

Cascades of Dynamical Transitions in an Adaptive Population

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In an adaptive population that models financial markets and distributed control, we consider how the dynamics depends on the diversity of the agents' initial preferences of strategies. When the diversity decreases, more agents tend to adapt their strategies together. This change in the environment results in dynamical transitions from vanishing to non-vanishing step sizes. When the diversity decreases further, we find a cascade of dynamical transitions for the different signal dimensions, which is supported by good agreement between simulations and theory. Besides, the signal of the largest step size at steady state is likely to be the initial signal.

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I. INTRODUCTION

Many natural and artificial systems consist of a population of agents with coupled dynamics. Through their mutual adaptation, they are able to exhibit interesting collective behavior. Although the individuals are competing to maximize their own payoffs, the system is able to self-organize itself to globally efficient states. Examples can be found in economic markets and communication networks [1–4].

An important factor affecting the behavior of an adaptive population is the dependence of the payoffs on the environment experienced by the individual agents. The payoffs help the agents to assess the preferences of their decisions, hence inducing them to take certain actions when they experience a similar dynamical environment in the future. Thus, the payoff function is crucial to the mechanism of adaptation.

As a prototype of an adaptive population, the Minority Game (MG) considers the dynamics of buyers and sellers in a model of the financial market in which the minority group is the winning one [2]. A good indicator of the mutual adaptation of the agents is the reduction of the variance of the buyer population with values below those of random fluctuations [2]. Furthermore, this variance has a universal dependence on the complexity of the strategies adopted by the agents, dropping to a minimum when the complexity is reduced to a universal critical value, and rapidly rising thereafter [5,6]. Theoretical studies using the replica method [7,8] and the generating functional [9,10] confirmed these general trends.

The agents in the original version of MG use a *step* payoff function [2,5,6], meaning that the payoffs received

by the winning group are the same, irrespective of the *winning margin* (the difference between the majority and the minority groups). Latter versions of MG use a *linear* payoff function [7–10], in which the payoffs increase with the winning margin. Other payoff functions yield the same macroscopic behaviors in their dependence of the population variance on the complexity of strategies [11, 12]. Thus, the behavior of the population is universal as long as the payoff function favors the minority group. A recent extension of the MG considers payoff functions that reward the minority agents only when they win by a large margin, but punish them when the winning margin is small [13]. The extended model displays a smooth crossover from a minority game to a majority game when the payoff function is tuned.

However, when one considers details beyond the population variance, one can find that the agents self-organize in different ways for different payoff functions. For a payoff function that favors a large winning margin, the distribution of the buyer population is double-peaked [2]. This shows that the dynamics of the population self-organizes to favor large winning margins of either buyers or sellers, because the agents have adapted themselves to maximize their payoffs.

In this paper, we compare the behavior of MGs using step and linear payoffs. Previously, we found that the population variance scaled as a power law of the *diversity* for a step payoff [14,15]. Diversity refers to the variance of the initial biases of the strategy payoffs of the agents. In a population with diverse preferences of strategies, the adaptation rate is slow, resulting in small fluctuations of the buyer or the seller population. As we shall see, when the payoff function becomes linear, the scaling relation between the variance and the diversity for the step payoff is replaced by a continuous dynamical transition from

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a vanishing variance at high diversity to a finite variance at low diversity. The dynamical transition is due to the payoffs being enhanced by large winning margins at low diversity. Furthermore, for systems with multi-dimensional signals feeding the strategies, the dynamical transition in each dimension does not take place at the same transition point. Rather, there is a cascade of dynamical transitions for different signal dimensions. This rich behavior demonstrates the flexibility of an adaptive population for self-organizing to states in which agents maximize their payoffs and is, hence, important in the modeling of economics and distributed control.

II. MINORITY GAME

The Minority Game model consists a population of N agents competing for limited resources, N being odd [2]. Each agent makes a decision 1 or 0 at each time step, and the minority group wins. For economic markets, the decisions 1 and 0 correspond to buying and selling respectively, so that the buyers can win by belonging to the minority group, which pushes the price down, and vice versa. For typical control tasks, such as the distribution of shared resources, the decisions 1 and 0 may represent two alternative resources so that fewer agents utilizing a resource implies more abundance. The decisions of each agent are responses to the environment of the game, described by *signal* $\mu^*(t)$ at time t , where $\mu^*(t) = 1, \dots, D$. These responses are prescribed by *strategies*, which are binary functions mapping the D signals to decisions 1 or 0. In this paper, we consider *endogenous* signals, which are the *history* of the winning bits in the most recent m steps. Thus, the strategies have an input dimension of $D = 2^m$, and the parameter $\alpha \equiv D/N$ is referred to as the *complexity*. Before the game starts, each agent randomly picks s strategies. Out of her s strategies, each agent makes decisions according to the most successful one at each step. The success of a strategy is measured by its cumulative payoff, as explained below.

Let $\xi_i^\mu(t) = \pm 1$ when the decisions of strategy a are 1 or 0, responding to signal μ . Let $a^*(i, t)$ be the strategy adopted by agent i at time t . Then, $A(t) \equiv \sum_i \xi_{a^*(i,t)}^{\mu^*(t)}/N$ is the excess demand of the game at time t . The payoff received by strategy a is then $-\xi_a^{\mu^*(t)} \varphi(\sqrt{N}A(t))$, where φ is the payoff function. The step and linear payoffs are $\varphi(\chi) = \text{sgn}\chi$ and χ , respectively. (Here, we have implicitly assumed that an agent does not consider the impact of adopting a strategy, although the excess demand depends only on the adopted ones.) Let $\Omega_a(t)$ be the cumulative payoff of strategy a at time t . Then, its updating dynamics is described by

$$\Omega_a(t+1) = \Omega_a(t) - \xi_a^{\mu^*(t)} \varphi(\sqrt{N}A(t)). \quad (1)$$

The diversity of initial preferences of strategies is introduced by adding random biases ω_{ia} to the cumulative

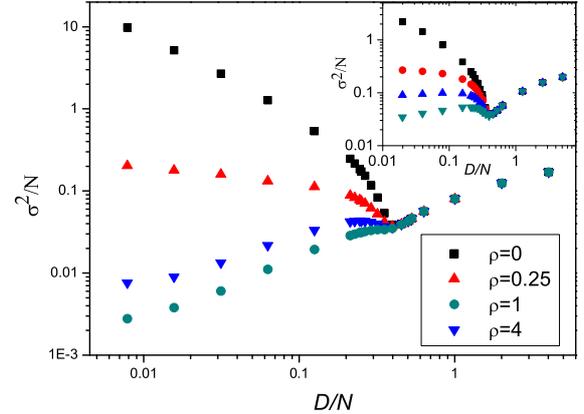


Fig. 1. σ^2/N versus α with linear payoffs for $\rho = 0, 0.25, 1, \text{ and } 4$, for $N = 251, s = 2$, and 1000 samples. Inset: The corresponding plot for step payoffs.

payoffs of strategy a ($a = 2, \dots, s$) of agent i with respect to her first one. The biases are drawn from a Gaussian or binomial distribution with mean 0 and variance R . The ratio $\rho \equiv R/N$ is referred to as the *diversity*. To monitor the mutual adaptive behavior of the population, we measure the variance σ^2/N of the population making decision 1, as defined by

$$\frac{\sigma^2}{N} \equiv \frac{N}{4} \langle [A^{\mu^*(t)}(t) - \langle A^{\mu^*(t)}(t) \rangle]^2 \rangle, \quad (2)$$

where the average is taken over time when the system reaches steady state and over the random distribution of strategies and biases.

III. DYNAMICAL TRANSITIONS

As Fig. 1 shows, the dependence of the variance σ^2/N on the complexity α for linear payoffs is very similar to that for step payoffs [14,15]. For α above a universal critical value α_c (≈ 0.3), the variance drops when α is reduced. The effect of introducing the diversity is also similar to that for step payoffs; namely, the variance remains unaffected when $\alpha > \alpha_c$, but decreases significantly with the diversity when $\alpha < \alpha_c$.

However, there are differences when one goes beyond this general trend. As Fig. 2 shows, the variance curves at different values of α cross at $\rho = \rho_c \approx 0.16$, indicating the existence of a continuous phase transition at ρ_c from a phase of vanishing variance at large ρ to a phase of finite variance at small ρ .

This behavior is very different from that for step payoffs, where the variance scales as ρ^{-1} and there are no dynamical transitions (Fig. 2 inset). The picture is confirmed by analyzing the dynamics of the game for small

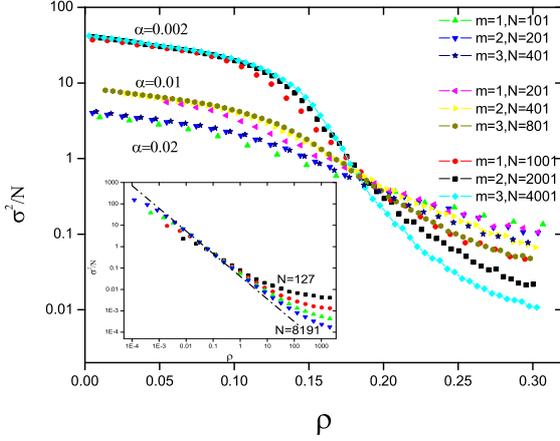


Fig. 2. σ^2/N versus ρ with linear payoffs of $\alpha = 0.002, 0.01$, and 0.02 for different m and N . Inset: σ^2/N versus ρ with step payoffs with $m = 1$, and $N = 127, 511, 2047$, and 8191 . The dashed-dotted line is the scaling prediction. For both payoffs, $s = 2$ and the number of samples is 1000 .

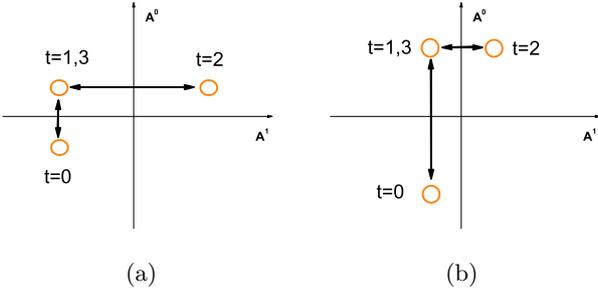


Fig. 3. Attractor dynamics when (a) $|\Delta A^1|$ is larger and (b) $|\Delta A^0|$ is larger.

m . The dynamics can be conveniently described by introducing the D -dimensional vector $A^\mu(t) \equiv \sum_i \xi_{a^*(i,t)}^\mu / N$. While only one of the D signals corresponds to the historical signal $\mu^*(t)$ of the game, the augmentation to D components is necessary to describe the attractor structure of the game dynamics. Fig. 3 illustrates the attractor structure in this phase space for the visualizable case of $m = 1$. The dynamics proceeds in the direction that tends to reduce the magnitude of the components of $A^\mu(t)$ [7]. However, the components of $A^\mu(t)$ overshoot, resulting in periodic attractors of period $2D$. For $m = 1$, the attractor is described by the sequence $\mu^*(t) = 0, 1, 1, 0$, and takes the L-shape as shown in Fig. 3 [15]. Note that the displacements in the two directions may not have the same amplitude.

Following steps similar to those in Ref. 15, we find that for m not too large and for convergence within time steps much less than \sqrt{R} ,

$$A^\mu(t+1) = A^\mu(t) - \sqrt{\frac{2}{\pi R}} \varphi(\sqrt{N} A^\mu(t)) \delta_{\mu \mu^*(t)}. \quad (3)$$

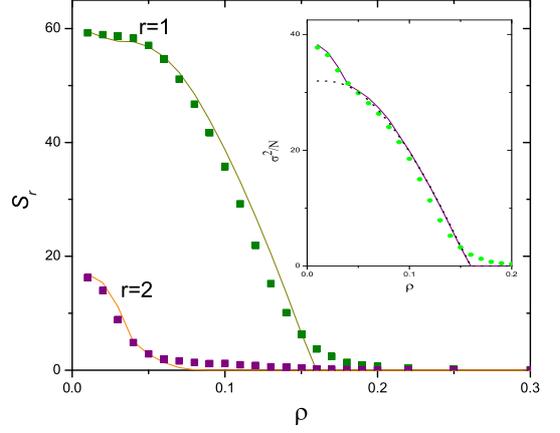


Fig. 4. S_r versus ρ . Inset: σ^2/N versus ρ . The symbols are the simulation, the dotted line the theory with one bifurcation, and the solid lines the theory with two bifurcations. $N = 1001$, $m = 1$, $s = 2$, and 1000 samples.

For step payoffs, Eq. (3) converges to an attractor confined in a D -dimensional hypercube of size $\sqrt{2/\pi R}$, irrespective of the value of R . On the other hand, for linear payoffs, $A^\mu(t+1)$ becomes a linear function of $A^\mu(t)$ with a slope of $1 - \sqrt{2/\pi\rho}$. Hence, for $\rho > \rho_c = 1/2\pi \sim 0.16$, the step sizes $|A^\mu(t+1) - A^\mu(t)|$ converge to zero, whereas for $\rho < \rho_c$, steps of vanishing sizes become unstable, resulting in a continuous dynamical transition at ρ_c .

IV. PHASE OF THE FINITE VARIANCE

However, when $\rho < \rho_c$, the step sizes for each of the D signals may not be equal. To see this, we monitor the variance for each of the D signals and rank them. The r th maximum variance is then given by

$$S_r = \text{large}_\mu \left(\frac{N}{4} [\langle (A^\mu)^2 \rangle_{\mu=\mu^*(t)} - \langle (A^\mu)_{\mu=\mu^*(t)} \rangle^2], r \right), \quad (4)$$

where $\text{large}_\mu(f(\mu), r)$ is the r th largest function $f(\mu)$ for $\mu = 1, \dots, D$.

As Figs. 4-5 show, the step sizes for the D signals do not bifurcate simultaneously at $\rho = \rho_c$. Rather, only the first maximum bifurcates from zero when ρ falls below ρ_c , while the step sizes for the remaining $D-1$ signals remain small. When the diversity further decreases to around 0.05 , the second maximum becomes unstable as well, and a further bifurcation takes place. For $m \geq 2$, there are further bifurcations of the third or higher order maxima, resulting in a *cascade* of dynamical transitions when the diversity decreases.

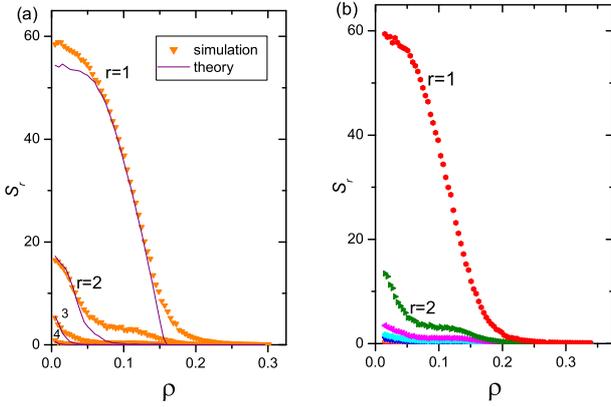


Fig. 5. S_r versus ρ for (a) $m = 2$ and (b) $m = 3$. In both cases, $N = 1001$, $s = 2$, and 1000 samples.

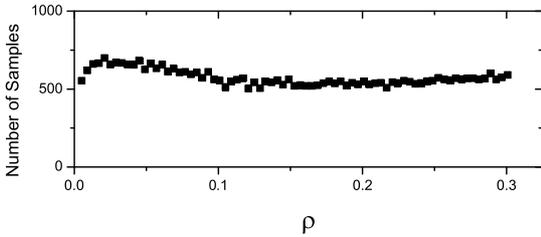


Fig. 6. Number of samples with step size responding to signal 1 being the maximum, out of 1000 samples, for $N = 1001$, $m = 1$, and $s = 2$. The initial signal is 1.

This cascade of transitions is confirmed by analysis. For $m = 1$, we can generalize Eq. (3) to convergence times of the order \sqrt{R} . Assuming, without loss of generality, that A^1 bifurcates while A^0 remains small, we find that the variance of the buyer population, as derived in Ref. 16, is

$$\frac{\sigma^2}{N} = \frac{N}{32}(\Delta A^1)^2, \quad \Delta A^1 = \text{erf}\left(\frac{\Delta A^1}{\sqrt{8\rho}}\right), \quad (5)$$

where ΔA^1 is the step size responding to signal 1. As inset in Fig. 4 shows, the analytical and the simulated results agree down to $\rho \sim 0.05$. However, when the diversity decreases further, this simple analysis implies that the variance will saturate to a constant $N/32$, but simulated results are clearly higher. This discrepancy is due to a further bifurcation of the minimum step size. This can be analyzed by considering the effect of a perturbation $\delta A^0(t)$ in the direction of A^0 . After a period of 4 steps, the accumulated perturbation becomes

$$\delta A^0(t+4) = \left[1 - \frac{1}{\sqrt{2\pi\rho}}(1 + e^{-\frac{(\Delta A^1)^2}{8\rho}})\right]^2 \delta A^0(t). \quad (6)$$

At $\rho = 0.0459$, where $\Delta A^1 = 0.9775$, the coefficient on the right-hand side of Eq. (6) reaches the value 1, and

$\delta A^0(t)$ diverges on further reduction of ρ . Numerical iterations of the analytical equations for $A^\mu(t)$, averaged over samples of different initial conditions, yield the theoretical curves in Fig. 4 and the inset, agreeing very well with the simulated results. Similarly, the agreement between analytical and simulated results are satisfactory for $m = 2$.

Since the attractors have asymmetric responses to different signals, we also study their dependence on the initial states. Letting the system start from a certain state (say, state 1 for $m = 1$) for a given sample, we show in Fig. 6 that the initial state is more likely to have the largest step size in the attractor for $m = 1$. Simulations show that higher values of m share the same trend.

V. CONCLUSION

We have studied the behavior of an adaptive population by using a payoff function that increases linearly with the winning margin. We found a continuous dynamical transition when the adaptation rate of the population was tuned by varying their diversity of preferences. This is in contrast with the case of payoff functions independent of the winning margin, in which no phase transitions are found. Furthermore, we found a cascade of dynamical transitions in the responses to different signals. This shows that an adaptive population has the ability to self-organize to globally efficient states and to display a rich behavior, although the individual agents make selfish decisions. Hence, despite the simplicity of the population models, they are able to capture the essential features of economic markets and distributed control.

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