HOLOMORPHIC TRIANGLE INVARIANTS AND THE TOPOLOGY OF SYMPLECTIC FOUR-MANIFOLDS

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Abstract
This article analyzes the interplay between symplectic geometry in dimension 4 and the invariants for smooth four-manifolds constructed using holomorphic triangles introduced in [20]. Specifically, we establish a nonvanishing result for the invariants of symplectic four-manifolds, which leads to new proofs of the indecomposability theorem for symplectic four-manifolds and the symplectic Thom conjecture. As a new application, we generalize the indecomposability theorem to splittings of four-manifolds along a certain class of three-manifolds obtained by plumbings of spheres. This leads to restrictions on the topology of Stein fillings of such three-manifolds.

Contents

1. Introduction

In [20], we constructed an invariant for smooth, closed four-manifolds (using holomorphic triangles and the Heegaard Floer homology theories defined in [19] and [18]). The aim of this article is to investigate this invariant in the case where X is a closed, symplectic four-manifold. Our first result is the following.

THEOREM 1.1
If (X, ω) is a closed, symplectic four-manifold with $b_2^+(X) > 1$, then for the canonical

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Spin^c *structure* k*, we have*

$$\Phi_{X,k} = \pm 1.$$

Moreover, if $\mathfrak{s} \in \mathrm{Spin}^c(X)$ *is any* Spin^c *structure for which* $\Phi_{X,\mathfrak{s}} \not\equiv 0$*, then we have the inequality*

$$\langle c_1(k) \cup \omega, [X] \rangle \leq \langle c_1(\mathfrak{s}) \cup \omega, [X] \rangle,$$

with equality if and only if $k = \mathfrak{s}$*.*

The above theorem can be seen as a direct analogue of a theorem of Taubes concerning the Seiberg-Witten invariants for symplectic manifolds (see [22] and [23]). However, the proof (given in Section 5) is quite different in flavor. While Taubes's theorem uses the interplay of the symplectic form with the Seiberg-Witten equations, our approach uses the topology of Lefschetz fibrations, together with general properties of HF^+. As such, our proof relies on a celebrated result of Donaldson [4], which constructs Lefschetz pencils on symplectic manifolds (see also [1] and [21]).

Combined with the general properties of Φ (see [20]), the above nonvanishing theorem has a number of consequences.

1.1. New proofs of known results

Theorem 1.1 can be used to re-prove the indecomposability theorem for symplectic four-manifolds, a theorem whose Kähler version was established by Donaldson using his polynomial invariants (see [3]) and whose symplectic version was established by Taubes using Seiberg-Witten invariants (see [22]).

COROLLARY 1.2 (Donaldson, Kähler case; Taubes, symplectic case)
If (X, ω) *is a closed symplectic four-manifold, then it admits no smooth decomposition as a connected sum* $X = X_1 \# X_2$ *into two pieces with* $b_2^+(X_1)$*,* $b_2^+(X_2) > 0$.

Proof
This follows immediately from the nonvanishing result in Theorem 1.1 together with the vanishing result for Φ for a connected sum (see [20, Theorem 1.3]) (which in turn follows easily from the definition of Φ). □

In the course of proving Theorem 1.1, we establish a certain "adjunction relation" that can be seen as an analogue of an earlier adjunction relation from Seiberg-Witten theory (see [7] and [15]). Together with Theorem 1.1, this relation gives a new proof of the symplectic Thom conjecture. Note that this question has a long history in gauge theory. Various versions were proved in [12], [11], and [14], and the general case (which we re-prove here) is contained in [15].

THEOREM 1.3
If (X, ω) *is a symplectic four-manifold and* $\Sigma \subset X$ *is an embedded, symplectic submanifold, then* Σ *is genus-minimizing in its homology class.*

1.2. Generalized indecomposability
We generalize the indecomposability theorem for symplectic four-manifolds (Corollary 1.2) to a large class of plumbed three-manifolds, in place of S^3.

By a *weighted graph* we mean a graph G equipped with an integer-valued function m on the vertices of G. Recall that for each weighted graph there is a uniquely associated three-manifold $Y(G, m)$, which is the boundary of the associated plumbing of disk bundles over spheres. (The integer multiplicities here record the Euler numbers of the disk bundles.) The *degree* of a vertex v in a graph G, denoted $d(v)$, is the number of edges that contain the given vertex.

THEOREM 1.4
Let $Y = Y(G, m)$ *be a plumbed three-manifold, where* (G, m) *satisfies the following conditions:*

- G *is a disjoint union of trees;*
- *at each vertex in* G*, we have*

$$m(v) \geq d(v). \tag{1}$$

Then no closed, symplectic four-manifold (X, ω) *can be decomposed along* Y *as a union*

$$X = X_1 \cup_Y X_2$$

into two pieces with $b_2^+(X_1) > 0$ *and* $b_2^+(X_2) > 0$.

Note that in the special cases where Y is $S^2 \times S^1$ or a lens space, the above theorem was known using Seiberg-Witten theory.

COROLLARY 1.5
Let G *be a weighted graph satisfying the hypothesis of Theorem 1.4. If* X *is any Stein four-manifold with* $\partial X = \pm Y(G)$*, then* $b_2^+(X) = 0$.

Proof
According to [13], such a Stein manifold W can always be embedded in a surface of general type X, so that $b_2^+(X - W) > 0$. Thus, the corollary follows from Theorem 1.4. □

Note that $-Y(G)$ always admits a Stein filling with $b_2^+(X) = 0$, using a theorem of Eliashberg [6] (see also [9]).

Theorem 1.4 follows from Theorem 1.1, coupled with a vanishing theorem for four-manifolds admitting a decomposition along $Y(G,m)$. In turn, this vanishing theorem follows from a Heegaard Floer homology calculation for plumbings along graphs which satisfy the hypotheses of Theorem 1.4. Of course, it is interesting to consider plumbing diagrams that do not satisfy inequality (1). For this more general case, one does not expect such a strong vanishing theorem—for instance, any Seifert fibered space with $b_1(Y)=0$ can be obtained as a plumbing along a tree. For more on the Heegaard Floer homology of three-manifolds obtained as plumbings along trees, see [17].

1.3. Organization

This paper is organized as follows. In Section 2 we rapidly review some of the basic notions used throughout this paper, specifically those regarding Lefschetz fibrations. We also extend the four-manifold invariant Φ defined in [20] to the case where the the four-manifold X has $b_2^+(X)=1$. In Section 3, we derive the adjunction relation Theorem 3.1, which is used later in the proofs of Theorems 1.1 and 1.3. In Section 4, we calculate Φ for the $K3$ surface. In Section 5, we prove Theorem 1.1 along with an auxiliary result about the Heegaard Floer homology groups of a three-manifold which fibers over the circle. One ingredient in this proof is the $K3$ calculation in the previous section. In Section 6, we deduce Theorem 1.3 from Theorems 1.1 and 3.1. In Section 7, we provide the Floer homology calculations that lead to Theorem 1.4.

This paper, of course, is built on the theory developed in [19], [18], and [20], and it is written assuming familiarity with those papers. Important properties of the 4-dimensional invariant Φ (which is used repeatedly here) are summarized in [20, Section 3]. Moreover, at two important points in this paper (when calculating the invariant for the $K3$ surface and when finding examples of three-manifolds with non-trivial Floer homology which fiber over the circle), we rely on some of the calculations of Floer homology groups given in [16] (see especially [16, Section 8]).

1.4. Further remarks

For the purposes of proving Theorem 1.3, we extend the invariant Φ to four-manifolds with $b_2^+(X)=1$. As one expects from the analogy with gauge theory, the invariant in that case has additional structure. For our purposes, it suffices to construct Φ as the invariant of a four-manifold equipped with a line L inside $H_2(X;\mathbb{Q})$ consisting of vectors with square zero. This line corresponds to a choice of a "chamber at infinity" (cf. [2]). We hope to return to this topic in a future paper.

The pseudoholomorphic triangles in the g-fold symmetric product of the Heegaard surface implicit in the statement of Theorem 1.1 naturally give rise to a locus inside X. It is quite interesting to compare this object with the pseudoholomorphic curve constructed by Taubes in [24]. This may also provide a link with the work of Donaldson and Smith (see [5]).

2. Preliminaries

We collect here some of the preliminaries for the proof of Theorem 1.1. In Section 2.1 we review some standard properties of Lefschetz fibrations, mainly to set up the terminology that is used later. For a thorough discussion of this topic, we refer the reader to [9]. We then return to some properties of $HF^{\pm}$, building on the results from [20].

2.1. Lefschetz fibrations

Let C be an oriented two-manifold (possibly with boundary). A *Lefschetz fibration over* C is a smooth four-manifold W and a map $\pi\colon W \longrightarrow C$ with finitely many critical points, each of which admits an orientation-preserving chart modeled on $(w,z) \in \mathbb{C}^2$, where the map π is modeled on the map $\mathbb{C}^2 \longrightarrow \mathbb{C}$ given by $(w,z) \mapsto w^2+z^2$. Moreover, we always assume that any two critical points map to different values under π.

If $\pi\colon W \longrightarrow C$ has no critical points, then the fibration endows W with a canonical almost-complex structure characterized by the property that the fibers of π are J-holomorphic. Since a Spin^c structure over a four-manifold is specified by an almost-complex structure in the complement of finitely many points, a Lefschetz fibration endows W with a canonical Spin^c structure, which we denote by k. We adopt here the conventions of [22]: the first Chern class of the canonical Spin^c structure agrees with the first Chern class of the complex *tangent* bundle (on the locus where the latter is defined).

A Lefschetz fibration is said to be *relatively minimal* if none of the fibers of π contain exceptional spheres, that is, spheres whose self-intersection number is -1.

Lefschetz fibrations over the disk D,

$$\pi\colon W \longrightarrow D$$

(with n critical points), can be specified by an ordered n-tuple of simple, embedded curves $\tau_1,\dots,\tau_n$ in F. The space W then has the homotopy type of the two-complex by attaching disks to F along the curves. Homologies between the $[\tau_i]$ give rise to homology classes in W. More precisely, we can identify

$$H_2(W;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Ker}\big(\mathbb{Z}^n \longrightarrow H_1(F;\mathbb{Z})\big),$$

where the first $\mathbb{Z}$-factor is generated by the homology class of the fiber F and the map $\mathbb{Z}^n \longrightarrow H_1(F;\mathbb{Z})$ is the map generated by taking multiples of the homology classes of $[\tau_1],\dots,[\tau_n]$ in $H_1(F;\mathbb{Z})$.

Relative minimality in this case is equivalent to the condition that none of these distinguished curves in F bound disks in F.

LEMMA 2.1

Suppose that $P \subset F$ is a 2-dimensional manifold-with-boundary whose boundary is some collection of curves among the $\{\tau_1, \dots, \tau_n\}$ (each with multiplicity one). Let $\widehat{P}$ denote the closed surface in W obtained by attaching copies of vanishing cycles to P. Then

$$\begin{aligned} g(\widehat{P}) &= g(P), \\ \widehat{P} \cdot \widehat{P} &= -\left(\#\text{ of boundary components of } P\right), \\ \langle c_1(k), [\widehat{P}]\rangle - \widehat{P} \cdot \widehat{P} &= 2 - 2g(\widehat{P}). \end{aligned}$$

Proof

The equality on the genus is obvious. The self-intersection number of $\widehat{P}$ follows from the fact that the vanishing cycles are finished off with disks with framing -1. The final equation is a local calculation in view of the fact that the determinant bundle of the canonical Spin^c structure is identified, in the complement of the singular locus, with the bundle of fiber-wise tangent vectors. □

A Lefschetz fibration over a disk bounds a three-manifold that is a surface bundle over the circle. Such a bundle is uniquely given by the mapping class of its monodromy. (A mapping class of a two-manifold is an orientation-preserving diffeomorphism, modulo isotopy.) Recall that a (*right-handed*) Dehn twist of the annulus (using the conventions of [9]) is a diffeomorphism Ψ of $[0,1] \times S^1$ which fixes the boundary pointwise and satisfies the additional property that

$$\#\big([0,1] \times \{x\} \cap \Psi([0,1] \times \{x\})\big) = -1.$$

More generally, a (right-handed) Dehn twist about a curve $\tau \subset F$ is a self-diffeomorphism D_τ of F whose restriction to some annular neighborhood of τ is a right-handed Dehn twist of the annulus, and which fixes all points in the complement in F of the annular neighborhood. If the Lefschetz fibration has a unique critical point, then its monodromy is a Dehn twist about some curve τ in the fiber F. More generally, if the fibration has critical values $\{x_1, \dots, x_n\}$, then we can find the tuple of curves $(\tau_1, \dots, \tau_n)$ by embedding a bouquet of n circles in $D - \{x_1, \dots, x_n\}$, so that the winding number of τ_i around x_j is $\delta_{i,j}$. Then the monodromy about the ith circle is a Dehn twist about τ_i. Thus, the monodromy map around the boundary of the disk is given as the product of Dehn twists $D_{\tau_1} \circ \cdots \circ D_{\tau_n}$.

Note that the curves $(\tau_1, \dots, \tau_n)$ obtained from a Lefschetz fibration as above depend on the embedding of the bouquet of circles. By changing the homotopy classes of the embedded circles, we can vary the curves $(\tau_1, \dots, \tau_n)$ by *Hurwitz moves*, moves that carry the tuple $(\tau_1, \dots, \tau_i, \tau_{i+1}, \dots, \tau_n)$ to $(\tau_1, \dots, \tau_{i+1}, D^{-1}_{\tau_{i+1}}(\tau_i), \dots, \tau_n)$.

It is well known that any orientation-preserving automorphism of F extends to a Lefschetz fibration over the disk. Indeed, we find it convenient to formulate this fact as follows.

THEOREM 2.2 (see [10])
The mapping class group is generated as a monoid by Dehn twists along finitely many nonseparating curves. Indeed, we can choose the generating set $\{\tau_1, \ldots, \tau_m\}$ *so that their homology classes span* $H_1(\Sigma;\mathbb{Z})$*, all homological relations between the curves are generated (over* $\mathbb{Z}$*) by special relations in which the homology classes* τ_i *appear with multiplicities zero or* ± 1*, and the curves that appear with nonzero multiplicities in these relations can be chosen to be disjoint from one another.*

Proof
It is a theorem of Humphries [10] that the mapping class group is generated (as a group) by the $2g+1$ curves $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \delta\}$ which are pictured in Figure 1.

Now it is easy to see that if we include in addition the curve ϵ, then we can express the inverses of Dehn twists along all of the α_i and β_j as positive multiples of Dehn twists along copies of all the α_i, β_j, and ϵ. This can be seen, for example, from the identity

$$1 = \left(\left(\prod_{i=1}^{g} D_{\alpha_i} \cdot D_{\beta_i}\right) \cdot D_\epsilon^2 \cdot \left(\prod_{i=1}^{g} D_{\beta_{g-i+1}} \cdot D_{\alpha_{g-i+1}}\right)\right)^4,$$

which in turn can be obtained by exhibiting a Lefschetz fibration over the two-sphere whose monodromy representation is given by the above curves. (That Lefschetz fibration is obtained by viewing the elliptic surface $E(2g)$ as a genus $2g$ fibration over the two-sphere; see [9, Chapter 8] for an extensive discussion.) It remains to capture δ^{-1}. To this end, we observe that F has a rotational symmetry $\phi\colon F \longrightarrow F$ with the property that we can introduce a new curve α_{g+1} so that for

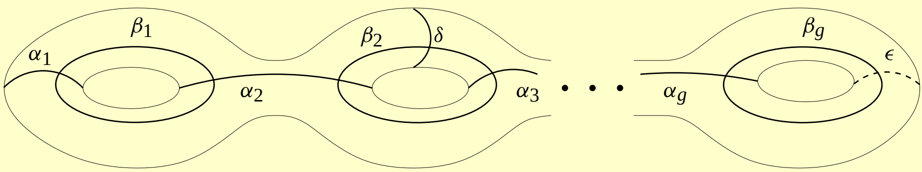

Figure 1. Generators of the mapping class group. Dehn twists about the pictured curves $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \delta\}$ generate the mapping class group. The additional curve ϵ is discussed in the proof of Theorem 2.2.

$i = 1, \ldots, g$, $\Psi(\beta_i) = \beta_j$ where $j \equiv i+1 \pmod{g}$ for $i = 2, \ldots, g$, $\Psi(\alpha_i) = \alpha_{i+1}$, $\Psi(\alpha_{g+1}) = \alpha_2$, $\Psi(\epsilon) = \alpha_1$, and finally $\Psi(\alpha_1) = \delta$. It is now clear that the mapping class group is generated as a monoid by Dehn twists about the $2g+3$ curves $\{\alpha_1, \ldots, \alpha_{g+1}, \beta_1, \ldots, \beta_g, \delta, \epsilon\}$. For homological relations between these curves, observe that the homology classes of the $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ span $H_1(\Sigma; \mathbb{Z})$. It follows that the three relations

$$[\alpha_1] + [\alpha_2] + [\delta] = 0,$$
$$[\epsilon] + [\alpha_{g+1}] + [\alpha_1] = 0,$$
$$[\alpha_2] + \cdots + [\alpha_{g+1}] = 0$$

span all relations (see Figure 2 for an illustration in the case where $g = 4$). □

Recall that a Spin^c structure over a three-manifold Y is a suitable equivalence class of nowhere vanishing vector fields over Y. A three-manifold that fibers over the circle has a canonical Spin^c structure induced by a vector field which is everywhere transverse to the fibers (oriented in the same direction as the base circle). When Y bounds a Lefschetz fibration over a disk, this Spin^c structure is the restriction of the canonical Spin^c structure of the Lefschetz fibration.

2.2. Symplectic manifolds and Lefschetz fibrations

A symplectic structure on a four-manifold (X, ω) gives the manifold an isotopy class of almost-complex structures and hence a canonical Spin^c structure. Symplectic manifolds can be blown up to construct a new four-manifold $\widehat{X}$ which is diffeomorphic to the connected sum of X with the complex projective plane given the opposite of its complex orientation. Symplectically, $\widehat{X}$ is obtained by gluing the complement of a ball in X to a neighborhood of a symplectic two-sphere E with self-intersection number -1. Note that the canonical Spin^c structure $\widehat{k}$ is the Spin^c structure that agrees with k in the complement of E and satisfies

$$\langle c_1(\widehat{k}), [E] \rangle = +1.$$

In [4], Donaldson showed that if (X, ω) is a symplectic four-manifold, then after blowing up X sufficiently many times, one obtains a new symplectic four-manifold $(\widehat{X}, \widehat{\omega})$ which admits a Lefschetz fibration

$$\pi : \widehat{X} \longrightarrow S^2.$$

In fact, the fibers of π are symplectic, and hence, the canonical Spin^c structure of the symplectic form agrees with the canonical class of the Lefschetz fibration in the sense of Section 2.1.

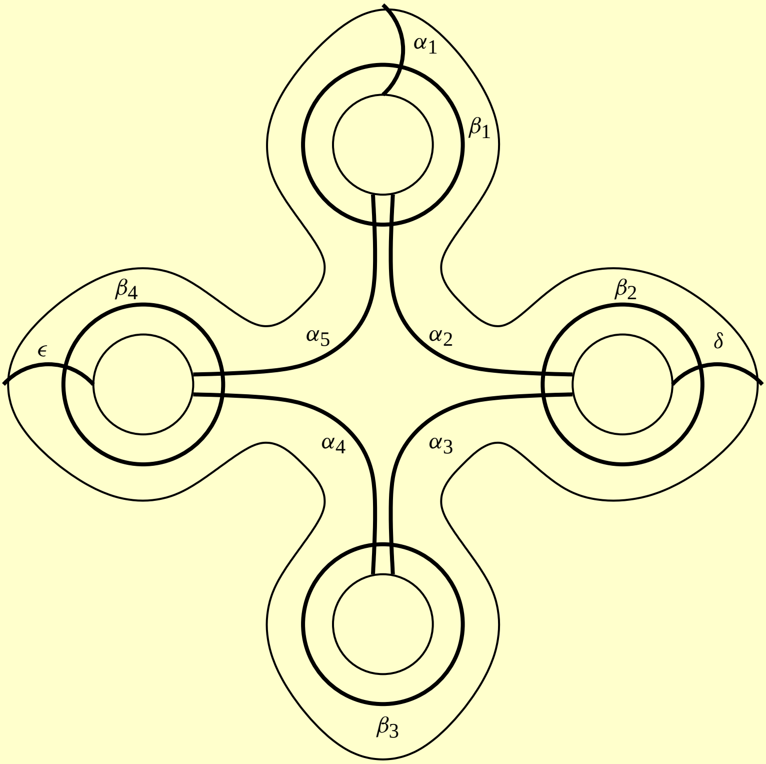

Figure 2. Monoid generators of the mapping class group $g = 4$. Dehn twists about the pictured curves $\{\alpha_1, \ldots, \alpha_5, \beta_1, \ldots, \beta_4, \epsilon, \delta\}$ generate the mapping class group as a monoid. The symmetry Ψ described in the proof of Theorem 2.2 is realized by a 90° clockwise rotation of this picture.

2.3. Preliminaries on HF^+

Let $\mathfrak{t}$ be a Spin^c structure on an oriented three-manifold Y. If $c_1(\mathfrak{t})$ is a torsion class, we simply call $\mathfrak{t}$ a torsion Spin^c structure. The *divisibility* of a nontorsion Spin^c structure $\mathfrak{t}$ is the integer given by

$$\mathfrak{d}(\mathfrak{t}) = \gcd_{\xi \in H^1(Y;\mathbb{Z})} \langle c_1(\mathfrak{t}) \cup \xi, [Y] \rangle.$$

LEMMA 2.3

Let Y be a three-manifold equipped with a nontorsion Spin^c *structure* $\mathfrak{t}$*, and let d be any positive multiple of* $\mathfrak{d}(\mathfrak{t})$*; then there is an integer ℓ with the property*

$$(1 - U^{d/2})^{\ell} HF^{\infty}(Y, \mathfrak{t}) = 0.$$

Proof
This is an easy consequence of the material in [18, Section 10]. Specifically, it is shown in [18, Theorem 10.12] that the twisted version of HF^∞, $\underline{HF}^\infty(Y,\mathfrak{t})$, is a free $\mathbb{Z}[U,U^{-1}]$-module, endowed with the $\mathbb{Z}[H^1(Y;\mathbb{Z})]$-action where e^h ($h\in H^1(Y;\mathbb{Z})$) acts as multiplication by $U^{\langle h\cup c_1(\mathfrak{t}),[Y]\rangle/2}$. There is a universal coefficients spectral sequence converging to the untwisted version $HF^\infty(Y)$ (as a $\mathbb{Z}[U,U^{-1}]$ module) and whose E_2-term is given by

$$\mathrm{Tor}^i_{\mathbb{Z}[H^1(Y;\mathbb{Z})]}\big(\underline{HF}^\infty_j(Y,\mathfrak{t}),\mathbb{Z}[U,U^{-1}]\big),$$

where here the $\mathbb{Z}[U,U^{-1}]$ is given a trivial action by $\mathbb{Z}[H^1(Y;\mathbb{Z})]$. Observe that we have a free resolution of $\underline{HF}^\infty_j(Y,\mathfrak{t})$ as a module over $\mathbb{A}=\mathbb{Z}[U,U^{-1}]\otimes_{\mathbb{Z}}\mathbb{Z}[H^1(Y;\mathbb{Z})]$, given by

$$\bigotimes_{i=1}^{b_1(Y)}\big(\mathbb{A}\xrightarrow{e^{h_i}-U^{n_i/2}}\mathbb{A}\big),$$

where h_i is a basis for $H^1(Y;\mathbb{Z})$ and $n_i=\langle c_1(\mathfrak{t})\cup h_i,[Y]\rangle$. So, the E_2-term of the above sequence is simply calculated by the homology of

$$\bigotimes_{i=1}^{b_1(Y)}\big(\mathbb{Z}[U,U^{-1}]\xrightarrow{1-U^{n_i/2}}\mathbb{Z}[U,U^{-1}]\big).$$

Bearing in mind that

$$\begin{aligned}\Big(\frac{\mathbb{Z}[U]}{U^a-1}\Big)\otimes_{\mathbb{Z}[U,U^{-1}]}\Big(\frac{\mathbb{Z}[U]}{U^b-1}\Big)&\cong\mathbb{Z}[U]/(U^c-1)\\&\cong\mathrm{Tor}^1_{\mathbb{Z}[U,U^{-1}]}\Big(\frac{\mathbb{Z}[U]}{U^a-1},\frac{\mathbb{Z}[U]}{U^b-1}\Big)\end{aligned}$$

(and all higher Tor^i vanish), where here $c=\gcd(a,b)$, it follows easily that $U^{d/2}-1$ annihilates this E_2-term (in view of the fact that $\mathfrak{d}(\mathfrak{t})$ is the greatest common divisor of the integers $\langle c_1(\mathfrak{t})\cup h_i,[Y]\rangle/2$ for $i=1,\dots,b_1(Y)$). It follows readily that $U^{d/2}-1$ acts nilpotently on $HF^\infty(Y)$ with untwisted coefficients. □

COROLLARY 2.4
For each nontorsion Spin^c *structure* $\mathfrak{t}$ *over a three-manifold* Y*, we have* $HF^+(Y,\mathfrak{t})\cong HF^+_{\mathrm{red}}(Y,\mathfrak{t})$.

Proof
Recall that $HF^+_{\mathrm{red}}(Y,\mathfrak{t})$ is the quotient of $HF^+(Y,\mathfrak{t})$ by the image of $HF^\infty(Y,\mathfrak{t})$. Now according to Lemma 2.3, since $\mathfrak{t}$ is nontorsion, we can find some polynomial $P(U)\in\mathbb{Z}[U]$ with $P(0)=0$ and with the property that if $\xi\in HF^\infty(Y,\mathfrak{t})$, then

$(1-P(U))\xi = 0$. On the other hand, $(1-P(U))$ is invertible on $HF^+(Y,\mathfrak{t})$ since for each fixed $\xi \in HF^+(Y,\mathfrak{t})$, all sufficiently high powers of U annihilate ξ $\big($i.e., the inverse is given by $\sum_{i=0}^{\infty} P(U)^i\big)$. It now follows that the image of $HF^\infty(Y,\mathfrak{t})$ in $HF^+(Y,\mathfrak{t})$ is trivial. □

Let Y be a closed, oriented three-manifold. It follows from Lemma 2.3 and the fact that $HF_{\mathrm{red}}(Y)$ is finitely generated that we can find arbitrarily large integers d and ℓ with the property that

$$(1-U^d)^\ell \colon HF^-(Y,\mathfrak{t}) \longrightarrow HF^-_{\mathrm{red}}(Y,\mathfrak{t})$$

defines a projection map of $HF^-(Y,\mathfrak{t})$ onto $HF^-_{\mathrm{red}}(Y,\mathfrak{t})$. While this projection does depend on the choice of d and ℓ, it is easy to see that if $d = m!$ and m and ℓ are suffiently large, then the projection is independent of the choices of m and ℓ. In fact, by composing with the inverse of the coboundary map

$$\tau\colon HF^+_{\mathrm{red}}(Y,\mathfrak{t}) \longrightarrow HF^-_{\mathrm{red}}(Y,\mathfrak{t}),$$

this gives a map

$$\Pi^{\mathrm{red}}_Y\colon HF^-(Y,\mathfrak{t}) \longrightarrow HF^+_{\mathrm{red}}(Y,\mathfrak{t}).$$

Using a decomposition of W along a three-manifold N, and using a Spin^c structure $\mathfrak{s}$ over W whose restriction to N is nontorsion, is analogous to the "admissible cuts" of [20]. Indeed, the comparison with the mixed invariants defined there is given by the following.

PROPOSITION 2.5

Suppose that W is a cobordism from Y_1 to Y_2 with $b_2^+(W) > 1$ which is separated by a three-manifold N into a pair of cobordisms $W_1 \cup_N W_2$. Given any pair of Spin^c *structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ over W_1 and W_2, respectively, whose restrictions to N agree and are nontorsion, we have*

$$F^+_{W_2,\mathfrak{s}_2} \circ \Pi^{\mathrm{red}}_N \circ F^-_{W_1,\mathfrak{s}_1}(\xi) = \sum_{\{\mathfrak{s}\in\mathrm{Spin}^c(W) \mid \mathfrak{s}|W_1=\mathfrak{s}_1,\, \mathfrak{s}|W_2=\mathfrak{s}_2\}} \pm F^{\mathrm{mix}}_{W,\mathfrak{s}}(\xi).$$

Proof

Since $c_1(\mathfrak{s})|N$ is nontorsion, we can find an embedded surface $F \subset N$ with $\langle c_1(\mathfrak{s}),[F]\rangle \neq 0$. Now, we can cut W in two along $N' = Y_1\#(S^1\times F)$, giving $W = W_1' \cup_{N'} W_2'$. By naturality of the exact sequences (relating HF^-, HF^∞, and HF^+) and the usual composition laws, we see that

$$F^+_{W_2,\mathfrak{s}|W_2} \circ \Pi^{\mathrm{red}}_N \circ F^-_{W_1,\mathfrak{s}|W_1}(\xi) = \sum_{\eta\in\delta H^1(N)} F^+_{W_2',\mathfrak{s}+\eta|W_2'} \circ \Pi^{\mathrm{red}}_{N'} \circ F^-_{W_1',\mathfrak{s}+\eta|W_1'}(\xi).$$

Next, we find some embedded surface $\Sigma \subset W$ of positive square which is disjoint from F, and we let Q denote its tubular neighborhood. Then $Q\#Y_2$ naturally gives a cut of W which we can arrange to be disjoint from the cut N' used above (by making the tubular neighborhoods sufficiently small). It now follows easily from the composition laws that

$$\sum_{\eta\in\delta H^1(N)} F^+_{W_2',\mathfrak{s}+\eta|W_2'} \circ \Pi^{\mathrm{red}}_{N'} \circ F^-_{W_1',\mathfrak{s}+\eta|W_1'}(\xi) = \sum_{\eta\in\delta H^1(N)} F^{\mathrm{mix}}_{W,\mathfrak{s}+\delta\eta}\left((1-U^d)^{\ell}\,\xi\right).$$

The equation now follows for d and ℓ large enough. □

2.4. The case where $b_2^+(X)=1$

The construction of closed invariants defined in [20] works only in the case where the four-manifold has $b_2^+(X)>1$. However, Proposition 2.5 suggests a construction that can be used even when $b_2^+(X)=1$. Rather than setting up the general theory at present, we content ourselves with developing enough of it to allow us to establish Theorem 1.3 in the case where $b_2^+(X)=1$.

Definition 2.6

Let X be a closed, smooth four-manifold, and choose a line $L \subset H_2(X;\mathbb{Q})$ with the property that each vector $v\in L$ has $v\cdot v=0$. Choose a cut $X = X_1\#_N X_2$ for which the image of $H_2(N;\mathbb{Q})$ inside $H_2(X;\mathbb{Q})$ is L. Then, for each Spin^c structure $\mathfrak{s}\in \mathrm{Spin}^c(X)$ for which $c_1(\mathfrak{s})$ evaluates nontrivially on L, we can define

$$\Phi_{W,\mathfrak{s},L}\colon \mathbb{Z}[U]\otimes \Lambda^*\big(H_1(X)/\mathrm{Tors}\big) \longrightarrow \mathbb{Z}/\pm 1$$

to be nonzero on only those homogeneous elements of $\mathbb{Z}[U]\otimes\Lambda^*(H_1(X)/\mathrm{Tors})$ whose degree is given by

$$d(\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X)}{4},$$

where $\chi(X)$ denotes the Euler characteristic of X and $\sigma(X)$ denotes the signature of its intersection form. On those elements, the invariant is the coefficient of $\Theta^+\in HF^+(S^3,\mathfrak{s})$ in the expression

$$F^+_{W_2,\mathfrak{s}|W_2}\circ \Pi^{\mathrm{red}}_N \circ F^-_{W_1,\mathfrak{s}|W_1}(U^n\cdot\Theta^-\otimes\zeta).$$

Here, Θ^+ and Θ^- are bottom- and top-dimensional generators of $HF^+(S^3)$ and $HF^-(S^3)$, respectively.

PROPOSITION 2.7

The invariant $\Phi_{W,\mathfrak{s},L}$ depends on the cut only through the choice of line $L\in H_2(X;\mathbb{Q})$.

Proof

An embedded surface $F \subset X$ whose homology class is in the line L always gives rise to a cut as in Definition 2.6. Specifically, let $F \subset X$ be a smoothly embedded, connected submanifold with $[F] \in L$. Then we decompose

$$X = \big(X - \mathrm{nd}(F)\big) \cup_{S^1 \times F} (F \times D).$$

Next, suppose that F_1 and F_2 are two embedded surfaces whose homology classes lie inside L. Then we claim that there is a third embedded surface F_3 which is disjoint from both F_1 and F_2 and whose homology class also lies inside L. This is easily constructed by starting with some initial surface Σ and then adding handles along canceling pairs of intersection points between Σ and F_1 (and then Σ and F_2). It follows now from the usual arguments that the invariant calculated by using the cut determined by F_1 (or F_2) agrees with the invariant calculated using the cut determined by F_3; that is, the invariant using any such embedded surface is independent of the choice of homology class and surface.

Finally, if $X = W_1 \cup_N W_2$ is an arbitrary cut as in Definition 2.6, then we can find an embedded surface $F \subset X$ disjoint from N whose homology class lies in the line L. Indeed, letting F_0 be any surface representing an element of $H_2(N; \mathbb{Z})$ with nontrivial image in $H_2(X; \mathbb{Z})$, we let F be a surface obtained by pushing F_0 out of N, using some vector field normal to N inside X. Since F is disjoint from N, the usual arguments again show that the invariant calculated using the cut N agrees with the invariants calculated using the cut determined by any embedded surface whose homology class lies in L. □

3. The adjunction relation

We prove here the following adjunction relation. (For the Seiberg-Witten analogues, compare [7] when $g = 0$, and compare [15] when $g > 0$.)

THEOREM 3.1

For each genus g, there is an element $\xi \in \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H_1(\Sigma)$ of degree $2g$ with the following significance. Given any smooth, oriented, 4-dimensional cobordism W from Y_1 to Y_2 (both of which are connected three-manifolds), any smoothly embedded, connected, oriented submanifold $\Sigma \subset W$ of genus g, and any $\mathfrak{s} \in \mathrm{Spin}^c(W)$ satisfying the constraint that

$$\langle c_1(\mathfrak{s}), [\Sigma]\rangle - [\Sigma] \cdot [\Sigma] = -2g(\Sigma), \tag{2}$$

we have the relation

$$F^\circ_{W,\mathfrak{s}}(\cdot) = F^\circ_{W,\mathfrak{s}+\epsilon\,\mathrm{PD}[\Sigma]}\big(i_*(\xi(\Sigma)) \otimes \cdot\big), \tag{3}$$

where ϵ is the sign of $\langle c_1(\mathfrak{s}), [\Sigma]\rangle$, and $i_* \colon \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H_1(\Sigma) \longrightarrow \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H_1(W)/\mathrm{Tors}$ is the map induced by the inclusion $i \colon \Sigma \longrightarrow W$.

Before proceeding to the proof of Theorem 3.1, we make a few general observations. Note that if

$$\left|\langle c_1(\mathfrak{s}), [\Sigma]\rangle\right| \geq 2g - \Sigma \cdot \Sigma,$$

the above theorem always gives relations of the form of equation (3), which can be obtained by reversing the orientation of Σ and adding extra null-homologous handles, if necessary, to achieve the hypotheses of Theorem 3.1.

It is not important for our present purposes to identify the particular word $\xi(\Sigma)$. However, it is easy to see that for a genus g surface,

$$\xi(\Sigma) \equiv U^g \pmod{\Lambda^* H_1(\Sigma)}$$

by observing that surfaces and Spin^c structures satisfying the hypotheses of Theorem 3.1 can be found in a tubular neighborhood of a two-sphere of arbitrary negative self-intersection number, where all the maps on HF^∞ are nontrivial. Indeed, it is natural to expect from the analogy with Seiberg-Witten theory that $\xi(\Sigma)$ is given by the formula

$$\xi(\Sigma) = \prod_{i=1}^{g} (U - A_i \cdot B_i)$$

(see [15]).

With these remarks in place, we turn our attention to the proof of Theorem 3.1. One ingredient in this proof is the behavior of HF° under connected sums (where here HF° denotes any one of HF^-, HF^∞, or HF^+), as we recall presently. In [16, Section 4] we defined a product

$$\bigotimes\colon\ HF^\circ(Y, \mathfrak{t}) \otimes_{\mathbb{Z}[U]} HF^{\leq 0}(Z, \mathfrak{u}) \longrightarrow HF^\circ(Y \# Z, \mathfrak{t} \# \mathfrak{u}),$$

which is an isomorphism in the case where $Z \cong S^3$. (Indeed, it is the canonical isomorphism obtained from the diffeomorphism $Y \# S^3$ with Y.) This product is functorial under cobordisms (see [16, Proposition 4.4]) in the sense that if W is a cobordism from Z_1 to Z_2 equipped with the Spin^c structure $\mathfrak{s}$, then the following diagram commutes:

$$\begin{array}{ccc} HF^\circ(Y) \otimes_{\mathbb{Z}[U]} HF^{\leq 0}(Z_1) & \xrightarrow{\otimes} & HF^\circ(Y \# Z_1, \mathfrak{t}\#\mathfrak{u}_1) \\ \downarrow{\scriptstyle \mathrm{Id} \otimes F^{\leq 0}_{W,\mathfrak{s}}} & & \downarrow{\scriptstyle F^\circ_{([0,1]\times Y)\# W, \mathfrak{t}\#\mathfrak{u}}} \\ HF^\circ(Y) \otimes_{\mathbb{Z}[U]} HF^{\leq 0}(Z_2) & \xrightarrow{\otimes} & HF^\circ(Y \# Z_2, \mathfrak{t}\#\mathfrak{u}_2) \end{array} \tag{4}$$

In the above diagram, $([0,1] \times Y) \# W$ denotes the boundary connected sum.

Proof of Theorem 3.1
By the blow-up formula, it suffices to consider the case where

$$\Sigma \cdot \Sigma = -n,$$

where $n \geq 2g$.

Now, let N be a tubular neighborhood of an oriented two-manifold of genus g with self-intersection number $-n \leq -2g$, and let $\mathfrak{u}$ denote the Spin^c structure over N with

$$\langle c_1(\mathfrak{u}), [\Sigma]\rangle = -n - 2g.$$

An easy application of the long exact sequence for integral surgeries, together with the adjunction inequality for three-manifolds (see [18, Theorem 9.19] and [18, Theorem 7.1], resp.), gives us

$$HF^+(Z, \mathfrak{u}|Z) \cong \mathbb{Z}[U^{-1}] \otimes \Lambda^* H^1(\Sigma_g).$$

(Details are given in [16, Lemma 9.17], where the absolute grading on $HF^\circ(Z, \mathfrak{u}|Z)$ is also calculated.) In particular, $HF^+_{\mathrm{red}}(Z, \mathfrak{u}|Z) = 0$, and hence,

$$HF^{\leq 0}(Z, \mathfrak{u}|Z) \cong \mathbb{Z}[U] \otimes \Lambda^* H^1(\Sigma_g).$$

Indeed, since

$$\left\langle c_1(\mathfrak{u} - \mathrm{PD}[\Sigma])^2, [N]\right\rangle > \langle c_1(\mathfrak{s}')^2, [N]\rangle$$

for any $\mathfrak{s}' \in \mathrm{Spin}^c(N)$ with $\mathfrak{s}' \neq \mathfrak{u} - \mathrm{PD}[\Sigma]$ and $\mathfrak{s}'|Z = \mathfrak{u}|Z$, we have that the map

$$F^{\leq 0}_{N,\mathfrak{u}-\mathrm{PD}[\Sigma]} \colon HF^{\leq 0}(S^3) \longrightarrow HF^{\leq 0}(Z, \mathfrak{u})$$

takes a top-dimensional Θ_{S^3} of $HF^{\leq 0}(S^3)$ to a top-dimensional generator Θ_Z of $HF^{\leq 0}(Z, \mathfrak{u}|Z)$. Moreover, according to the dimension formula, the grading of $F^{\leq 0}_{N,\mathfrak{u}}(\Theta_{S^3})$ is $2g$ less than the grading of this element, so (since $HF^{\leq 0}(Z, \mathfrak{u}|Z)$ is generated by Θ_Z as a module over the ring $\mathbb{Z}[U] \otimes \Lambda^* H_1(Z)/\operatorname{Tors} \cong \mathbb{Z}[U] \otimes \Lambda^* H_1(\Sigma)$) we can find an element $\xi(\Sigma)$ of degree $2g$ in the graded algebra $\xi(\Sigma) \in \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H_1(\Sigma)$ with the property that

$$F^{\leq 0}_{N,\mathfrak{u}}(\Theta_{S^3}) = \xi(\Sigma) \cdot F^{\leq 0}_{N,\mathfrak{u}-\mathrm{PD}[\Sigma]}(\Theta_{S^3}).$$

Next, suppose that Y_1 is a three-manifold equipped with the Spin^c structure $\mathfrak{t}_1$ and that W_1 is the connected sum $([0,1] \times Y_1) \# N$; then the naturality of the product map (diagram (4)) shows that

$$\begin{aligned} F^\circ_{W_1,\mathfrak{u}}(\zeta) &= \zeta \otimes F^{\leq 0}_{N,\mathfrak{u}}(\Theta_{S^3}) \\ &= \zeta \otimes \left(\xi(\Sigma) \cdot F^{\leq 0}_{N,\mathfrak{u}-\mathrm{PD}[\Sigma]}(\Theta_{S^3})\right) \\ &= F^\circ_{W_1,\mathfrak{u}-\mathrm{PD}[\Sigma]}\left(\xi(\Sigma) \otimes \zeta\right). \end{aligned}$$

Finally, if W is a cobordism as in the statement of the theorem, we can decompose it into a union of W_1 (the connected sum of a collar neighborhood of Y_1 with a tubular neighborhood N of Σ) and its complement W_2. Both Spin^c structures $\mathfrak{s}$ and $\mathfrak{s} - \mathrm{PD}[\Sigma]$ agree over W_2, so the theorem follows from the above equation, together with the composition law for the cobordism invariant. □

4. The invariant for the $K3$ surface

In proving the nonvanishing theorem for symplectic four-manifolds in general, it is helpful to have one explicit example. The aim of this section is such a calculation for the $K3$ surface. Recall that the $K3$ surface is the simply connected smooth four-manifold that can be given the structure of a compact algebraic surface whose canonical class is trivial; that is, if k is the canonical Spin^c structure coming from the almost-complex structure, then $c_1(k) = 0$.

PROPOSITION 4.1
The invariants for the $K3$ surface are given by

$$\Phi_{K3,\mathfrak{s}} = \begin{cases} 1 & \text{if } c_1(\mathfrak{s}) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We model this calculation on a paper by Fintushel and Stern [8], where they calculate a Donaldson invariant for $K3$ using Floer's exact triangle. In particular, they employ the following handle decomposition of $K3$.

Following the notation of [8], let $M\{p, q, r\}$ denote the three-manifold obtained by surgeries on the Borromean rings, with integer coefficients p, q, and r. There is a cobordism X from the Poincaré homology three-sphere $\Sigma(2, 3, 5) \cong M\{-1, -1, -1\}$ to itself with the opposite orientation, $-\Sigma(2, 3, 5) = M\{1, 1, 1\}$, composed of six two-handles apiece, which we break up as the following composition:

$$\begin{aligned} M\{-1,-1,-1\} &\Rightarrow M\{-1,-1,0\} \Rightarrow M\{-1,-1,1\} \\ &\Rightarrow M\{-1,0,1\} \Rightarrow M\{-1,1,1\} \Rightarrow M\{0,1,1\} \Rightarrow M\{1,1,1\}. \end{aligned}$$

The two-handles are attached in the obvious manner; for example, to go from $M\{p, q, r\}$ to $M\{p + 1, q, r\}$, we attach a two-handle along an unknot with framing -1 which links the first ring once. Let E denote the negative-definite manifold obtained as a plumbing of two-spheres according to the $E8$ Dynkin diagram; then $\partial E = M\{-1, -1, -1\}$. There is a decomposition of $K3$ as

$$K3 \cong E\#X\#E.$$

To obtain an admissible cut of the $K3$ as required in the definition of Φ (cf. [20, Definition 8.3]), we cut the surface along $N = M\{-1, -1, 1\}$ to get the decomposition of $K3 - B^4 - B^4$ as

$$X_1 = \big(S^3 \Rightarrow M\{-1,-1,-1\} \Rightarrow M\{-1,-1,0\} \Rightarrow M\{-1,-1,1\}\big)$$

and

$$X_2 = \big(M\{-1,-1,1\} \Rightarrow M\{-1,0,1\} \Rightarrow M\{-1,1,1\} \Rightarrow M\{0,1,1\} \Rightarrow M\{1,1,1\} \Rightarrow S^3\big).$$

Our goal now is to determine the maps on Floer homology induced by these two-handle additions. Indeed, the Floer homology groups themselves, as absolutely graded groups, were calculated in [16, Section 8]. In particular, it is shown there that

$$HF_k^+(M\{-1,-1,-1\}) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ is even and } k \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$HF_k^+(M\{0,-1,-1\}) \cong \begin{cases} \mathbb{Z} & \text{if } k \equiv \frac{1}{2} \pmod 1 \text{ and } k \geq \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

$$HF_k^+(M\{1,-1,-1\}) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} & \text{if } k \text{ is even and } k > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$HF_k^+(M\{-1,0,1\}) \cong \begin{cases} \mathbb{Z} & \text{if } k \equiv \frac{1}{2} \pmod 1 \text{ and } k \geq \frac{1}{2}, \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } k = -\frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

It is also shown there that the $\mathbb{Z}[U]$-action is surjective for the first two examples, while it has a 1-dimensional cokernel for the second two. The groups HF_k^- for these three-manifolds can be immediately deduced by the long exact sequence relating HF^- and HF^+ (see [18]), and the groups for the remaining three-manifolds are determined by the duality of $HF^\pm$ under orientation reversal and the observation that $-M\{p,q,r\} \cong M\{-p,-q,-r\}$.

In the above statements, we are using the absolute gradings on the Floer homology groups for Y equipped with a torsion Spin^c structure $\mathfrak{t}$ defined in [20, Section 7]. This absolute grading has the property that if W is a cobordism from Y_1 to Y_2 (endowed with a Spin^c structure $\mathfrak{s}$ whose restrictions $\mathfrak{t}_1$ and $\mathfrak{t}_2$, resp., are both torsion), then

$$\mathrm{gr}\big(F_{W,\mathfrak{s}}(\xi)\big) - \mathrm{gr}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\,\mathrm{sgn}(W)}{4} \tag{5}$$

(cf. [20, Theorem 7.1]).

LEMMA 4.2
For the cobordism $E - B^4$ from S^3 to $M\{-1,-1,-1\}$, endowed with the Spin^c *structure obtained by restricting k, the generator in $HF^-_{-2}(S^3)$ is mapped to the generator $HF^-_0(\Sigma(2,3,5))$.*

Proof

For a negative-definite cobordism between integral homology spheres, the map induced on HF^{∞} is always an isomorphism (see [16, Proposition 9.4]). From the dimension formula (equation (5)), it follows that the degree is raised by two. □

LEMMA 4.3

For the cobordism W,

$$M\{-1,-1,-1\} \Rightarrow M\{-1,-1,0\} \Rightarrow M\{-1,-1,1\}$$

(endowed with the Spin^c *structure obtained by restricting k), the induced map*

$$\mathbb{Z} \cong HF_0^-(M\{-1,-1,-1\}) \xrightarrow{F^-_{W,\mathfrak{s}}} HF^-_{-1}(M\{-1,-1,1\}) \cong \mathbb{Z}$$

is an isomorphism. Moreover, if we equip W with any other Spin^c *structure, the induced map is trivial.*

Proof

Let F denote the map given by the cobordism and summing over all Spin^c structures in $\delta H^1(M\{-1,-1,0\})$. Dualizing (i.e., applying [20, Theorem 3.5] and the graded version of the duality isomorphism; cf. [20, Proposition 7.11]), we get the following diagram:

$$\begin{array}{ccc}
\mathrm{H}F_-^{-1}(M\{1,-1,-1\}) & \xrightarrow{F_-} & \mathrm{H}F_-^{0}(M\{-1,-1,-1\}) \\
\cong \downarrow & & \cong \downarrow \\
HF^+_{-1}(M\{1,1,-1\}) & \xrightarrow{F^+} & HF^+_{-2}(M\{1,1,1\})
\end{array}$$

From the calculations of the Floer homology groups restated above, we see that

$$HF^+_{-1}(M\{1,1,-1\}) \cong \mathbb{Z} \cong HF^+_{-2}(M\{1,1,1\}).$$

Indeed, an isomorphism is given by composing the maps in the surgery exact sequence (see the proof of [16, Proposition 8.2]). But this composition is precisely F^+. It follows that the map F_- (the map on cohomology) above is an isomorphism, and hence (since there is no torsion present), its dual, the map

$$F^- : HF_0^-(M\{-1,-1,-1\}) \longrightarrow HF^-_{-1}(M\{-1,1,1\}),$$

induces an isomorphism (between two groups that are isomorphic to $\mathbb{Z}$).

The cobordism W has $b_2(W) = 2$. Indeed, we can find an embedded torus $T_1 \subset W$ which generates the image of $H_2(M\{-1,-1,0\};\mathbb{Z})$ inside W, and another embedded torus $T_2 \subset W$ with square zero with $T_1 \cdot T_2 = 1$. Applying the adjunction inequality for the cobordism invariant (see [20, Theorem 1.5]) to the embedded surface T_2, it follows that the only Spin^c structure $\mathfrak{s} \in k + \delta H^1(M\{-1,-1,0\})$ whose associated map $F^+_{W,\mathfrak{s}}$ is nontrivial is the restriction of k itself. □

Proof of Proposition 4.1
According to Lemmas 4.2 and 4.3, the generator of $HF^-_{-2}(S^3)$ is mapped to the generator of $HF^-_{-1}(M\{-1,-1,1\}) \cong \mathbb{Z}$. Now, δ^{-1} of that generator is the generator of $HF^+_{\mathrm{red},0}(M\{-1,-1,1\}) \cong \mathbb{Z}$. Investigating the four exact sequences connecting

$$\begin{array}{lll} \big(M\{-1,-1,1\}, & M\{-1,0,1\}, & M\{-1,\infty,1\} \cong S^3\big), \\ \big(M\{-1,\infty,1\} \cong S^3, & M\{-1,0,1\}, & M\{-1,1,1\}\big), \\ \big(M\{-1,1,1\}, & M\{0,1,1\}, & M\{\infty,1,1\} \cong S^3\big), \\ \big(M\{\infty,1,1\} \cong S^3, & M\{0,1,1\}, & M\{1,1,1\}\big), \end{array}$$

we see that the map

$$\mathbb{Z} \cong HF^+_{\mathrm{red},0}(M\{-1,-1,1\}) \longrightarrow HF^+_{-2}(M\{1,1,1\}) \cong \mathbb{Z}$$

induced by summing the maps induced by all Spin^c structures on the composite cobordism from $X_2 - N$ is an isomorphism. In fact, by finding square zero tori that intersect the homology classes coming from $H_2(M\{-1,0,1\};\mathbb{Z})$ and $H_2(M\{0,1,1\};\mathbb{Z})$ in $X_2 - N$ and applying the adjunction inequality (as in the proof of Lemma 4.3), we see that the only Spin^c structure that contributes to this sum is the one with trivial first Chern class. Finally, the map

$$HF^+_{-2}(M\{1,1,1\}) \longrightarrow HF^+_0(S^3)$$

is an isomorphism (for the given Spin^c structure) once again, in view of the dimension formula and the fact that $N - B^4$ has negative-definite intersection form (see [16, Proposition 9.4]). □

5. The nonvanishing theorem for symplectic four-manifolds

The aim of this section is to prove Theorem 1.1.* Via Donaldson's construction of Lefschetz pencils, we reduce this theorem to the following more manifestly topological variant.

THEOREM 5.1
Let $\pi : X \longrightarrow S^2$ be a relatively minimal Lefschetz fibration over the sphere with

*It is perhaps worth noting that some extra work is invested in identifying the Spin^c structure with nonvanishing invariant with the canonical Spin^c structure in the usual sense (see especially Lemma 5.7). If one is interested in a less precise nonvanishing result, some of this work can be avoided.

$b_2^+(X) > 1$ whose generic fiber F has genus $g > 1$. Then, for the canonical Spin^c *structure, we have*

$$\langle c_1(k), [F]\rangle = 2 - 2g,$$
$$\Phi_{X,k} = \pm 1.$$

Moreover, for any other Spin^c *structure $\mathfrak{s} \neq k$ with $\Phi_{X,\mathfrak{s}} \neq 0$, we have*

$$\langle c_1(k), [F]\rangle = 2 - 2g < \langle c_1(\mathfrak{s}), [F]\rangle.$$

One ingredient in the above proof is a related result for three-manifolds that fiber over the circle. To state it, recall that a three-manifold Y that admits a fibration

$$\pi : Y \longrightarrow S^1$$

has a canonical Spin^c structure that is obtained as the (integrable) two-plane field, which is the kernel of the differential of π. If F is a fiber of π, then the evaluation

$$\langle c_1(\ell), [F]\rangle = 2 - 2g.$$

THEOREM 5.2
Let Y be a three-manifold that fibers over the circle, with fiber genus $g > 1$, and let $\mathfrak{t}$ be a Spin^c *structure over Y with*

$$\langle c_1(\mathfrak{t}), [F]\rangle = 2 - 2g.$$

Then, for $\mathfrak{t} \neq \ell$, we have

$$HF^+(Y, \mathfrak{t}) = 0,$$

while

$$HF^+(Y, \ell) \cong \mathbb{Z}.$$

Indeed, we also establish the following result, which bridges Theorems 5.1 and 5.2.

THEOREM 5.3
Let $\pi : W \longrightarrow D$ be a relatively minimal Lefschetz fibration over the disk with fiber genus $g > 1$, and let $Y = -\partial W$. Then there is a unique Spin^c *structure $\mathfrak{s}$ over W for which*

$$\langle c_1(\mathfrak{s}), [F]\rangle = 2 - 2g,$$

and the induced map

$$F^+_{W-B^4,\mathfrak{s}} : HF^+(Y, \mathfrak{s}|Y) \longrightarrow HF^+(S^3)$$

is nontrivial; and that is the canonical Spin^c *structure* k*. Indeed, the induced map*

$$F^+_{W-B^4,k}\colon HF^+(Y,k|Y)\longrightarrow HF^+_0(S^3)\cong\mathbb{Z}$$

is an isomorphism.

We prove Theorems 5.1, 5.2, and 5.3 in reverse order.

In fact, we prove several special cases of these theorems first. It is convenient to fix some notation. Suppose that W is some four-manifold that admits a Lefschetz fibration π (over some two-manifold, possibly with boundary). Then we let

$$\mathfrak{S}(W)=\left\{\mathfrak{s}\in\mathrm{Spin}^c(W)\,\middle|\,\langle c_1(\mathfrak{t}),[F]\rangle=2-2g\right\}.$$

(This is a slight abuse of notation: $\mathfrak{S}(W)$ depends on the Lefschetz fibration π, not just the four-manifold W.) Similarly, if Y is a three-manifold that fibers over the circle, we let

$$\mathfrak{T}(Y)=\left\{\mathfrak{t}\in\mathrm{Spin}^c(Y)\,\middle|\,\langle c_1(\mathfrak{t}),[F]\rangle=2-2g\right\}.$$

We also let $HF^+\big(Y,\mathfrak{T}(Y)\big)$ denote the direct sum

$$HF^+\big(Y,\mathfrak{T}(Y)\big)=\bigoplus_{\mathfrak{t}\in\mathfrak{T}(Y)}HF^+(Y,\mathfrak{t}).$$

LEMMA 5.4
Let $\pi\colon W\longrightarrow[1,2]\times S^1$ *be a relatively minimal Lefschetz fibration with fiber genus* $g>1$ *over the annulus, which connects a pair of three-manifolds* Y_1 *and* Y_2 *(which fiber over the circle); then for some choice of signs, the map*

$$\sum_{\mathfrak{s}\in\mathfrak{S}(W)}\pm F^+_{W,\mathfrak{s}}\colon HF^+\big(Y_1,\mathfrak{T}(Y_1)\big)\longrightarrow HF^+\big(Y_2,\mathfrak{T}(Y_2)\big)$$

induces an isomorphism.

Proof
Note that, whereas $\mathfrak{S}(W)$ can easily be infinite, according to the finiteness properties for the maps associated to cobordisms (see [20, Theorem 3.3]), there are only finitely many $\mathfrak{s}\in\mathfrak{S}(W)$ for which $F^+_{W,\mathfrak{s}}$ is nontrivial.

First, assume that the Lefschetz fibration π has a single node. In this case, W can be viewed as the cobordism obtained by attaching a single two-handle to $Y=Y_1$ along a curve K in the fiber of π, with framing -1 (with respect to the framing K inherits from the fiber $F\subset Y$); in particular, $Y_2=Y_{-1}(K)$. Moreover, since the Lefschetz fibration is relatively minimal, the curve K is homotopically nontrivial as a curve in F. Now, if $Y_0(K)$ is the three-manifold obtained as zero-surgery along

K, then the cobordism from Y to Y_0 also maps to the circle (by a map π_0 which is no longer a fibration but which extends the map π from Y to S^1). Clearly, if $\mathfrak{s}$ is any Spin^c structure that extends over W_0, the restriction of $c_1(\mathfrak{s})$ to a generic fiber of $\pi_0\colon Y_0(K) \longrightarrow S^1$ is also $2-2g$. However, since K is homotopically nontrivial, the Thurston norm of the homology class of this fiber in $Y_0(K)$ is smaller than $2-2g$, so the adjunction inequality for HF^+ (see [18, Theorem 7.1]) ensures that $HF^+(Y_0, \mathfrak{s}|Y_0) = 0$. Thus, the lemma follows immediately from the surgery long exact sequence for HF^+ (see [18, Theorem 9.12]):

$$\cdots \longrightarrow HF^+\big(Y, \mathfrak{T}(Y)\big) \longrightarrow HF^+\big(Y_{-1}(K), \mathfrak{T}(Y_{-1}(K))\big) \longrightarrow HF^+\big(Y_0(K), \mathfrak{T}(Y_0)\big) = 0 \longrightarrow \cdots .$$

In the above sequence, $\mathfrak{T}(Y_0)$ denotes those Spin^c structures whose evaluation on the homology class of a fiber of π_0 (which is no longer a fibration) is given by $2-2g$, where g still denotes the genus of the fibration for Y.

The case of multiple nodes follows immediately by the composition law. □

LEMMA 5.5
If $\pi : Y \longrightarrow S^1$ is a surface bundle over S^1, with fiber genus $g > 1$, then there is a unique Spin^c *structure $\mathfrak{t} \in \mathfrak{T}(Y)$ with $HF^+(Y, \mathfrak{t}) \neq 0$. In fact,*

$$HF^+(Y, \mathfrak{t}) \cong \mathbb{Z}.$$

Proof
Note that the mapping class group is generated as a monoid by (right-handed) Dehn twists. This is equivalent to the claim that if $p_1\colon Y_1 \longrightarrow S^1$ and $p_2\colon Y_2 \longrightarrow S^1$, any two fibrations over the circle whose fiber has the same genus, then we can extend the two fibrations to form a relatively minimal Lefschetz fibration over the annulus. It follows from Lemma 5.4 that for a genus g fibration over the circle, $HF^+(Y, \mathfrak{T}(Y))$ is independent of the monodromy map and depends only on the genus g.

Thus, for each $g > 1$, it suffices to find some fibered three-manifold for which the lemma is known to be true. For this purpose, let $Y = Y(g)$ be the zero-surgery on the torus knot K of type $(2, 2g+1)$. This is a fibered three-manifold whose fiber has genus g. Writing the symmetrized Alexander polynomial of K as

$$\Delta_K(T) = (-1)^{g/2} \sum_{i=-g}^{g} (-T)^i = a_0 + \sum_{i=1}^{d} a_i (T^i + T^{-i}),$$

it is shown in [16, Proposition 8.1] that if $\mathfrak{t}$ is a Spin^c structure over Y with

$$\langle c_1(\mathfrak{t}), [F]\rangle = 2i \neq 0,$$

then $HF^+(Y, \mathfrak{t})$ is a free Abelian group of rank

$$\sum_{j=1}^{\infty} j a_{|i|+j}.$$

In particular, when $\langle c_1(\mathfrak{t}), [F]\rangle = 2 - 2g$, it follows immediately that $HF^+(Y, \mathfrak{t}) \cong \mathbb{Z}$. □

LEMMA 5.6
Let F be an oriented surface of genus $g > 0$, and consider the cobordism W from S^3 to $F \times S^1$ obtained by puncturing the product $F \times D^2$ in a single point. Let k denote the Spin^c *structure over W with $\langle c_1(k), [F]\rangle = 2 - 2g$. Then the induced map*

$$F^+_{W,k} \colon HF^+(F \times S^1, \ell) \longrightarrow HF^+_0(S^3) \cong \mathbb{Z}$$

is an isomorphism, as is the induced map

$$\begin{aligned}(1 - U^{g-1})F^-_{W,k} \colon \mathbb{Z} \cong HF^-_{-2}(S^3) &\longrightarrow HF^-_{\mathrm{red}}(F \times S^1, \ell)\\ &\cong \mathbb{Z} \subset HF^-(F \times S^1, \ell).\end{aligned}$$

Proof
To show the claim about $F^+_{W,k}$, it suffices to embed the cobordism (W, k) into a closed four-manifold $(X, \mathfrak{s})$ with $b_2^+(X) > 1$, so that $\mathfrak{s}|W = k$ and $\Phi_{X,\mathfrak{s}} = \pm 1$. To see why this suffices, observe that $U \cdot HF^+(F \times S^1, \ell) = 0$; so $F^+_{W,\mathfrak{s}}$ must take $HF^+(F \times S^1, \ell)$ into $HF^+_0(S^3) \cong \mathbb{Z}$. In general, the image of such a map consists of multiples of some integer d. Now, take an admissible cut of $X = X_1 \#_N X_2$ which is disjoint from F and such that $F \subset X_2$. (Such a cut is found by taking any embedded surface Σ of positive square which is disjoint from F.) It then follows that for each Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$ which restricts to W as k, the sum of invariants

$$\sum_{n\in\mathbb{Z}} \Phi_{X,\mathfrak{s}+n\,\mathrm{PD}[\Sigma]}$$

is divisible by d. In fact, it is a straightforward consequence of the dimension formula that the part of this sum which is homogeneous of degree zero is the invariant $\Phi_{X,\mathfrak{s}}$, and this in turn forces $d = \pm 1$, so that the claimed map is an isomorphism.

Now, such four-manifolds can be found for all possible genera g in the blow-ups of the $K3$ surface in light of the blow-up formula and the $K3$ calculation. Specifically, for each genus g, we can find an embedded surface $\Sigma \subset K3$ with $\Sigma \cdot \Sigma = 2g - 2$, for instance, by taking a single section of an elliptic fibration of $K3$, which is a sphere of self-intersection number -2, and attaching g copies of the fiber. In the $(2g - 2)$-fold blow-up, Σ has a proper transform $\widehat{\Sigma}$ with $\widehat{\Sigma} \cdot \widehat{\Sigma} = 0$. Consider the Spin^c structure

$\widehat{\mathfrak{s}}$ with $c_1(\widehat{\mathfrak{s}}) = -\mathrm{PD}[E_1] - \cdots - \mathrm{PD}[E_{2g-2}]$, so that $\langle c_1(\widehat{\mathfrak{s}}), [\widehat{\Sigma}]\rangle = 2 - 2g$; that is, the tubular neighborhood of Σ is W, and $\widehat{\mathfrak{s}}$ is an extension of k. According to Proposition 4.1 and the blow-up formula (see [20, Theorem 1.4]), $\Phi_{\widehat{X},\widehat{\mathfrak{s}}} = \pm 1$.

The statement about HF^- follows similarly if the cut for X is chosen so that the surface F lies in X_1. □

LEMMA 5.7
Let $\pi : W \longrightarrow D$ be a Lefschetz fibration over the disk whose singular fibers are all nonseparating nodes. Then $\pi : W \longrightarrow D$ can be embedded in a Lefschetz fibration V over a larger disk with the property that the canonical Spin^c *structure k is the only* Spin^c *structure in $\mathfrak{s} \in \mathfrak{S}(V)$ for which*

$$F^+_{V,\mathfrak{s}} : HF^+\big(\partial V, \mathfrak{T}(\partial V)\big) \longrightarrow HF^+_0(S^3) \cong \mathbb{Z}$$

is nontrivial; and indeed, $F^+_{V,k}$ is an isomorphism.

Proof
We claim that any Lefschetz fibration over the disk with nonseparating fibers can be embedded into a Lefschetz fibration over the disk with nodes corresponding to (isotopic translates) of the standard curves $\{\tau_1, \ldots, \tau_m\}$ described in Theorem 2.2. This is constructed as follows. Suppose that W is described by monodromies that are Dehn twists around curves $(C_1, \ldots, C_n)$. Then we can find automorphisms of F, $\phi_1,\ldots,\phi_n$, so that $\phi_i(\tau_1) = C_i$. We then express each $\phi_i = D(\tau_{m_{i,1}}) \cdot \ldots \cdot D(\tau_{m_{i,\ell_i}})$. We let V be the Lefschetz fibration over the disk with monodromies obtained by juxtaposing $\tau_{m_{i,1}}, \ldots, \tau_{m_{i,\ell_i}}, \tau_1$ for $i = 1, \ldots, n$, with union as many τ_i as it takes to span all of $H_1(\Sigma; \mathbb{Z})$. By performing Hurwitz moves, we obtain a subfibration with monodromies $(\phi_1(\tau_1), \ldots, \phi_n(\tau_1))$; that is, we have embedded W in V.

Next, we argue that V has the required form. According to Lemmas 5.5, 5.6, and 5.4, we see that

$$\sum_{\mathfrak{s}\in\mathfrak{S}(V)} F^+_{V-B^4,\mathfrak{s}} : HF^+(\partial V, \mathfrak{t}) \cong \mathbb{Z} \longrightarrow HF^+_0(S^3)$$

is an isomorphism. We claim that k is the only Spin^c structure in the sum with nonzero contribution.

Note that $H_1(V; \mathbb{Z})$ is the quotient of $\mathbb{Z}^{2g}$ by the homology classes of the vanishing cycles for V. So we have arranged that $H_1(V; \mathbb{Z}) = 0$; in particular, $H^2(V; \mathbb{Z})$ has no torsion. It follows that the Spin^c structure k is uniquely determined by the evaluation of its first Chern class on the various 2-dimensional homology classes in V. Moreover, if we choose the translates of the various τ_i carefully, so that parallel copies of the same τ_i remain disjoint, then we can find a basis for $H_2(V; \mathbb{Z})$ consisting of

$[F]$ and surfaces $\widehat{P}$ obtained by "capping off" submanifolds-with-boundary $P \subset F$ whose boundaries consist of copies of the vanishing cycles. Next, suppose that $\widehat{P_1}$ is induced from a relation P_1 in F with this form, and let m denote the number of its boundary components. Then the relation $F - P_1 = P_2$ also has this form (and has the same number of boundary components), and its closed extension $\widehat{P_2}$ satisfies the following elementary properties (see Lemma 2.1):

$$\begin{aligned}[F] &= [\widehat{P_1}] + [\widehat{P_2}],\\ g(F) &= g(\widehat{P_1}) + g(\widehat{P_2}) + m - 1,\\ m &= -[\widehat{P_1}]^2 = -[\widehat{P_1}]^2 = [\widehat{P_1}] \cdot [\widehat{P_2}].\end{aligned}$$

Now suppose that $\mathfrak{s} \in \mathfrak{S}(V)$ is a Spin^c structure for which $F^+_{W,\mathfrak{s}}$ is nontrivial. Then the above equations and the condition $\langle c_1(\mathfrak{s}), [F]\rangle = 2 - 2g$ say that

$$\begin{aligned}&\big(\langle c_1(\mathfrak{s}), [\widehat{P_1}]\rangle - [\widehat{P_1}] \cdot [\widehat{P_1}]\big) + \big(\langle c_1(\mathfrak{s}), [\widehat{P_2}]\rangle - [\widehat{P_2}] \cdot [\widehat{P_2}]\big)\\ &\qquad\qquad = \big(2 - 2g(\widehat{P_1})\big) + \big(2 - 2g(\widehat{P_2})\big).\end{aligned}$$

Now, either

$$\langle c_1(\mathfrak{s}), [\widehat{P_1}]\rangle - [\widehat{P_1}] \cdot [\widehat{P_1}] = 2 - 2g([\widehat{P_1}]),$$

in which case (according to Lemma 2.1)

$$\langle c_1(\mathfrak{s}), [\widehat{P_1}]\rangle = \langle c_1(k), [\widehat{P_1}]\rangle, \tag{6}$$

or, after possibly switching the roles of $\widehat{P_1}$ and $\widehat{P_2}$, we have

$$\langle c_1(\mathfrak{s}), [\widehat{P_1}]\rangle - [\widehat{P_1}] \cdot [\widehat{P_1}] \leq -2g([\widehat{P_1}]). \tag{7}$$

Inequality (7) is ruled out by the adjunction relation, Theorem 3.1, as follows. By adding trivial two-handles to $\widehat{P_1}$ if necessary, we obtain an embedded surface with $\langle c_1(\mathfrak{s}), [\Sigma]\rangle = -2g + \Sigma \cdot \Sigma$. There are two cases, according to whether $g(\Sigma) = 0$ or $g(\Sigma) > 0$. In the latter case, the adjunction relation gives some word $\xi(\Sigma)$ of degree $2g(\Sigma) > 0$ in $\mathbb{A}(\Sigma)$ with the property that

$$F^+_{V,\mathfrak{s}}(\cdot) = F^+_{V,\mathfrak{s}+\epsilon\,\mathrm{PD}[\Sigma]}\big(\xi(\Sigma) \otimes \cdot\big).$$

Observe that homology classes in Σ are all homologous to classes in the fiber F in ∂V, so the action by $\xi(\Sigma)$ appearing above can be interpreted as the action by an element of positive degree in $\mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^*(H_1(Y)/\mathrm{Tors})$ on $HF^+(\partial V, \ell)$. But all such elements annihilate $HF^+(\partial V, \ell)$ (since it is supported in a single dimension). Thus, the only remaining possibility is that $g(\Sigma) = 0$, in which case no handles were added to $\widehat{P}_1$. In this case, the adjunction relation ensures that the Spin^c structure $\mathfrak{s} - \mathrm{PD}[\widehat{P}_1]$ has nontrivial invariant, while

$$\langle c_1(\mathfrak{s}), [\widehat{P}_2]\rangle - [\widehat{P}_2] \cdot [\widehat{P}_2] = 4 - 2g(\widehat{P}_2).$$

But then

$$\big\langle c_1(\mathfrak{s} - \mathrm{PD}[\widehat{P}_1]), [\widehat{P}_2]\big\rangle - [\widehat{P}_2] \cdot [\widehat{P}_2] = 4 - 2g(\widehat{P}_2) - 2m.$$

Next, observe that $m > 1$ since the vanishing cycles for V are all homotopically nontrivial. Moreover, if $m = 2$, then $g(\widehat{P}_2) = g(F)$. Thus, using $\widehat{P}_2$ in place of $\widehat{P}_1$, and $\mathfrak{s} - \mathrm{PD}[\widehat{P}_1]$ in place of $\mathfrak{s}$, we obtain the same contradiction as before.

The contradiction to inequality (7) leads to the conclusion that equation (6) holds for all choices of $\widehat{P}_1$. But these surfaces, together with $[F]$, generate the homology of V. Thus, we have shown that $\mathfrak{s} = k$, as claimed. □

Proof of Theorem 5.2
According to Lemma 5.5, there is a unique $\mathfrak{t} \in \mathfrak{T}(Y)$ with $HF^+(Y, \mathfrak{t}) \neq 0$, and for $\mathfrak{t}$ we have $HF^+(Y, \mathfrak{t}) \cong \mathbb{Z}$. It remains to identify $\mathfrak{t}$ with the canonical Spin^c structure. As in the proof of Lemma 5.5, we constructed a Lefschetz fibration over the annulus which connects Y with $S^1 \times \Sigma$. By attaching $D \times \Sigma$ to the $S^1 \times \Sigma$ boundary component, we obtain a Lefschetz fibration W over the disk. Indeed, since the mapping class group is generated by Dehn twists along nonseparating curves, we can choose W so that Lemma 5.7 applies to W. In particular, in this case, the canonical Spin^c structure k in $\mathfrak{S}(W)$ induces a nontrivial map $F^+_{W,k}$. The result follows since $k|Y = \ell$. □

LEMMA 5.8
Let W be a relatively minimal Lefschetz fibration over the annulus, all of whose nodes are separating. Then the only Spin^c *structure $\mathfrak{s} \in \mathfrak{S}(W)$ for which the map*

$$F^+_{W,\mathfrak{s}} \colon HF^+(Y_1, \mathfrak{s}|Y_1) \cong \mathbb{Z} \longrightarrow HF^+(Y_2, \mathfrak{s}|Y_2) \cong \mathbb{Z}$$

is nontrivial is the canonical Spin^c *structure. And for that* Spin^c *structure, the induced map is an isomorphism.*

Proof
According to Lemmas 5.5, 5.6, and 5.4, we see that

$$\sum_{\mathfrak{s} \in \mathfrak{S}(W)} F^+_{W,\mathfrak{s}} \colon \sum_{\mathfrak{t} \in \mathfrak{T}(Y)} HF^+(Y, \mathfrak{t}) \longrightarrow \sum_{\mathfrak{t} \in \mathfrak{T}(S^1 \times F)} HF^+(S^1 \times F, \mathfrak{t})$$
$$\cong HF^+(S^1 \times F, \ell) \cong \mathbb{Z}$$

is an isomorphism.

Now, observe that W is a cobordism that is obtained by attaching a sequence of two-handles along null-homologous curves. Thus, a Spin^c structure over W is uniquely characterized by its restriction to one of its boundary components, and its evaluations on the 2-dimensional homology classes introduced by the two-handles. According to Theorem 5.2, the restriction to the boundary must agree with the canonical Spin^c structure. Each node has, as fiber, a union of two surfaces meeting at a point; that is, we obtain a pair of embedded surfaces $g(\widehat{P}_1) + g(\widehat{P}_2) = g(F)$ and $\widehat{P}_1^2 = \widehat{P}_2^2 = -1$. Moreover, since the fibration is assumed to be relatively minimal, $g(\widehat{P}_1) > 0$ and $g(\widehat{P}_2) > 0$. Thus, applying the adjunction relation as in the proof of Lemma 5.7, we see that

$$\langle c_1(\mathfrak{s}), [\widehat{P}_1]\rangle = \langle c_1(k), [\widehat{P}_1]\rangle.$$

It is easy to see that the homology classes of the form $[\widehat{P}_1]$ (one for each node) generate $H_2(W;\mathbb{Z})/H_2(Y;\mathbb{Z})$. Thus, it follows that $\mathfrak{s} = k$. □

Proof of Theorem 5.3
Let $\pi\colon X \longrightarrow D$ be the Lefschetz fibration. By combining Lemmas 5.4, 5.5, and 5.6, we see that the map

$$\sum_{\mathfrak{s}\in\mathfrak{S}(X)} F^+_{X,\mathfrak{s}}\colon \bigoplus_{\mathfrak{t}\in\mathfrak{T}(Y)} HF^+(Y,\mathfrak{t}) \longrightarrow HF^+_0(S^3)$$

induces an isomorphism.

We can find a subdisk $D_0 \subset D$ which contains all the fibers with nonseparating nodes. Let $X_0 \subset X$ denote its preimage. According to Lemmas 5.4, 5.5, and 5.6, there must be at least one Spin^c structure $\mathfrak{s} \in \mathfrak{S}(X)$ for which the map

$$F^+_{X,\mathfrak{s}}\colon HF^+(Y,\mathfrak{s}|Y) \cong \mathbb{Z} \longrightarrow HF^+(S^3)$$

is nontrivial. According to Lemma 5.7, its restriction $\mathfrak{s}|X_0$ is the canonical Spin^c structure; according to Lemma 5.8, its restriction $\mathfrak{s}|X - X_0$ is also the canonical Spin^c structure. Now, the map $H^1(X - X_0) \longrightarrow H^1(Y;\mathbb{Z})$ is an isomorphism since $X - X_0$ is obtained from $Y \times [0,1]$ by attaching two-handles along null-homologous curves. Thus, the only Spin^c structure whose restrictions to both $X - X_0$ and X_0 agree with k is the canonical Spin^c structure k itself. □

Proof of Theorem 5.1
We decompose $X = X_1 \#_{S^1\times\Sigma_g} X_2$, where X_1 is the preimage of a disk in the Lefschetz fibration which contains no singular points (in particular, $X_1 = D \times F$). Ac-

cording to Proposition 2.5,

$$F^+_{W_2,\mathfrak{s}_2} \circ \Pi^{\mathrm{red}}_N \circ F^-_{W_1,\mathfrak{s}_1} = \sum_{\{\mathfrak{s}\in \mathrm{Spin}^c(W) | \mathfrak{s}|W_1=\mathfrak{s}_1, \mathfrak{s}|W_2=\mathfrak{s}_2\}} F^{\mathrm{mix}}_{W,\mathfrak{s}},$$

where here $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are the restrictions of k. Now, by Lemma 5.6,

$$\Pi^{\mathrm{red}} \circ F^-_{W_1,\mathfrak{s}_1} : \mathbb{Z} \cong HF^-_{-2}(S^3) \longrightarrow HF^+_{\mathrm{red}}(S^1 \times \Sigma_g) \cong \mathbb{Z}$$

is an isomorphism. Similarly, according to Theorem 5.3,

$$F^+_{W_2,\mathfrak{s}_2} : HF^+_{\mathrm{red}}(S^1 \times \Sigma_g) \cong \mathbb{Z} \longrightarrow HF^+_0(S^3) \cong \mathbb{Z}$$

is an isomorphism. Thus, we conclude that

$$1 = \sum_{\eta \in \delta H^1(\Sigma\times S^1)} \pm \Phi_{X,k+\eta}.$$

Observe, however, that $\delta H^1(\Sigma \times S^1)$ is 1-dimensional; in fact, the Spin^c structures in the $\delta H^1(\Sigma \times S^1)$-orbit of k are of the form $k + \mathbb{Z}\,\mathrm{PD}[F]$. By the dimension formula, the only such Spin^c structure that has degree zero is k (using the adjunction formula and the fact that the fiber genus $g > 1$). If $m > 0$, we see that $F^{\mathrm{mix}}_{W,k-m\,\mathrm{PD}[F]}$ is zero. If $F^{\mathrm{mix}}_{W,k+m\,\mathrm{PD}[F]}$ were nonzero, the expression $F^+_{W_2,\mathfrak{s}_2} \circ \Pi^{\mathrm{red}}_N \circ F^-_{W_1,\mathfrak{s}_1}(U)$ would have to be nonzero. But this is impossible, since U annihilates $HF^+_{\mathrm{red}}(S^1 \times \Sigma_g, \ell)$.

Finally, we observe that the usual adjunction inequality for surfaces with square zero (see [20, Theorem 1.5]) ensures that if

$$\langle c_1(k), [F]\rangle = 2 - 2g > \langle c_1(\mathfrak{s}), [F]\rangle,$$

then $\Phi_{X,\mathfrak{s}} \equiv 0$. □

Proof of Theorem 1.1

First, observe that the conditions on ω in Theorem 1.1 are all open conditions; so it suffices to prove the theorem in the case where ω has rational periods. According to Donaldson's theorem, any sufficiently large multiple $N\omega$ gives rise to a Lefschetz pencil. Specifically, if we blow up X sufficiently many times, we get a new symplectic manifold $(\widehat{X}, \widehat{\omega})$ with the property that

$$N\omega - \sum_{i=1}^{m} \mathrm{PD}[E_i]$$

is Poincaré dual to the fiber of a Lefschetz fibration over S^2. Here, $\{E_i\}_{i=1}^m$ are the exceptional spheres in $\widehat{X}$. In particular, for any Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$, we have

$$\langle c_1(\widehat{\mathfrak{s}}), [F]\rangle = \langle c_1(\mathfrak{s}), N\omega\rangle - m. \tag{8}$$

Clearly, the canonical Spin^c structure of $(\widehat{X}, \widehat{\omega})$ is the blow-up of the canonical Spin^c structure of (X, ω), so according to the blow-up formula for Φ, it follows that $\Phi_{X,\ell} = \pm 1$ if and only if $\Phi_{\widehat{X},\widehat{\ell}} = \pm 1$. But the latter equation follows according to Theorem 5.1.

For a suitable choice of N, we can arrange for the Lefschetz fibration to be relatively minimal (see [21], [1]). In this case, if $\mathfrak{s} \in \mathrm{Spin}^c(X)$ is any structure with $\Phi_{X,\mathfrak{s}} \neq 0$, then its blow-up $\widehat{\mathfrak{s}}$ satisfies $\Phi_{\widehat{X},\widehat{\mathfrak{s}}} \neq 0$. Thus, the inequality stated in this theorem is equivalent to the corresponding inequality from Theorem 5.1, in view of equation (8). □

6. The genus-minimizing properties of symplectic submanifolds

In the case where $b_2^+(X) > 1$, Theorem 1.3 is now an easy consequence of Theorems 3.1 and 1.1. For this implication, we follow [15].

Proof of Theorem 1.3 when $b_2^+(X) > 1$
If the theorem were false, we could find a symplectic manifold (X, ω) and a pair $\Sigma, \Sigma' \subset X$ of homologous, smoothly-embedded submanifolds, with Σ symplectic, and $g(\Sigma') < g(\Sigma)$. By blowing up X and taking the proper transform of Σ as necessary, we can assume that $\langle c_1(k), [\Sigma]\rangle < 0$. By attaching handles to Σ' as necessary, we can arrange for $g(\Sigma') = g(\Sigma) - 1$. Then the adjunction formula for Σ gives us

$$\langle c_1(k), [\Sigma']\rangle - [\Sigma'] \cdot [\Sigma'] = -2g(\Sigma').$$

Theorem 1.1 says that $\Phi_{X,k}$ is nontrivial, so according to Theorem 3.1, $\Phi_{X,k-\mathrm{PD}[\Sigma']}$ is nontrivial as well. But since

$$\big\langle \omega, c_1(k - \mathrm{PD}[\Sigma'])\big\rangle = \langle \omega, c_1(k)\rangle - 2\langle \omega, [\Sigma]\rangle < \langle \omega, c_1(k)\rangle,$$

we obtain the desired contradiction to Theorem 1.1. □

For the case where $b_2^+(X) = 1$, we appeal directly to the analogue of Theorem 5.1.

Specifically, recall that if $\pi : X \longrightarrow S^2$ is a Lefschetz fibration with genus $g > 1$, then $\langle c_1(k), [F]\rangle = 2-2g \neq 0$; so we have an invariant $\Phi_{X,\mathfrak{s},L}$ in the sense of Section 2.4, where L is the line containing F in $H^2(X; \mathbb{Q})$. The proof of Theorem 5.1 gives the following.

THEOREM 6.1
Let $\pi : X \longrightarrow S^2$ be a relatively minimal Lefschetz fibration over the sphere with $b_2^+(X) = 1$ whose generic fiber F has genus $g > 1$. Then, for the canonical Spin^c

structure, we have

$$\langle c_1(k), [F]\rangle = 2 - 2g,$$
$$\Phi_{X,k,L} = \pm 1,$$

where L *denotes the line in* $H^2(X;\mathbb{Q})$ *containing* $[F]$. *Moreover, for any other* Spin^c *structure* $\mathfrak{s} \neq k$ *with* $\Phi_{X,\mathfrak{s},L} \neq 0$, *we have*

$$\langle c_1(k), [F]\rangle = 2 - 2g < \langle c_1(\mathfrak{s}), [F]\rangle. \tag{9}$$

Proof of Theorem 1.3 when $b_2^+(X) = 1$
Once again, if the theorem were false, we would be able to find homologous surfaces Σ and Σ' in (X, ω) with Σ symplectic and $g(\Sigma') = g(\Sigma) - 1$. We claim that for sufficiently large N, we can find a relatively minimal Lefschetz fibration on some blow-up $\widehat{X}$ whose fiber F satisfies $F \cdot \widehat{\Sigma} = 0$, where $\widehat{\Sigma}$ is some suitable proper transform of Σ. Specifically, if $\omega \cdot \Sigma = c$ (which we can assume is an integer), then provided that $N\omega^2 > c$, we can let $\widehat{\Sigma}$ represent the homology class

$$[\widehat{\Sigma}] = [\Sigma] - [E_1] - \cdots - [E_{Nc}]$$

inside the Lefschetz fibration obtained by blowing up the Lefschetz pencil for $N\omega$. The homology class of the fiber here is given by

$$[F] = N[\omega] - [E_1] - \cdots - [E_M],$$

where $M = N^2\omega^2$. Of course, Theorem 6.1 ensures that $\Phi_{X,k,L} \not\equiv 0$. We can then find a new embedded surface F' representing F, but disjoint from Σ', and cut X along $F' \times S^1$ into two pieces, one of which is a tubular neighborhood of F'. For this cut, Theorem 3.1 shows that $\Phi_{X,k\pm\mathrm{PD}[\Sigma],L}$ is also nontrivial. But since

$$\langle c_1(k \pm \mathrm{PD}[\Sigma]), [F]\rangle = \langle c_1(k), [F]\rangle,$$

this violates inequality (9). □

7. A class of three-manifolds with $HF^+_{\mathrm{red}}(Y) = 0$

We now prove the following.

THEOREM 7.1
Let Y *be a three-manifold that can be obtained as a plumbing of spheres specified by a weighted graph* (G, m) *which satisfies the following conditions:*

- G *is a disjoint union of trees;*

- *at each vertex in G, we have*

$$m(v) \geq d(v). \tag{10}$$

Then $HF^+_{\mathrm{red}}(Y) = 0$.

Note that any lens space can be expressed as a plumbing of two-spheres along a graph (G, m) satisfying the above hypotheses. (Indeed, the graph is linear: it is connected and each vertex has degree at most 2 and multiplicity at least 2.)

Any Seifert fibered space Y with $b_1(Y) \leq 1$ which is not a lens space is obtained as a plumbing along a star-like graph: the graph is connected and has a unique vertex (the "central node") with degree $n > 2$, and all other vertices have degree at most 2 and multiplicity at least 2. The degree of the central node agrees with the number of "singular fibers" of the Seifert fibration, and its multiplicity b is one of the Seifert invariants of the fibration. Thus, a Seifert fibration satisfies the hypotheses of the above theorem when $b \geq n$.

Remark 7.2

An easy inductive argument similar to the proof given below also gives the absolute grading. Suppose that (G, m) is a weighted graph satisfying the hypotheses of Theorem 7.1, with the additional hypothesis that $Y = -Y(G, m)$ is a rational homology three-sphere (this in turn is equivalent to the hypothesis that each component of G contains at least one vertex for which inequality (10) is strict), let $W(G, m)$ be the four-manifold obtained by plumbing two-sphere bundles according to a weighted graph (G, m), and let $W = -W(G, m)$ be the plumbing with negative-definite intersection form. Then for each $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, letting $\mathfrak{K}(\mathfrak{t})$ denote the set of characteristic vectors $K \in H^2(W; \mathbb{Z})$ for which $K|Y = c_1(\mathfrak{t})$, we have

$$d(Y, \mathfrak{t}) = \min_{K \in \mathfrak{K}(\mathfrak{t})} \frac{K^2 + |G|}{4}, \tag{11}$$

where $|G| = \mathrm{rk}(H_2(W))$ denotes the number of vertices in G. Indeed, equation (11) remains true even in the case where the graph has a single vertex where inequality (10) fails, which includes all Seifert fibered rational homology three-spheres. For more on HF^+ of plumbing manifolds, see [17].

Proof of Theorem 7.1

In view of the Künneth decomposition for connected sums (see [18, Theorem 6.2]), it suffices to consider the case where G is a connected graph.

We prove inductively that if there is some vertex v in G where $m(v) > d(v)$, then Y is a rational homology sphere and $\widehat{HF}(Y)$ has rank given by the number of elements in $H_1(Y;\mathbb{Z})$. (Observe that if this is not the case, and if equality holds everywhere, then it is easy to see by repeated blow-downs that the three-manifold in question is $S^2\times S^1$, and it is easy to see that $HF^+_{\mathrm{red}}(S^2\times S^1)=0$; cf. [18].)

Next, we induct on the number of vertices. Clearly, if the number of vertices is one, the three-manifold in question is a lens space; for lens spaces, the conclusion of the theorem follows easily from the genus 1 Heegaard diagram (cf. [18, Proposition 3.1]).

For the inductive step on the number of vertices, we use induction on $m(v)$, where v is some leaf (vertex with $d(v)=1$). Suppose that $m(v)=1$. In this case, it is easy to see that $-Y(G)=-Y(G')$, where G' is the weighted tree obtained from G by deleting the leaf v and decreasing the weight of the neighbor of v (thought of as a vertex in G') by one. Observe that G' also satisfies the hypothesis of the theorem. Thus, the case where $m(v)=1$ follows from the inductive hypothesis on the number of vertices. More generally, suppose that G_1 is a weighted graph and that we have a leaf v with $m(v)=k$. In this case, we can form two other weighted graphs G_2 and G_3, where G_2 is obtained from G_1 by deleting the leaf v, and G_3 is obtained from G_1 by increasing the weight of v by one. We then have the following long exact sequence (see [18, Theorem 9.12]):

$$\cdots\longrightarrow \widehat{HF}\big(-Y(G_2)\big)\longrightarrow \widehat{HF}\big(-Y(G_3)\big)\longrightarrow \widehat{HF}\big(-Y(G_1)\big)\longrightarrow\cdots.$$

By the inductive hypothesis, we know that the theorem is true for the weighted graphs G_1 and G_2. Now cobordisms from $-Y(G_2)$ to $-Y(G_3)$ and from $-Y(G_3)$ and $-Y(G_1)$ (which induce two of the maps in the above long exact sequence) are clearly negative-definite. So it follows that $-Y(G_3)$ is a rational homology sphere with

$$\big|H_1(Y(G_3);\mathbb{Z})\big| = \big|H_1(Y(G_1);\mathbb{Z})\big| + \big|H_1(Y(G_2);\mathbb{Z})\big|.$$

Moreover, by the induction hypothesis, $\widehat{HF}(-Y(G_1)$ and $\widehat{HF}(-Y(G_2))$ have no odd-dimensional generators. Since the map from $\widehat{HF}(-Y(G_1))$ to $\widehat{HF}(-Y(G_2))$ changes the $\mathbb{Z}/2\mathbb{Z}$ grading, it follows that this map is zero, so that the above long exact sequence is actually a short exact sequences. This implies that $\widehat{HF}(Y(G_3))$ is a free Abelian group with rank

$$\operatorname{rk}\widehat{HF}\big(Y(G_3)\big) = \operatorname{rk}\widehat{HF}\big(Y(G_1)\big) + \operatorname{rk}\widehat{HF}\big(Y(G_2)\big).$$

The induction hypothesis is equivalent to the statement that for $i=1,2$, $\widehat{HF}(Y(G_i))$ are free and $\operatorname{rk}\widehat{HF}(Y(G_i)) = |H_1(Y(G_i);\mathbb{Z})|$, which in turn gives the corresponding equation for the graph G_3. □

Proof of Theorem 1.4

According to the definition of Φ, if X is a smooth four-manifold that can be separated along a rational homology three-sphere Y into

$$X = X_1 \cup_Y X_2$$

so that $b_2^+(X_i) > 0$, then Y constitutes an admissible cut for the definition of Φ. If $HF^+_{\mathrm{red}}(Y) = 0$, then the invariant Φ must vanish identically. Thus, in this case, the existence of such a decomposition along a graph manifold satisfying the hypotheses of Theorem 7.1 gives a vanishing result that is inconsistent with Theorem 1.1.

In the case where Y is not a rational homology three-sphere, it is formed as a connected sum of a rational homology three-sphere (as in Theorem 7.1) with some copies of $S^2 \times S^1$. It follows from the behaviour of Floer homology under connected sums (cf. [18]) that $\underline{HF}^+_{\mathrm{red}}(Y, M) = 0$ for any choice of twisted coefficient system M over Y, so we again get a vanishing result for Φ for any smooth four-manifold which admits the hypothesized decomposition along Y. □

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