_{i \in I} \left[u_i(\Xi) + \sum_{\omega \in \Xi_{i \rightarrow}} p_\omega - \sum_{\omega \in \Xi_{\rightarrow i}} p_\omega \right]. \end{aligned} \tag{3}$$

In light of the fact that $[\Xi; p]$ is a competitive equilibrium, so that

$$\begin{aligned} u_i(\Xi) + \sum_{\omega \in \Xi_{i \rightarrow}} p_\omega - \sum_{\omega \in \Xi_{\rightarrow i}} p_\omega &= U_i([\Xi; p]) \\ &\geq U_i([\Psi; p]) = u_i(\Psi) + \sum_{\omega \in \Psi_{i \rightarrow}} p_\omega - \sum_{\omega \in \Psi_{\rightarrow i}} p_\omega \end{aligned}$$

for every $i \in I$, we see that (3) can hold only if $U_i([\Xi; p]) = U_i([\Psi; p])$ for every $i \in I$. Hence (as $\Xi \in D_i(p)$ for all $i \in I$) we have $\Psi_i \in D_i(p)$ for all $i \in I$, and therefore $[\Psi; p]$ is a competitive equilibrium.

Proof of Theorem 5

The proof is analogous to the proof of Theorem 3 of Sun and Yang (2009), applied to the network setting. Given a price vector p, let $V(p) \equiv \sum_{i \in I} V_i(p)$. Let $\Psi^* \subseteq \Omega$ be any efficient set of trades and let $U^* = \sum_{i \in I} u_i(\Psi^*)$. Note that for any competitive equilibrium price vector p^*, $V(p^*) = U^*$.

We first prove an analogue of Lemma 1 of Sun and Yang (2009).

Lemma B.2. *A price vector p' is a competitive equilibrium price vector if and only if $p' \in \operatorname{argmin}_p V(p)$.*

Proof. To prove the first implication of the lemma, we let p' be a competitive equilibrium price vector and let p be an arbitrary price vector. For each agent i, consider some arbitrary $\Psi^i \in D_i(p)$. By construction, we have

$$\begin{aligned} V(p) = \sum_{i \in I} V_i(p) &= \sum_{i \in I} \left[u_i(\Psi^i) + \sum_{\omega \in \Psi^i_{i \to}} p_\omega - \sum_{\omega \in \Psi^i_{\to i}} p_\omega \right] \\ &\geq \sum_{i \in I} \left[u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \to}} p_\omega - \sum_{\omega \in \Psi^*_{\to i}} p_\omega \right] \\ &= \sum_{i \in I} u_i(\Psi^*) = U^* = V(p'), \end{aligned}$$

where the inequality follows from utility maximization.

Now, to prove the other implication of the lemma, let p' be any price vector that minimizes V (and thus satisfies $V(p') = U^*$). We claim that $[\Psi^*; p']$ is a competitive equilibrium. To see this, note that the definition of V_i implies that

$$V_i(p') \geq u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \to}} p'_\omega - \sum_{\omega \in \Psi^*_{\to i}} p'_\omega. \tag{4}$$

Summing (4) across $i \in I$ gives

$$\sum_{i \in I} V_i(p') \geq \sum_{i \in I} \left[u_i(\Psi^*) + \sum_{\omega \in \Psi^*_{i \to}} p'_\omega - \sum_{\omega \in \Psi^*_{\to i}} p'_\omega \right] = \sum_{i \in I} u_i(\Psi^*) = U^*, \tag{5}$$

with equality holding exactly when (4) holds with equality for every i. If (4) were strict for any i, we would obtain $V(p') > U^*$ from (5), contradicting the assumption that p' minimizes

V and thus satisfies $V(p') = U^*$. Thus, for all $i \in I$, equality holds in (4), and thus $[\Psi^*; p]$ is a competitive equilibrium. $\square$

Now, suppose p and q are two competitive equilibrium price vectors, and let $p \wedge q$ and $p \vee q$ denote their meet and join, respectively. Note that

$$\begin{aligned} 2U^* &\leq V(p \wedge q) + V(p \vee q) \\ &\leq V(p) + V(q) = 2U^*, \end{aligned}$$

where the first inequality follows because (by Lemma B.2) U^* is the minimal value of V, the second follows from the submodularity of V (which holds because by Theorem 1, V_i is submodular for every $i \in I$), and the equality follows from Lemma B.2, because p and q are competitive equilibrium price vectors. Since we also know that $V(p \wedge q) \geq U^*$ and $V(p \vee q) \geq U^*$, it has to be the case that $V(p \wedge q) = V(p \vee q) = U^*$, and so by Lemma B.2, $p \wedge q$ and $p \vee q$ are competitive equilibrium price vectors.

Proof of Theorem 6

Let $A \equiv \kappa([\Psi; p])$. Suppose A is not stable; then either it is not individually rational or there exists a blocking set.

If A is not individually rational, then $A_i \notin C_i(A)$ for some $i \in I$. Hence, $A_i \notin \operatorname{argmax}_{Z \subseteq A_i} U_i(Z)$, and therefore $\tau(A_i) = \Psi_i \notin D_i(p)$, contradicting the assumption that $[\Psi; p]$ is a competitive equilibrium.

Suppose now that there exists a set Z blocking A, and let $J = a(Z)$ be the set of agents involved in contracts in Z. For any trade ω involved in a contract in Z, let $\tilde{p}_\omega$ be the price for which $(\omega, \tilde{p}_\omega) \in Z$. For each $j \in J$, pick a set $Y^j \in C_j(Z \cup A)$. As Z blocks A, (by definition) we have $Z_j \subseteq Y^j$. Since $Z \cap A = \varnothing$, and for all $Y \in C_j(Z \cup A)$ we have that $Z_j \subseteq Y$ and $A_j \notin C_j(Z \cup A)$. Hence, for all $j \in J$,

$$U_j(A) < U_j(Y^j) = \begin{bmatrix} u_j(\tau(Y^j)) + \\ \sum_{\omega \in \tau(Z)_{j\to}} \tilde{p}_\omega - \sum_{\omega \in \tau(Z)_{\to j}} \tilde{p}_\omega + \\ \sum_{\omega \in \tau(Y^j - Z)_{j\to}} p_\omega - \sum_{\omega \in \tau(Y^j - Z)_{\to j}} p_\omega \end{bmatrix}.$$

Summing these inequalities over all $j \in J$, we have

$$\sum_{j \in J} U_j(A) < \sum_{j \in J} \left[\begin{array}{c} u_j(\tau(Y^j))+ \\ \sum_{\omega \in \tau(Y^j - Z)_{j\to}} p_\omega - \sum_{\omega \in \tau(Y^j - Z)_{\to j}} p_\omega \end{array} \right]$$

$$= \sum_{j \in J} \left[\begin{array}{c} u_j(\tau(Y^j))+ \\ \sum_{\omega \in \tau(Z)_{j\to}} p_\omega - \sum_{\omega \in \tau(Z)_{\to j}} p_\omega + \\ \sum_{\omega \in \tau(Y^j - Z)_{j\to}} p_\omega - \sum_{\omega \in \tau(Y^j - Z)_{\to j}} p_\omega \end{array} \right]$$

$$= \sum_{j \in J} \left[\begin{array}{c} u_j(\tau(Y^j))+ \\ \sum_{\omega \in \tau(Y^j)_{j\to}} p_\omega - \sum_{\omega \in \tau(Y^j)_{\to j}} p_\omega \end{array} \right] = \sum_{j \in J} U_j(Y^j),$$

where we repeatedly apply the fact that for every trade ω in $\tau(Z)$, the price (first $\tilde{p}_\omega$ and then p_ω) of this trade is added exactly once and subtracted exactly once in the summation over all agents.

Now, the preceding inequality says that the sum of the utilities of agents in J given prices p would be strictly higher if each $j \in J$ chose Y^j instead of A_j. It therefore must be the case that for some $j \in J$, we have $U_j([\tau(Y^j); p]) > U_j([A; p])$. It follows that $A_j \notin D_j(p)$, and therefore $[\Psi; p]$ cannot be a competitive equilibrium.

Proof of Theorem 7

Consider a stable set $A \subseteq X$. For every agent $i \in I$, define a modified valuation function $\hat{u}_i$, on sets of trades $\Psi \subseteq \Omega - \tau(A)$:

$$\hat{u}_i(\Psi) = \max_{Y \subseteq A_i} \left[u_i(\Psi \cup \tau(Y)) + \sum_{(\omega, \bar{p}_\omega) \in Y_{i\to}} \bar{p}_\omega - \sum_{(\omega, \bar{p}_\omega) \in Y_{\to i}} \bar{p}_\omega \right].$$

In other words, the modified valuation $\hat{u}_i(\Psi)$ of Ψ is equal to the maximal value attainable by agent i by combining the trades in Ψ_i with various subsets of A_i. We denote the utility function associated to $\hat{u}_i$ by $\hat{U}_i$. Since the original utilities were fully substitutable, and thus the choice correspondences C_i satisfied (CEFS), the choice correspondences $\hat{C}_i$ for utility functions $\hat{U}_i$ also satisfy (CEFS) and thus every $\hat{U}_i$ is also fully substitutable.

Now, consider a modified economy for the set of agents I: The set of trades is $\Omega - \tau(A)$, and utilities are given by $\hat{U}$. If there is a competitive equilibrium of the modified economy of the form $[\varnothing; \hat{p}_{\Omega - \tau(A)}]$, i.e., involving no trades, then we are done. We combine the prices in this competitive equilibrium with the prices in A to obtain the price vector p as

$$p_\omega = \begin{cases} \bar{p}_\omega & (\omega, \bar{p}) \in A \\ \hat{p}_\omega & \text{otherwise.} \end{cases}$$

It is clear that $[\tau(A); p]$ is a competitive equilibrium of the original economy: since $\varnothing \in \hat{D}_i(\hat{p})$ for every i, no agent strictly prefers to add trades not in $\tau(A)$, and by the individual rationality of A, no agent strictly prefers to drop any trades in $\tau(A)$.

Now suppose there is no competitive equilibrium of this modified economy in which no trades occur. By Theorem 2, this economy has at least one competitive equilibrium $[\hat{\Psi}; \hat{p}]$. By Theorems 3 and 4, we know that $\hat{\Psi}$ is efficient and $\varnothing$ is not. It follows that

$$\frac{\sum_{i\in I} \hat{u}_i(\hat{\Psi}) - \sum_{i\in I} \hat{u}_i(\varnothing)}{2|\Omega|} > 0;$$

we denote this value by δ.

Now, consider a second modification of the valuation functions:

$$\tilde{u}_i(\Psi) = \hat{u}_i(\Psi) - \delta|\Psi_i|.$$

We show next that utility functions $\tilde{U}_i$ corresponding to $\tilde{u}_i$ are fully substitutable. Take agent i. Take any two price vectors p' and p''. Construct a new price vector $\tilde{p}'$ as follows. For every trade $\omega \in \Omega - \tau(A)$, $\tilde{p}'_\omega = p_\omega + \delta$ if $b(\omega) = i$, $\tilde{p}'_\omega = p_\omega - \delta$ if $s(\omega) = i$, and $\tilde{p}'_\omega = 0$ if $\omega \notin \Omega_i$. Construct price vector $\tilde{p}''$ analogously, starting with p''. Note that for any set of trades $\Psi \subset \Omega - \tau(A)$, we have $\tilde{U}_i([\Psi; p']) = \hat{U}_i([\Psi; \tilde{p}'])$ and $\tilde{U}_i([\Psi; p'']) = \hat{U}_i([\Psi; \tilde{p}''])$, and therefore, for the corresponding indirect utility functions, we have $\tilde{V}_i(p') = \hat{V}_i(\tilde{p}')$ and $\tilde{V}_i(p'') = \hat{V}_i(\tilde{p}'')$.

Now, by the submodularity of $\hat{V}_i$, we have

$$\hat{V}_i(\tilde{p}' \wedge \tilde{p}'') + \hat{V}_i(\tilde{p}' \vee \tilde{p}'') \leq \hat{V}_i(\tilde{p}') + \hat{V}_i(\tilde{p}''),$$

and therefore

$$\tilde{V}_i(p' \wedge p'') + \tilde{V}_i(p' \vee p'') \leq \tilde{V}_i(p') + \tilde{V}_i(p'').$$

Hence, $\tilde{V}_i$ is submodular, and therefore $\tilde{U}_i$ is fully substitutable.

Now, by our choice of δ, $\sum_{i\in I} \tilde{u}_i(\hat{\Psi}) > \sum_{i\in I} \tilde{u}_i(\varnothing)$. Thus, $\varnothing$ is not efficient under the valuations $\tilde{u}$ and therefore cannot be supported in a competitive equilibrium under those valuations. Take any competitive equilibrium $[\tilde{\Psi}, q]$ of the economy with agents I, trades $\Omega - \tau(A)$, and utilities $\tilde{U}$. We know that $\tilde{\Psi} \neq \varnothing$. Moreover, since $\tilde{\Psi} \in \tilde{D}_i(q)$ for every i (where $\tilde{D}$ is the demand correspondence induced by $\tilde{U}$), we know that $\tilde{U}([\tilde{\Psi}; q]) \geq \tilde{U}([\Phi; q])$ for any

$\Phi \subsetneq \tilde{\Psi}$, which in turn implies $\hat{U}([\tilde{\Psi}; q]) > \hat{U}([\Phi; q])$. This, in turn, implies that for all i, in the original economy with trades Ω and utility functions U_i, the set of trades $\{(\psi, q_\psi) : \psi \in \tilde{\Psi}_i\}$ is a subset of every $Y \in C_i(A \cup \{(\psi, q_\psi) : \psi \in \tilde{\Psi}_i\})$. Thus, $\{(\psi, q_\psi) : \psi \in \tilde{\Psi}_i\}$ is a blocking set for A, contradicting the assumption that A is stable.

Proof of Theorem 8

Suppose that the preferences of i are not fully substitutable, and in particular fail the first part of the definition of (DFS) with unique demands (Definition 4). Then there exist price vectors p and p' and trades ω and ψ where $b(\omega) = i$ and $p'_{-\omega} = p_{-\omega}, p'_\omega > p_\omega$ and $\{\Psi\} = D^i(p)$ and $\{\Psi'\} = D^i(p')$ and either

Case 1. $b(\psi) = i$ and $\psi \in \Psi$ but $\psi \notin \Psi'$, or

Case 2. $s(\psi) = i$ and $\psi \notin \Psi$ but $\psi \in \Psi'$.

For every trade $\xi \in \Omega_i - \{\psi, \omega\}$, if $b(\xi) = i$ we let

$$u^{s(\xi)}(\Xi - \{\xi\}) - u^{s(\xi)}(\Xi \cup \{\xi\}) = p_\xi$$

for all $\Xi \subseteq \Omega$ and similarly, if $s(\xi) = i$, then we let

$$u^{b(\xi)}(\Xi \cup \{\xi\}) - u^{b(\xi)}(\Xi - \{\xi\}) = p_\xi$$

for all $\Xi \subseteq \Omega$. It is clear that these preferences, restricted to $\Omega_i - \{\psi, \omega\}$, are simple for every agent.

Without further specification of agents' preferences we can infer that whenever a stable outcome A exists, the outcome

$$\bar{A} = (A - \{(\xi, q_\xi) : \xi \in [\tau(A_i) - \{\psi, \omega\}]\, ; (\xi, q_\xi) \in A\}) \cup \{(\xi, p_\xi) : \xi \in [\tau(A_i) - \{\psi, \omega\}]\}$$

is also stable. To see this, note that if $(\xi, q_\xi) \in A$ for some $\xi \neq \psi, \omega$ such that $b(\xi) = i$ then $q_\xi \geq p_\xi$. If $q_\xi > p_\xi$ then $\tilde{A} \equiv [A - (\xi, q_\xi)] \cup \{(\xi, p_\xi)\}$ is also a stable match, as it is clearly individually rational as A is individually rational, and if Z was a blocking set for $\tilde{A}$, it would also be a blocking set for A. Similarly, if $(\xi, q_\xi) \in A$, $s(\xi) = i$, and $\xi \neq \psi, \omega$ then $q_\xi \leq p_\xi$ and so $[A - (\xi, q_\xi)] \cup \{(\xi, p_\xi)\}$ is also a stable match. The above claim now follows by induction.

It will be helpful to define the marginal utility agent i obtains from having available trades in some set $\Phi \subseteq \{\psi, \omega\}$ in addition to having trades in $\Omega_i - \{\psi, \omega\}$ at their prices according to the price vector p by

$$v^i(\Phi) \equiv \max_{\substack{\Xi \subseteq \Omega_i - \{\psi,\omega\} \\ \hat{\Phi} \subseteq \Phi}} \left\{ u^i(\Xi \cup \hat{\Phi}) + \sum_{\xi \in \Xi_{i\to}} p_\xi - \sum_{\xi \in \Xi_{\to i}} p_\xi \right\}.$$

We now proceed to discuss the two possible cases.

Case 1: $b(\psi) = i$ and $\psi \in \Psi$ but $\psi \notin \Psi'$.

Note that

$$v^i(\{\psi,\omega\}) - v^i(\{\omega\}) > v^i(\{\psi\}) - v^i(\varnothing) \geq 0,$$

as otherwise we would have that $\psi \in \Psi'$, as if

$$v^i(\{\psi,\omega\}) - v^i(\{\omega\}) \leq v^i(\{\psi\}) - v^i(\varnothing),$$

then i must demand ψ at prices $(p_{-\omega}, p'_\omega)$ as i demanded ψ at prices p.

Now let $\hat{\psi}, \hat{\omega}$ be two trades such that $s(\psi) = s(\hat{\psi})$, $s(\omega) = s(\hat{\omega})$, and $b(\hat{\psi}) = b(\hat{\omega}) = j \neq i$ (such a trade must exist as there are at least four agents and the set of trades is exhaustive). Let $s(\psi), s(\omega)$ have preferences such that

$$\begin{aligned} u^{s(\psi)}(\Xi \cup \{\psi\}) - u^{s(\psi)}(\Xi) = u^{s(\psi)}\left(\Xi \cup \left\{\hat{\psi}\right\}\right) - u^{s(\psi)}(\Xi) = 0, \\ u^{s(\psi)}\left(\Xi \cup \left\{\psi, \hat{\psi}\right\}\right) = -\infty \end{aligned}$$

for all $\Xi \subseteq \Omega - \left\{\psi, \hat{\psi}\right\}$ and

$$\begin{aligned} u^{s(\omega)}(\Xi \cup \left\{\omega\right\}) - u^{s(\omega)}(\Xi) = u^{s(\omega)}(\Xi \cup \left\{\hat{\omega}\right\}) - u^{s(\omega)}(\Xi) = 0, \\ u^{s(\omega)}(\Xi \cup \left\{\omega, \hat{\omega}\right\}) = -\infty \end{aligned}$$

for all $\Xi \subseteq \Omega - \{\omega, \hat{\omega}\}$. It is possible that $s(\psi) = s(\omega)$.

Let j's preferences satisfy

$$\begin{aligned} u^j\left(\left\{\hat{\psi}\right\} \cup \Xi\right) - u^j(\Xi) &= \frac{2\left[v^i(\{\psi,\omega\}) - v^i(\{\omega\})\right] + \left[v^i(\{\psi\}) - v^i(\varnothing)\right]}{3} \equiv w(\psi), \\ u^j(\{\hat{\omega}\} \cup \Xi) - u^j(\Xi) &= \frac{2\left[v^i(\{\psi,\omega\}) - v^i(\{\psi\})\right] + \left[v^i(\{\omega\}) - v^i(\varnothing)\right]}{3} \equiv w(\omega), \\ u^j\left(\left\{\hat{\psi}, \hat{\omega}\right\} \cup \Xi\right) - u^j(\Xi) &= -\infty \end{aligned}$$

for all $\Xi \subseteq \Omega - \{\hat{\psi}, \hat{\omega}\}$. Then, by the above inequality, we must have

$$0 < w(\psi) < v^i(\{\psi, \omega\}) - v^i(\{\omega\}),$$
$$0 < w(\omega) < v^i(\{\psi, \omega\}) - v^i(\{\psi\}).$$

Clearly, the preferences of all agents but i can be extended to simple preferences on Ω.[13] There are four subcases to consider to show that $\bar{A}$ cannot be stable:

Subcase 1: $\tau(\bar{A}) \cap \{\psi, \omega\} = \emptyset$. If both $\hat{\psi}$ and $\hat{\omega} \in \tau(\bar{A})$, then $\bar{A}$ is not individually rational for j. If $\hat{\psi}, \hat{\omega} \notin \tau(\bar{A})$, then $\{(\hat{\psi}, \epsilon)\}$ is a block. Hence, exactly one of $\hat{\psi}$ and $\hat{\omega}$ is in $\tau(\bar{A})$. Suppose $(\hat{\psi}, q_{\hat{\psi}}) \in \bar{A}$ for some $q_{\hat{\psi}} \in \mathbb{R}_+$. Individual rationality for j requires that

$$q_{\hat{\psi}} \leq w(\psi) < v^i(\{\psi, \omega\}) - v^i(\{\omega\}).$$

But then

$$Z \equiv \left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi - \tau(\bar{A}_i)\right\} \cup \left\{(\psi, q_{\hat{\psi}} + \epsilon), (\omega, \epsilon)\right\}$$

is a blocking set for some small $\epsilon > 0$. Note that (ω, ϵ) strictly increases by ϵ the utility of $s(\omega)$, no matter what other contracts $s(\omega)$ chooses. Similarly, for all $\xi \in \Psi - \tau(\bar{A})$, $(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i})$ strictly increases by ϵ the utility of the agent other than i associated with this contract, no matter what other contracts that agent chooses. Agent $s(\hat{\psi})$ will choose contract $(\psi, q_{\hat{\psi}} + \epsilon)$ and not choose $(\hat{\psi}, q_{\hat{\psi}})$, regardless of other contracts he chooses. Finally, agent i's choice $\bar{A} \cup Z$ is single valued and includes Z, as the above inequality implies that if i chooses (ω, ϵ), he must also choose $(\psi, q_{\hat{\psi}} + \epsilon)$. We also have that $v^i(\{\omega\}) \geq v^i(\varnothing)$, implying that for ϵ small enough i will choose both $(\psi, q_{\hat{\psi}} + \epsilon)$ and (ω, ϵ) from $\bar{A} \cup Z$, and hence i will choose all of the contracts associated with trades in Ψ as such contracts are optimal at prices p.

If $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_+$, we obtain a similar contradiction since individual rationality for j requires that $q_{\hat{\omega}} \leq w(\omega) < v^i(\{\psi, \omega\}) - v^i(\{\psi\})$.

Subcase 2: $(\psi, q_\psi) \in \bar{A}$ for some $q_\psi \in \mathbb{R}_+$ and $\omega \notin \tau(\bar{A})$. In this case we must have $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_+$, as otherwise $\{(\hat{\omega}, \epsilon)\}$ for some small $\epsilon > 0$ would be a blocking set since j's incremental utility of signing $\hat{\omega}$ is $w(\omega) > 0$. Individual rationality for j requires

$$q_{\hat{\omega}} \leq w(\omega) < v^i(\{\psi, \omega\}) - v^i(\{\omega\}).$$

[13] Throughout the proof we only specify the parts of the preferences that are important to show that no stable outcome can exist.

Furthermore, we must have

$$q_\psi \leq v^i(\{\psi\}) - v^i(\varnothing)$$

as otherwise either $\bar{A}$ is not individually rational for i or

$$\left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi' - \tau(\bar{A})\right\}$$

is a blocking set for $\epsilon > 0$ sufficiently small. However, these inequalities imply that

$$\left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi - \tau(\bar{A})\right\} \cup \{(\omega, p_{\hat{\omega}} + \epsilon)\}$$

is a blocking set for some small $\epsilon > 0$.

Subcase 3: $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_+$ and $\psi \notin \tau(\bar{A})$. The reasoning is analogous to that of the previous subcase.

Subcase 4: $\{(\psi, q_\psi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\psi, q_\omega \in \mathbb{R}_{\geq 0}$. It must be the case that

$$q_\psi + q_\omega \leq v^i(\{\psi, \omega\}) - v^i(\varnothing),$$

as otherwise $\bar{A}$ is not individually rational for i or

$$\left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi' - \tau(\bar{A})\right\}$$

is a blocking set for some small $\epsilon > 0$. In order to prevent a block by $s(\psi)$ and j (using $(\hat{\psi}, q_\psi + \epsilon)$ for some small $\epsilon > 0$), we must have $q_\psi \geq w(\psi)$. Similarly, to prevent a block by $s(\omega)$ and j, we must have $q_\omega \geq w(\omega)$. Simple algebra shows that $w(\psi) + w(\omega) > v^i(\{\psi, \omega\}) - v^i(\emptyset)$ is equivalent to the inequality $v^i(\{\psi, \omega\}) + v^i(\emptyset) > v^i(\{\psi\}) + v^i(\{\omega\})$, which holds as explained in the beginning of the proof of Case 1. Hence, we must have $q_\psi + q_\omega > v^i(\{\psi, \omega\}) - v^i(\emptyset)$, contradicting our earlier statement.

Case 2: $s(\psi) = i$ and $\psi \notin \Psi$ but $\psi \in \Psi'$.

Note that

$$v^i(\{\omega\}) - v^i(\varnothing) > v^i(\{\psi, \omega\}) - v^i(\{\psi\}),$$

as otherwise we would have $\psi \in \Psi'$, as if

$$v^i(\{\omega\}) - v^i(\{\psi, \omega\}) \leq v^i(\varnothing) - v^i(\{\psi\}),$$

then i must demand to sell ψ at prices p if i demanded to sell ψ at prices $(p_{-\omega}, p'_\omega)$.

As in Case 1, we use the following conventions to simplify notation:

$$\frac{2\left[v^i(\{\omega\}) - v^i(\{\psi,\omega\})\right] + \left[v^i(\emptyset) - v^i(\{\psi\})\right]}{3} \equiv w(\psi),$$
$$\frac{2\left[v^i(\{\omega\}) - v^i(\varnothing)\right] + \left[v^i(\{\psi,\omega\}) - v^i(\{\psi\})\right]}{3} \equiv w(\omega).$$

By the above inequality, we must have

$$0 < w(\psi) < v^i(\{\omega\}) - v^i(\{\psi,\omega\}),$$
$$0 < w(\omega) < v^i(\{\omega\}) - v^i(\emptyset).$$

We have to consider two subcases, depending on whether $s(\omega)$ is equal to $b(\psi)$.

Subcase 1: $s(\omega) \neq b(\psi)$. Consider the trade $\hat{\omega}$ (which must exist by exhaustivity), where $s(\hat{\omega}) = s(\omega)$ and $b(\hat{\omega}) = b(\psi) \equiv j$. Let $s(\omega)$ have preferences such that

$$u^{s(\omega)}(\Xi \cup \{\omega\}) - u^{s(\omega)}(\Xi) = u^{s(\omega)}(\Xi \cup \{\hat{\omega}\}) - u^{s(\omega)}(\Xi) = 0,$$
$$u^{s(\omega)}(\Xi \cup \{\omega, \hat{\omega}\}) = -\infty$$

for all $\Xi \subseteq \Omega - \{\omega, \hat{\omega}\}$.

Now let j's preferences satisfy

$$u^j(\{\hat{\omega}\} \cup \Xi) - u^j(\Xi) = w(\omega),$$
$$u^j(\{\psi\} \cup \Xi) - u^j(\Xi), = w(\psi)$$
$$u^j(\{\psi, \hat{\omega}\} \cup \Xi) - u^j(\Xi) = -\infty,$$

for all $\Xi \subseteq \Omega - \{\psi, \hat{\omega}\}$.

We note that preferences of the above type can be extended to simple preferences over all sets of trades. Now, we show that no stable match can exist if preferences satisfy the above properties by distinguishing four cases.

(a) $\{\psi, \omega, \hat{\omega}\} \cap \tau(\bar{A}) = \emptyset$: In this case

$$\left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi - \tau(\bar{A}) - \{\omega\}\right\} \cup \{(\omega, w(\omega))\}$$

is a blocking set for sufficiently small $\epsilon > 0$, as it is increases the utility of each agent except i and $s(\omega)$ by at least ϵ, increases the utility of $s(\omega)$ by at least $w(\omega) > 0$, and increases i's utility , since $w(\omega) < v^i(\{\omega\}) - v^i(\emptyset)$.

(b) $(\hat{\omega}, q_{\hat{\omega}}) \in \bar{A}$ for some $q_{\hat{\omega}} \in \mathbb{R}_{\geq 0}$: Given our assumptions about preferences, individual rationality (for $s(\omega)$ and j) requires that $\psi, \omega \notin \tau(\bar{A})$ and $q_{\hat{\omega}} \leq w(\omega)$. Since $w(\omega) <$

$v^i(\{\omega\}) - v^i(\emptyset)$, this implies that

$$\left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi - \tau(\bar{A}) - \{\omega\}\right\} \cup \{(\omega, q_{\hat{\omega}} + \epsilon)\}$$

is a blocking set for sufficiently small $\epsilon > 0$; this shows that we cannot have $\hat{\omega} \in \tau(\bar{A})$.

(c) $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_+$ and $\psi \notin \tau(\bar{A})$: In this case j obtains a utility of zero under $\bar{A}$ and in order to prevent a block by $s(\omega)$ and j, we must have $q_\omega \geq w(\omega)$. But then

$$Z \equiv \left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi' - \tau(\bar{A}) - \{\psi\}\right\} \cup \{(\psi, w(\psi) - \epsilon)\}$$

is a blocking set for sufficiently small $\epsilon > 0$. To see this note first that j will clearly accept all of his contracts in the blocking set, since each of these contracts increases his utility by $\epsilon > 0$. Note that i's utility after the block is $v^i(\{\psi\}) + w(\psi) - |Z|\epsilon$, while his utility before the block is at most $v^i(\{\omega\}) - w(\omega)$. Subtracting the former expression from the latter, we obtain

$$\frac{[v^i(\{\omega\}) - v^i(\varnothing)] - [v^i(\{\omega, \psi\}) - v^i(\{\psi\})]}{3} - |Z|\epsilon,$$

which is positive for $\epsilon > 0$ sufficiently small.

(d) $\{(\psi, q_\psi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\psi, q_\omega \in \mathbb{R}_{\geq 0}$: We must have that

$$q_\psi \geq v^i(\{\omega\}) - v^i(\{\psi, \omega\}),$$

since otherwise either $\bar{A}$ would not be individually rational for i, or

$$\left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi - \tau(\bar{A})\right\}$$

would be a blocking set. Similarly, we must have

$$q_\omega \leq v^i(\{\omega, \psi\}) - v^i(\{\psi\}).$$

We claim that $\{(\hat{\omega}, q_\omega + \epsilon)\}$ is a blocking set for $\epsilon > 0$ sufficiently small. It will clearly be chosen by $s(\omega)$, and $b(\psi)$ obtains a utility increase of at least

$$[w(\omega) - (q_\omega + \epsilon)] - [w(\psi) - q_\psi].$$

Substituting and using the price inequalities we just derived, we find that this expression is greater than or equal to

$$[v^i(\{\omega\}) - v^i(\emptyset)] - [v^i(\{\psi, \omega\}) - v^i(\{\psi\})] - \epsilon,$$

which is positive for ϵ sufficiently small.

(e) $(\psi, q_\psi) \in \bar{A}$ for some $q_\psi \in \mathbb{R}$ and $\omega \notin \tau(\bar{A})$: Then we must have

$$\begin{aligned} q_\psi &\leq w(\psi) - w(\omega) \\ &\leq v^i(\varnothing) - v^i(\{\psi, \omega\}) \end{aligned}$$

for $\{(\hat{\omega}, \epsilon)\}$ to not be a blocking set for $\bar{A}$. But then

$$Z \equiv \left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi - \tau(\bar{A}) - \{\omega\}\right\} \cup \{(\omega, \epsilon)\}$$

is a blocking set for $\epsilon > 0$ sufficiently small, as $s(\omega)$ will clearly accept this contract, and i's utility before is at most

$$v^i(\{\psi\}) + v^i(\varnothing) - v^i(\{\psi, \omega\})$$

and choosing from $\bar{A} \cup Z$, i obtains

$$v^i(\{\omega\}) - |Z|\epsilon.$$

Subtracting the former expression from the latter, we obtain

$$\left[v^i(\{\omega\}) - v^i(\varnothing)\right] - \left[v^i(\{\psi, \omega\}) - v^i(\{\psi\})\right] - |Z|\epsilon$$

which is positive for $\epsilon > 0$ sufficiently small.

Subcase 2: $s(\omega) = b(\psi) \equiv j$**.** Let j's preferences satisfy, for all $\Xi \subseteq \Omega - \{\omega, \psi\}$,

$$\begin{aligned} u^j(\{\omega\} \cup \Xi) - u^j(\Xi) &= -w(\omega), \\ u^j(\{\psi, \omega\} \cup \Xi) - u^j(\Xi) &= w(\psi) - w(\omega), \\ u^j(\{\psi\} \cup \Xi) - u^j(\Xi) &= -\infty. \end{aligned}$$

There are four subcases to consider to show that $\bar{A}$ cannot be stable:

(a) $\tau(\bar{A}) \cap \{\psi, \omega\} = \varnothing$: The argument from Case 2(a) can be used to show that i and $j = s(\omega)$ have an incentive to deviate.

(b) $(\omega, q_\omega) \in \bar{A}$ for some $q_\omega \in \mathbb{R}_{\geq 0}$: Suppose that $\psi \notin \tau(\bar{A})$. Individual rationality for j requires that $q_\omega \geq w(\omega)$. The argument from Case 2(c) can then be used to establish that

$$\left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi' - \tau(\bar{A}) - \{\psi\}\right\} \cup \{(\psi, w(\psi) - \epsilon)\}$$

is a blocking set for sufficiently small $\epsilon > 0$.

(c) $\{(\psi, q_\psi), (\omega, q_\omega)\} \subseteq \bar{A}$ for some $q_\psi, q_\omega \in \mathbb{R}_{\geq 0}$: We must have

$$q_\psi \geq v^i(\{\omega\}) - v^i(\{\psi, \omega\}),$$

as otherwise either $\bar{A}$ would not be individually rational for i, or

$$\left\{(\xi, p_\xi + \epsilon \mathbb{I}_{b(\xi)=i} - \epsilon \mathbb{I}_{s(\xi)=i}) : \xi \in \Psi - \tau(\bar{A})\right\}$$

would be a blocking set. Similarly, we must have

$$q_\omega \leq v^i(\{\omega, \psi\}) - v^i(\{\psi\}).$$

The first inequality implies that $\bar{A}$ cannot be individually rational for j since the incremental utility of signing ψ on top of ω is

$$w(\psi) - q_\psi \leq w(\psi) - [v^i(\{\omega\}) - v^i(\{\psi, \omega\})] < 0.$$

(d) $\psi \in \tau(\bar{A})$ and $\omega \notin \tau(\bar{A})$: This clearly cannot be individually rational for j, given that he obtains $-\infty$ utility if he signs ψ but not ω.

The argument in the case that the preferences of i do not satisfy the second part of Definition 4 is analogous to that presented above for the first part.

Proof of Theorem 10

Suppose A is a stable outcome that is not strongly group stable. Let Z be a set that strongly blocks A. By the second part of the proof of Theorem 6, there is no vector of prices p such that $[\tau(A)\,; p]$ is a competitive equilibrium. This contradicts Theorem 7.

To see that for any core outcome A there is a stable outcome $\hat{A}$ such that $\tau(A) = \tau(\hat{A})$, note that by Theorem 9, every core outcome induces an efficient set of trades. By Theorem 4, we can find a competitive equilibrium corresponding to any efficient set of trades. Finally, by Theorem 6, the competitive equilibrium induces a stable outcome.

Proof of Theorem 11

We prove the following result, which implies Theorem 11 by a natural inductive argument:

Lemma B.3. *For any feasible outcome A blocked by a nonempty set Z, if Z is not itself a chain, then there exists a nonempty chain $W \subsetneq Z$ such that set A is blocked by $Z - W$ and set $A \cup (Z - W)$ is blocked by W.*

Proof. The second part of the statement, that $A \cup (Z - W)$ is blocked by W, is true for any $W \subseteq Z$: Set Z blocks A, and thus for every agent i involved in W (and hence in Z), every choice of i from $A \cup Z = (A \cup (Z - W)) \cup W$ contains every contract in Z_i, and thus contains every contract in $W_i \subset Z_i$. So we only need to prove the first part of the statement, that we can "remove" some chain W from Z so that the remaining set $Z - W$ still blocks A.

Also, without loss of generality, we can assume that set A is empty: As in the proof of Theorem 7, consider a modified economy with the set of available trades equal to $\Omega - \tau(A)$ and agents' valuations over subsets Ψ of this set given by

$$\hat{u}_i(\Psi) = \max_{\Xi \subseteq A_i} \left[u_i(\Psi \cup \tau(\Xi)) + \sum_{(\omega, p_\omega) \in \Xi_{i\rightarrow}} p_\omega - \sum_{(\omega, p_\omega) \in \Xi_{\rightarrow i}} p_\omega \right].$$

Agents' preferences in this modified economy are fully substitutable, and blocking of outcome A by any set of contracts Y in the original economy is equivalent to blocking of the empty set of contracts by the same set Y in the modified economy. Note that by definition, set Y blocks the empty set if and only if for every i involved in Y, the unique optimal choice of agent i from Y_i is equal to Y_i itself.

Consider now any contract $y \in Z$, and let $y^0 = y$. We will algorithmically "grow" a chain

$$W^{\ell_s, \ell_b} \equiv \{y^{-\ell_s}, \ldots, y^{\ell_b}\}$$

by applying (generally in both directions from y^0) the iterative procedure below, starting with $\ell_s = \ell_b = 0$ and $W^{0,0} = \{y^0\}$. We will ensure that at every step of the procedure, the following four conditions hold for every agent i. If $i \neq b(y^{\ell_b})$, $i \neq s(y^{\ell_s})$, then $C_i(Z - W^{\ell_s, \ell_b}) = \{(Z - W^{\ell_s, \ell_b})_i\}$. If $i = b(y^{\ell_b}) \neq s(y^{\ell_s})$, then $C_i((Z - W^{\ell_s, \ell_b}) \cup \{y^{\ell_b}\}) = \{(Z - W^{\ell_s, \ell_b})_i \cup \{y^{\ell_b}\}\}$, and if $i = s(y^{\ell_s}) \neq b(y^{\ell_b})$, then $C_i((Z - W^{\ell_s, \ell_b}) \cup \{y^{\ell_s}\}) = \{(Z - W^{\ell_s, \ell_b})_i \cup \{y^{\ell_s}\}\}$. Finally, if $i = s(y^{\ell_s}) = b(y^{\ell_b})$, then $C_i((Z - W^{\ell_s, \ell_b}) \cup \{y^{\ell_b}, y^{\ell_s}\}) = \{(Z - W^{\ell_s, \ell_b})_i \cup \{y^{\ell_b}, y^{\ell_s}\}\}$. Clearly, these conditions are satisfied in the beginning. Our chain-growing algorithm has two main steps:

Buyer Step: Let $j = b(y^{\ell_b}) \neq s(y^{\ell_s})$. If $C_j(Z - W^{\ell_s, \ell_b}) = \{(Z - W^{\ell_s, \ell_b})_j\}$, stop. Otherwise, recall that $C_j((Z - W^{\ell_s, \ell_b}) \cup \{y^{\ell_b}\}) = \{(Z - W^{\ell_s, \ell_b})_j \cup \{y^{\ell_b}\}\}$. This implies (by full substitutability) that each $Y \in C_j(Z - W^{\ell_s, \ell_b})$ must contain $(Z - W^{\ell_s, \ell_b})_{\rightarrow j}$ as a subset, and also (by the Law of Aggregate Demand; see Section A.3) that each such Y excludes at most one contract in $(Z - W^{\ell_s, \ell_b})_{j\rightarrow}$. Pick any such Y that excludes exactly one contract, and denote that excluded contract by y^{ℓ_b+1}. Note that by construction, the unique optimal choice of j from $(Z - W^{\ell_s, \ell_b+1})$ is $(Z - W^{\ell_s, \ell_b+1})_j$, and the four conditions are satisfied for W^{ℓ_s, ℓ_b+1}.

Seller Step: Let $k = s(y^{\ell_s}) \neq b(y^{\ell_b})$. Consider its optimal choices from $Z - W^{\ell_s, \ell_b}$. If the unique optimal choice is $(Z - W^{\ell_s,\ell_b})_k$, stop. Otherwise, by analogy with the Buyer Step, pick contract y^{ℓ_s-1}.

As long as $s(y^{\ell_s}) \neq b(y^{\ell_b})$, we sequentially iterate these two steps for W^{ℓ_s,ℓ_b}, decrementing ℓ_s and incrementing ℓ_b with each iteration.

Over the course of this process, it may happen that $s(y^{\ell_s}) = b(y^{\ell_b}) \equiv h$. If $C_h(Z - W^{\ell_s,\ell_b}) = \{(Z - W^{\ell_s,\ell_b})_h\}$, we are done—the chain W^{ℓ_s,ℓ_b} suffices for the result. Otherwise, note that as before, by full substitutability and the Laws of Aggregate Supply and Demand, each $Y \in C_h(Z - W^{\ell_s,\ell_b})$ excludes at most one upstream contract and at most one downstream contract from $(Z - W^{\ell_s,\ell_b})_h$. Take any such excluded contract y, and suppose it is a downstream contract for agent h (the argument if it is an upstream contract for h is completely analogous). Let $y^{\ell_b+1} = y$. Each $Y \in C_h((Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})$ is, by construction, also in $C_h(Z - W^{\ell_s,\ell_b})$. Each such Y contains $(Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})_{h\to}$ as a subset and excludes at most one contract from $(Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})_{\to h}$. If it so happens that each Y contains $(Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})_{\to h}$ as a subset, then we continue with the Buyer Step on W^{ℓ_s,ℓ_b+1}. Otherwise, we select any excluded contract in $(Z - W^{\ell_s,\ell_b}) - \{y^{\ell_b+1}\})_{\to h}$, take it to be y^{ℓ_s-1}, and continue with the Buyer Step on W^{ℓ_s-1,ℓ_b+1}.

Since set Z is finite, this algorithm must terminate, resulting in some chain W^{ℓ_s,ℓ_b}. At every iteration, we ensured that for each agent $i \notin \{b(y_b^\ell), s(y_s^\ell)\}$, $C_i(Z - W^{\ell_s,\ell_b}) = \{(Z - W^{\ell_s,\ell_b})_i\}$. The algorithm's stopping conditions ensure that the same equality also holds for $i \in \{b(y_b^\ell), s(y_s^\ell)\}$. Thus, the empty set is blocked by $Z - W^{\ell_s,\ell_b}$. □

Proof of Theorem 12

We prove part (a); the proof of part (b) is completely analogous. We show first that $\Xi_i \in D_i(q)$. In the following, let $\bar{p} = \max_{\omega\in\Psi} p_\omega$ and let $\xi \in \Psi$ be any trade such that $p_\xi = \bar{p}$. Note that $|\Xi_i \cap \Psi| \in \{0,1\}$ due to mutual incompatibility, and if $\omega \in \Xi_i \cap \Psi$, then $p_\omega = \bar{p}$: Otherwise, perfect substitutability would imply $U_i([\Xi - \{\omega\} \cup \{\xi\}; p]) > U_i([\Xi; p])$, contradicting the assumption that $[\Xi; p]$ is a competitive equilibrium.

This implies that prices for trades in Ξ_i have not been changed in going from p to q, and in particular implies that $U_i([\Xi_i; p]) = U_i([\Xi_i; q])$. Since prices for trades in $\Omega - \Psi$ have also not been changed, we must have $U_i([\Xi_i; q]) = U_i([\Xi_i; p]) \geq U_i([\Phi; p]) = U_i([\Phi; q])$ for all $\Phi \subseteq \Omega - \Psi$. Now, take any set $\Phi \subseteq \Omega - \Psi$ and any $\omega \in \Psi$. By perfect substitutability, $U_i([\Phi \cup \{\omega\}; q]) \leq U_i([\Phi \cup \{\xi\}; q]) = U_i([\Phi \cup \{\xi\}; p]) \leq U_i([\Xi_i; p]) = U_i([\Xi_i; q])$. Hence, $\Xi_i \in D_i(q)$.

Finally, consider an arbitrary agent $j \neq i$. If $\Xi_j \cap \Psi = \{\omega\}$, we must have $p_\omega = \bar{p}$, implying $U_j([\Xi_j; q]) = U_j([\Xi_j; p])$. If $\Xi_j \cap \Psi = \emptyset$, the last statement is evidently true as well. Let $\Phi \subseteq \Omega_j$ be arbitrary and note that $U_j([\Phi; q]) \leq U_j([\Phi; p])$, since trades in $\Phi \cap \Psi_{\rightarrow j}$ have become weakly more expensive. Since $U_j([\Xi_j; q]) = U_j([\Xi_j; p]) \geq U_j([\Phi; p])$, we obtain $\Xi_j \in D_j(q)$. This completes the proof.

Appendix C: Example Omitted from Section 5.1

In this appendix, we provide an example of an outcome that is stable and in the core, but is not strongly group stable.

Example C.1. Let $I = \{i, j\}$, $\Omega = \{\chi, \psi, \omega\}$, and $s(\chi) = s(\psi) = s(\omega) = i$ and $b(\chi) = b(\psi) = b(\omega) = j$. Furthermore, let agents' valuations be given by:

Ψ	$\varnothing$	$\{\chi\}$	$\{\psi\}$	$\{\omega\}$	$\{\chi, \psi\}$	$\{\chi, \omega\}$	$\{\psi, \omega\}$	$\{\chi, \psi, \omega\}$
$u_i(\Psi)$	0	0	-2	-2	-2	-2	-9	-20
$u_j(\Psi)$	0	2	1	1	3	3	2	15

.

In this case, any outcome of the form $\{(\chi, p_\chi)\}$ such that $0 \leq p_\chi \leq 2$ is both stable and in the core. At the same time, any such outcome is not strongly group stable, as $\{(\psi, 6), (\omega, 6)\}$ constitutes a block.

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