On the State Space and Frequency Domain Characterization of H^∞-Norm of Sampled-Data Systems

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Abstract

This paper studies the problem of characterization and computation of the H^∞-norm of sampled-data systems using the time-invariant function space model via lifting. With the advantage of time-invariance, the treatment gives an eigenvalue type characterization, first in the operator form in the frequency domain, and then in the Hamiltonian-type finite-dimensional form. The obtained form can be adopted to the bisection algorithm for actual computation.

Keywords: sampled-data systems, H^∞-norm, singular value equation, generalized eigenvalue problem

1 Introduction

The H^∞-type problem for sampled-data systems attracts recent research interest: [3], [5], [17], [8, 9], [6], [15], [1], [14], [10], [7], just to name a few. Among them, Francis [5], Chen and Francis [3], gave a way of computing the L^2-induced norms of certain types of systems; for the H^∞ design problem, Toivonen [17], Hara and Kabamba [6], Kabamba and Hara [8, 9], Bamieh and Pearson [1], Hayakawa et al. [7] derived norm-equivalent finite-dimensional discrete-time systems for a given standard H^∞ control problem. In contrast

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to these, Sun et al. [15], Sivashankar and Khargonekar [14] derived directly a Riccati equation type solution through a limiting procedure of finite-horizon Riccati differential equations.

A somewhat different but closely related problem is that of computing the L^2-induced (or H^∞) norm of sampled-data systems [3], [5]. While a time-domain solution is obtained by Hara and Kabamba [6], Hara and Kabamba [8, 9], this paper attempts a direct characterization of the H^∞ norm of a sampled-data system in the frequency domain, using a time-invariant function space model via lifting [18], [17], [1]. The resulting advantages due to time-invariance are

- transfer matrix becomes available so that H^∞-norm is directly characterized as the maximal singular value in the frequency domain;
- the singular value equation, along with the time-invariance of the adjoint system, naturally leads to a Hamiltonian characteristic equation

As we will see below, these advantages due to time-invariance turn out to be a crucial asset for clarifying various technical but delicate issues involved in the eigenvalue and singular value problems in the characterization of H^∞-norms.

NOTATION AND CONVENTION

All the function spaces such as $L^2[0,h]$, $L^2[0,\infty)$ etc. are standard with usual notation for norms. Only when precise distinction is desirable, we may write $\|\cdot\|_2$ or $\|\cdot\|_{L^2[0,h]}$ for the L^2-norm.

Since we have to deal with eigenvalue problems with complex eigenvalues, we must introduce the inner product over $\mathbb{C}$. Let x and y be n-vectors with entries in $L^2[0,h]$. Their inner product (x,y) is defined by

$$(x,y) := \int_0^h y^*(\theta)x(\theta)d\theta, \tag{1}$$

where $*$ denotes conjugate transposition. Note that the order of arguments is reversed on the right; this follows from the requirement that the inner product (x,y) must be, by definition, conjugate-linear in the second variable.

2 Sampled-Data System via Lifting

We employ the function space model of sampled-data systems via lifting, as introduced by [18], [17], [1].

Let h be a fixed sampling period throughout. The idea of *lifting* consists in associating, to each function φ defined on $[0,\infty)$, a function-space valued *sequence* $\{\varphi_k\}_{k=0}^{\infty}$ via the correspondence:

$$\mathcal{S} : \varphi \mapsto \{\varphi_k\}_{k=0}^{\infty} : \varphi_k(\theta) := \varphi(kh+\theta). \tag{2}$$

The k-th element represents in general an intersample signal at the k-th step. When considered over $L^2[0,\infty)$, this mapping gives rise to a norm-preserving isomorphism between $L^2[0,\infty)$ and $\ell^2_{L^2[0,h]}$, where the latter is equipped with the norm

$$\|\{\varphi_k\}\| := \left\{\sum_{k=0}^{\infty} \|\varphi_k\|^2_{L^2[0,h]}\right\}^{1/2}.$$

In what follows, the function space valued signal $\varphi_k(\theta)$ will be denoted by $\varphi[k](\theta)$, etc. to clarify the distinction between the digital and analog variables. In accord with this, pure discrete-time signals may be denoted as $x[k]$, etc.

Let

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u(t), \\ y(t) &= C_c x_c(t) + D_c u(t) \end{aligned}$$

be a given continuous-time system. By taking $x_c[k] := x_c(kh)$, this system is easily seen to be expressed by

$$\begin{aligned} x_c[k+1] &= e^{A_c h} x_c[k] + \int_0^h e^{A_c(h-\tau)} B_c u[k](\tau) d\tau \\ y[k](\theta) &= C_c e^{A_c\theta} x_c[k] + \int_0^\theta (D_c\delta(\theta-\tau) + C_c e^{A_c(\theta-\tau)} B_c) u[k](\tau) d\tau, \end{aligned}$$

where $\delta(t)$ is the delta function. This is a *time-invariant* discrete-time system where the intersample (continuous-time) variable θ comes into play only as a parameter describing operators. Combining this with a digital controller, we consider in this paper a generalized sampled-data system in the following form:

$$\begin{bmatrix} x_c[k+1] \\ x_d[k+1] \end{bmatrix} = \begin{bmatrix} A_{cs} & A_{cd} \\ A_{ds} & A_d \end{bmatrix} \begin{bmatrix} x_c[k] \\ x_d[k] \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} u[k](\cdot) \\ 0 \end{bmatrix} \tag{3}$$

$$y[k](\theta) = \begin{bmatrix} C_1(\theta) & C_2(\theta) \end{bmatrix} \begin{bmatrix} x_c[k] \\ x_d[k] \end{bmatrix} + \boldsymbol{D} u[k](\theta) \tag{4}$$

where $x_c[k]$ and $x_d[k]$ denote, respectively, the continuous and discrete state variables and belong to $\mathbb{C}^{n_c}$ and $\mathbb{C}^{n_d}$. The matrices $C_1(\theta)$ and $C_1(\theta)$ are continuous functions in θ, and operators $\boldsymbol{B}, \boldsymbol{D}$ are of the following form:

$$\boldsymbol{B}: \quad L^2[0,h] \to \mathbb{C}^{n_c} : u(\cdot) \mapsto \int_0^h K(h-\tau)u(\tau)d\tau \tag{5}$$

$$\boldsymbol{D}: \quad L^2[0,h] \to L^2[0,h] : u(\cdot) \mapsto \int_0^\theta W(\theta-\tau)u(\tau)d\tau. \tag{6}$$

Here we have also made, and will make, the following assumptions:

1. the input term to the discrete state is zero; this is necessary to assure the boundedness of the H^∞-norm of the system; see, e.g., [6], [8], etc.;

2. $\boldsymbol{B} : L^2[0,h] \to L^2[0,h]$ arises from an integral operator with an L^2 kernel function $K(\cdot)$, and there is no directly sampled input to the continuous state $x_c[k]$. This is also usually required to assure the existence of the H^∞-norm (the input term must be preceded by a low-pass filter before sampling) [3], [6], [8], [9]; note that $\boldsymbol{B}$ has finite-dimensional range, so it is a compact operator.

3. the system is exponentially stable, i.e., the sample-time transition matrix

$$A := \begin{bmatrix} A_{cs} & A_{cd} \\ A_{ds} & A_d \end{bmatrix} \tag{7}$$

   should satisfy $A^n \to 0$ as $n \to \infty$, i.e., the eigenvalues of A must be all inside the unit circle. This is also equivalent to the poles of the transfer function (as appropriately defined) of (3), (4) being all inside the unit circle (assuming no pole-zero cancellation); see, e.g., [18].

Although seemingly a little restrictive, the above assumptions are general enough to handle the H^∞-norm problem of all sampled-data systems of interest. Typically, A, $C_i(\theta)$, $K(\theta), W(\theta)$ are of the following form:

$$\begin{aligned}
A_{cs} &= e^{A_c h} + \int_0^h e^{A_c(h-\tau)} H_{cs}(\tau)d\tau \\
A_{cd} &= \int_0^h e^{A_c(h-\tau)} H_{cd}(\tau)d\tau \\
C_1(\theta) &= C_c(e^{A_c\theta} + \int_0^\theta e^{A_c(\theta-\tau)} H_{cs}(\tau)d\tau) + C_{cs}(\theta) \\
C_2(\theta) &= C_c \int_0^\theta e^{A_c(\theta-\tau)} H_{cd}(\tau)d\tau + C_d(\theta) \\
K(\theta) &= e^{A_c\theta} B_c \\
W(\theta) &= D_c\delta(\theta) + C_c e^{A_c\theta} B_c
\end{aligned}$$

where $H_{cs}(\tau)$ and $H_{cd}(\tau)$ are generalized hold functions.

We denote the above system simply as

$$x_{k+1} = Ax_k + Bu_k \tag{8}$$

$$y_k = Cx_k + Du_k \tag{9}$$

(note $D = \boldsymbol{D}$). Introducing the z-transform

$$\mathcal{Z}[\{\varphi_k\}_{k=0}^{\infty}] := \sum_{k=0}^{\infty} \varphi_k z^{-k}, \tag{10}$$

we can also define the z-transform of (8)–(9) as $D + C(zI - A)^{-1}B$. This admits the Neumann series expansion

$$G(z) = D + \sum_{k=1}^{\infty} CA^{k-1}Bz^{-k} =: \boldsymbol{D} + G_0(z), \tag{11}$$

at least for sufficiently small z^{-1}. When the assumptions 1.-3. above are further satisfied, for each fixed z with $|z^{-1}| \leq 1$, $G(z)$ is a bounded linear operator from $L^2[0,h]$ to $L^2[0,h]$. Since A is stable and since operators C and B are continuous (as a consequence of assumption 2 above), the operator $G(z)$ is uniformly bounded for $|z^{-1}| < 1$, and according to [11] (see also [1]), its H^{∞}-norm

$$\|G\|_{\infty} := \sup_{|z^{-1}|<1} \left\{ \sup_{v \in L^2[0,h]} \frac{\|G(z)v\|_2}{\|v\|_2} \right\} \tag{12}$$

is finite. It is also known that this norm is equal to the L^2 induced norm in the time domain. That is, if $u \in L^2[0,\infty)$ is an L^2 input, and if we denote by Gu its corresponding output, [1] then (12) is known ([11], [1]) to be equal to the L^2 induced norm

$$\|G\| = \left\{ \sup_{v \in L^2[0,\infty)} \frac{\|Gu\|_2}{\|u\|_2} \right\}. \tag{13}$$

Furthermore, by the exponential stability, its domain of analyticity extends to $|z^{-1}| \leq 1$, and by the maximum modulus principle, (12) is equal to

$$\|G\|_{\infty} := \sup_{0 \leq \omega \leq 2\pi} \left\{ \sup_{v \in L^2[0,h]} \frac{\|G(e^{j\omega})v\|_2}{\|v\|_2} \right\}. \tag{14}$$

Also, in this case, for each fixed z with $|z^{-1}| \leq 1$, $G_0(z)$ in (11) converges in norm because $A^k \to 0$. Since B is a compact operator as noted above, each CA^kB is also

[1] The notation is a little sloppy, but we here regard G as an operator in the time domain also.

compact, so that as a uniform limit of compact operators, $G_0(z)$ is compact (but $\boldsymbol{D}$ is *never* compact unless D_c is zero), so that $G(z)$ is the sum of $\boldsymbol{D}$ and a compact operator. This decomposition plays a crucial role in what follows.

Remark 2.1 We have here employed the finite-dimensional state space model as in [1] as opposed to [18] where one also takes state trajectory as a state, so that the state space is also infinite-dimensional. On the other hand, in the finite-dimensional model here it is necessary to introduce an infinite-dimensional direct transmission operator $\boldsymbol{D}$ to describe the continuous-time plant. However, from the external viewpoint they describe the same behavior in lifting, so both lead to the same notion of transfer functions.

3 Characterization of the H^∞-Norm of $G(z)$

As seen in the preceding section, the H^∞-norm of G is the supremum of the norm $\|G(e^{j\omega})\|$ which is the induced norm of $G(e^{j\omega})$ from that of $L^2[0,h]$. In the finite-dimensional context, this would lead to the computation of the maximal singular value $\bar{\sigma}(G(e^{j\omega}))$, but in the present (infinite-dimensional) context, the situation is not that simple, since the spectral radius need not be attained as an eigenvalue. [2]

Let $G(e^{j\omega})$ be the transfer matrix operator of (3)–(4) evaluated at $z = e^{j\omega}$. Its operator norm is the square root of the norm of the operator $V := G^*G(e^{j\omega})$, where G^* is the adjoint operator of G. It is clear that V is a self-adjoint operator, but its norm need not be attained as the maximum eigenvalue, because it is not necessarily a compact operator; so the norm of $G(e^{j\omega})$ may not be realized as the maximum singular value. However, under some reasonable condition, we can assure that this is indeed the case.

As noted earlier in (11), G is decomposable as $G(e^{j\omega}) = \boldsymbol{D} + G_0(e^{j\omega})$ with $G_0(e^{j\omega})$ being a compact operator. Since composition of a bounded operator with a compact operator is again compact, we can decompose V as

$$V = \boldsymbol{D}^*\boldsymbol{D} + W. \tag{15}$$

where W is compact.

We now prove the following:

[2] The author is indebted to S. Hara and P. P. Khargonekar for valuable comments and discussions in the subsequent materials.

Lemma 3.1 *Suppose* $\|V\| = \gamma^2 > \|\boldsymbol{D}\|^2$. *Then* γ^2 *is the maximum eigenvalue of* V. *Hence* $\|G(e^{j\omega})\|$ *is also attained as the maximum singular value.* [3]

Proof Put

$$R_\gamma := \gamma^2 I - \boldsymbol{D}^*\boldsymbol{D}. \tag{16}$$

This is self-adjoint and positive definite, so admits the spectral resolution

$$R_\gamma = \int_m^M \mu dS_\mu, \quad 0 < m \leq M.$$

for a spectral measure S_μ (cf. Taylor[16]). This also yields the square root

$$R_\gamma{}^{1/2} = \int_m^M \sqrt{\mu} dS_\mu.$$

Since m is positive, this square root satisfies

$$\|R_\gamma{}^{1/2} v\|^2 \geq m\|v\|^2. \tag{17}$$

Hence $R_\gamma{}^{1/2}$ is continuously invertible and we denote the inverse by $R_\gamma{}^{-1/2}$.

Now observe the identity

$$\gamma^2 I - V = \gamma^2 I - \boldsymbol{D}^*\boldsymbol{D} - W = R_\gamma{}^{1/2}[I - R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}] R_\gamma{}^{1/2}. \tag{18}$$

Here $R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}$ is a compact operator since the composition of continuous and compact operators is again compact. Since γ^2 is the norm of V, the left-hand side of (18) is nonnegative definite, and so is $[I - R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}]$. Therefore $\|R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}\| \leq 1$.

We claim $\|R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}\| = 1$. Assume contrary, i.e., $\|R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}\| = \delta < 1$. We have

$$(x, R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2} x) \leq \delta\|x\|^2 \tag{19}$$

[16, Theorem 6.11-C]. It follows that

$$(x, [I - R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}]x) \geq (1 - \delta)\|x\|^2.$$

Putting $v := R_\gamma{}^{-1/2} x$ ($x = R_\gamma{}^{1/2} v$) yields

$$(v, [\gamma^2 I - \boldsymbol{D}^*\boldsymbol{D} - W]v) \geq (1 - \delta)\|x\|^2 = (1 - \delta)\|R_\gamma{}^{1/2} v\|^2 \geq (1 - \delta)m\|v\|^2$$

[3] If the assumption here is not satisfied, then the conclusion here need not follow. The author is indebted to P. P. Khargonekar for this, and the relevant discussions here.

But this clearly implies $\|V\| < \gamma^2$, which is a contradiction.

Therefore, $\|R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}\| = 1$. Since $R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}$ is compact and self-adjoint, its norm is equal to the spectral radius, and this value must be attained as an eigenvalue [13, Theorem 4.25]. Hence there must exist x_0 such that

$$[I - R_\gamma{}^{-1/2} W R_\gamma{}^{-1/2}] x_0 = 0.$$

Then by putting $v_0 := R_\gamma{}^{-1/2} x_0$, we have

$$[\gamma^2 I - \boldsymbol{D}^* \boldsymbol{D} - W] v_0 = 0. \tag{20}$$

This means that γ^2 is indeed an eigenvalue of V. □

This lemma yields the following:

Lemma 3.2 *There exists a z_0 on the unit circle such that $\|G\|_\infty^2 = \|V(z_0)\|$.*

Proof By (14), there exists a sequence $\{z_k\}$, all of modulus 1, such that $\|V(z_k)\|$ approaches $\|G\|_\infty^2$ from below. Since the unit circle is compact, there exists a convergent subsequence; without loss of generality, assume z_k itself is convergent, to z_0. Clearly $\|V(z_0)\| \leq \|G\|_\infty^2$.

Now let v_k, v_0 be normalized eigenvectors that give rise to the eigenvalues λ_k $(= \|V(z_k)\|)$, λ_0 $(= \|V(z_0)\|)$. Since $\|V(e^{j\omega})\|$ is continuous in ω by the stability of G, for any $\epsilon > 0$, there exists N such that for all $k \geq N$,

$$\lambda_0 = \|V(z_0)(v_0)\| \geq \|V(z_0)(v_k)\| \geq \|V(z_k)(v_k)\| - \epsilon \geq \|G\|_\infty^2 - 2\epsilon.$$

Since ϵ is arbitrary, this yields,

$$\|G\|_\infty^2 \leq \lambda_0 = \|V(z_0)\|,$$

so that the equality holds, completing the proof. □

These two lemmas now imply the following theorem.

Theorem 3.3 *Let $G(e^{j\omega}) = \boldsymbol{D} + G_0(e^{j\omega})$ and $V(e^{j\omega})$ be as above, and suppose $\gamma > \max\{\|\boldsymbol{D}\|, \mu := \inf_{0 \leq \omega < 2\pi} \|G(e^{j\omega})\|\}$. Then $\|G\|_\infty < \gamma$ if and only if there exists no real ω_0 such that*

$$\lambda_{max}(G^* G(e^{j\omega_0})) = \gamma^2. \tag{21}$$

Proof The necessity is obvious from Lemma 3.1.

Conversely, suppose that γ is less than or equal to $\|G\|_\infty$. Either $\gamma = \|G\|_\infty$ or $\gamma < \|G\|_\infty$. In the first case, we have already seen in Lemma 3.2 that (21) holds. So suppose $\gamma < \|G\|_\infty$. By the stability of G, $V(e^{j\omega})$ is continuous in ω, and since $\|\cdot\|$ is continuous in its argument, we see that $\omega \mapsto \|V(e^{j\omega})\|$ is a continuous function. Then its range clearly contains the interval $(\mu^2, \|G\|_\infty^2)$. Hence if $\mu < \gamma < \|G\|_\infty$, this γ^2 belongs to this range, and $\gamma^2 = \|V(e^{j\omega})\|$ for some ω and by Lemma 3.1 the conclusion follows. □

Theorem 3.3 gives a discrete-time counterpart of [2, Theorem 2], so that the bisection algorithm presented there can be adopted to compute the H^∞-norm here. However, the eigenvalue problem here is still infinite-dimensional. We will convert this to a finite-dimensional problem in the next section.

4 Reduction to Finite-Dimension

We have thus seen that the H^∞-norm problem is reduced to that of checking the existence of a nontrivial solution to the eigenvalue problem

$$(\gamma^2 I - G^* G(e^{j\omega}))u = 0 \tag{22}$$

To reduce (22) to a finite-dimensional equation, we first need the state space form for the adjoint G^*. So we start with the following lemma, which is a rather straightforward consequence of the standard duality for discrete-time systems:

Lemma 4.1 *The dual system of system (3)–(4) is given by*

$$\Sigma^* : \begin{cases} \begin{bmatrix} p_c[k] \\ p_d[k] \end{bmatrix} = \begin{bmatrix} A_{cs}^* & A_{ds}^* \\ A_{cd}^* & A_d^* \end{bmatrix} \begin{bmatrix} p_c[k+1] \\ p_d[k+1] \end{bmatrix} + \begin{bmatrix} (C_1, v[k](\theta))^* \\ (C_2, v[k](\theta))^* \end{bmatrix} \\ z[k](\theta) = \begin{bmatrix} K^*(h-\theta) \\ 0 \end{bmatrix} p_c[k+1] + \int_\theta^h W^*(\tau-\theta) v_k(\tau) d\tau \end{cases} \tag{23}$$

where the costate vector $\begin{bmatrix} p_c[k]^T & p_d[k]^T \end{bmatrix}^T \in \mathbb{C}^{n_c+n_d}$.

Proof We readily see that the dual system of (8)-(9) with input $v[k]$ and output $z[k]$ is

$$\begin{bmatrix} p_c[k] \\ p_d[k] \end{bmatrix} = \begin{bmatrix} A_{cs}^* & A_{ds}^* \\ A_{cd}^* & A_d^* \end{bmatrix} \begin{bmatrix} p_c[k+1] \\ p_d[k+1] \end{bmatrix} + C^* v[k](\theta) \tag{24}$$

$$z[k] \quad = \quad B^* \begin{bmatrix} p_c[k+1] \\ p_d[k+1] \end{bmatrix} + D^* v[k]. \tag{25}$$

To exhibit this dual system in a concrete form, we need only compute the dual operators B^*, C^* and D^*. By Lemma A.1 in Appendix, we have

$$C^* v(\theta) = \begin{bmatrix} (C_1, v(\theta))^* \\ (C_2, v(\theta))^* \end{bmatrix} \in \mathbb{C}^{n_c + n_d}, \tag{26}$$

$$B^* \begin{bmatrix} p_c \\ p_d \end{bmatrix} = K^*(h-\theta) p_c, \quad \begin{bmatrix} p_c \\ p_d \end{bmatrix} \in \mathbb{C}^{n_c + n_d}, \tag{27}$$

and

$$(D^* v)(\theta) = \int_\theta^h W^*(\tau - \theta) v(\tau) d\tau. \tag{28}$$

Substituting these into (24)–(25) yields (23). $\square$

We now return to our eigenvalue problem

$$(\gamma^2 I - G^* G(e^{j\omega})) u = 0. \tag{29}$$

To express this in terms of the state space equations, let us write down $y = G(e^{j\omega})u$ and $v = G^*(e^{j\omega})y$, $u, y, v \in L^2[0, h]$, and set $v = \gamma^2 u$. With state space equations (3), (4) and (23), (29) admits a nontrivial solution if and only if there exist u, v, y, not all zero, such that

$$G(e^{j\omega})u: \qquad e^{j\omega} x = Ax + Bu, \ y = Cx + \boldsymbol{D} u \tag{30}$$

$$G^*(e^{j\omega})y: \qquad p = e^{j\omega} A^* p + C^* y, \ v = \gamma^2 u = e^{j\omega} B^* p + \boldsymbol{D}^* y \tag{31}$$

where $x := [x_c^T, x_d^T]^T$, $p := [p_c^T, p_d^T]^T$ (recall also $D = \boldsymbol{D}$).

Combining them together leads to

$$\gamma^2 u = \boldsymbol{D}^* \boldsymbol{D} u + e^{j\omega} B^* p + \boldsymbol{D}^* C x, \tag{32}$$

so that

$$R_\gamma u = (\gamma^2 I - \boldsymbol{D}^* \boldsymbol{D}) u = e^{j\omega} B^* p + \boldsymbol{D}^* C x. \tag{33}$$

Suppose $\gamma > \|\boldsymbol{D}\|$ from now on. Then R_γ is invertible, and we can solve (33) as

$$u(\theta) \quad = \quad R_\gamma^{-1} (e^{j\omega} B^* p + \boldsymbol{D}^* C x) \tag{34}$$

$$= \quad R_\gamma^{-1} \Big(e^{j\omega} K^*(h-\theta) p_c + \begin{bmatrix} \int_\theta^h W^*(\tau-\theta) C_1(\tau) d\tau & \int_\theta^h W^*(\tau-\theta) C_2(\tau) d\tau \end{bmatrix} \begin{bmatrix} x_c \\ x_d \end{bmatrix} \Big). \tag{35}$$

Substituting this for u in (30)–(31), and eliminating y in there will complete the picture. This yields

$$\begin{aligned} e^{j\omega}x &= Ax + e^{j\omega}BR_\gamma^{-1}\boldsymbol{B}^*p_c + BR_\gamma^{-1}\boldsymbol{D}^*Cx & (36)\\ p &= e^{j\omega}A^*p + e^{j\omega}C^*\boldsymbol{D}R_\gamma^{-1}\boldsymbol{B}^*p_c + C^*(I+\boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)Cx. & (37)\end{aligned}$$

Now write $\xi := [x^T\ p^T]^T = [x_c^T\ x_d^T\ p_c^T\ p_d^T]^T$. Then by (3), (4) and (23), we get the generalized eigenequation

$$e^{j\omega}\mathcal{E}\xi = \mathcal{A}\xi \tag{38}$$

where

$$\mathcal{E} := \begin{bmatrix} I & 0 & \mathcal{E}_{13} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \mathcal{E}_{33} & A_{ds}^* \\ 0 & 0 & \mathcal{E}_{43} & A_d^* \end{bmatrix},\ \mathcal{A} := \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & 0 & 0 \\ A_{ds} & A_d & 0 & 0 \\ \mathcal{A}_{31} & \mathcal{A}_{32} & I & 0 \\ \mathcal{A}_{41} & \mathcal{A}_{42} & 0 & I \end{bmatrix} \tag{39}$$

$$\begin{aligned} \mathcal{E}_{13} &= -\boldsymbol{B}R_\gamma^{-1}K^*(h-\cdot)\\ \mathcal{E}_{33} &= A_{cs}^* + \int_0^h C_1^*(\theta)\boldsymbol{D}R_\gamma^{-1}K^*(h-\cdot)d\theta\\ \mathcal{E}_{43} &= A_{cd}^* + \int_0^h C_2^*(\theta)\boldsymbol{D}R_\gamma^{-1}K^*(h-\cdot)d\theta\\ \mathcal{A}_{11} &= A_{cs} + \boldsymbol{B}R_\gamma^{-1}\boldsymbol{D}^*C_1(\cdot)\\ \mathcal{A}_{12} &= A_{cd} + \boldsymbol{B}R_\gamma^{-1}\boldsymbol{D}^*C_2(\cdot)\\ \mathcal{A}_{31} &= -\int_0^h C_1^*(\theta)(I+\boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)C_1(\theta)d\theta\\ \mathcal{A}_{32} &= -\int_0^h C_1^*(\theta)(I+\boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)C_2(\theta)d\theta\\ \mathcal{A}_{41} &= -\int_0^h C_2^*(\theta)(I+\boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)C_1(\theta)d\theta\\ \mathcal{A}_{42} &= -\int_0^h C_2^*(\theta)(I+\boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)C_2(\theta)d\theta \end{aligned}$$

We now see that this eigenvalue problem admits a nontrivial solution if and only if the same is true of (22):

Theorem 4.2 *Assume $\gamma > \|\boldsymbol{D}\|$. There exists a nontrivial solution u to the equation*

$$(\gamma^2 I - G^*G(e^{j\omega}))u = 0 \tag{40}$$

if and only if

$$\det(e^{j\omega}\mathcal{E} - \mathcal{A}) = 0 \tag{41}$$

where $\mathcal{E}$ and $\mathcal{A}$ are given by (39).

Proof If there is a nonzero solution u to the first equation, then by the discussion above, it is clear that there must exist a nonzero ξ (otherwise by (34), $u(\theta)$ derived from ξ is easily seen to be 0) that satisfies (38), so that (41) follows.

Conversely, suppose there exists a nonzero ξ that satisfies (38). We must assure that $u(\theta)$ induced by ξ is not identically zero. Suppose contrary. Then all the integrals involving R_γ^{-1} are zero, and hence we must have

$$e^{j\omega}\begin{bmatrix} I & 0 \\ 0 & A^* \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C^*C & I \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} \tag{42}$$

for $\xi = \begin{bmatrix} x^T & p^T \end{bmatrix}^T$. Then

- either $e^{j\omega}$ must be an eigenvalue of A (in which case x is nonzero); or
- $e^{-j\omega}$ must be an eigenvalue of A^*.

In any case, this contradicts exponential stability of our system (3), (4). Hence $u(\theta)$ cannot be identically zero. □

Combining this with Theorem 3.3 readily yields the following:

Theorem 4.3 *Let $G(z)$ be the transfer function of (3)–(4), and suppose $\gamma > \max\{\|\boldsymbol{D}\|, \mu := \inf_{0\le\omega<2\pi}\|G(e^{j\omega})\|\}$. Then $\|G\|_\infty < \gamma$ if and only if there exists no λ of modulus one such that*

$$\det(\lambda\mathcal{E} - \mathcal{A}) = 0. \tag{43}$$

This theorem gives a sampled-data version of [2, Theorem 2]; since the characteristic equation here is finite-dimensional, we can readily adopt the bisection algorithm combined with an effective check of the existence of an eigenvalue of modulus one, e.g., by converting this to "s-domain" via bilinear transformation, and employ the well-known Sturm method ([2]; see also [12]). However, to guarantee the condition $\gamma > \|\boldsymbol{D}\|$, we need the value of $\|\boldsymbol{D}\|$ in advance. This is related to an H^∞-problem for a delay system, and a computational procedure is also obtained. See [4], [19] and [1] for details. Observe also how the equation above has the structure of Hamiltonian system, i.e., if λ is an eigenvalue to the above, then so is $1/\lambda$ (this is clear from the form of (30) and (31)).

A similar but different Hamiltonian eigenvalue problem has been obtained by [8, 9]. The difference is mainly due to the fact that here operator R_γ^{-1} must be present as a result of one step integration while in their setting, the computation of R_γ^{-1} is embedded into the framework of their hybrid state space model. The mutual relationship is, however, not yet entirely clear.

The entries appearing in the characteristic equation above contain terms involving R_γ^{-1} and the integrals containing it. However, as shown in [1], this can be explicitly computed, and when the hold function is the zero-order hold, all the integrals can be obtained via exponentiations (see, e.g. [1], [7]). Details along with generalizations to a more general context will be reported in a subsequent work.

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A Appendix

Lemma A.1 *Let $K(\theta)$ be a fixed (continuous) matrix function on $[0, h]$. Define the operators T_i, $i = 1, 2, 3$ as follows:*

1. $T_1 : \mathbb{C}^n \to L^2[0, h] : x \mapsto K(\cdot)x$
2. $T_2 : L^2[0, h] \to \mathbb{C}^n : u(\cdot) \mapsto \int_0^h K(h - \tau)u(\tau)d\tau$
3. $T_3 : L^2[0, h] \to L^2[0, h] : u(\cdot) \mapsto \int_0^\theta K(\theta - \tau)u(\tau)d\tau$

Then their adjoints are given as follows:

1. $T_1^* : L^2[0, h] \to \mathbb{C}^n : T_1^*\psi = (K, \psi)^* = \int_0^h K^*(\theta)\psi(\theta)d\theta.$
2. $T_2^* : \mathbb{C}^n \to L^2[0, h] : (T_2x)(\theta) := K^*(h - \theta)x.$
3. $T_3^* : L^2[0, h] \to L^2[0, h] : (T_3^*\psi)(\theta) = \int_\theta^h K^*(\tau - \theta)\psi(\tau)d\tau.$

Proof The proof for T_1 is immediate from

$$(x, T_1^*\psi) = (T_1x, \psi) = (K(\cdot)x, \psi(\cdot)) = \int_0^h \psi^*(\theta)K(\theta)xd\theta$$

$$= \left(x, \left(\int_0^h \psi^*(\theta)K(\theta)d\theta\right)^*\right) = (x, (K,\psi)^*).$$

Formula for T_2 follows from

$$\begin{aligned}(u, T_2^* x) &= (T_2 u, x) = (\int_0^h K(h-\tau)u(\tau)d\tau, x) \\ &= \int_0^h (K(h-\tau)u(\tau), x)d\tau = \int_0^h (u(\tau), K^*(h-\tau)x)d\tau\end{aligned}$$

The case of T_3 follows from

$$\begin{aligned}(u, T_3^*\psi) &= (T_3 u, \psi) = \int_0^h \psi(\theta)^* \left(\int_0^\theta K(\theta-\tau)u(\tau)d\tau\right) d\theta \\ &= \int_0^h d\tau \int_\tau^h \psi(\theta)^* K(\theta-\tau)u(\tau)d\theta = \int_0^h \left\{\int_\tau^h \psi(\theta)^* K(\theta-\tau)d\theta\right\} u(\tau)d\tau \\ &= \left(u, \left(\int_\tau^h \psi(\theta)^* K(\theta-\tau)d\theta\right)^*\right)\end{aligned}$$

□

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