A Model Theoretic Analysis of Non-Provability in Arithmetic

Shai Ben-David*and Michal Dvir
Department of Computer Science
The Technion
Haifa 32000, ISRAEL

1 Definitions

This section contains the basic definitions and notations used throughout this paper.

1.1 Partitions

Definition 1.1
- *For $m \in \mathbb{N}$ we denote by $[m]$ the set $\{0, \ldots, m-1\}$. We shall often identify m with $[m]$.*
- *Given $m, n, c \in \mathbb{N}$, $F : [m]^{[n]} \to c$ denotes a partition - that is, a function. The domain set is the set of all n-size subsets of $[m]$, and the range is the set $[c]$.*
- *Given a partition $F : [m]^{[n]} \to c$ and a set $H \subseteq [m]$, H is said to be homogeneous for F if there exist $k < c$ such that for all $h_0 < \cdots < h_{n-1} \in H : F(h_0, \ldots, h_{n-1}) = k$ (F is k-constant on H).*

We now recall the Ramsey partitions theorems:

Finite Ramsey theorem: *For every $n, c, k \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that for every partition $F : [m]^{[n]} \to c$ there exist a set $H \subseteq [m]$ with cardinality k, which is homogeneous for F.*
We denote this by : $m \to (k)^n_c$

Infinite Ramsey theorem: *For every $n, c \in \mathbb{N}$ and for every $F : \mathbb{N}^{[n]} \to c$ there is an infinite set $H \subseteq \mathbb{N}$, which is homogeneous for F.*

Definition 1.2 *A set $A \subseteq \mathbb{N}$ is **large** if $|A| > min(A)$.*

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Theorem 1.1 *[PH77] For every $n, c, k \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that for every partition $P : [m]^n \to c$ there exist a set $H \subseteq [m]$ such that the following holds:*
H is large, H is homogeneous for P, $min(H) \geq k$ and $|H| > n$.
We denote this by $m \underset{*}{\to} (k)^n_c$.

Definition 1.3 *A function $f : [m]^n \to \mathbb{N}$ is said to be* ***regressive*** *if for every $x_0 < \cdots < x_{n-1} \in [m]$ such that $x_0 > 0$, $f(x_0, \ldots, x_{n-1}) < x_0$.*

Definition 1.4 *Given a function $f : [m]^n \to \mathbb{N}$ and a set $H \subseteq [m]$, H is said to be* ***min-homogeneous*** *for f if for every $x_0 < \cdots < x_{n-1} \in H$ and $y_0 < \cdots < y_{n-1} \in H$ such that $x_0 = y_0$:*

$$f(x_0, \ldots, x_{n-1}) = f(y_0, \ldots, y_{n-1})$$

Theorem 1.2 *[KM87] For every $n, k \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that for every regressive function $f : [m]^n \to N$ there is a set $H \subseteq [m]$ which is min-homogeneous for f.*

1.2 Ordinals and proof theory

Definition 1.5 *We say that a function $f : \mathbb{N} \to \mathbb{N}$ is* ***provably recursive in*** *PA if f is a total recursive function and*

$$PA \vdash \forall x \exists y f(x) = y$$

Definition 1.6 *Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we say that f* ***dominates*** *g ($f \prec g$) if there exist $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$: $f(k) > g(k)$.*

Recall that ε_0 is the first ordinal α such that $\omega^\alpha = \alpha$.
It is known that every ordinal $\alpha > 0$ has a unique representation (Cantor's normal form):

$$\alpha = k_1\omega^{\alpha_1} + \cdots + k_t\omega^{\alpha_t}$$

such that $t, k_1, \ldots, k_t$ are positive integers and $\alpha_1 > \alpha_2 > \cdots > \alpha_t$.

We are about to define an hierarchy of functions - the Wainer's hierarchy - but first we define for a limit ordinal α a series of ordinals - $\{\alpha\}(n)$ - such that the limit of this series is α. We will use this series in order to define the Wainer's hierarchy.

Definition 1.7 *Let $\alpha \leq \varepsilon_0$ be a limit ordinal. $\{\alpha\}(n)$ is the series of ordinals, with limit α, defined as follows:*
Let $k_1\omega^{\alpha_1} + k_2\omega^{\alpha_2} + \cdots + k_t\omega^{\alpha_t}$ be the Cantor's normal form of α.

a. If α_t is a successor ordinal, $\alpha_t = \beta_t + 1$:

$$\{\alpha\}(n) = k_1\omega^{\alpha_1} + \cdots + k_{t-1}\omega^{\alpha_{t-1}} + (k_t - 1)\omega^{\alpha_t} + n\omega^{\beta_t}$$

b. If $\alpha_t < \varepsilon_0$ is a limit ordinal:

$$\{\alpha\}(n) = k_1\omega^{\alpha_1} + \cdots + k_{t-1}\omega^{\alpha_{t-1}} + \omega^{\{\alpha_t\}(n)}$$

c. If $\alpha = \varepsilon_0$:

$$\begin{aligned} \{\varepsilon_0\}(0) &= 0 \\ \{\varepsilon_0\}(n+1) &= \omega^{\{\varepsilon_0\}(n)} \end{aligned}$$

We can now define the Wainer's hierarchy.

Definition 1.8 *The Wainer's hierarchy is the sequence $\{F_\alpha : \mathbb{N} \to \mathbb{N}\}_{\alpha \le \varepsilon_0}$ of functions, defined (recursively) by:*

$$F_\alpha(x) = \begin{cases} x+1 & \text{if } \alpha = 0 \\ F_\beta^{(x+1)}(x) & \text{if } \alpha = \beta + 1 \\ F_{\{\alpha\}(x)}(x) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

were $F^{(k)}(x)$ is the k'th iterate of $F(x)$.

Theorem 1.3 (Wainer [?]) *The hierarchy has the following properties:*

- *For every $\alpha < \varepsilon_0$ F_α is provably recursive in PA.*
- *Let f be a function which is provably recursive in PA. Then there is $\alpha < \varepsilon_0$ such that f is dominated by F_α.*

Definition 1.9 *Given $n, c \in \mathbb{N}$ $\sigma_{n,c} : \mathbb{N} \to \mathbb{N}$ is the following function:*

$$\sigma_{n,c}(k) = \text{the minimal } m \text{ such that } m \underset{*}{\rightarrow} (k)^n_c$$

Definition 1.10 *We define a monotone increasing series of ordinals - $\{\gamma_n\}_{n\in\mathbb{N}}$ - as follows:*

$$\begin{aligned} \gamma_0 &= \omega \\ \gamma_{n+1} &= \omega^{\gamma_n} \end{aligned}$$

The limit of this series is ε_0.

Ketonen and Solovay established the following relation between Wainer's hierarchy and the family of functions $\{\sigma_{n,c} : n, c \in \mathbb{N}\}$.

Theorem 1.4 (Ketonen and Solovay [?]) *Let $n \ge 2$ and let $f : \mathbb{N} \to \mathbb{N}$ be a weakly monotone increasing function. Then the following statements are equivalent:*

- *f is dominated by $\sigma_{n,c}$ for some $c \in \mathbb{N}$.*
- *f is dominated by F_α for some $\alpha < \gamma_{n-2}$.*

Thus, $\{F_\alpha : \alpha < \varepsilon_0\}$ and $\{\sigma_{n,c} : n, c \in \mathbb{N}\}$ are cofinal, that is: For every weakly monotone function $f : \mathbb{N} \to \mathbb{N}$ there is $c, n \in \mathbb{N}$ such that $f \prec \sigma_{n,c}$ if and only if there exist $\alpha < \varepsilon_0$ such that $f \prec F_\alpha$.

1.3 Filters

In this section we list some basic definitions and well known facts concerning the notions of a filter and an ultra-filter and their applications to Model Theory. For further elaboration we refer the reader to standard textbooks on Model Theory, such as, Bell and Slomson [1] or Chang and Keisler [2].

Definition 1.11 *Given a set $I \neq \emptyset$, let $S(I)$ denote its power set.*

- *A* ***filter*** *D over I is a subset of $S(I)$ satisfying the following conditions:*
  1. *$I \in D$, $\emptyset \notin D$.*
  2. *If $X, Y \in D$ then $X \cap Y \in D$.*
  3. *If $X \in D$ and $X \subset Z \subset I$ then $Z \in D$.*
- *$D \subset S(I)$ is an* ***ultra-filter*** *over I if D is a filter over I and, on top of it,*
  4. *For every $X \in S(I)$, $X \in D$ if and only if $(I - X) \notin D$.*

Definition 1.12 *A set, $E \subset S(I)$, is said to have the* ***finite intersection property****, if the intersection of any finite number of elements from E is not empty.*

Theorem 1.5 *If $E \subset S(I)$ and E has the finite intersection property, then there exist an ultra-filter D over I such that $E \subset D$.*

Definition 1.13 *Let I be a non-empty set, let $\{A_i\}_{i \in I}$ be a collection of non-empty sets, and let D be an ultra-filter over I.*

- *For each $i \in I$ let $A_i \neq \emptyset$ be a given set. We denote by $C = \prod_{i \in I} A_I$ the set of all functions f with domain I such that, for every $i \in I$, $f(i) \in A_i$.*
- *Two functions $f, g \in C$ are D-equivalent ($f \equiv_D g$) if*

  $$\{i \in I : f(i) = g(i)\} \in D.$$

  Let f_D denote the equivalence class of f in the relation $\equiv_D$.
- *The* ***ultra-product*** *of $\{A_i\}_{i \in I}$ modulo D, is the set of all equivalence classes of the relation $\equiv_D$:*

  $$\prod_D A_i = \{f_D : f \in \prod_{i \in I} A_i\}$$
- *If, for some set A, for all $i \in I$, $A_i = A$, then the ultra-product is denoted by $\prod_D A$ or A^I/D and it is called the* ***ultra-power*** *of A modulo D.*

In the following definition we'll use the following notation:
If $\{x_i\}_{i\in I}$ is such that, for all $i \in I$, $x_i \in A_i$ then, we denote by $\langle x_i : i \in I\rangle$ the function in $\prod_{i\in I} A_i$ which maps x_i to i.

Definition 1.14 *Let I be a non-empty set. Let D be an ultra-filter over I. Let L be a first order language and for each $i \in I$ let U_i be a model for L.*
We denote by $\hat{R}_i$ the relation in U_i which is the interpretation of the relation symbol R from L. $\hat{f}_i$ denotes the interpretation in U_i of the function symbol f, and $\hat{a}_i$ denotes the interpretation in U_i of the constant symbol a. The universe of U_i is denoted by A_i.

*The **ultra-product model** $\prod_D U_i$ is the following model for L:*

- *The universe of $\prod_D U_i$ is $\prod_D A_i$.*
- *Let R be n-ary relation symbol of L. The interpretation of R in $\prod_D U_i$ is the following relation $\hat{R}$:*
$$\hat{R}(f_D^1, \ldots, f_D^n) \Longleftrightarrow \{i \in I : \hat{R}_i(f^1(i), \ldots, f^n(i))\} \in D$$
- *Let g be n-ary function symbol of L. The interpretation of g in $\prod_D U_i$ is the following function $\hat{g}$:*
$$\hat{g}(f_D^1, \ldots, f_D^n) = \langle \hat{g}_i(f^1(i), \ldots, f^n(i)) : i \in I\rangle_D$$
- *Let a be a constant symbol of L. a is interpreted in $\prod_D U_i$ by the constant $\hat{a} \in \prod_D A_i$ where $\hat{a}$ is defined as follows:*
$$\hat{a} = \langle \hat{a}_i : i \in I\rangle_D$$

Theorem 1.6 (The Fundamental Theorem of Ultra-Products) *Let I be a non-empty set, let D be an ultra-filter over I, and let $\mathcal{B}$ be an ultra-product model $\prod_D U_i$ of a first order language L. then:*

A. For every term $t[x_1, \ldots, x_n]$ of L and elements $f_D^1, \ldots, f_D^n \in \mathcal{B}$:
$$t_{\mathcal{B}}[f_D^1, \ldots, f_D^n] = \langle t_{U_i}[f^1(i), \ldots, f^n(i)] : i \in I\rangle_D$$
where $t_{\mathcal{B}}$ and t_{U_i} $(i \in I)$ are the interpretations of t in $\mathcal{B}$ and U_i $(i \in I)$ respectively.

B. For every formula $\varphi(x_1, \ldots, x_n)$ of L and elements $f_D^1, \ldots, f_D^n \in \mathcal{B}$:
$$\mathcal{B} \models \varphi(f_D^1, \ldots, f_D^n) \Longleftrightarrow \{i \in I : U_i \models \varphi(f^1(i), \ldots, f^n(i))\} \in D$$

C. For every sentence φ of L:
$$\mathcal{B} \models \varphi \Longleftrightarrow \{i \in I : U_i \models \varphi\} \in D$$

Corollary 1.7 *Let D be an ultra-filter over $\mathbb{N}$. Then the ultra-power model $\mathbb{N}^{\mathbb{N}}/D$ is elementary equivalent to the standard model of the natural numbers.*

2

We are interested in functions over the natural numbers that are recursive but not provably recursive in PA. Functions that are not dominated by the hierarchy are such functions. In this section we deal with a subset of the functions not dominated by the hierarchy - the functions that dominate it - all the functions f for which it holds that for all $\alpha < \varepsilon_0$ there is k_α in $\mathbb{N}$ such that for all $k > k_\alpha$: $f(k) < F_\alpha(k)$. In the first part of this section we will show that given a function f that dominates the hierarchy we can construct a non-standard model of PA in which f is not total. In the second part of the section we will proof that there exists a model of PA in which all the functions that dominate the hierarchy are not total.

2.1

Let f be a function on the natural numbers that dominates Wainer's hierarchy. Since the hierarchy characterizes all the functions that are provably recursive in PA it follows that f is not provably recursive - that is : $PA \nvdash \forall x \exists y f(x) = y$. By completeness theorem it follows that there exists a model of PA in which this statement fails. We will now construct such a model.

The construction is a generalization of the construction presented in [KM87], for a specific function. Most of the technical lemmas used here appear in [KM87] or in [PH77] - implicitly or explicitly, as parts of proofs presented there. These lemmas are given here - rephrased in the terms of this work - with their complete proofs, to achieve a continuous representation

Construction 2.1 *Given a function $f : \mathbb{N} \to \mathbb{N}$ that is total and dominates the Wainer's hierarchy, we construct a model non-standard model I of PA such that: $I \not{models} \forall x \exists y f(x) = y$.*

Proof: We shall start with a non-standard model of the natural numbers. Given in this model a non-standard element - d - we will find an initial segment of this model that contains d but not $f(d)$, and satisfies Peano axioms.

Let M be a non-standard model of the natural numbers. since f dominates Wainer's hierarchy, then for all n, c f dominates $\sigma_{n,c}$, that is - there exists $k_{n,c} \in \mathbb{N}$ such that for every $k > k_{n,c}$, $f(k) > \sigma_{n,c}(k)$. But, every non-standard element in M is greater then $k_{n,c}$. Thus, for every non-standard d in M and for every natural n, c we have: $\sigma_{n,c}(d) < f(d)$.

Given a non-standard element d in M, we will examine the interval $[d, f(d))$, and show that there exists $d < I < f(d)$ such that $I \models PA$. The technique is very much like the one used in [KM87]. we would like to prove that for every natural k, there are k indiscernibles for the first k Σ_0-formulas in $[d, f(d))$. It will follow, that there is a countable series of indiscernibles for all Σ_0-formulas, and this property of the interval will help us define I and prove that it is closed under addition and multiplication and satisfies the induction scheme.

For every natural k the desired k indiscernibles will be found in the interval $[d, \sigma_{r,c}(d))$, where c will be a constant and r will be defined in terms of k. This will be done using a series of partitions over the domain $\sigma_{r,c}$. Homogeneous sets for these partition will eventually define the indiscernibles' set. Since $\sigma_{n,c}(d) < f(d)$ for all n, c, the existence of k indiscernibles in the interval $[d, f(d))$ will follow.

Proposition 2.1 *let k be a natural number and let $\psi_1, \ldots, \psi_k$ be k Σ_0-formulas. There exists a set $H_0 \in [d, f(d))^k$ that constitutes indiscernibles for these formulas.*

Proof: let n be an upper bound on the number of free variables in $\psi_1, \ldots, \psi_k$. we can assume that $k \geq 2n+1$ (otherwise, we can expand the set of formulas to include $2n+1$ Σ_0-formulas. For the expanded set, we will have a set of $2n+1$ indiscernibles, and any subset of size k will constitute indiscernibles for the original set).

Let $\ell \in \mathbb{N}$ be such that $\ell \to (k+n)^{2n+1}_{k+2}$. Let $c = 49$, $r = 2n+2$. We shall define two partitions over the domain $[\sigma_{r,c}(d)]^{2n+1}$ - Q and f (as in [KM87]):

Given $x_0 < \cdots < x_{2n} < \sigma_{r,c}$:

if

there are $p < x_0$ and $i \leq k$ such that $\psi_i(p, x_1, \ldots, x_n) \neq \psi_i(p, x_{n+1}, \ldots, x_{2n})$

then

$$\begin{aligned} f(x_0, \ldots, x_{2n}) &= \text{the least such } p \\ Q(x_0, \ldots, x_{2n}) &= \text{the least such } i \text{ for that } p \end{aligned}$$

otherwise

$$\begin{aligned} f(x_0, \ldots, x_{2n}) &= 0 \\ Q(x_0, \ldots, x_{2n}) &= k+1 \end{aligned}$$

Clearly, f is a regressive function.
For now, let us assume that there exists $H_0 \in [d, \sigma_{r,c}(d)]^{\ell}$ that is minimal-homogeneous for f (to be proven later - lemmas 2.4-2.5). The indiscernibles' set will be a subset of H_0. By hypothesis on ℓ, for the partition Q reduced to the domain H_0^{2n+1} there is an homogeneous subset of H_0 with cardinality $k+n$. that is - there exist $H_1 \in [H_0]^{k+n}$ and $i_0 \leq k+1$ such that Q is i_0-constant on H_1.

Lemma 2.2 *the partition Q is $k+1$-constant on H_1.*

Proof: assume $i_0 \leq k$:
The cardinality of H_1 is $k+n$. By hypothesis on k : $k \geq 2n+1$, therefore $|H_1| \geq 2n+1+n \geq 3n+1$. Let $x_0 < \cdots < x_{3n}$ be $3n+1$ elements from H_1. Since H_0 is minimal-homogeneous for

f, every series of $2n+1$ elements from H_0 beginning with x_0 has the same value, say - $p < x_0$. Therefore:

$$\begin{aligned} p &= f(x_0, x_1, \ldots, x_{2n}) && (1)\\ p &= f(x_0, x_{n+1}, \ldots, x_{3n}) && (2)\\ p &= f(x_0, x_1, \ldots, x_n, x_{2n+1}, \ldots, x_{3n}) && (3)\end{aligned}$$

by (1) we have, according to f's definition:

$$\psi_{i_0}(p, x_1, \ldots, x_n) \neq \psi_{i_0}(p, x_{n+1}, \ldots, x_{2n})$$

by (2) we have:

$$\psi_{i_0}(p, x_{n+1}, \ldots, x_{2n}) \neq \psi_{i_0}(p, x_{2n+1}, \ldots, x_{3n})$$

but then:

$$\psi_{i_0}(p, x_1, \ldots, x_n) = \psi_{i_0}(p, x_{2n+1}, \ldots, x_{3n})$$

but by (3) we have:

$$\psi_{i_0}(p, x_1, \ldots, x_n) \neq \psi_{i_0}(p, x_{2n+1}, \ldots, x_{3n})$$

Contradiction. Thus, we conclude that $i_0 = k+1$. ■

Now we are able to complete the proof of proposition 2.1. Let $z_1 < \cdots < z_n$ be the last n elements of H_1. Let H be the first k elements of H_1 (recall that the cardinality of H_1 is $k+n$). Clearly $H \subseteq [d, \sigma_{r,c}(d)] \subseteq [d, f(d))$. Now, H is the set of indiscernibles we looked for: Let $c_0, c_1, \ldots, c_n, d_1, \ldots, d_n$ be $2n+1$ elements in H such that $c_0 < c_1 < \cdots < c_n$ and $c_0 < d_1 < \cdots < d_n$. Q is $k+1$-constant on H, therefore:

$$Q(c_0, d_1, \ldots, d_n, z_1, \ldots, z_n) = Q(c_0, c_1, \ldots, c_n, z_1, \ldots, z_n)$$

(Q is well defined on these vectors since $z_1 > max\{d_n, c_n\}$).
Therefore, by definition of Q - for any $i \leq k$ and for any $p < c_0$:

$$\begin{aligned} \psi_i(p, c_1, \ldots, c_n) &= \psi_i(p, z_1, \ldots, z_n)\\ \psi_i(p, d_1, \ldots, d_n) &= \psi_i(p, z_1, \ldots, z_n)\end{aligned}$$

Therefore:

$$\psi_i(p, c_1, \ldots, c_n) = \psi_i(p, d_1, \ldots, d_n)$$

Thus, to really complete the proof we still have to show the existence of $H_0 \in [d, \sigma_{r,c}(d)]^\ell$ that is minimal-homogeneous for f.

We shall define a partition G over the domain $[\sigma_{r,c}(d)]^{2n+2}$:

$$G_(x_0, \ldots, x_{2n+1}) = \begin{cases} 0 & f(x_0, \ldots, x_{2n}) = f(x_0, x_2, \ldots, x_{2n+1})\\ 1 & f(x_0, \ldots, x_{2n}) < f(x_0, x_2, \ldots, x_{2n+1})\\ f & f(x_0, \ldots, x_{2n}) > f(x_0, x_2, \ldots, x_{2n+1})\end{cases}$$

The next two lemmas will show us how to derive the desired H_0 using the new-defined partition G.

Lemma 2.3 *Let $X \subseteq [d-2n, \sigma_{r,c}(d)]$ be an homogeneous set for G, such that $|X| > min(X) + 2n$. Then the value of G on $[X]^{2n+2}$ is 0.*

Lemma 2.4 *Given a set X such that: $X \subseteq [d-2n, \sigma_{r,c}(d)]$, G is 0-constant on $[X]^{2n+2}$ and $|X| > d$, there exists $H_0 \in [d, \sigma_{r,c}(d)]^\ell$ that is minimal-homogeneous for f.*

We will first prove lemma 2.4:

Proof of lemma 2.4: Let H_0 be the set of the first ℓ elements of X which are greater than d. Obviously $H_0 \in [d, \sigma_{r,c}(d)]^\ell$. Let $x_0, x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}$ be elements from H_0 such that:

$$x_0 < y_1 < \cdots < y_{2n} \text{ and } x_0 < x_1 < \cdots < x_{2n}$$

and let $z_1 < \cdots < z_{2n}$ be $2n$ elements from X such that $z_1 > max\{y_{2n}, x_{2n}\}$. Now:

$$f(x_0, x_1, \ldots, x_{2n}) = f(x_0, x_2, \ldots, x_{2n}, z_1)$$

since $G(x_0, x_1, \ldots, x_{2n}, z_1) = 0$. Similarly:

$$f(x_0, x_2, \ldots, x_{2n}, z_1) = f(x_0, x_3, \ldots, x_{2n}, z_1, z_2) = \cdots =$$

$$= \cdots = f(x_0, z_1, \ldots, z_{2n}) = \cdots = f(x_0, y_3, \ldots, y_{2n}, z_1, z_2) =$$

$$= f(x_0, y_2, \ldots, y_{2n}, z_1) = f(x_0, y_1, \ldots, y_{2n})$$

and therefore:

$$f(x_0, x_1, \ldots, x_{2n}) = f(x_0, y_1, \ldots, y_{2n})$$

that is - H_0 is minimal-homogeneous for f. ∎

Now, we must find a set X with the desired properties. To have such an X on which G is 0-constant we must find an appropriate set X, homogeneous for G such that $|X| > min(X) + 2n$. The lemma 2.3 shows why it is necessary.

Proof of lemma 2.3: Assume by contradiction that the value of G on $[X]^{2n+2}$ is 1: By the hypothesis on X, $|X| > min(X) + 2n$, and thus, there exists a set of $min(X) + 2n$ elements from X not including $min(X)$. Let A be such a set and let $x_1, x_2, \ldots, x_{min(X)+2n}$ be elements from A. Now we will create a set D that contains $min(X) + 1$ different vectors, of length $2n + 1$, in which the first element is $min(X)$, and the other elements are from A. These vectors constitute a chain in which for every two incident vectors $(min(x), a_1, \ldots, a_{2n})$ and $(min(X), b_1, \ldots, b_{2n})$: $a_2, \ldots, a_{2n} = b_1, \ldots, b_{2n-1}$.

By the contradiction assumption the value of G on $[X]^{2n+2}$ is 1. Thus, by definition of G, given two incident vectors on this chain $(min(X), a_1, a_2, \ldots, a_{2n})$ and $(min(X), a_2, \ldots, a_{2n}, b_{2n})$ $G(min(X), a_1, a_2, \ldots, a_{2n}, b_{2n}) = 1$, and hence

$$f(min(X), a_1, a_2, \ldots, a_{2n}) < f(min(X), a_2, \ldots, a_{2n}, b_{2n})$$

Therefore all $min(X)+1$ vectors of D are mapped by f to different values. Thus, the cardinality of the set of f's values on D's vectors is like the cardinality of D, that is - if B is the set of f's values on D's vectors, then:

$$|B| = |\{f(z) : z \in D\}| = min(X) + 1$$

But, f is regressive, and therefore, for every $z \in D : f(z) < min(z) = min(X)$ (since $min(X) \neq 0$). Hence, $B \subseteq \{z : z < min(X)\}$, that is :

$$1 + min(X) = |B| \leq |\{z : z < min(X)\}| = min(X)$$

Contradiction, and therefore the assumption is false and the value of G on $[X]^{2n+2}$ is not 1 and symmetrically is not 2, and the claim follows. ■

If, for the creation of X, we would have used the fact that G is a partition from $[\sigma_{r,c}(d)]^{2n+2}$ to 3 (and therefore, also to 49), then by definition of $\sigma_{r,c}(d)$ we would result with a set X which is large, homogeneous for G with cardinality greater than $2n+2$, and with minimal element greater than d. Thus, $|X| > min(X)$, but not necessarily $|X| > min(X) + 2n$.

We will have to go through some more partitions that will take care for the appropriate shift of $2n$ (intuitively, we wont change the cardinality of X, but lower its minimum element by $2n$).

We shall define the first partition:

$$P(x_0, x_1) = \begin{cases} 0 & x_0 + 2n < x_1 \\ 1 & x_0 + 2n \geq x_1 \end{cases}$$

Thus, $P : [\sigma_{r,c}(d)]^2 \to 2$

and, we shall define a function :

$$h(a) = \begin{cases} a - 2n & a \geq 2n \\ 0 & \text{otherwise} \end{cases}$$

That is, a shift backward of a by $2n$ or just down to 0 if $a < 2n$.
For $\bar{a} = (a_0, \ldots, a_{2n+1})$ we shall use the notation: $h(\bar{a}) = (h(a_0), \ldots, h(a_{2n}))$.

We shall define another partition: $S : [\sigma_{r,c}(d)]^{2n+2} \to 4$. S should be the composition of G on h, but such a composition is not always possible, since not always after evaluating h we come up with a sorted vector. Therefore S will be defined in the following way:

$$S(\bar{a}) = S(a_0, \ldots, a_{2n+1}) = \begin{cases} G(h(\bar{a})) & \text{if } h(\bar{a}) \text{ is sorted,} \\ & \text{that is, } h(a_0) < \cdots < h(a_{2n+1}). \\ 3 & \text{otherwise} \end{cases}$$

As done in [PH77] - lemma 2.9, we now multiply P and S to get a new partition
$P^* : [\sigma_{r,c}(d)]^{2n+2} \to 49$ (this now explains our choice of r and c), such that any set X with cardinality greater than $2n+2$ is homogeneous for P^* iff it is homogeneous for P and S.

By definition of $\sigma_{r,c}(d)$ there exists a set $Y \subseteq [0, \sigma_{r,c}(d)]$ that is large, with cardinality greater than $2n+2$, with a minimal element greater than d and homogeneous for P^*. Through this set Y we will be able to get the desired set X:

Lemma 2.5 *There is a set $X \subseteq [d-2n, \sigma_{r,c}(d)]$, homogeneous for G such that $|X| > min(X)+2n$, $|X| > d$ and $min(X) > d - 2n$.*

Proof: Y is homogeneous for P^*, hence by definition of P^* Y is also homogeneous for P. Y is large, that is - its cardinality is greater than its minimal element and that is greater than d. Therefore Y contains at least d elements, and certainly more than $2n+2$ elements. Therefore, there exist in Y two elements $x_0 < x_1$ such that $x_0 + 2n < x_1$.

By definition of P: $P(x_0, x_1) = 0$ and by the homogeneity of Y for P we have : for any $x_0 < x_1$ in Y : $x_0 + 2n < x_1$. Let $a_1 < a_2$ be two elements in Y. It follows that $a_2 > 2n$, and thus, by definition of h : $h(a_2) = a_2 - 2n > a_1$. Also, it is obvious, by definition of h, that for every a: $h(a) \leq a$. hence, we have : $h(a_1) \leq a_1 < h(a_2)$ and therefore $h(a_1) < h(a_2)$.

Thus, if we define X to be $\{h(a) : a \in Y\}$, we have that h is one to one from Y onto X, that is : $|Y| = |X|$. Also, h is order preservative while going from Y to X. Hence : $min(X) = h(min(Y))$.

We already know that $min(Y) > d$ and hence surely $min(Y) > 2n$. Thus, by definition of h: $h(min(Y)) = min(Y) - 2n$. We thus have: $min(X) = min(Y) - 2n > d - 2n$, $|X| = |Y| > d$ and $|X| = |Y| > min(Y) = min(X) + 2n$, that is : $|X| > min(X) + 2n$.

We still have to show that X is homogeneous for G: Let $\bar{a} = (a_0, \ldots, a_{2n+1})$ and $\bar{b} = (b_0, \ldots, b_{2n+1})$ be two vectors of length $2n+2$ over X. By definition of X we have: $a_i = h(z_i)$ and $b_i = h(u_i)$ $(0 \leq i \leq 2n+1)$, where $z_0 < \cdots < z_{2n+1} \in Y$ and $u_0 < \cdots < u_{2n+1} \in Y$.

We will denote $(z_0, \ldots, z_{2n+1})$ and $(u_0, \ldots, u_{2n+1})$ by $\bar{z}$ and $\bar{u}$ respectively.
Y is homogeneous for P^* and hence also for S. Therefore, $S(\bar{u}) = S(\bar{z})$. Now, $h(\bar{z}) = \bar{a}$ and $h(\bar{u}) = \bar{b}$ are sorted vectors, and thus, by definition of S we have : $S(\bar{z}) = G(h(\bar{z}))$ and similarly $s(\bar{u}) = G(h(\bar{u}))$. Hence, $G(h(\bar{z})) = G(h(\bar{u}))$ and therefore $G(\bar{a}) = G(\bar{b})$, and X is homogeneous for G. ∎

Since we have found the desired X, the proof of proposition 2.5 now follows. ∎

We can now go back to the proof of theorem 2.1. We know, so far, that for every $k \in \mathbb{N}$ there exist in the interval $[d, f(d))$ k indiscernibles for every set of Σ_0-formulas with cardinality k. In particular, this holds for the first k Σ_0-formulas. Let $\sigma(k)$ be a (primitive-recursive) statement saying there are at least k indiscernibles for the first k Σ_0-formulas in the interval $[d, f(d))$. Then, for every natural number k $M \models \sigma(k)$. Hence, necessarily, there exists a non-standard element t in M such that $M \models \sigma(t)$. Let us take the first ω indiscernibles out of these t indiscernibles : $\{c_i : i \in \omega\}$. We now have in the interval $[d, f(d))$ a series of indiscernibles for all Σ_0-formulas.

We shall now define I in the following way: $I = \{x \in M : \exists i(x < c_i)\}$.
Clearly, $d < I < f(d)$.

Proposition 2.6 *[KM87] Let M be a non-standard model of the natural numbers. Let $\{c_i : i \in \omega\}$ be a series of elements from M which is a series of indiscernibles for all Σ_0-formulas. Let I be an initial segment of M defined by $I = \{x \in M : \exists i(x < c_i)\}$. Then I satisfies Peano's axioms.*

It now follows that I is a model of PA in which the statement $\forall x \exists y(f(x) = y)$ is false, since for $d \in I$ there is no $y \in I$ such that $f(d) = y$. ■

2.2

A simple compactness argument shows that we can found a non-standard model of PA in which **all** the functions that dominate Wainer's hierarchy are not complete

Theorem 2.7 *Let A be the following set of formulas (in the language of arithmetic):*

$$A = \{\varphi : \varphi \text{ satisfies the 3 following properties}\}$$

Property 1: φ is of the form $\forall x \exists y P_\varphi(x, y)$.

Property 2: φ is valid in the standard model of the natural numbers.

Property 3: The function $f_\varphi(x) = min\{y : P_\varphi(x, y)\}$ dominates F_α for every $\alpha < \varepsilon_0$.

Then there exists a non-standard model of PA - I_u - such that for all $\varphi \in A : I_u \not\models \varphi$.

Proof: We will show that the set $Y = PA \cup \bar{A}$ is satisfiable, where PA is the set of Peano's axioms and $\bar{A}$ is the set $\{\neg\varphi : \varphi \in A\}$. This will be done using the compactness theorem:

Let $Z \subseteq Y$ be a finite set. By compactness it is enough to show that Z is satisfiable. We shall expand Z to $\hat{Z}$ that contains $Z \cap \bar{A}$ and PA. We will show that $\hat{Z}$ is satisfiable and hence it will follow that so is Z.
$\hat{Z}$ contains a finite number of formulas from $\bar{A}$, say:

$$\hat{Z} \cap \bar{A} = \{\neg\varphi_{i_1}, \ldots, \neg\varphi_{i_n}\}$$

Let M be a non-standard model of the natural numbers, let d be a non-standard element in M and let $1 \leq \ell \leq n$ be a natural number such that:

$$f_{\varphi_{i_\ell}}(d) = min_j\{f_{\varphi_{i_j}}(d)\}$$

Let us now concentrate on the interval $[d, f_{\varphi_{i_\ell}}(d))$. By the construction in the proof of theorem 2.1, we already know that there exists $d < I_Z < f_{\varphi_{i_\ell}}(d)$ such that $I_Z \models PA$. Obviously, for all $1 \leq j \leq n$, $f_{\varphi_{i_j}}(d)$ is not in I_Z and hence, for all $1 \leq j \leq n$:

$$I_Z \not\models \forall x \exists y P_{\varphi_{i_j}}(x, y)$$

Therefore I_Z satisfies PA and $Z \cap \bar{A}$, that is - I_Z satisfies $\hat{Z}$ and hence it satisfies Z as well. Thus, we have shown that every finite $Z \subseteq Y$ is satisfiable, and by compactness we have that Y is satisfiable - that is : there exists a non-standard model of PA - I_u such that for all $\varphi \in A$: $I_u \not\models \varphi$ ■

3

We turn to deal with functions that are not dominated by the hierarchy but not necessarily dominate it. For such functions we will show a result similar to theorem 2.1. Given a function not dominated by the hierarchy we construct a non-standard model of PA in which it is not total. We will also show that unlike the previous case, there is no non-standard model of PA in which all the functions not dominated by the hierarchy are not total.

3.1

In order to construct, given a function f not dominate by the hierarchy, a model of PA in which it is not total, we will use the results of the previous section and the way the model presented there was constructed. The difference will be in the non-standard model of the natural numbers from which the desired model is derived.

Theorem 3.1 *Let $\psi = \forall x \exists y P(x, y)$ be a formula that is valid in the standard model of the natural numbers, such that the function f defined by $f(x) = min\{y : P(x, y)\}$ is not dominated by F_α for any $\alpha < \varepsilon_0$. Then there exists a non-standard model of PA - I_ψ such that $I_\psi \not\models \psi$.*

Proof: If we examine a function f that is not dominated by the hierarchy but does not dominate it either, we see that the construction used in the proof of theorem 2.1 does not apply in this case. f that is not dominated by the hierarchy satisfies: For any $\alpha < \varepsilon_0$ the set $Y_\alpha = \{k : F_\alpha(k) < f(k)\}$ is infinite, and similarly - for any natural numbers n, c the set $Y_{n,c} = \{k : \sigma_{n,c}(k) < f(k)\}$ is infinite.

Hence, in a non-standard model of the natural numbers, for any n, c there **exists** a non-standard element d such that $\sigma_{n,c}(d) < f(d)$, but it is not necessarily the same d for all n, c. Thus, the main principle in the former proof - that all the $\sigma_{n,c}(d)$ were beneath one non-standard element d is not applicable here.

Instead, we will change slightly the construction of the model. Instead of taking as a basic model - from which the construction starts - any non-standard model of the natural numbers, we will take a specific model that contains an appropriate non-standard element d.

This model will be an ultra-power model. We shall first define the ultra-filter on which it is based. We want to construct an ultra-filter that contains all the sets Y_α for $\alpha < \varepsilon_0$. Let E be the following set:

$$E = \{Y_\alpha : \alpha < \varepsilon_0\}$$

Lemma 3.2 *E has the finite intersection property.*

Proof: We need to show that the intersection of a finite subset of E is not empty. Let $Y_{\alpha_1}, \ldots, Y_{\alpha_\ell}$ be ℓ sets from E, such that $\alpha_1 < \cdots < \alpha_\ell$. By properties of the hierarchy it follows that $F_{\alpha_1} \prec \cdots \prec F_{\alpha_\ell}$, that is - for any $1 < i \leq \ell$ there exists k_i such that for all $k > k_i$: $F_{\alpha_{i-1}}(k) < F_{\alpha_i}(k)$. Let $\hat{k}$ be $max\{k_i\}$. Hence, for any $k > k_i$:

$$F_{\alpha_1}(k) < \cdots < F_{\alpha_\ell}(k)$$

Now, the set Y_{α_ℓ} is infinite and therefore there exists $\hat{k} < m \in Y$. By definition of Y_{α_ℓ} we have $F_{\alpha_\ell}(m) < f(m)$, and therefore for any $1 \leq i \leq \ell$: $F_{\alpha_i}(m) < f(m)$ and thus, for any $1 \leq i \leq \ell$ $m \in Y_{\alpha_i}$. We have, therefore : $\bigcap_i Y_{\alpha_i} \neq \emptyset$. ■

Since E has the finite intersection property, the filter generated by E is a proper filter and can be extended to an ultra-filter. Let U be such an ultra-filter. U contains all the sets of E.

We shall construct the ultra-power model $M = \mathbb{N}^{\mathbb{N}}/U$. We know that it is a non-standard model of the natural numbers.

Lemma 3.3 *There exists in the model M a non-standard element d such that for any $\alpha < \varepsilon_0$ $F_\alpha(d) < f(d)$.*

Proof: Let ID be the identity function ($\forall n \in \mathbb{N} : ID(n) = n$) and let ID_U be the equivalence class of ID in relation to the ultra-filter U. We will show:

$$\forall \alpha < \varepsilon_0 : M \models F_\alpha(ID_U) < f(ID_U)$$

that is -

$$\{k : F_\alpha(ID(k)) < f(ID(k))\} \in U$$

But

$$\{k : F_\alpha(ID(k)) < f(ID(k))\} = \{k : F_\alpha(k) < f(k)\} = Y_\alpha$$

and by definition of U : $Y_\alpha \in U$. Therefore $d = ID_U$ satisfies the requirements of the lemma. ■

Lemma 3.4 *Let d' be an element in M such that for any $\alpha < \varepsilon_0 : F_\alpha(d') < f(d')$. Then for all two natural numbers $n, c : \sigma_{n,c}(d') < f(d')$.*

Proof: By [KS81], for any n, c there exists $\alpha_{n,c} < \varepsilon_0$, such that $F_{\alpha_{n,c}}$ dominates $\sigma_{n,c}$, that is - there exists $k_{n,c}$ such that for all $k > k_{n,c} : \sigma_{n,c}(k) < F_{\alpha_{n,c}}(k)$.
Hence, for any non-standard element d in M: $\sigma_{n,c}(d) < F_{\alpha_{n,c}}(d)$. In particular, for d': $\sigma_{n,c}(d') < F_{\alpha_{n,c}}(d') < f(d')$, hence - $\sigma_{n,c}(d') < f(d')$. ■

Now we can complete the proof of the theorem: M is a non-standard model of the natural numbers in which - by lemmas 3.3 and 3.4 - there exists a non-standard element d' such that for all $n, c \in \mathbb{N} : \sigma_{n,c}(d') < f(d')$. By the construction done for functions that dominate the hierarchy, given such d', we can construct $d' < I_\psi < f(d')$ that satisfies PA, such that $I_\psi \not\models \forall x \exists y P(x, y)$ ■

3.2

If we try again to use the compactness theorem in order to get a universal model in which all the functions that are not dominated by the hierarchy are not complete, we see that it does not work. Even if we take only finite set of such functions : $f_1, \ldots, f_\ell$, in order to be able to construct an appropriate model, we wish to construct an ultra-filter U that contains for any $\alpha < \varepsilon_0$ and for any $1 \le i \le \ell$ the set

$$Y_{i,\alpha} = \{k : F_\alpha(k) < f_i(k)\}$$

The resulting set:

$$E = \{Y_{i,\alpha} : \alpha < \varepsilon_0, 1 \le i \le \ell\}$$

does not necessarily satisfy the finite intersection property - since it is possible to find two functions f_i and f_j that are not dominated by the hierarchy, and for some $\alpha < \varepsilon_0 : Y_{i,\alpha} \cap Y_{j,\alpha} = \emptyset$. Thus, the resulting E for the finite set $\{f_i, f_j\}$ will not satisfy the finite intersection property.

such f_i and f_j are for example:

$$f_i(n) = \begin{cases} 0 & n \text{ is even} \\ F_{\varepsilon_0}(n) & n \text{ is odd} \end{cases}$$

$$f_j(n) = \begin{cases} 0 & n \text{ is odd} \\ F_{\varepsilon_0}(n) & n \text{ is even} \end{cases}$$

Since F_{ε_0} dominates the hierarchy, we have for these f_i and f_j that $Y_{i,\alpha}$ and $Y_{j,\alpha}$ are both infinite - i.e. f_i and f_j are not dominated by the hierarchy, but on the other hand $Y_{i,\alpha} \cap Y_{j,\alpha} = \emptyset$.

We will now show that such a model does not exist - there is no non-standard model of the natural numbers in which all the functions that are not dominated by the hierarchy are not complete.

Theorem 3.5 *There exist formulas* $\varphi_0 = \forall x \exists y P_0(x,y)$ *and* $\varphi_1 = \forall x \exists y P_1(x,y)$ *such that the functions* f_1 *and* f_2 *defined by* $f_i(x) = min\{y : P_i(x,y)\}$ *(*$i = 1,2$*) are total in the standard model of the natural numbers and are not dominated by the hierarchy but there is no non-standard model of* PA *-* M *- such that* $M \not\models \varphi_1$ *and* $M \not\models \varphi_2$*.*

Proof: Let g be a recursive function that dominates the hierarchy (for example, PH-function or F_{ε_0}). h will be the g's partial sums function:

$$\begin{aligned} h(0) &= g(0) \\ h(i+1) &= h(i) + g(i) \end{aligned}$$

Clearly, h is a recursive function as well. Also, for every $i : h(i) \geq g(i)$ and hence h dominates the hierarchy.

We shall now define a boolean function D_g:

$$D_g(m) = \begin{cases} 0 & \text{if the maximal } i \text{ such that } h(i) \leq m \text{ is even} \\ 1 & \text{otherwise (the maximal such } i \text{ is odd)} \end{cases}$$

D_g is recursive too (obviously, there is a simple algorithm that computes $D_g(m)$ given m).
How does the output of D_g look like? There are alternating blocks of 0 and 1, where the length of the ith block is $h(i+1) - h(i) = g(i)$:

Let n be a natural number and let $h(n)$ and $h(n+1)$ be denoted by x and y respectively. For every $x \leq m < y$ the maximal i such that $h(i) \leq m$ is n. For y - the maximal i such that $h(i) \leq y$ is $i+1$, and for $x-1$ the maximal i such that $h(i) \leq x-1$ is $n-1$.

If we assume, without loss of generality, that n is even we have:

$$\begin{aligned} D_g(x-1) &= 1 \\ D_g(m) &= 0 \ (x \leq m < y\ ) \\ D_g(y) &= 1 \end{aligned}$$

Thus, the length of this block (the nth block) is

$$y - x = h(n+1) - h(n) = g(n)$$

We now define, for a boolean function $B : \mathbb{N} \to \mathbb{N}$ a function $Len_B : \mathbb{N} \to \mathbb{N}$ as follows:

$$Len_B(n) = \text{the length of the } n\text{th block of } B$$

That is, for D_g we have - $Len_{D_g} = g$.
Let φ be the following formula:

$$\varphi = \forall x \exists y (D_g(h(x)) \neq D_g(h(x) + y))$$

such a formula exists since h and D_g are recursive and hence representable.
How is this formula interpreted in the standard model of the natural numbers? For every natural number n, if we examine $h(n)$ and the value D_g has on $h(n)$ - which is, without loss of generality, 0 - then there exists another natural number - y - that if we add it to $h(n)$ we result with a new number on which D_g has a different value - 1.

Clearly, for every natural number n there exists such y - according to the analysis done before : $h(n)$ is where the nth block starts, $D_g(h(n))$ is the parity of n. The next place where D_g has a different value is where the next block $(n+1)$ starts, and the distance of that place from $h(n)$ is as the length of the nth block. Therefore, for every n there exists an appropriate y and the minimal such y is the length of the nth block.

Thus, the statement φ is valid in the standard model of the natural numbers. φ is of the form $\forall x \exists y P(x,y)$. The function defined by φ is $t(x) = min\{y : P(x,y)\}$, and as mentioned earlier:

$$t(n) = \text{ the length of the } n\text{th block } (= Len_{D_g}(n)) = h(n+1) - h(n) = g(n)$$

Hence, the function defined by φ dominates the hierarchy.

Let M be a non-standard model of the natural numbers. Let d be a non-standard element in M. Without loss of generality, $D_g(h(d)) = 0$. We already know, by the proof of theorem 2.1, that there is $d < I_0 < t(d)$ such that $I_0 \models PA$. Hence,

$$I_0 \not\models \forall x \exists y (D_g(h(x)) \neq D_g(h(x)+y))$$

since there is an element in this model - d - for which:

$$I_0 \not\models \exists y (D_g(h(d)) \neq D_g(h(d)+y))$$

Therefore,

$$I_0 \models \forall y (D_g(h(d)) = D_g(h(d)+y))$$

That is, In I for every $z > h(d)$ $D_g(z) = 0$. Let φ be the following statement:

$$\varphi_0 = \forall x \exists y ((D_g(h(x)) = 0) \rightarrow (D_g(h(x)) \neq D_g(h(x)+y))) \equiv \forall x \exists y P_0(x,y)$$

Clearly : $I_0 \not\models \varphi_0$ since there exists an x in this model - which is the element d - for which there is no appropriate y: For every y $D_g(h(d)) = 0$ and still $D_g(h(d)) = D_g(h(d)+y)$.
The function defined by φ_0 - $f_0(x) = min\{y : P_0(x,y)\}$ - is the following:

$$f_0(x) = \begin{cases} t(x) & D_g(h(x)) = 0 \\ 0 & \text{otherwise (since then every } y \text{ satisfies it trivially, and the minimal is 0)} \end{cases}$$

Since $t(x)$ dominates the hierarchy, and for infinite number of x's: $D_g(h(x)) = 0$ then f_0 is greater than every F_α in infinite numbers of points. Hence, f_0 is not dominated by the hierarchy, but does not dominate it either (since its value in infinite number of points is 0).

Now, since M is elementary equivalent to the standard model of the natural numbers, there exists a y in M such that:

$$D_g(h(d)) \neq D_g(h(d)+y)$$

Let d' be $h(d)+y$. By definition of y -

$$D_g(d') = D_g(h(d)+y) = 1$$

We know, again - by the proof of theorem 2.1, that there is $d' < I_1 < t(d')$ such that $I_1 \models PA$ and therefore:

$$I_1 \not\models \forall x \exists y (D_g(h(x)) \neq D_g(h(x)+y))$$

since for d':

$$I_1 \not\models \exists y (D_g(h(d')) \neq D_g(h(d')+y))$$

that is:

$$I_1 \models \forall y (D_g(h(d')) = D_g(h(d')+y))$$

Hence, in I_1, for every $z > h(d')$ $D_g(z) = 1$. Thus if we examine the dual formula:

$$\varphi_1 = \forall x \exists y ((D_g(h(x)) = 1) \rightarrow (D_g(h(x)) \neq D_g(h(x)+y))) \equiv \forall x \exists y P_1(x,y)$$

we have : $I_1 \not\models \varphi_1$.
The function defined by φ_1 - $f_1(x) = min\{y : P_1(x,y)\}$ - is a sort of a completion of f_0 in relation to t:

$$f_1(x) = \begin{cases} 0 & D_g(h(x)) = 0 \\ t(x) & \text{otherwise} \end{cases}$$

Like f_0, f_1 is also a function that does not dominate the hierarchy and is not dominated by it either.

Assume by contradiction that there is a model M such that $M \not\models \varphi_0$ and $M \not\models \varphi_1$. φ_0 is false in M and therefore there is x_0 such that $D_g(h(x_0)) = 0$ and there is no y such that $D_g(h(x_0+y) = 1$.
φ_1 is false in M and therefore there is x_1 such that $D_g(h(x_1)) = 1$ and there is no y such that $D_g(h(x_1+y) = 0$.
Without loss of generality $h(x_0) < h(x_1)$. Therefore, there is y such that $h(x_0)+y = h(x_1)$. Hence,

$$D_g(h(x_0)+y) = D_g(h(x_1)) = 1$$

Thus, there is y such that $D_g(h(x_0)+y) = 1$ - a contradiction. Therefore, there is no such M.
■

Corollary 3.6 *There is no non-standard model M of PA in which all the functions that are not dominated by the hierarchy are not complete*

Proof: If such a model existed, then in particular, f_0 and f_1 defined during the proof of theorem 3.5 would have been incomplete in this model, and thus we would have had : $M \not\models \varphi_0$ and $M \not\models \varphi_1$ in contradiction with theorem 3.5. ∎

Corollary 3.7 *Given a recursive function $g : \mathbb{N} \to \mathbb{N}$ that dominates the hierarchy, there is a function $D_g : \mathbb{N} \to \{0,1\}$ that satisfies the following:*

1. *D_g is recursive.*
2. *$PA \vdash \forall x \exists y D_g(x) = y$ (that is, The completeness of D_g is provable in PA).*
3. *$PA \not\vdash \forall x \exists y > x D_g(y) = 1$.*
4. *$PA \not\vdash \forall x \exists y > x D_g(y) = 0$.*

Also, for any boolean function $D' : \mathbb{N} \to \{0,1\}$ for which $Len_{D'}$ is recursive and $Len_{D_g} \prec Len_{D'}$:

a. $PA \not\vdash \forall x \exists y > x D'(y) = 1$.

b. $PA \not\vdash \forall x \exists y > x D'(y) = 0$.

Proof: Given g we define D_g as defined during the proof of theorem 3.5. As already stated, D_g is recursive, and it therefore satisfies *1.*

D_g is defined as the parity of a maximal element in a set which is definable and is provable in PA to be bounded. Therefore it is provable in PA that D_g is well defined everywhere, and thus, it satisfies em 2. as well.

During the proof of theorem 3.5 we have shown that:

$$I_0 \models \exists x \forall y((D_g(h(x)) = 0) \to (D_g(h(x)) \neq D_g(h(x) + y)))$$

That is, there is an element in I_0 - which is that $h(x)$ - such that the value D_g has on it is 0, and on all greater elements D_g has the same value - 0. Therefore,

$$I_0 \models \exists x \forall y > x D_g(y) = 0$$

and hence:

$$I_0 \not\models \forall x \exists y > x D_g(y) = 1$$

Now, since I_0 is a model of PA, we have

$$PA \not\vdash \forall x \exists y > x D_g(y) = 1$$

Symmetrically, we have shown that

$$I_1 \models \exists x \forall y((D_g(h(x)) = 1) \to (D_g(h(x)) \neq D_g(h(x) + y)))$$

and therefore,

$$I_1 \not\models \forall x \exists y > x D_g(y) = 0$$

Since I_1 is also a model of PA we have:

$$PA \not\vdash \forall x \exists y > x D_g(y) = 0$$

Thus, requirements em 3. and em 4. are satisfied as well.

Let D' a boolean function - as described. We shall define a new function $s : \mathbb{N} \to \mathbb{N}$:

$$s(n) = \sum_{m<n} Len_{D'}(m)$$

Given n, $s(n)$ brings us to the beginning of the nth block of D' (recall that $Len_{D'}(m)$ is the length of the mth block of D'). Since $Len_{D'}$ is recursive then so is s. Let φ' be the following formula:

$$\varphi' = \forall x \exists y (D'(s(x)) \neq D'(s(x) + y))$$

This formula is, of course, valid in the standard model of the natural numbers, and y is the length of the xth block of D'. Therefore, the function defined by φ' is $Len_{D'}$. This function dominates Len_{D_g}, and thus it also dominates the Wainer Hierarchy. We now construct - as done before - two non-standard models of PA: We begin with any non-standard model of the natural numbers - M. Let d be a non-standard element in M such that $D'(s(d)) = 0$. Let $d < I'_0 < Len_{D'}(d)$ be a model of PA. We have:

$$I'_0 \models \exists x \forall y (D'(s(x)) = D'(s(x) + y) = 0)$$

(x is of course d). Therefore:

$$I'_0 \models \exists x \forall y > x D'(y) = 0$$

that is,

$$I'_0 \not\models \forall x \exists y > x D'(y) = 1$$

We now go back to M, and find in it a non-standard element d' such that $D'(s(d')) = 1$. Let $d' < I'_1 < Len_{D'}(d')$ be a model of PA. We have:

$$I'_1 \models \exists x \forall y (D'(s(x)) = D'(s(x)y) = 1)$$

(now x is d'). Therefore,

$$I'_1 \models \exists x \forall y > x D'(y) = 0$$

That is,

$$I'_1 \not\models \forall x \exists y > x D'(y) = 1$$

Since I'_0 and I'_1 are both models of PA, we have:

$$PA \not\vdash \forall x \exists y > x D'(y) = 0$$

$$PA \not\vdash \forall x \exists y > x D'(y) = 1$$

and that satisfies requirements $a.$ and $b.$ ∎

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