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Pricing the Risks of Default

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Pricing the Risks of Default [1]

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Abstract: This paper models default risk as composed of arrival and magnitude risks. In our model the two default components are explicitly priced as if they were traded in the futures market and the spot price of risky debt is derived as a consequence. We develop estimation strategies to evaluate the magnitude risks which are then employed to construct implicit prices of pure arrival risk contingent securities. The latter prices are used to estimate the structure of arrival risks. The models are estimated on monthly data for rates on certificates of deposit offered by institutions in the Savings and Loan Industry, during the 1987-1991 period. Empirical results support market expectations of lower likelihoods of default after 1989.

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PRICING THE RISKS OF DEFAULT

The event of default has two underlying risk components, one associated with the timing of the event and the other with its magnitude. Ex-ante, default occurs at an uncertain future time when there is some, as yet unknown, reduction in the value of creditor claims. The event may be triggered by many sources that include the occurrence of large unsustainable operational losses and the development of superior alternative technologies that question the economic viability of the firm and its ability to continue to generate earnings. Once the event is triggered the magnitude depends on the relative value of assets in place to the value of creditor claims.

A description of the risks of default, typically requires a description of these two risks. The evolution over time of the likelihood that default occurs, is termed the *arrival rate of default.* The conditional density of the discount on creditor claims in the event of default, is termed the *magnitude risk of default.* Market prices and models of such prices, value both these risks. In addition to these timing and magnitude risks imbedded in the prices of defaultable debt claims, the spot prices for risky debt incorporate interest rate risk as well. The current literature, by focusing attention on modeling the spot price of risky debt implicitly prices all three risks simultaneously. This paper follows a new approach of focusing attention on the timing and magnitude risks separately. We avoid term structure modeling by developing a model for the arbitrage free complete markets forward/ futures price of a defaultable claim.

The timing risk is modeled in the current literature collectively by defining the default event, typically as occurring when the firm value reaches a threshold. Beginning with Merton (1974, 1977) and followed by Black and Cox (1976), Lee (1981), Ho and Singer (1982), Pitts and Selby (1983), Johnson and Stulz (1987), Chance (1990), and Cooper and

Mello (1990, 1991), default is modeled as occurring at debt maturity if firm value falls short of debt value at this time. More recently, Hull and White (1992) and Longstaff and Schwartz (1993) allow for a random time of default at a fixed magnitude by modeling default as occurring at the first time the firm value reaches a prespecified default boundary. This yields a random default time that is referred to in the literature on random times as a predictable stopping time (for further details we refer the reader to Jacod and Shiryaev (1980), henceforth JS).[1] Briefly, the predictability is a consequence of the property that continuous processes approach smooth boundaries with sufficient forewarning.

We deviate from this literature by not defining precisely when default occurs, but focusing instead on the determinants of its likelihood. This approach reflects the common observation that at times default may not occur when formal boundary conditions are satisfied and at other times it may occur well before boundary conditions defining its certainty are met.[2] Furthermore. because default is a complex event, attempts at defining its precise location in time can easily be subject to misspecification. This also has the empirical consequence of theoretically forced low spreads, on low maturity debt, when one is away from the boundary and dramatically high spreads in the vicinity of the default boundary.

We focus on modeling the likelihood of default, and leave the actual time of occurrence unspecified. Such a modeling strategy allows us to specify the instantaneous likelihood of default as depending on a cumulated excess equity return index, where the excess is relative to a money market account. In this regard we view the relativized equity as a market determined sufficient statistic on the financial health and well being of the firm and this provides us with a one factor model for the arrival risk.

When arrival rates of default are responsive to such Markovian economic information, we show that one may solve for the arbitrage free futures price of a survival contingent pure discount bond using the methods of Cox, Ingersoll and Ross (1985).[3] This is because, under a relevant martingale measure the relationship between this futures price and the arrival rates of default is shown to mirror that between Treasury bond prices and the instantaneous spot rate of interest.[4] The arrival rates of default therefore account for additional instantaneous discounting in survival contingent zero coupon bonds, compared to the discounting implicit in Treasury instruments.

Our general approach, is shared with a number of recent investigations into the default process. Artzner and Delbaen (1994) investigate the market structure required for the unique pricing by arbitrage of a complex structure of claims contingent on the default event. Their results are an important precursor for the applicability of our methods. Jarrow and Turnbull (1995) model the pricing of options on bonds subject to credit risk using a similar approach. Duffie, Schroder and Skiadas (1994) model market sensitivity to the timing of the resolution of uncertainty. Duffie and Singleton (1994) exploit the flexibility of this approach to consider directly time series models for the spreads on defaultable instruments, while Nielsen and Ronn (1995) employ a two factor diffusion model for describing the arrival risk process.

On the magnitude risk of default the current literature offers fairly limited models. In the Merton type models, the default magnitude is random by design, its distribution is predetermined. Hull and White (1992) and Longstaff and Schwartz (1995) on the other hand take the default magnitude to be fixed. For the first time, this paper models the conditional risk neutral density for the default magnitude. We show that under certain assumptions, information on the default magnitude risk is imbedded in the relative spreads of two pairs of securities with matching maturities, facing common default arrival risks, but differing in their payout characteristics. Examples of such securities are debt instruments of a single corporation with differing seniority status. Our formulation enables us to recover a parameterized form of the risk neutral distribution of payouts conditional on default, provided that the two securities face common default arrival risks.

We first estimate the parameters of our default magnitude model on data for thrift certificate of deposits (CDs) viewed as defaultable pure discount bonds. Shifts in this distribution are also estimated by employing switching regression methods. In this application the uninsured CDs are treated as the junior claimant and the insured CDs are viewed as the senior claimant. We argue that insured CD holders face default risk stemming from the possibility of the government insuring agency, e.g. the Federal Savings and Loan Insurance Corporation reneging on its committment.

Parameter estimates of the magnitude risk model are then used to infer the prices of pure arrival risk contingent securities that form the input for the estimation of arrival risk parameters. The model for the arrival risk is used to derive the conditional expectation for the spread of CD yields over Treasury yields for matching maturities. As noted earlier, we use equity returns relative to a money market return as the primary determinant of the default premium associated with the arrival risk of default. The resulting model is estimated by generalized method of moments (GMM) to control for the possibility of contemporaneous correlations in equity returns and CD spreads. Both the arrival and magnitude risk models are estimated using monthly data for average CD rates offered by our sample thrifts for three maturities. Our sample consists of roughly 200 thrift institutions which are listed on the CRSP NYSE, AMEX and NASDAQ tapes, covering the period January 1987 to December 1991.

Results for the magnitude risk model indicate that the conditional density of payouts shows two significant changes between the passage of two legislations: The Financial Institutions Reform Recovery and Enforcement Act of 1989 (FIRREA) and The Federal Deposit Insurance Corporation Improvement Act of 1991 (FDICIA). We observe that the mean payout rates are decreasing over this period. This finding is consistent with the argument that the bank legislations reduced the possibility of the government reneging. Results for the arrival risk model also provides strong support for this argument. We observe that following the recapitalization of the insurance fund, insured CD holders perceived the arrival rates of default to diminish.

A framework for modeling default risk, both in a two period and continuous time setting is presented in Section I. Section II provides specific models for obtaining closed form solutions for arrival and magnitude risks of default. Section III discusses the default risk embedded in CDs and describes data. The econometric estimation procedure and results for the arrival and magnitude risk models are presented in section IV. Section V concludes.

I. A GENERAL FRAMEWORK FOR MODELING THE RISKS OF DEFAULT

In this section we explain our strategy for pricing defaultable bonds and assessing, from market price data, component risks of default. The two component risks are the risk of default occurring, the *arrival risk*, and the risk of the severity of default if it were to occur, the *magnitude risk.* Subsection A presents and discusses the assumptions justifying our procedure in the context of a simple finite state discrete time model. The results are generalized to a continuous time and state setting in the subsection B. In subsection C we show how we separate the estimation problem of assessing these two components of default.

A. A finite state two period framework for default pricing

We present in this subsection a two period model of an economy with a money market account, reflecting a stochastic interest rate, and trading defaultable pure discount bonds subject to both an arrival and magnitude risk of default. Under certain assumptions, we develop an expression for the forward and futures prices of these bonds that separate out the two component risks of interest.

Consider a two period, three date economy with dates 0, 1 and 2. Traded in this economy is a money market account paying a one period risk free return and a pure discount bond of unit face maturing at time 2, that is subject to default which may occur at either time 1 or time 2. The evolution of the payoff structure of the defaultable bond is presented in Figure 1.[5] The arrival risk of default is modeled by specifying the probabilities of default occurrence. The magnitude risk is modeled by specifying the conditional density of recovery rates, conditional on the occurrence of default.

Let the probability of default at time 1 be ϕ_1 and let the first period risk free interest rate be r_1. Contingent on default occurring, there are two possibilities for the magnitude of default or the level of recovery. We term these the high and low recovery states with payouts of H_1 and L_1 that occur with probabilities q_{H1} and q_{L1} respectively. If there is no default in the first period we move to the second period and interest rates and firm specific information (x) evolve. Specifically we allow for an upward and downward move in the equity of the firm, relativized by the money market account (x), that we term the up and down states. We also allow for positive and negative interest rate moves that we term the positive and negative states. In general movements in relativized equity and interest are correlated. In all there are four states denoted up, ud, dp and dn with the respective joint probabilities q_{up}, q_{un}, q_{dp}, and q_{dn}. The relativized equity outcomes are denoted x_{2up}, x_{2un}, x_{2dp} and x_{2dn}. The interest rate outcomes in the four states are denoted by r_{2up}, r_{2un}, r_{2dp} and r_{2dn}. The second period default probabilities are denoted by ϕ_{2up}, ϕ_{2un}, ϕ_{2dp} and ϕ_{2dn}.

If there is default in the second period then as in the case of first period default, there are four state contingent recovery rate possibilities of a high state with payouts H_{2up}, H_{2un}, H_{dp} and H_{2dn}, with the corresponding low state payouts being L_{2up}, L_{2un}, L_{2dp} and L_{2dn}. The probabilities of the high payouts are q_{H2up}, q_{H2un}, q_{H2dp} and q_{H2dn} while those of the low payout states are q_{L2up}, q_{L2un}, q_{L2dp} and q_{L2dn}.

The spot price, v, of the defaultable bond may be easily written as

(I.A.1)
$$v = \left\{ \frac{1-\phi_1}{1+r_1} [q_{up} \frac{1-\phi_{2up}}{1+r_{2up}} + \cdots + q_{dn} \frac{1-\phi_{2dn}}{1+r_{2dn}}] \right\} +$$
$$\left\{ \frac{1-\phi_1}{1+r_1} [q_{up} \frac{\phi_{2up}}{1+r_{2up}} [q_{H2up}H_{2up} + \cdots + q_{L2up}L_{2up}] + \right.$$
$$\cdots + q_{dn} \frac{\phi_{2dn}}{1+r_{2dn}} [q_{H2dn}H_{2dn} + \cdots + q_{L2dn}L_{2dn}]]$$
$$\left. \frac{\phi_1}{1+r_1} [q_{H1}H_1 + q_{L1}L_1] \right\}.$$

The first term in brackets accounts for full payment on no default and the second term in brackets prices all possible default payouts. The problem of pricing a defaultable bond represented by equation (I.A.1) has been simplified in the literature by placing specific assumptions on the interaction between the two components of default.

The Merton (1974,1977) approach supposes that default occurs only at maturity and hence in the terms of equation (I.A.1), ϕ_1 is zero. At maturity, the default event is defined by the condition that firm value falls short of the promised payment and occurs if and only if this condition is met. Hence ϕ_2 is a function of the debt equity ratio at maturity and takes on just the values 0 or 1 depending on whether the default condition is met. Since the Merton model has constant interest rates, in terms of equation (I.A.1) the positive and negative interest rate states coincide. Regarding the default magnitude, the density is independent of interest rate movements, depends on the firm value process but has a predetermined distribution given by the shortfall of firm value relative to the promised payment.

Longstaff and Schwartz (1995) on the other hand extend Merton by allowing prematurity default. Hence ϕ_1 may be positive. However, they follow Merton in defining default as occurring at the first time a default boundary is reached. At each instant in the Longstaff and Schwartz (1995) model, one knows for sure whether the default condition is met or not. Hence ϕ_1 and ϕ_2 take on just the values 0 and 1. They also add stochastic interest rates to the Merton framework. In this framework, the default event can be viewed as a state contingent digital option in the context of a two factor model where stochastic interest rates and firm values are correlated. The magnitude of default is however a constant or, in the terms of equation (I.A.1), the high and low states coincide.

Jarrow and Turnbull (1995), were the first to allow the ϕ's to take on values strictly between 0 and 1. They provide a general framework, making an analogy between default and exchange devaluation. The default event is not defined and occurs at a random time that is surprise with the ϕ's allowed to be general processes potentially dependent on all past and current information.[6] However, the pricing equation for defaultable bonds in the paper supposes constant ϕ's. Furthermore, the magnitude of default is also a constant as in Longstaff and Schwartz (1995). To contrast these approaches to that of Merton's option theoretic approach, note that in the latter case claims typically valued have the property that their payoff uncertainty is resolved at date of exercise. For example in the case of a European put option, at date of exercise or maturity, both the uncertainties about whether there will be exercise or not and the value of the put if there is exercise are known with certainty. In Jarrow and Turnbull because the ϕ's are always between 0 and 1 the default event is a possibility at each date and hence the exercise or default is uncertain, however conditional on default the payoff is known. The option theoretic analogy can be made with the embedded prepayment option in mortgages, in that there is a positive probability of prepayment at all times that is strictly between 0 and 1, but conditional on prepayment the payoff is known.

This paper extends the literature on pricing defaultable bonds in three ways. First we extend the Jarrow and Turnbull (1995) pricing equation, that supposes constant ϕ's, by explicitly modeling the ϕ's as parametric functions of the equity level. Second, rather than assuming a constant magnitude of default, as in the previous literature, we allow for a stationary distribution for the payout contingent upon the occurrence of default. In other words, conditional on default we assume that there still exists uncertainty regarding the level of the payment to creditors.

Third, our approach to computing the value of the defaultable claim differs from the previous literature. Merton, Longstaff and Schwartz, and Jarrow and Turnbull use the valuation principle in equation (I.A.1) in that they evaluate the spot price of risky debt as the expectation under the risk neutral measure of all payoffs discounted by the money market account. In contrast, we evaluate directly the forward/futures price of risky debt. In this context we partition the payoff into two categories, full and partial payment. The forward/futures price of the states where full payment occurs is akin to the price of a security termed a pure survival bond that pays unity if there is no default and nothing otherwise. In terms of equation (I.A.1) this accounts for all nodes ending at unit payouts. We show in the paper with respect to the pure survival bond that its spot price may be determined as if it was default free and one adjusts the discount rate to reflect the ϕ's or arrival rates of default. For the value of the default nodes we use the valuation principle of equation (I.A.1).

Duffie, Schroder and Skiadas (1995), Duffie and Singleton (1995) generalize our result for pure survival bonds and show in particular how to adjust the discount rate in general and treat all defaultable claims in valuation as if they were default free. Specifically, they do not separately account for default and non-default states.

Five simplifying assumptions are employed in developing expressions for the prices of defaultable bonds that avoid term structure modeling and separate out the estimation of the magnitude and arrival risk process parameters.

ASSUMPTION A.l: The second period default payouts are independently and identically distributed across all the four states.

This assumption allows us to define H_2, L_2, q_{H2} and q_{L2} as $H_2 = H_{2up} = H_{2un} = H_{dp} = H_{2dn}$; $L_2 = L_{2up} = L_{2un} = L_{dp} = L_{2dn}$; $q_{H2} = q_{H2up} = q_{H2un} = q_{H2dp} = q_{H2dn}$; and $q_{L2} = q_{L2up} = q_{L2un} = q_{L2dp} = 2_{L2dn}$. A number of papers (e.g. Jarrow and Turnbull (1995), and Longstaff and Schwartz (1995)) have treated the default magnitude as a constant, or equivalently that $H_2 = L_2$. We allow for a distribution and seek to extract this information from market prices. In this paper we refrain from making magnitude risk a full fledged and rich stochastic process responding to market information. Under this assumption we may rewrite the spot price of the defaultable bond as

(I.A.2)

$$v = \frac{1-\phi_1}{1+r_1}\left[q_{up}\frac{1-\phi_{2up}}{1+r_{2up}} + \cdots + q_{dn}\frac{1-\phi_{2dn}}{1+r_{2dn}}\right] +$$

$$\frac{1-\phi_1}{1+r_1}\left[q_{up}\frac{\phi_{2up}}{1+r_{2up}} + \cdots + q_{dn}\frac{\phi_{2dn}}{1+r_{2dn}}\right][q_{H2}H_2 + q_{L2}L_2] +$$

$$\frac{\phi_1}{1+r_1}[q_{H1}H_1 + q_{L1}L_1].$$

ASSUMPTION A.2: Arrival rates of default are functions of just the relativized equity value or the ratio of firm value to the money market account.

This assumption provides us with a one factor model for the arrival rate of default in which the level of the relativized equity serves as a sufficient statistic for assessing the risk neutral hazard of default. This extends Jarrow and Turnbull (1995) by making ϕ functionally dependent on specific economic information that evolves over time. Note that the impact of interest rates on arrival rates of default is transmitted through the impact on relativized equity. In this sense we view the relativized equity value as a sufficient statistic for the default arrival rate. There is no further dependence on the interest rates than is already captured by the movement in relativized equity. Additionally we note that this assumption is shared with Jarrow and Turnbull (1995), and with other authors like Merton (1974) where interest rates were constant.

This assumption allows us to define ϕ_{2u} and ϕ_{2d} as $\phi_{2u}=\phi_{2up}=\phi_{2un}$ and $\phi_{2d}=\phi_{2dp}=\phi_{2dn}$.

ASSUMPTION A.3: The relativized equity and interest rate processes are independent.

Relativized equity is a cumulation of excess returns and its independence from the interest rate process is analogous to assuming the independence of market risk premia from interest rates. This is a fairly standard assumption in asset pricing theory that is in particular implicit in the literature assuming market risk premia constant over time.

This assumption allows us to define r_{2p} and r_{2n} as $r_{2p}=r_{2up}=r_{2dp}$ and $r_{2n}=r_{2un}=r_{2dn}$. We may also define the marginal probabilities of the up (q_u) and down (q_d) states for relativized equity and the positive (q_p) and negative (q_n) interest rate states as $q_u=q_{up}$ + q_{un}, $q_d=q_{dp}+q_{dn}$, $q_p=q_{up}+q_{dp}$, and $q_n=q_{un}+q_{dn}$. Under this assumption the joint probabilities are given by the product of the appropriate marginal probabilities, specifically $q_{up}=q_u q_p, \cdots$.

Under assumptions 2 and 3 the spot price of the defaultable bond given in equation (I.A.2) can be rewritten as

(I.A.3)
$$v = \frac{1-\phi_1}{1+r_1}[q_u(1-\phi_{2u}) + q_d(1-\phi_{2d})][q_p \frac{1}{1+r_{2p}} + q_n \frac{1}{1+r_{2n}}] + \frac{1-\phi_1}{1+r_1}[q_u(1-\phi_{2u}) + q_d(1-\phi_{2d})][q_p \frac{1}{1+r_{2p}} + q_n \frac{1}{1+r_{2n}}][q_{H2}H_2 + q_{L2}L_2] + \frac{\phi_1}{1+r_1}[q_{H1}H_1 + q_{L1}L_1].$$

Equation (I.A.3) follows from (I.A.2) on noting that $E[(1-\phi_2)/(1+r_2)]=E[(1-\phi_2)]E[(1/(1+r_2)]$ under the independence assumptions on ϕ_2 and r_2.

Our next set of simplifying assumptions allow us to put together the prematurity default payouts with the default payouts at maturity. This assumption essentially defines the sense of recovery rate in our model.

ASSUMPTION A.4: The recovery rate of the model is defined as the ratio of time 2 money paid out on a claim relative to the time 2 promised payment.

Hence if the payout is H_1 at time 1 then the equivalent time 2 payout is $H_1(1+r_{2p})$ if interest rates have had a positive move and the recovery rate is this value or the ratio of this value to the promised unity payout at time 2. It is the payout distribution at time 2 measured in time 2 monies that is fundamental in our formulation of recovery for a contract of maturity two.

By way of contrast, Duffie, Schroder and Skiadas (1994) and Duffie and Singleton (1994) define recovery rates as the ratio of the payout to the value of the defaultable instrument at time of default, whenever this may be. They then show that on this definition one may put together the arrival rate and the payout rate and adjust the arrival rate of default downwards by the extent of the recovery rate so defined. On the other hand we now show that our formulation has the effect of lumping together all payouts at maturity even if they occurred earlier.

The distribution that is the focus of our model is that of H_2 and L_2 the time 2 payouts. The distribution for H_1 and L_1 is derived in the following way. If at the time 1 default we have had a positive interest rate movement then we suppose that in the high payout state $H_1 = H_2/(1+r_{2p})$ and in the low payout state $L_1 = L_2/(1+r_{2p})$. Similarly, if the interest rate move has been negative then the time one payouts are $H_1 = H_2/(1+r_{2n})$ and $L_1 = L_2/(1+r_{2n})$.

ASSUMPTION A.5: We assume that the time 1 high and low state probabilities agree with those for time 2 or specifically that $q_{H1} = q_{H2} = q_H$ and $q_{L1} = q_{L2} = q_L$.

Under assumptions 4 and 5 the spot price of the defaultable bond may be rewritten as

(I.A.4)
$$v = \frac{1-\phi_1}{1+r_1}[q_u(1-\phi_{2u}) + q_d(1-\phi_{2d})][q_p \frac{1}{1+r_{2p}} + q_n \frac{1}{1+r_{2n}}] +$$

$$\frac{1-\phi_1}{1+r_1}[q_u(1-\phi_{2u}) + q_d(1-\phi_{2d})][q_p \frac{1}{1+r_{2p}} + q_n \frac{1}{1+r_{2n}}][q_H H_2 + q_L L_2] +$$

$$\frac{\phi_1}{1+r_1}[q_p \frac{1}{1+r_{2p}} + q_n \frac{1}{1+r_{2n}}][q_H H_2 + q_L L_2]$$

We next consider the forward price for time 2 delivery, V, and the marked to market price, W, of a futures contract for time 2 delivery. These are the prices that would prevail for such contracts in an arbitrage free complete markets economy. The actual forward and futures contracts need not be trading in the economy. We refer the reader to Artzner and Delbaen (1994) for a more detailed description of market structure permitting the unique pricing of such contracts even in the context of incomplete markets.

By focusing attention on the forward and futures prices, V and W, and developing expressions for them we show that under the maintained assumptions, it is possible to avoid the necessity of specifying a term structure model. Hence attention can be focused on modeling just the arrival and magnitude risks. Further one may also separate out the focus on these two aspects of the default process.

Proposition: Under assumptions A.1 through A.5 the forward price of the defaultable pure discount bond of maturity 2 equals its futures price and

(I.A.5)
$$V = W = (1-\phi_1)\,[q_u(1-\phi_{2u}) + q_d(1-\phi_{2d})] + (1-\phi_1)\,[q_u\phi_{2u} + q_d\phi_{2d}][q_H H_2 + q_L L_2] + \phi_1\,[q_H H_2 + q_L L_2].$$

Proof: See Appendix A.

Defining the probability of no default to time 2 by $F=[q_u(1-\phi_1)\,(1-\phi_{2u}) + q_d(1-\phi_1)\,(1-\phi_{2d})]$, and defining by $R=[q_{H2}H_2 + q_{L2}L_2]$, the expected recovery on default in terms of time 2 monies, we may write the forward and futures prices for the two period defaultable bond more compactly as

(I.A.6)
$$V = W = F + (1 - F)\,R\,.$$

Equation (I.A.6) involves the parameters of the arrival risk process ϕ_1,ϕ_{2u}, and ϕ_{2d} just in the expression for the survival probability, F and the parameters q_{H2},q_{L2} for the magnitude of default appear only in the expression for expected recovery, R. In section I.C below we exploit this separation property to separate the problem of estimating the parameters of the arrival and magnitude risk processes.

B. A continuous time and state framework for default pricing

Continuous time and state modeling is used to obtain more realistic pricing models. We allow for the occurrence of default at any time in the interval [t,T] with the magnitude of default now being measured by a recovery or payout rate y that takes any value in the interval [0, 1]. Hence both the default time and state take values in a continuum. For this context we develop, under assumptions comparable to those of subsection A, the continuous time and state generalization of equation (I.A.6).

In the continuous time and state context default may be represented by a simple jump process $\Delta(t)$, that has a single jump at a random time Z, of a random magnitude x. The default time is Z, the magnitude of default is x and creditors recover a proportion y=(l-x) of the amount owed. We let D(t) represent the default time and D(t) is 0 for $t<Z$ and equals 1 for $t \geq Z$.[7]

In addition to the defaultable claim we suppose the economy has a money market account paying the accumulation on a dollar to time t, of $B(t)=\exp(\int_0^t r(u)du)$, where r(t) is the spot interest rate process. Unit face default free pure discount bonds of maturity T also trade, at time t prices of P(t,T). We suppose complete markets and let Q be the unique risk neutral measure.[8]

Applying the general principles of risk neutral valuation we may write the continuous time equivalent of equation (I.A.1), for the spot price at time t, v(t,T) of the defaultable bond maturing at time T as

(I.B.1) $$v(t,T) = B(t)E_t^Q[\mathbf{1}_{Z>T}/B(T)] + E_t^Q\left[\int_t^T \frac{B(t)}{B(u)}\, y(u)\, dD(u)\right],$$

where $\mathbf{1}_{Z>T}$ equals 1 if there is no default prior to T and zero otherwise. The first term of equation (I.B.1), as in equation (I.A.1), accounts for full payment on no default. The second term accounts for all possible default payouts, including default prior to maturity. In this formulation the state probabilities and the default arrival probabilities (q's and ϕ's of equation (I.A.1)) are imbedded in the measure Q. Relative money market accounts accomplish the time discounting in (I.B.1). The variation in recovery rates across default states at time u is represented by the random variable y(u). Note that dD(u) is unity for at most one time u and so there is at most one default payout.

We now develop the continuous time equivalent of equation of (I.A.6), including the equality of forward and futures prices under assumptions comparable to those section I.A. In the interests of expediency we begin directly with the forward price. The forward price, V(t,T), of the defaultable bond is obtained by spot forward arbitrage on division of (I.B.1) by the bond price and is

(I.B.2) $$V(t,T) = \frac{B(t)E_t^Q[\mathbf{1}_{Z>T}/B(T)]}{P(t,T)} + \frac{1}{P(t,T)} E_t^Q\left[\int_t^T \frac{B(t)}{B(u)} y(u)dD(u) \right].$$

Three assumptions are invoked to develop the equivalent of equation (I.A.6) for the continuous time and state model.

ASSUMPTION B.1: The default event and recovery rates are independent of the interest rate process.

Under this assumption we may simplify (I.B.2) by conditioning on the evolution of interest rates and then noting that under independence the conditional and unconditional expectations of the default event and recovery rates are equal. Further, we also note that for s>t, $P(t,s)=E_t^Q[B(t)/B(s)]$. This assumption on the default event is comparable to assumption A.2 and A.3 of the finite state discrete time model. Regarding the payout rates it is comparable to A.1. Hence, invoking assumption B. 1 we may write the forward price as

(I.B.3) $$V(t,T) = E_t^Q[\mathbf{1}_{Z>T}] + E_t^Q\left[\int_t^T \frac{P(t,u)}{P(t,T)} y(u)dD(u) \right].$$

ASSUMPTION B.2: The recovery rate is defined as the ratio of time T money paid out on a claim relative to the promised time T payment.

Under this assumption if the payout at time u is y(u) on a unit face promise at time T, then the recovery rate y=y(T) is the ratio of the payout of time T monies of y(u)B(T)/B(u) to the promised unit face. Hence y(T)=y(u)B(T)/B(u), or equivalently Y(u)=Y(T)B(u)B(T). This assumption is comparable to assumption A.4 of the finite state discrete time model. Invoking assumption B.2 we may write the forward price of equation (I.B.3) as

(I.B.4) $$V(t,T) = E_t^Q[\mathbf{1}_{Z>T}] + E_t^Q\left[\int_t^T \frac{P(t,u)}{P(t,T)}\frac{B(u)}{B(T)}\, y(T)dD(u)\right].$$

Invoking assumption B.1 once again we may write

(I.B.5) $$V(t,T) = E_t^Q[\mathbf{1}_{Z>T}] + E_t^Q\left[\int_t^T y(T)dD(u)\right].$$

ASSUMPTION B.3: The time T recovery rates are independently and identically distributed across states with density q(y).

This assumption is comparable to assumption A.1 and A.5 of the finite state discrete time model. Under this assumption one may write

(I.B.6) $$V(t,T) = E_t^Q[\mathbf{1}_{Z>T}] + E_t^Q\left[\int_t^T dD(u)\right]\int_0^1 q(y)dy.$$

Defining the probability of no default as $F(t,T) = E_t^Q[\mathbf{1}_{Z>T}]$ and noting that $E_t^Q[\int_t^T dD(u)]$ is just l-F(t,T) we obtain the equivalent of (I.A.6) for the forward price and

(I.B.7) $$V(t,T) = F(t,T) + (1-F(t,T))\ R$$

where R is the expected recovery and $R=\int_0^1 q(y)dy$.

The futures price on the other hand is given by the undiscounted expectation of the payoff, carrying early payments to the delivery date at the money market account, and is

(I.B.8)
$$W(t,T) = E_t^Q[\mathbf{1}_{Z>T}] + E_t^Q\left[\int_t^T \frac{B(T)}{B(u)} y(u)\ dD(u)\right]$$

Under assumptions B.2 and B.3 we obtain that futures and forward prices are equal.

C. Separating the arrival and magnitude risk components

We explain in this subsection our method for separating the problem of estimating the arrival and magnitude risk elements imbedded in market prices. This separation is possible if there are securities facing the same arrival risk but differing in their payout characteristics. In this case we show that one may use the relative prices of such securities to first estimate the conditional density of recovery rates, q(y). Once this is done then Equation (I.B.7) may be used to construct proxies for the prices of pure arrival risk claims or claims that pay zero on default. Parameters of the arrival risk process may then be estimated in a second stage estimation using these proxied prices.[9]

We present first the method for inferring the conditional density of the default magnitude q(y) from data on the prices of risky bonds. In general, a direct estimation of equation (I.B.7) permits one to infer just the mean payout rate. However, if there are two securities facing the same arrival risk and having different payoff responses to the level of default then one may be able to infer higher moments of the conditional payout density. Credit risk derivatives and indirectly, debt with differing seniority status are examples of such securities.[10]

Consider in this regard the case of senior and junior debt. Let S and J denote a senior and junior debt claims respectively with the former having a recovery rate of S(y) while the latter has a recovery rate of J(y), with y being the aggregate recovery rate and

(I.C.1) $$y = p_S S(y) + (1-p_S)J(y)$$

when the proportion of senior debt in the firm is p_S.[11] The futures prices of senior, $W_S(t,T)$, and junior, $W_J(t,T)$, debt are given by equations (I.B.7) applied now to their respective payoffs and we have that

(I.C.2a) $$W_S(t,T) = F(t,T) + (1 - F(t,T)) \int_0^1 S(y)q(y)dy,$$

and

(I.C.2b) $$W_J(t,T) = F(t,T) + (1 - F(t,T)) \int_0^1 J(y)q(y)dy.$$

Subtraction of (I.C.2b) from (I.C.2a) yields on division by $1-W_J(t,T)$ that

(I.C.3) $$\frac{W_S(t,T) - W_J(t,T)}{1 - W_J(t,T)} = \frac{\int_0^1 (S(y) - J(y))q(y)dy}{\int_0^1 (1 - J(y))q(y)dy}.$$

Equation (I.C.3) may be simplfied further by substituting for S(y) from (I.C.1) to obtain

(I.C.4) $$\frac{W_S(t,T) - W_J(t,T)}{1 - W_J(t,T)} = \frac{\int_0^1 (y - J(y))q(y)dy}{p_S \int_0^1 (1 - J(y))q(y)dy}.$$

The left hand side of equation (I.C.3) is the ratio of two spreads. The spread of senior over junior debt prices to the spread of default free over junior debt prices. The left hand side variable can be constructed from data on debt prices. The right hand side is a pure magnitude risk model involving just the parameters of the density q(y) and the payoff functions S(y) and J(y). Importantly, parameters describing the arrival risk enter only the specification of the function F(t,T) and these do not appear in equation (I.C.3).

Potentially one could formulate a model for the density of the magnitude of default and estimate the parameters by estimating equation (I.C.3). Once these parameters have been estimated one may then construct proxies for the prices of the pure arrival risk contingent claims by solving (I.C.2a) or (I.C.2b) for F to obtain

(I.C.5a)
$$\hat{F}_S(t,T) = \frac{W_S(t,T) - \int_0^1 S(y)q(y)dy}{1 - \int_0^1 S(y)q(y)dy}$$

or

(I.C.5b)
$$\hat{F}_J(t,T) = \frac{W_J(t,T) - \int_0^1 J(y)q(y)dy}{1 - \int_0^1 J(y)q(y)dy}$$

These proxy prices for pure arrival risk claims may be used to estimate the arrival risk model once a specific model has been developed for F(t,T). The formulation of such a model is taken up in the next section.

The two stage estimation strategy proposed here for the estimation of arrival and magnitude characteristics involves the estimation of (I.C.4), inference of implied pure arrival risk contingent claim prices by (I.C.5) and the estimation of the arrival risk parameters by modeling the probability F(t,T) and fitting the model to the proxy price data obtained from (I.C.5). Note especially that though we have a two stage procedure here, we do not have the typical errors in variable problems of lack of consistency. This is because in the second stage the output of the first stage only impacts the left hand side variable of the second stage. Hence, though we would expect to lose some efficiency, the second pass estimates can still be consistent. Further, there is an advantage of the two stage approach, in that the first stage is informative and can be estimated free of the impact of any possible modeling misspecifications associated with the second stage model.

II. A MODEL OF THE ARRIVAL AND MAGNITUDE RISK OF DEFAULT

A. The arrival risk of default

We develop here a specific model for the forward/futures price of a pure default arrival risk contingent bond, or a model for the probability of no default $F(t,T)=E_t^Q[\mathbf{1}_{Z>T}]$. We proceed in three steps. First, we establish the general relationship between the probability of no default, $F(t,T)$, and the arrival rate of default process $\phi(t)$, that generalizes the default probabilities ϕ_1, ϕ_2 of section I.A to the continuous time context. Next we develop a partial differential equation for the no default probability under a general Markov model specification for the arrival rate of default, ϕ. Finally, under a specific Markov formulation of the arrival rate process, we solve the partial differential equation for F, obtaining a specific model for the futures/forward price of no default.

In continuous time, the analog of the discrete time default probabilities, ϕ_1, and ϕ_2 in section I.A, is a process $\phi(t)$ that can be interpreted as the instantaneous likelihood of default or the arrival rate of default, with $\phi(t)dt$ being the probability of default occureing in the interval [t,t+dt]. Formally the process ϕ is obtained as the unique predictable process such that $D(t)-\int_0^t \mathbf{1}_{Z\geq u}\phi(u)du$ is a martingale.[12] It follows from this martingale condition that the probability of no default is given by $F(t,T) = 1 - E_t^Q[\int_t^T \mathbf{1}_{Z\geq u}\phi(u)du]$.

We now develop an alternative relationship between $F(t,T)$ and the process $\phi(t)$ that mirrors the relationship between risk free pure discount bond prices $P(t,T)$ and the spot interest rate process $r(t)$. This result makes it possible to apply the methods of default free term structure modeling towards the solution of models for the pricing of default risk.

ASSUMPTION 2.1 The arrival rate of default at time t, $\phi(t)$, is a function of the level of m state-variables at time t, s(t), specifically

$$\phi(t) = \phi(t,s(t)). \tag{II.A.1}$$

This assumption can be contrasted with a simple Poisson process model for the arrival rate of default, where the arrival rates are constant as in Jarrow and Turnbull. We allow the arrival rate to depend on evolving economic information as represented by s(t). The example in section 1 has arrival rates varying with the relativized equity of the firm and the specific continuous time model developed here makes the same assumption.

ASSUMPTION 2.2 The state variables s(t) follow the Markov diffusion process

$$ds(t) = \alpha(t,s(t))dt + \beta(t,s(t))^T dW(t), \tag{II.A.2}$$

for an m-dimensional standard Brownian motion W(t), where $\alpha(t,s(t)$ is the m-dimensional vector of drift coefficient functions, and $\beta(t,s)$ is the almost surely nonsingular matrix of diffusion coefficient functions.

THEOREM 1 Under assumption 2.1 and 2.2 the time t conditional probability of no default till time T given no default prior to t, F(t,T) is given by

$$F(t,T) = E_t^Q[e^{-\int_t^T \phi(u)du}]. \tag{II.A.3}$$

PROOF See Appendix B.

This theorem establishes a useful result relating the probability of no default to a discount factor in which discounting is done at the arrival rate of default, $\phi(t)$. The spot price of default free pure discount bonds discounts the promised payoff for reasons related to the time value of money as captured by the interest rate risk process r(t). In contrast the futures price of the defaultable pure discount bond, F(t,T), "discounts" the promised payoff for reasons related to exposure to the hazard of default as captured by the process for the arrival rate of default risk, $\phi(t)$.[13]

We obtain the spot price of risky debt by first obtaining the spot price of the pure survival bond on further discounting the futures price in equation (II.A.3) by the interest rate and adding the value of the partial payouts in the default states. Duffie, Schroder and Skiadis, and Duffie and Singleton show how to adjust discount rates and obtain the spot price of risky debt using a generalization of equation (II.A.3). They show that one may adjust the discount rate by adding to the interest rate the arrival rate of default less the recovery rate measured as a proportion to market value and then value just the default free states.

Theorem 1 completes the first of our three steps. Next, Theorem 2 develops a partial differential equation for the futures price of a defaultable bond.[14]

THEOREM 2. Under assumptions 2.1 and 2.2, there exists a function ψ solving the partial differential equation

(II.A.6) $$\psi_t(t,s,T) + \psi_s'\alpha(t,s) + \frac{1}{2}\,\mathrm{Trace}[\ \psi_{ss}(t,s,T)\beta(t,s)\beta'(t,s)\ ] = \phi(t,x)\psi(t,s,T),$$

subject to the boundary condition $\psi(t,x,t) = 1$, such that.

(II.A.7) $$F(t,T) = \psi(t,s(t),T).$$

PROOF See Appendix B.

To solve equation (II.A.6) and obtain the futures price $F(t,T)$, we need to specify the process s and the functional form for ϕ.

ASSUMPTION 2.3 The arrival rate of default is a function of the firm's equity value relativized by the money market account, $s(t)=S(t)/B(t)$, $S(t)$ is the equity value of the firm and $B(t)$ is the money market account.

This implies a one factor model for the arrival rate of default. The firm's equity value is a forward looking market based assessment on its financial well being and as such we would expect it to reflect variations in default probabilities. However, equity value must be relativized before it is an indicator of financial well being. Note that this structure allows for a dependence between arrival rates of default and interest rates via $s(t)$.

ASSUMPTION 2.4 The relativized equity value of the firm $s(t)$ satisfies the stochastic differential equation

(II.A.8) $$ds = \sigma s dW(t),$$

where W is a standard Brownian motion.

Consistent with the initial Black-Scholes assumption for the equity value process we suppose that s is a geometric Brownian motion with volatility given by a constant σ.

ASSUMPTION 2.5 The arrival rate of default for the firm is related to the level of relativized equity by the function $\phi(s)$, where

(II.A.9) $$\phi(s) = \frac{c}{(\ln(s/\delta))^2}.$$

This specification provides an analytically tractable solution for F(t,T) and has the following properties. It is time homogeneous, non-negative, and does not impose a restriction on the direction of the relationship between arrival rates and relativized equity values. The level of the arrival rates of default and the significance of the relationship between arrival rates and s is captured by the parameter c. In addition (II.A.6) allows for a critical value of s denoted δ, at which instantaneous spreads rise to infinity. Note that in the specification of ϕ, the threat of default is measured in terms of the distance of relativized equity from the critical value of δ. In that a movement of s by the percentage of $(\delta/s-1)$ triggers default.

A typical graph of the function ϕ is presented in Figure 1 below for a δ value of unity and a value of c=.0003. The relationship between ϕ and s is positive for s below δ, and negative for s exceeding δ. The position of the data relative to the estimated δ value determines the type of relationship relevant in the data. Furthermore if the value of s on average is 1.5 then losses in equity relative to the money market in the magnitude of 33 per cent trigger default.

INSERT FIGURE 1 HERE

Under the further Assumptions 2.4 and 2.5, the partial differential equation (II.A.6) for the futures price of the defaultable bond, simplifies to

(II.A.10) $$\psi_t(t,s,T) + \frac{1}{2}\sigma^2 s^2 \psi_{ss}(t,s,T) = \phi(s)\psi(t,s,T),$$

subject to the boundary condition $\psi(t,s,t)=1$.

It is instructive to note that the partial differential equation (II.A.10) reduces to the Black-Scholes partial differential equation for the futures price of a claim written on an underlying futures price if $\phi(s)$ is identically zero or there is no discounting. However, the presence of an exposure to default risk results in the futures price being discounted by $\phi(s)$.

The third step involves the solution of equation (II.A.10) with the specification of ϕ given by (II.A.9). It is shown in Appendix C that the solution is given by

(II.A.11) $$F(t,T) = \psi(t,s,T) = G_a(2/d^2(s,\tau)),$$

where $a=c/(2\sigma^2)$, $\tau=(T-t)$,

$$d(s,\tau) = \frac{\ln(s/\delta)}{\sigma\sqrt{\tau}} - \frac{\sigma\sqrt{\tau}}{2}$$

and the function $G_a(u)$ is a solution to the linear second differential equation

(II.A.12) $$u^2G_a'' + (3u/2 - 1)G_a' - aG_a = 0,$$

with the boundary conditions $G_a(0)=1$, $G_a'(0)=-a$ and $G_a(\infty)=0$.

The critical value of d resembles the critical value in the Black-Scholes option pricing model for the probability of a call option being in the money at maturity. The role of the strike in our model is played by the critical value, δ, of s that triggers default. In this context δ serves as the critical point insuring certain default on the passage of s to δ. However, as noted earlier, default has a positive probability of occurring at all times. The magnitude d measures the distance of s from δ in standardized units. The default probability is defined in terms of the inverse squared distance that we show below to be essentially exponentially distributed. The greater this distance the higher the probability of no default. Note in this regard that the event of default is not tied in our model to the security under consideration but is an event associated with the operation of the firm. In fact, firm default can occur either before or after the maturity of a specific instrument.

The function $G_a(u)$ is a complementary distribution function. It is shown in Appendix D that the function $G_a(u)$ is well approximated for the lower u values by

(II.A.13) $$G_a(u) \cong e^{-au}.$$

The distribution function for $(1/d^2)$ is then given by $G_a(2/d^2)=e^{-2a/d^2}$ for the larger standardized distances of s from δ or low default probabilities and short maturities.[15] For the more general case of longer maturities one may follow Appendix D and construct a table of values of the function $G_a(u)$ for a discretized range of (a,u) values. Such a table may then be used to evaluate $G_a(u)$ by interpolation.

For estimation it is useful to convert the futures/forrward prices of no default to yields. The resulting model for these yields, y(t,T)=-ln(F(t,T)/(T-t), using the approximation (II.A.13), is

(II.A.13) $$y(t,T) = \frac{c}{\sigma^2 d^2 \tau}.$$

Substituting for d we obtain

(II.A.14) $$y(t,T) = \frac{c}{(\ln(s/\delta) - \sigma^2\tau/2)^2}.$$

Equation (II.A.14) forms the basis of a regression equation from which the parameters of the arrival rate process are estimated. Section V develops the estimation procedure and presents the results.

B. The magnitude risk of default

The general structure for estimating the conditional density of the magnitude of default q(y) is given by equation (I.C.3). An estimable model is obtained on specifying the payout functions to senior debt, S(y), and junior debt, J(y) and on defining parametrically the density of the magnitude of default q(y).

Our choice for the functions S(y) and J(y) reflects strict priority rules. We suppose that senior debt is fully paid off before junior debt receives any distribution. Let S and J denote respectively the face value of senior and junior debt. The proportion of senior debt is then p_s=S/(S+J) and the functions S(y) and J(y) must satisfy equation (I.C.1).

To determine the functions S(y) and J(y), consider first the case when the firm's market value M falls short of S. In this case, senior debt receives the payout M/S, junior debt receives 0, with y=M/(S+J). Once M exceeds S, senior debt has a payout of unity while junior debt now receives a payout of (M-S)/J. Hence we may express the payouts to senior and junior debt in terms of the average payout y and the time varying proportion of senior debt p_{St}, as follows,

(II.B.1)
$$S(y,p_{St}) = \mathrm{Min}\left(\frac{y}{p_{St}},\ 1\right)$$

and

(II.B.2)
$$J(y,p_{St}) = \mathrm{Max}\left(\frac{y-p_{St}}{1-p_{St}},\ 0\right).$$

It follows from these expressions for the payouts to senior and junior debt that these instruments are equivalent to payouts of a put option, and a call option, respectively, written on the firm's average payout ratio. This structure of payoffs to creditors is consistent with the Black and Cox (1976) specification of differential creditor claims.

From equations (II.B.1) and (II.B.2) one may establish some useful inequalities between the mean payout rate and the ratio of relative spreads, the dependent variable of equation (I.C.4). Note first that the priority of senior to junior payouts implies that expected payouts to the senior claimant exceed the mean payout rate, m or that $W_S \geq m$. Second, the payout to the junior claimant is a call option struck at the proportion of senior debt and has maximum value at maximum volatility. For the case here of a bounded random variable y, the volatility is capped and corresponds to a uniform distribution on y. Hence the payout to the junior claimant is less than the proportion of junior debt or that $W_J \leq p_J$. It follows that the mean payout rate is dominated by the ratio of relative spreads plus the proportion of junior debt or that $m \leq (W_S - W_J)/(1-W_J) + p_J$. Therefore, for low levels of p_J together with low relative spreads, the implied mean payout rates conditional on default are low. An interpretation of this result is that default is expected to occur only when the asset backing is sufficiently seriously eroded so as to be consistent with the implied low default conditional payout rates. Hence, a lower relative spread is indicative of the default event occurring in such weak conditions of asset backing that the payoffs to the two claimants are not much different from each other. Conversely, when default conditional mean payout rates are high, the payoffs to the junior and senior claimants are substantially differentiated to require a significant relative spread.

The aggregate magnitude of default, y is taken to be the ratio of assets to the value of outstanding debt claims. Hence at any default time y lies between 0 and 1. The mean and variance of y must be related for as the mean approaches unity, (100% recovery) or zero (no recovery) the variance of y is zero. This dependence between the mean and the variance is captured in the two parameter class of Beta distributions. This family of Beta distributions is described by two positive parameters α and β. Such a distribution is obtained on normalizing the function $y^{\alpha-1}(1-y)^{\beta-1}$. Specifically we have, (II.B.3) $$q(y;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1} \quad \text{for } 0 < y < 1,$$

where Γ is the Gamma function. The mean of this density is $m=\alpha/(\alpha+\beta)$. It is useful to take m as a parameter as it is the expected aggregate payout rate conditional on default. The variance however, is $v=m(1-m)/(1+\alpha+\beta)$. Note that the variance goes to zero as the mean approaches zero or unity. It is difficult to take the variance v as a parameter given the nonlinear relationship between v and m. Instead we take $\lambda=(\alpha+\beta)$ as the other parameter which is inversely related to the variance. The density q can then be rewritten in terms of m and λ.

We obtain on substitution into (I.C.4) that,

(II.B.6) $$\frac{W_S(t,T) - W_J(t,T)}{1 - W_J(t,T)} = \frac{m - \int_{p_{St}}^{1} J(y,p_{St})q(y;m\lambda,(1-m)\lambda)dy}{p_S\left(1 - \int_{p_{St}}^{1} J(y,p_{St})q(y;m\lambda,(1-m)\lambda)dy\right)}.$$

Note that the integration of the junior payout rate begins at p_{St} as this payout rate is a call option on the aggregate payout rate with a strike of the proportion of senior debt, p_{St}. An explicit closed form model is obtained on integrating $J(y,p_{St})$ with respect to the density q. Defining the expected payout to the junior claimant as, $\Phi_J(p_{St},m,\lambda)=\int_{p_{St}}^{1} J(y,p_{St})q(y;m\lambda,(1-m)\lambda)dy$, one obtains that

(II.B.8) $$\Phi_J(p_{St},m,\lambda) = \frac{(m - p_{St})}{1-p_{St}} + \frac{p_{St}}{1-p_{St}} \mathbb{B}(p_{St};m\lambda,(1-m)\lambda) - \frac{m}{1-p_{St}} \mathbb{B}(p_{St};m\lambda+1,(1-m)\lambda),$$

where $\mathbb{B}(u;m,\lambda)$ is the cummulative Beta distribution function.

Figure 2 displays the expected payouts to the senior and junior claimants as functions of the proportion of senior debt for a mean payout, m, conditional on default, of .4 and a λ value of 20, consistent with a volatility of y of .1. The junior payout function is constructed in accordance with equation (II.B.8) and the senior payout, $\Phi_S(p_{St},m,\lambda)$, is then obtained in accordance with (I.C.1), or $\Phi_S(p_{St},m,\lambda) = m/p_S - (1-p_S)/p_S\Phi_J(p_{St},m,\lambda)$. The senior and junior claimants receive the mean payout when their share in the capital structure is close to 100 percent. For low levels of senior debt the figure shows that senior claimants are protected. This is plausible given the assumed volatility of the payout rate of .1, with a mean of .4. As the proportion of senior debt rises, they are at higher risk with declining expected payouts that converge to the mean payout for high levels of senior debt. The junior payout reflects the effects of increased subordination. The difference between the senior and junior payouts rises initially and then falls past a critical level.

The estimated model is

$$\frac{W_S(t,T) - W_J(t,T)}{1 - W_J(t,T)} = \frac{m - \Phi_J(p_{St},m,\lambda)}{p_{St}(1 - \Phi_J(p_{St},m,\lambda))}. \tag{II.B.9}$$

Note that equation (II.B.9) uses the same parameter values for all maturities. This is a consequence of supposing that payouts depend just on the structure of seniority and are independent of maturity.

It is instructive to consider the behavior of the model (II.B.9) for the relative spreads as a function of the proportion of senior debt p_s, for different values of the parameters m and λ. Figures 3 and 4 demonstrate that there is an inverse relationship between the proportion of senior debt and the relative spread. Though, the spread of the senior to junior payout may rise at low levels of senior debt, it does not rise as fast as the increase in the spread of Treasury over the junior claimant.

Figure 3 presents the effect of different levels of the mean payout. We observe that for all levels of the proportion of senior debt, a lowering of the mean payout rate reduces the relative spread. Equivalently, falling relative spreads, in the presence of a stable capital structure, are indicative of declining mean payout rates conditional on default. This reflects the fact that when mean payout rates are low, the call option of the junior claimant is out of the money, and hence the senior payout is reduced, bringing it closer to the junior payout. This relationship between relative spreads and mean payout rates is more pronounced for high levels of senior debt as then the junior claimant is forced out of the money sooner.

Figure 4 presents the effect of varying λ or indirectly the variance of payout rates on the relative spreads. As the variance of payout rates is increased or λ is reduced, the value of the junior claimant's call option rises and this lowers the relative spread. However, this effect is negligible when the call option is far out of the money and the proportion of senior debt is high.

Equation (II.B.9) may be used to estimate by maximum likelihood the parameters m and λ of the conditional risk neutral density of the default magnitude. Note that identification requires time series data on the futures prices of two classes of defaultable discount bonds (W_S and W_J) and the proportion of one class of bond in the debt structure. Section IV reports on the results of the estimation.

III. THRIFT CERTIFICATES OF DEPOSIT AND DEFAULT RISK

Estimation of the magnitude default risk model developed in this paper requires two securities of a single firm. By virtue of being securities of a single firm, they face the same arrival risk of default, which is the risk that the firm goes into default. The two securities must also differ in the payouts conditional on default. We take insured (senior) and uninsured (junior) CDs as an example of such securities.

There exists extensive literature examining the rates offered on insured/uninsured CDs and thrift default risk. This literature can be partitioned into two groups. One group relates rates offered on uninsured CDs to the risk of thrift default, while the other focuses on insured CD rates and their relationship to this risk. The default risk embedded in uninsured CDs is quite straightforward as these securities are to receive residual value once the insuring agency, the Federal Savings and Loan Corporation in the case of thrifts, resolves payment to insured depositors at time of default. Indeed, James (1988), Hannan and Hanweck (1988) and Ellis and Flannery (1992) report a significant relationship between rates offered on large bank and thrift CDs (over $100,000 and uninsured) and firm risk. Hence evidence is strong indicating that uninsured CD rates reflect default premia.

The existence of default risk premia in insured CDs is not that clear. These securities have payoffs that are fully-insured in all states of the world in which the deposit guarantee remains in place. However, if insured CD holders develop expectations that FSLIC would renege on its contractual obligations, then these investors could require risk premiums that reflect the risk neutral default probability on the part of FSLIC. This hypothesis has been rigorously tested recently by Strahan (1995). He first provides evidence that the yield on FSLIC-insured thrift CDs were significantly higher than FDIC (Federal Deposit Insurance Corporation - the insuring agency for banks) insured bank CDs during the 1987-1989 period. Second, he shows that well-capitalized thrifts offered lower rates on their insured CDs than their poorly capitalized competitors during the same period. His empirical analysis provides strong evidence that depositors lost confidence in the ability of FSLIC to fulfill its contractual obligations prior to the passage of FIRREA in the summer of 1989.[16] Indeed, the rates offered on fully-insured thrift CDs peaked just prior to the passage of FIRREA and significantly decreased following the capitalization of the insuring agency.

The hypothesis that insured CDs reflect FSLIC's default risk is also supported by Cool and Spellman (1996). They also attribute increased CD premiums on insured CDs to increases in FSLIC default risk. Additionally, Cooperman, Lee and Wolfe (1992) provide evidence that at the height of the state of Ohio deposit insurance crisis, fully insured CD rates at Ohio thrifts and banks decreased as the financial health of the issuing firm increased.

Hence, we argue that, whatever the source of risk, rates on fully insured CDs reflected magnitude risk conditional on thrift default during our sample period, 1987-1991. The source of variations in this risk can either be due to changes in the conditions of the guarantor or changes in the thrift itself. Our objective in this paper is not to identify the source of this risk. Rather, we wish to capture the risk neutral average payout rate to firm securities facing the timing risk of default, but differing in the payout rates conditional on default. This is achieved by focusing on insured and uninsured thrift CD rates which are presumed to reflect investor expectations of loss in both cases.

One caveat applies. The moral hazard model of the thrift crisis of the 1980's (Kane (1988)) would argue that weak thrifts pursued a risk shifting strategy to exploit deposit insurance. Such thrifts aggressively marketed fully insured deposits to attract large flows of funds to pursue a high growth high risk investment strategy. Hence, increased CD rates by these thrifts may also reflect an increased demand for funds. Strahan (1995) provides evidence that CD rates reflected more than thrift risk and high growth thrifts paid higher rates of deposits. Our model does not allow for such additional factors in explaining CD spreads. Hence, our empirical results should be interpreted in the light of such omitted variables.

The four sets of variables used in the estimation of our model are i) rates offered on insured and uninsured CDs for various maturities by thrifts, ii) stock prices for the thrifts, iii) Treasury yields for maturities that match the CD rates, and iv) the proportion of insured deposits to total deposits. The sources of the data are as follows.

Every Federally insured Savings and Loan institution is required to report to the Office of Thrift Supervision, the rates on CDs issued during the last five working days of each month. From these reports we obtained data on two size classes of CDs for three maturities. The size classes are for amounts between $80,000 and $100,000 and above $100,000. The maturities are up to one month, one to two months and two to three months.[17] Data cover the period January 1987 to December 1991. During this period the number of institutions reporting the CD information ranged from 945 to 1324. Out of this sample we identified thrifts that had active stock market trading and for which monthly return data are provided in the CRSP NASDAQ, and CRSP NYSE/AMEX tapes, compiled by the University of Chicago. This resulted in a sample of roughly 300 thrifts. Data for Treasury yields with maturities of one, two and three months come from the Fama-Bliss tapes compiled by the University of Chicago. Aggregate data on the proportion of insured deposits was also obtained from the Office of Thrift Supervision.

We estimate our default risk model at a representative firm level. For this purpose we constructed monthly average spread data for each of the three maturities. For each month we averaged spreads across all thrifts that had available stock price data for that month. This allows us to include in our average spreads data for firms that defaulted during the sample period. A model of the type proposed here could be estimated at the individual firm level to assess the impact of relativized equity values on individual firm CD spreads. However, we expect to be able to better identify this relationship at an aggregate level. Hence, parameter estimates of the model should reflect values that are appropriate for a randomly selected firm from the sample.

For the average level of relativized equity s_t, we took the ratio of a simple average of the end of month stock values of our sample thrifts to the accumulation in the money market account. The stock values were obtained by cumulating dividend inclusive monthly returns. The money market accumulation is at the one month Treasury Bill rate, from the start of the data period.

Panel A of Table 1 presents annual average CD spreads reported by thrifts for the month of December over the five years, 1987 -1991, in the two amount categories. Also included are the respective means and standard deviations of the spreads for each maturity category for the full sample period. These spreads are obtained by averaging across the sample of thrifts, with the number of thrifts in the sample reported in the last row of the Table. Panel B of Table 1 presents corresponding values for the relativized equity and the proportion of insured to total deposits.

INSERT TABLE 1 HERE

The subsample of averages reported in Tables 1 provides spread data for the representative sample thrift. Averages similar to those reported in Tables 1 were computed for each month giving in all 60 observations on each of three maturities. There are in all 180 spread for the representative thrift in each of the insured ($80,000 to $100,000) and uninsured (over $100,000) categories.

IV. ECONOMETRIC METHODOLOGY AND RESULTS

A. Econometric specification and estimation of the risk neutral conditional density of the magnitude of default.

To estimate equation (II.B.9) we frost need to construct futures prices of CDs viewed as pure discount bonds, from the data on the rates offered on these CDs. The spot price of the unit face CD of maturity τ, given its yield y, is $e^{-y\tau}$. The forward/futures price is then obtained on dividing the spot price by the price of a unit face Treasury bond of equivalent maturity. We construct the futures prices for each of three maturities, in both the insured and uninsured categories for all 60 months of the sample period. For each of the three maturities (τ=1, 2, and 3 months) at each month we construct the ratio of the spread of insured over uninsured futures prices to the spread of the Treasury futures price over the uninsured.

Figure 5 presents a graph of the relative spreads for the three maturities over the data period March 1987 to December 1991. As can be observed the relative spread peaks prior to 1989 and consistently falls thereafter. We denote this spread ratio in month t for maturity τ, by $SR_{t\tau}$. In this notation the regression equation of equation (II.B.9) is

(IV.A.1)
$$SR_{t\tau} = \frac{m - \Phi_J(p_{St}, m, \lambda)}{p_{St}(1 - \Phi_J(p_{St}, m, \lambda))} + \varepsilon_{t\tau}$$

where $\varepsilon_{t\tau}$ is a zero mean, normally distributed error term, independent across t and τ with variance σ^2_τ.

Time series estimation of equation (IV.A.1) assumes constancy of the parameters m and λ across the sample period. However, regulatory developments during the 1987-1991 period have the potential to cause shifts in these parameters. For example, as alluded to before, the passage of FIRREA and FIDICIA may have impacted market expectations of payout rates. Rather than splitting the sample into ad-hoc subperiods, our estimation procedure endogenizes the possible unknown parameter switch dates.

The number of switch points is determined by a likelihood ratio criterion. We first illustrate the construction of the likelihood function for two switch points and three periods. For three periods, there are three sets of parameter values m_i, λ_i that pertain to the three periods i=l, 2, and 3, respectively. Conditional on time t being in period i the model for the spread ratio $SR_{t\tau}$ is

(IV.A.2)
$$SR_{t\tau} = \frac{m_i - \Phi_J(p_{St}, m_i, \lambda_i)}{p_{St}(1 - \Phi_J(p_{St}, m_i, \lambda_i))} + \varepsilon^i_{t\tau}$$

where $\varepsilon^i_{t\tau}$ has variance $(\sigma^i_\tau)^2$. The conditional likelihood of the spread ratio $SR_{t\tau}$, conditional on t in period i is then

(IV.A.3)
$$L^i_{t\tau} = \frac{e^{-1/2(\varepsilon^i_{t\tau})^2/(\sigma^i_\tau)^2}}{\sqrt{2\pi}\ \sigma^i_\tau}.$$

In addition to these parameters there are switch point parameters that give the mean switch points and their standard deviations. We let Z_1 and Z_2 be the mean switch times, with standard deviations of ζ_1 and ζ_2 respectively. Any particular time point t can in principle be in any of the three regimes, as the exact switch times are unknown. The probability that t is in the first period is given by the distance between t and Z_1 in standardized units. If this is large and negative then we are still in period 1, while if it is large and positive then we have passed out of period 1 and into either period two or three.

The probability of being in period one is modeled by

(IV.A.4)
$$p_{1t} = 1-N((t-Z_1)/\zeta_1),$$

where N(x) is the standard normal distribution function. The total probability of being in periods two or three is $1-p_1 = N((t-Z_1)/\zeta_1)$. In a similar manner we obtain the probability of being in period 2 as

(IV.A.5) $$p_{2t} = N((t-Z_1)/\zeta_1)\ (1-N((t-Z_2)/\zeta_2)),$$

and that of being in period three as

(IV.A.6) $$p_{3t} = N((t-Z_1)/\zeta_1)\ N((t-Z_2)/\zeta_2)).$$

Note by construction that the three probabilities sum to unity as we are in the three period case.

The likelihood at time t for the spread ratio $SR_{t\tau}$ of maturity τ conditional on t being from period i is given by $L^i_{t\tau}$ and the unconditional likelihood is

(IV.A.7) $$L_{t\tau} = \Sigma_i\ p_{it} L^i_{t\tau}.$$

The full likelihood is

(IV.A.8) $$L = \prod_{t,\tau} L_{t\tau}.$$

The model is estimated by maximizing L over the parameter space $(m_i, \lambda_i, (\sigma^i_\tau)^2$, i=1,2,3 and τ=1,2,3), Z_1, Z_2, ζ_1 and ζ_2.

In addition to estimating (IV.A.8) we estimate the equivalent of (IV.A.8) for one, and three switch points. To conclude that we have two switch points two likelihood ratio tests are conducted, for two versus one switch point, and for three versus two switch points. The likelihood ratios are distributed χ^2 with degrees freedom equal to the number of restrictions.

INSERT TABLE 2 HERE

The results are reported in Table 2. We observe that there are two sharp regime switches ($\zeta \approx 0$), one in the March of 1990 (Z_1=37.35) and a second switch in November of 1990.[18] The estimated mean payout rates fall from 36 cents in the dollar, to 29 cents in the second period and 16 cents in the third period. The standard deviations for the conditional density of the magnitude of default may be inferred from the m and λ parameters, for the two periods and these are, .0992, .0865 and .1256 respectively. The two switch points fall between two landmark legislations: the Financial Institutions Reform and Recovery and Enforcement Act of 1989 (FIRREA) and Federal Deposit Insurance Corporation Improvement Act of 1991 (FDICIA).

Note from Figure 2, that for high levels of senior debt for the estimated parameter values the payout to the junior claimant is zero. Hence, the estimated mean payouts reflect expected payouts to the senior claimant conditional on default. The estimated value for the mean payout is 36 cents in the dollar for the pre FIRREA period and appears at first glance to be too low for supposedly insured claimants who have never taken a loss. However, though the senior claimants are insured, their claims are at risk if the government is forced by circumstances to renege on its promise. In more or less normal circumstances, this is not likely to occur, but is a possibility in catastrophic states of the world. Such states are associated with extremely low levels of asset backing and depleted levels of the insurance fund. Hence conditional on default, the expected payouts in such states would be low.

Furthermore we observe that following FIRREA the expected payout rates dropped to 16 cents in the dollar. This is consistent with the capitalization of the insurance fund with the passage of FIRREA. In this environment, the recapitalized fund can shoulder greater depletion of the asset backing. Hence the insured depositors associate government reneging at levels of expected payouts as low as 16 percent. We infer from these results that the markets expectation of default occurring or the government reneging on its promise is substantially reduced.

B. Econometric specification and the estimation of the arrival rate process

For the parameters of the arrival rate process we estimate equation (II.A.14). The left hand side variable, $y(t,T) = -\ln(F(t,T))/(T-t)$, where $F(t,T)$ is the probability of no default in the time interval [t,T]. We use equations (I.C.5a) and (I.C.5b) to obtain time series estimates of $F(t,T)$. This requires market data on forward prices of defaultable bonds and estimates of payoffs to the senior and junior claimants. The expected payoffs to the senior and junior claimants are obtained from the parameter estimates of the magnitude model. In this way we obtain, for the insured and uninsured categories, for each of three maturities, time series on $y_S(t,T)$ and $y_J(t,T)$ for T=1,2 and 3 months. Figure 6 presents a graph of $y_S(t,3)$ and $y_J(t,3)$ for the sample period.

This time series of adjusted yields is related to the level of relativized equity (x) by equation (II.A.14). Though the insured and uninsured categories face the same arrival risk on the null hypothesis of the model, in estimation we allow for parameter differences between them. The estimated model then is

$$y_U(t,T) = \frac{c_U}{(\ln x_t + \gamma - \sigma_U^2\tau/2)^2} + \varepsilon^U_{tT}, \qquad \text{for } U=S,J \tag{IV.B.1}$$

where $\gamma=-\ln\delta$ and $\tau=T-t$.

The parameters to be estimated are γ_u, c_u and σ^2_u for U=S,J. The variable ε^U_{tT} is a random error term that allows for errors in variables induced by the first stage estimation of the left hand side variable. We suppose that these yields estimates are conditionally unbiased and hence that $E_{t-1}[\varepsilon^U_{tT}] = 0$. Further, we suppose that $E[\varepsilon^2_{tT}]<\infty$. Our estimation strategy also allows for the possible simultaneity in the determination of $y_U(t,T)$ and x_t.

Equation (IV.B.1) is estimated using the generalized method of moments procedure. For the instrumental variable, Z_t, the specific orthogonality tested by the GMM procedure is of the form

(IV.B.2)
$$E_{t-1}\, Z_t\Big(y_{t,\tau} - c_u \frac{1}{((\ln x_t + \gamma_u) - \sigma_u^2\tau/2)^2} \Big) = 0,$$

for U=S,J for the insured and uninsured deposits and τ=1,2 and 3 months. For the choice of instruments we use a constant term, all six lagged yields, the lagged excess return on the equity value, and its square. In each of the two cases of U=S, J we have three maturities and nine instruments yielding 27 orthogonalities to be simultaneously tested.

The estimation of the magnitude model indicated parameter shifts in March 1990 and November 1990. In the estimation of the arrival risk model we allowed for parameters shifts at both these dates, however we could not achieve convergence for the rather short second regime of 8 months. Hence we estimated the arrival risk model for two subsample periods, March 87 to October 1990, and November 1990 to December 1991. We therefore have for each category six parameters to be estimated.

Table 4 reports the results. For both the insured and uninsured CDs the model is not rejected, as indicated by the p-values of .81 and .78 associated with the χ^2_{22} statistics of 17.92 and 17.96, respectively.

INSERT TABLE 4 HERE

The coefficients are significant and the parameter estimates for the default arrival rate process (c, γ) for both the insured and uninsured cases are of similar magnitude. This is consistent with the model specification. The hazard rate of default is negatively related to equity values. We observe that there is a significant reduction in the arrival rates of default between the subsample periods. This complements the finding of the magnitude risk model in that markets perceive default probabilities of these institutions coming down on average after the capitalization of the insurance fund.

V. CONCLUSION

This paper characterizes the risk neutral jump process of default in terms of two entities, i) an instantaneous arrival rate of default and ii) a conditional density of the magnitude of the proportionate reduction in the value of creditors claims. We propose models for default the arrival risk of default that varies over time in response to cumulated excess returns. The model for the risk in the magnitude of default is time homogeneous. These two default components are then explicitly priced in the futures market with the spot price of risky debt being derived as a consequence.

The resulting models for default arrival and magnitude risks are successfully estimated on monthly data for rates on certificates of deposit offered by institutions in the Savings and Loan Industry. The data period is January 1987 to December 1991. Our empirical results for the arrival and magnitude risk models provide support for the hypothesis that default has become a significantly less likely event for the insured CD holders after the passage of FIRREA. The conditional density for the magnitude of default is estimated to have undergone two shifts, which fall between the period bracketed by FIRREA and FDICIA. These shifts cause the mean payout rates to decrease. We interpret this indicative of how far asset backing must deteriorate before default becomes a possibility. The study arrival rates corroborates this view.

FOOTNOTES

[1]Stopping times are in fact decomposed into two categories termed accessible and totally inaccessible. The predictable stopping times are a subclass of the accessible stopping time category. A stopping time is accessible if it is almost surely equal to one of a number of predictable stopping times, but it fails to be predictable itself as we can not in general construct an announcing sequence for it, lacking knowledge on which of the many possible candidate predictable times is in fact relevant. An example helps illustrate the difference. Suppose that we wish to model the default of an agency insuring bank deposits. The default time of the insuring agency can be modeled as an accessible stopping time. This is because the agency insures a number of banks that are subject to default risk. Suppose that these banks are sufficiently large, so that failure of a number of them will lead to a failure of the insuring agency itself. If the banks have firm value processes that are continuous in time and each one of them defaults when firm value equals debt value, then bank default times are predictable (by virtue of their firm value continuity). However, the insuring agency default time is accessible, since an announcing sequence for the agency does not exist. On the other hand, a stopping time T is totally inaccessible if for any predictable stopping time S, the probability that T=S is zero, yet there is a positive probability that default may occur in the interval.

[2]A good example of this latter situation is the policy of forbearance followed by the Federal Savings and Loan Insurance Corporation toward insolvent thrifts.

[3]Using the PDE approach of Cox, Ingersoll and Ross (1985), to obtain conditional expectations under the martingale measure, we derive a closed form model in terms of a single integral for the futures price of a survival contingent pure discount bond.

[4]The relationship between futures prices of survival contingent bonds and the default arrival rate is a consequence of the simple payoff structure associated with pure discount bonds subject to default risk. For more complex instruments involving intermediate and possibly random cash flows the reader is referred to Artzner and Delbaen (1994) and Duffie, Schroder and Skiadas (1994).

[5]Al1 event probabilities in the paper are risk neutral probabilities with respect to the money market account as the discounting asset.

[6]The specification of ϕ's as in Jarrow and Turnbull leads to inaccessible stopping times of default, where every instant of time has a zero probability that default occurs at that time yet there is a positive probability that default occurs in any time interval. In contrast, the option theoretic approach uses a predictable stopping time for default, where there are instants of time at which the default probability is not only positive but 1. This occurs because the default condition is prespecified and so any instant at which the default condition is met has a unit default probability. Intuitively we have predictability measured in terms of distance from default boundaries.

[7] The default process $\Delta(t)$ may be written as the integral $\Delta(t) = \int_0^t x(u)dD(u)$.

[8]Q is the equivalent martingale measure under which asset prices, discounted by B(t), are martingales. Even if markets are incomplete, the values of default contingent claims may be uniquely determined provided the structure of traded assets is suitably rich. For further details on these issues the reader is referred to the important contribution of Artzner and Delbaen (1994).

[9]We thank Alan White for suggesting this approach.

[10]We thank John Hull for suggesting the use of credit derivatives in this context.

[11] Black and Cox (1976) were the first to expand Merton (1974) to allow for multiple debt claimants. We follow Black and Cox (1976) in defining the differential payoffs to these securities. This approach has also been followed by Gorton and Santomero (1990) to price bank subordinated debt.

[12] The requirement of predictability ensures that $\phi(t)$ is defined using information that is prior to time t. For a formal definition of predictability and results on the uniqueness of $\phi(t)$, the reader is referred to Jacod and Shiryaev (1980).

[13] The intuition behind this result can also be seen by considering equation (I.A.5) of section I.A, and noting that in continuous time the probability of no default requires the evaluation of the limit of the product of factors like $(1-\phi_1\Delta t)(1-\phi_2\Delta t)$. This limit is the negative exponential of the sum of the $\phi_i\Delta t$'s via the approximation $(1-\phi_i\Delta t)\cong e^{-\phi_i\Delta t}$. This approximation may be contrasted with the one used for pricing risk free discount bonds. In the case of risk free discount bonds the approximation used is $1/(1+r\Delta t)\cong e^{-r\Delta t}$.

[14] Note that this approach is analogous to obtaining the partial differential equation for the spot price of the pure discount bond, as initiated by Cox, Ingersoll and Ross (1985).

[15] In the empirical section we estimate the model for short maturity (less than 3 months) Savings and Loans certificates of deposit. The sample period also coincides with a period when the policy of forbearance was in effect. These considerations justify our use of the low probability approximation given by equation (III.A.13).

[16]Upon the passage of FIRREA, the thrift insurance fund of FSLIC was closed and the thrifts were brought under the control of the FDIC. In addition 50 billion dollars was provided to the Resolution Trust Corporation which was established for the purpose of closing insolvent thrifts, and thrifts were directed to display the disclaimer "insured deposits are backed by the full faith and credit of the U.S. Government. "

[17]Thrifts also report CD rates for six maturity buckets for amounts below $80,000 dollars. In addition they report rates for two additional maturity groups for amounts above $100,000. We do not use this information as these maturity buckets are too wide and do not match up across the insured and uninsured categories.

[18]The third switch was not significant and results are reported for a two switch three regime formulation. The χ^2_7 statistic for a third switch was 8.72 with a p-value of .2763.

APPENDIX A

Forward Futures Equality in the Finite State Discrete Time Model

Consider first the forward price of the defaultable bond. By spot forward arbitrage the forward price is obtained simply by dividing v, the spot price of the defaultable bond of expression (1.4) by the spot price of the two period risk free bond. We write the bond price in terms of forward rates as $[(1+r_1)(1+f_2)]^{-1}$, where f_2 is the forward rate for the second period. The forward price of the defaultable bond is therefore

(A.1)
$$V = (1-\phi_1)\ [q_u(1-\phi_{2u}) + q_d(1-\phi_{2d})][q_p \frac{1}{1+r_{2p}} + q_n \frac{1}{1+r_{2n}}]\ (1+f_2) +$$
$$(1-\phi_1)\ [q_u(1-\phi_{2u}) + q_d(1-\phi_{2d})][q_p \frac{1}{1+r_{2p}} + q_n \frac{1}{1+r_{2n}}]\ [q_H H_2 + q_L L_2]\ (1+f_2) +$$
$$\phi_1\ [q_p \frac{1}{1+r_{2p}} + q_n \frac{1}{1+r_{2n}}]\ [q_H H_2 + q_L L_2]\ (1+f_2).$$

The dependence of the forward price equation (A.1) on the interest rates may be eliminated by noting from the time 0 pricing equation for the two period risk free bond that

(A.2)
$$\frac{1}{(1+r_1)(1+f_2)} = \frac{1}{1+r_1}\ [q_{up} \frac{1}{1+r_{2up}} + \cdots + q_{dn} \frac{1}{1+r_{2dn}}]$$

or equivalently that

(A.3)
$$(1+f_2)\ [q_{up} \frac{1}{1+r_{2up}} + \cdots + q_{dn} \frac{1}{1+r_{2dn}}]=1.$$

Substitution of equation (A.3) into equation (A.1) leads to the forward price of

(A.4)
$$V = (1-\phi_1)\ [q_u(1-\phi_{2u}) + q_d(1-\phi_{2d})] +$$
$$(1-\phi_1)\ [q_u\phi_{2u} + q_d\phi_{2d}][q_H H_2 + q_L L_2] +$$
$$\phi_1\ [q_H H_2 + q_L L_2].$$

Next we consider the marked to market futures price. The futures price involves no discounting but time two delivery requires that we carry over time 1 payments to the end of the tree at the prevailing money market rates. Hence the futures price of the defaultable bond is given by

(A.5)
$$W = (1-\phi_1)\ [q_u(1-\phi_{2u}) + q_d\ (1-\phi_{2d})\ ] +$$
$$(1-\phi_1)\ [q_u\phi_{2u} + \cdots + q_d\phi_{2d}][q_H H_2 + q_L L_2] +$$
$$\phi_1\ [q_H H_1 + q_L L_1]\ [q_p\ (1+r_{2p}) + q_n\ (1+r_{2n})\ ].$$

By assumption 4 and 5 applied to H_1 and L_1 in (A.5) one observes that W equals V as given by (A.4).

APPENDIX B

Proof of Theorem 1

We first show that (III.A.3) is consequence of

(B.1) $$E^Q[\ (1-D(t))\ |\ \mathcal{G}_t] = e^{-\int_0^t \phi(u)du},$$

where $\mathbf{G}=\{\mathcal{G}_t; t\in[0,T]\}$, with $\mathcal{G}=\mathcal{G}_T$, is the filtration generated by the Ito process x(t) of assumption 3.2. To show this note that the probability of no default in [t,T] given no default by time t is

$$F(t,T) = E^Q_t[\ (1 - D(T))\ |\ \mathcal{G}_t \text{ and } Z>t\ ] = E^Q_t[Z>T\,|\,\mathcal{G}_t \text{ and } Z>t\ ].$$

It follows from (B.1) on employing Bayes theorem that

$$F(t,T) = \frac{E^Q[\ Z>T \text{ and } Z>t\ |\ \mathcal{G}_t]}{E^Q[\ Z>t\ |\ \mathcal{G}_t]} = \frac{E^Q_t[E^Q_T[(1 - D(T))\,|\,\mathcal{G}_T]\,|\,\mathcal{G}_t]}{E^Q[\ (1 - D(t))\ |\ \mathcal{G}_t]} = E^Q[\ e^{-\int_t^T \phi(u)du}\ |\ \mathcal{G}_t].$$

Hence the theorem follows on demonstrating that (B.1) or equivalently,

(B.2) $$X(t) = e^{\int_0^t \phi(u)du}(1-U(t)) = 1$$

holds, where $U(t) = E^Q_t[\ D(t)\ |\ \mathcal{G}_t]$.

Equation (B.2) follows on showing that X(t) is i) a process of bounded and integrable variation, ii) a predictable process, and iii) a martingale. Under these conditions X(t) equals X(0)=l for all t (see Elliott (1982) Corollary 7.32).

Proof of i): X(t) is a process of bounded and integrable variation.

Define the counting process, N(t), such that D(t) is just N(t) stopped at its first arrival time T_1. Let

(B.3) $$N(t) = \Sigma_{n\geq 1} \mathbf{1}_{T_n \leq t},$$

where $\{T_n\}_{n\geq 1}$ is a sequence of jump times with $T_0=0$ and $T_n<T_{n+1}$ if $T_n<\infty$. Let $\phi(t)$ denote the process for the arrival rate of the jumps T_n and let A(t) be the cumulated arrival rate process,

(B.4) $$A(t) = \int_0^t \phi(u)du.$$

By construction N(t) - A(t) is a martingale. The default time process D(t) can then be written as the process N stopped at the first jump $Z=T_1$. If we define α as the process A stopped at the frost jump time $Z=T_1$,

(B.5) $$\alpha(t) = \int_0^t (1-D(u\_))\phi(u)du = A(t\wedge Z),$$

then D(t) - $\alpha(t)$ is a martingale. The process $\alpha(t)$ in equation (B.5) cummulates the arrival rates only if the first arrival has not yet occurred and stops accumulating thereafter.

Let the counting process N be adapted to the larger filtration $\mathbf{F}=\{\mathcal{F}_t;t\in[0,T]\}$. If **F** is taken as the smallest filtration with $\mathcal{G}\subseteq\mathcal{F}_0$ such that N is adapted to **F** then it is shown in JS page 134, that the probability measure Q is the unique solution to the martingale problem of constructing N with the process A as its compensator. Note in this regard that since by construction N - A is a martingale starting at zero with $\mathcal{G}\subseteq\mathcal{F}_0$,

(B.6) $$E^Q[\ N(t) - A(t)\ |\ \mathcal{G}\ ] = 0,$$

or equivalently that

(B.7) $$E^Q[\ N(t)\ |\ \mathcal{G}\ ] = A(t).$$

Since A(t) is $\mathcal{G}_t$ measurable it follows that

(B.8) $$E^Q[\ N(t)\ |\ \mathcal{G}_t] = A(t),$$

where A(t) is an increasing process of bounded and integrable variation.

The process D(t) may be written as $N(t)\wedge 1$ and as N(t) is essentially defined upto t by its compensator that is $\mathcal{G}_t$ measurable it follows that the distribution of

N(t) conditional on $\mathcal{G}$ is the same as the distribution of N(t) conditional on $\mathcal{G}_t$ and so

(B.9) $$U(t) = E^Q[\, D(t) \mid \mathcal{G}_t] = E^Q[\, D(t) \mid \mathcal{G}].$$

Now since D(t) is an increasing process we must have that $E^Q[\, D(t) \mid \mathcal{G}]$ is an increasing process and so U is an increasing process. Also as D is dominated by N we must have that U is dominated by A. It follows that U is increasing and of bounded and integrable variation.

Proof of ii). X(t) is a predictable process.

The predictability of X(t) follows from that of U(t). For this we observe that U(t) is a conditional expectation process, conditioned on the evolution of the Ito process x(t). U(t) is then the filtered value of D(t), filtered from the observation process x(t). Theorem 18.11 of Elliott (1982) shows that U(t) is a continuous and hence predictable process.

Proof of iii). X(t) is a martingale.

To show that X(t) is a martingale, consider the negative of the martingale D(t) - α(t) or the martingale M(t) where,

$$M(t) = -\, D(t) + \int_0^{t \wedge Z} \phi(u)du.$$

Let Y(t) be the Doleans-Dade exponential of M(t). By construction this is the process

$$Y(t) = e^{\int_0^{t \wedge Z} \phi(u)du}(1 - D(t)) = e^{\int_0^{t} \phi(u)du}(1 - D(t)),$$

and Y(t) is an $\mathcal{F}_t$ adapted martingale. It follows that

$$Y(t) = E^Q[\, Y(\infty) \mid \mathcal{F}_t],$$

and hence that

$$X(t) = E^Q[\, Y(t) \mid \mathcal{G}_t] = E^Q[\, Y(\infty) \mid \mathcal{G}_t],$$

must be a $\mathcal{G}_t$ adapted martingale. We observe on evaluation using the definition of U(t) that X(t) in (B.2) is a martingale.■

Proof of Theorem 2

By theorem 1, F(t,T) is the conditional expectation of a functional $L_{t,T}(s)$ of the Markov process s. Specifically,

$$F(t,T) = E^Q[L_{t,T}(s) \mid \mathcal{G}_t],$$

where

$$L_{t,T}(s) = \exp(\, - \int_t^T \phi(u,s(u))du\,).$$

By the properties of conditional expectations of functionals of Markov Processes (Breiman (1968), Bismut (1981) and Kunita (1990), there exists a function $\psi(t,s,T)$, differentiable in t,T and twice continuously differentiable in s, such that

$$F(t,T) = E^Q[L_{t,T}(s) \mid s(t)] = \psi(t,s,T).$$

Observe that

$$e^{-\int_0^t \phi(u)du}\psi(t,s,T) = e^{-\int_0^t \phi(u)du}F(t,T) = e^{-\int_0^t \phi(u)du}E^Q[\, e^{-\int_t^T \phi(u)du} \mid \mathcal{G}_t].$$

It follows from the $\mathcal{G}_t$ measurability of $\int_0^t \phi(u)du$, that

(B.1O) $$e^{-\int_0^t \phi(u)du}\psi(t,s,T) = E^Q[e^{-\int_0^T \phi(u)du} \mid \mathcal{G}_t\,].$$

Since the right hand side of equation (B.10) is a martingale by virtue of being a conditional expectation process of a terminal random variable we must have that the left hand side is also a martingale. Equating the dt term of the stochastic differential of the left hand side of (B.10) to zero yields the partial differential equation (III.A.6). The boundary condition follows from $\psi(t,s,t) = F(t,t) = 1$.■

APPENDIX C

Solution for the futures price of the defaultable bond

We wish to solve the partial differential equation (III.10) for $\psi(t,s,T)$ with the boundary conditions $\psi(t,s,t)=1$. First consider ψ in the form

(C.1) $$\psi(t,s,T) = H(s,T-t) = H(s,\tau),$$

with boundary condition $H(s,0) = 1$. The partial differential equation in H is

(C.2) $$-H_\tau + \frac{1}{2}\sigma^2 s^2 H_{ss} = \phi(s)H .$$

In order to obtain an equation with constant coefficients for the second order term we introduce the transformation

(C.3) $$K(y,\tau) = H(e^y,\tau),$$

or equivalently that

(C.4) $$H(s,\tau) = K(\ln s,\tau).$$

The boundary condition is now $K(y,0) = 1$. Since $K_\tau = H_\tau$, $K_y = e^y H_s = sH_s$ and $K_{yy} = sH_s + s^2 H_{ss}$ the partial differential equation in K is

(C.5) $$K_\tau - \frac{\sigma^2}{2}(K_{yy} - K_y) + \phi(e^y)K = 0.$$

Consider the transformation

(C.6) $$U(y,\tau) = K(y + \frac{\sigma^2}{2}\tau, \tau),$$

with boundary condition $U(y,0) = 1$. Since $U_\tau = K_\tau + \frac{\sigma^2}{2} K_y$ the partial differential equation in U is

(C.7) $$U_\tau - \frac{\sigma^2}{2} U_{yy} + \phi(e^y)\, U = 0,$$

and we may recover K by

(C.8) $$K(y,\tau) = U(y - \frac{\sigma^2}{2}\tau, \tau).$$

Defining $\gamma = -\ln \delta$ and substituting for ϕ its explicit form from equation (15) we obtain that

$$U_\tau - \frac{\sigma^2}{2} U_{yy} + \left[\frac{c}{(y + \gamma)^2} \right] U = 0. \tag{C.9}$$

Now define

$$L(z,\tau) = U(z-\gamma,\tau), \tag{C.10}$$

with

$$U(y,\tau) = L(y+\gamma,\tau). \tag{C.11}$$

The partial differential equation in L is

$$L_\tau - \frac{\sigma^2}{2} L_{zz} + \frac{c}{z^2} L = 0, \tag{C.12}$$

with boundary condition $L(z,0) = 1$. To get back to ψ we have that

$$\psi(t,s,T) = L\left((\ln s) + \gamma - \frac{\sigma^2}{2} (T-t),\ T-t \right). \tag{C.13}$$

For a solution to (C.12) consider one in the form

$$L(z,\tau) = G_a\left((2\sigma^2\tau/z^2) \right), \tag{C.14}$$

where $a=c/(2\sigma^2)$. The validity of the form (C.14) can be derived by applying inverse Laplace transforms to series solutions for the Laplace transforms of (C.12). For details the reader is referred to Madan and Unal (1994).

Computing the partials of L from (C.14) and substituting into (C.12) we obtain a second order differential equation for G_a. The relevant partials are

$$L_\tau = G_a'(2\sigma^2/z^2). \tag{C.15}$$

$$L_z = - G_a'(4\sigma^2\tau/z^3), \tag{C.16}$$

and

$$L_{zz} = G_a''(16\sigma^4\tau^2/z^6) + G_a'(12\sigma^2\tau/z^4). \tag{C.17}$$

Substitution in (C.12) yields that

(C.18) $$-(8\sigma^6\tau^2/z^6)\ G_a'' + ((2\sigma^2/z^2) - (6\sigma^4\tau/z^4))\ G_a' + (c/z^2)\ G_a = 0,$$

and multiplication by $-(z^2/(2\sigma^2))$ gives

(C.19) $$(4\sigma^4\tau^2/z^4)\ G_a'' + ((3\sigma^2\tau/z^2) - 1)\ G_a' - (c/(2\sigma^2))\ G_a = 0.$$

The argument of G_a is $y = 2\sigma^2\tau/z^2$ and substituting this fact into (C.19) we obtain that

(C.20) $$y^2 G_a'' + (\tfrac{3}{2}y - 1)\ G_a' - aG_a = 0.$$

The initial conditions associated with G_a are that $G_a(0) = 1$, and that $G_a'(0)$ is such that the initial hazard rate is as required. For the initial arrival rate to be $\phi(s)$ as given by equation (III.A.9) we must have that

(C.21) $$G_a'(0) = -\frac{c}{2\sigma^2} = -a.$$

Hence we wish to solve the second order ordinary differential equation (C.20) subject to $G_a(0) = 1$ and (C.21) or $G_a'(0)=-a$ for $a=c/(2\sigma^2)$. We denote this solution by $G_a(y)$. We also note that as y tends to infinity the arrival rates of default diverge to infinity as s approaches δ and so $G_a(\infty)=0$.

The futures price of the defaultable bond, F(t,T) can then be written in terms of the function $G_a(y)$. The complete solution is obtained on sequentially substituting back through the equations (C.14), (C.11), (C.6), (C.4), and (C. 1) and yields the result (III.A.11) and defines the required futures prices.

APPENDIX D

Solution of the differential equation (III.A.12).

The differential equation to be solved, deleting the subscript a for notational convenience is given by

(D.1) $$y^2G'' + (3y/2-1)G' - aG = 0$$

subject to the boundary conditions $G(0)=1$, $G'(0)=-a$ and $G(\infty)=0$. We begin by transforming this equation to focus attention on the tail behavior by defining

(D.2) $$H(x) = G(1/x).$$

The differential equation in H is then given by

(D.3) $$x^2H'' + (x/2 + x^2)H' - aH = 0$$

subject to the boundary conditions $H(0)=0$, $H(\infty)=1$, and the limit as $x \longrightarrow \infty$ of $x^2H'(x)=a$.

The second order equation for $H(x)$ leads to a first order equation in $h(x)=H'(x)/H(x)$. We observe that $h'=H''/H - h^2$ and substitute from (D.3) to obtain

(D.4) $$h'(x) = a/x^2 - \left(\frac{1}{2x} + 1\right) h(x) - h^2,$$

with the boundary condition $\lim_{x\to\infty} x^2h(x) =a$. We recover H and hence G by the equation

(D.5) $$H(x) = \exp\left(- \int_x^\infty h(u)du \right),$$

where by construction $H(\infty)=1$ and $H(0)=0$ provided $\int_0^\infty h(u)du$ is infinite.

The boundary condition on $h(x)$ suggests that we consider the function $k(x)=x^2h(x)$ that has a limit of a as x tends to infinity. We then have that $h(x)=k(x)/x^2$, and provided $k(x)$ is bounded in the neighbourhood of zero we will also have that h integrates to infinity over the half line $(0,\infty)$. The equation for $k(x)$ may be obtained on differentiation of its definition and substitution from (D.6). This yields

(D.6) $$k'(x) = a + \left(\frac{3}{2x} - 1\right)k(x) - \frac{k^2}{x^2}.$$

We see from (D.5) that $k'(\infty)=0$.

The set $\{(x,k) \mid k'=0\}$ for $a=.1$ is graphed in figure 7 and we observe that k will start out from its value at zero decreasing sharply. It will then reach the region where it will start increasing. After that it will either tend toward a with a positive slope or cross over to where it begins to decrease towards a. Numerical solutions using NDSolve on Mathematica show that the latter is the case and figure 8 provides the solution for $k(x)$ for $a=.1$. Since our data is in the region of y near zero or x near infinity, where k' is essentially zero we may approximate $k(x)$ by the constant function a in this region. This gives $h(x)=a/x^2$, $H(x) = e^{-a/x}$ and $G(y)=e^{-ay}$.

REFERENCES

Abramowitz, M., and I. A. Stegun, 1972, *Handbook of Mathematical Functions,* Dover, New York.

Anderson, R. W. and S. Sundaresan, 1992, "Design and Valuation of Debt Contracts," Working Paper, *Columbia University,* New York.

Artzner, P. and F. Delbaen (1994), "Default Risk Insurance and Incomplete Markets," Working Paper, *Institut de Recherche Mathématique Avancée Université Louis Pasteur et CNRS, et Laboratoire de Recherche en Gestion,* 7 rue René Descartes, F 67084 Strasbourg Cedex, France.

Avery, R. B., T. M. Belton and M. A. Goldberg, 1988, "Market Discipline in Regulating Bank Risk: New Evidence from the Capital Markets," *Journal of Money Credit and Banking,* 20, 597-610.

Baer, H. and E. Brewer, 1986, "Uninsured deposits as a Source of Market Discipline: Some New Evidence," *Federal Reserve Bank of Chicago, Economic Perspectives,* September, 23-31.

Bismut Jean-Michel, 1981, *Mécanique A1éatoire, Lecture Notes in Mathematics No 866,* Springer Verlag, Berlin.

Black, F. and J. Cox, 1976, "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions," *Journal of Finance,* (May), 351-367.

Breiman, L, 1968, *Probability,* Addison-Wesley, Reading Massachusetts.

Cargill, T. F., 1989, "CAMEL Ratings and the CD Market," *Journal of Financial Services Research,* 3, 347-358.

Chance, D., 1990, "Default Risk and the Duration of Zero Coupon Bonds," *Journal of Finance,* 45 (l), 265-274.

Coddington, E. A., 1961, *An Introduction to Ordinary Differential Equations*, Prentice Hall, Englewood Cliffs, New Jersey.

Cook, D. O. and L. J. Spellman, 1991, "Federal Financing Guarantees and the Occasional Market Pricing of Default Risk: Evidence from Insured Deposits," *Journal of Banking and Finance,* 15, 1113-1130.

Cooper, I and A. Mello, 1991, "The Default risk of Swaps," *Journal of Finance,* 46 (2), 597-620.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1981, "The relation between forward and futures prices," *Journal of Financial Economics,* 9, 321-346.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1985. "A Theory of the Term Structure of Interest Rates," *Econometrica,* 53, 385-408.

Duffie (1992), *Dynamic Asset Pricing Theory,* Princeton University Press, Princeton.

Duffie, D., M. Schroder and C. Skiadas (1994), "Two Models of Price Dependence on the Timing of Resolution of Uncertainty," Working Paper, *Graduate School of Business, Stanford University,* Stanford CA.

Duffie, D and K. J. Singleton (1994), "Econometric Modeling of Term Structures of Defaultable Bonds," Working Paper, *Graduate School of Business, Stanford University,* Stanford CA.

Elliott, R., 1982, *Stochastic Calculus and Applications,* Springer-Verlag: New York.

Ellis, D. and M. J. Flannery, 1992, "Does the Debt Market assess large Banks' Risk," *Journal of Monetary Economics,* 30, 481-502.

Fama, E. F., 1986, "Term Premiums and Default premiums in Money Markets," *Journal of Financial Economics,* 17, 175-196.

Gorton, G. and A. M. Santomero, 1990, "Market Discipline and Bank Subordinated Debt," *Journal of Money Credit and Banking,* 1, 119-128.

Gradshetyn, I.S. and I.M. Ryzhik, 1980, *Table of Integrals, Series, and Products,* Academic Press, New York.

Hannan, T. H. and G. A. Hanweck, 1988, "Bank Insolvency Risk and the Market for Large Certificates of Deposit," *Journal of Money Credit and Banking,* 2, 203-212.

Heath, D., R. Jarrow, and A. Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica,* 60 (l), 77-105.

Hirschorn, E., 1992, "Certificates of Deposit as Sources of Market Discipline," Working Paper, Office of thrift Supervision, Department of the Treasury.

Hull, J. and A. White, 1992, "The Impact of Default Risk on Option Prices," Working Paper, University of Toronto.

Ho, T. and R. Singer, 1982, "Bond Indenture Provisions and the Risk of Corporate Debt," *Journal of Financial Economics,* 10, 375-406.

Jacod, J., and A. N. Shiryaev, 1987, *Limit Theorems for Stochastic Processes,* Springer-Verlag, New York.

Jarrow, R.A., 1987, "The pricing of commodity options with stochastic interest rates," *Advances in Futures and Options Research,* 2, 19-45.

Jarrow, R., and D. Madan, 1992, "Option Pricing Using the Term Structure of Interest Rates to Hedge Systematic Discontinuities in Asset Returns," forthcoming, *Mathematical Finance.*

Jarrow, R., and S. Turnbull, 1995, "Pricing Derivatives on Financial Securities Subject to Credit Risk," *Journal of Finance,* 50, 1, 53-85.

James, C., 1988, "The Use of Loan Sales and Standby Letters of Credit by Commercial Banks," *Journal of Monetary Economics,* 22, 395-422.

Johnson, H., and R. Stulz, 1987, "The Pricing of Options with Default Risk," *Journal of Finance,* 42 (2), 267-280,

Kane, Edward J., 1989, *The S&L Insurance Mess: How did It Happen? The* Urban Institute Press, Washington D.C.

Kunita, H., 1990, *Stochastic Flows and Stochastic Differential Equations,* Cambridge University Press, Cambridge, England.

Lee, C. J., 1981, "The Pricing of Corporate Debt: A Note," *Journal of Finance,* 36, 1187-1189.

Longstaff, F., and E. Schwartz, 1995, "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt," *Journal of Finance,* 50, 3, 789-819.

Madan, D., and E. Seneta, 1990, "The VG model for Share Market Returns," *Journal of Business*, 63, 511-524.

Merton, R. C., 1974, "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance,* (2), 449-470,

Merton, R. C., 1977, "On the Pricing of Contingent Claims and the Modigliani-Miller Theorem," *Journal of Financial Economics,* 5, 241-249.

Nielsen, S. S. and E. I. Ronn 1995, "The Valuation of Default Risk in Corporate Bonds and Interest Rate Swaps," *Working Paper,* University of Texas at Austin.

Pitts, C.G.C. and M.J.P. Selby, 1983, "The Pricing of Corporate Debt: A Further Note," *Journal of Finance,* 38, 1311-1313.

Ronn, E. I., and A. K. Verma, 1986, "Pricing Risk-Adjusted Deposit Insurance: An Option Based Model," *Journal of Finance*, 41 (4), 871-896.

Sarig, O. and A. Warga, 1989, "Some Empirical Estimates of the Risk Structure of Interest Rates," *Journal of Finance*, 44, 1351-1360.

Schwert, William G., 1990, "Stock Volatility and the crash of 1987," *Review of Financial Studies,* 3 (l), 77-102.

Sinkey, Joseph F., 1992, *Commercial Bank Financial Management In the Financial Services Industry,* Fourth Edition, Macmillan, New York.

Thomson, James B. ,1992, "Modeling the Bank Regulator's Closure Option: A Two-Step

Logit Regression Approach," *Journal of Financial Services Research*, 6, 5-23.

Table 1: Annualized CD - Treasury Spreads (in percentage points), Relativized Equity and Proportion of Insured Deposits averaged over the sample of thrifts for the month of December

Type of CD	1987	1988	1989	1990	1991
Insured Category					
$80,000 to $100,000 CD's					
up to 1 month	1.54	0.88	0.93	1.20	0.38
1 to 2 months	1.36	0.58	0.60	0.87	0.37
2 to 3 months	1.28	0.26	0.29	0.61	0.44
Uninsured Category					
Over $100,000 CD's					
up to 1 month	2.32	1.53	1.44	1.57	0.39
1 to 2 months	2.07	1.04	0.94	1.16	0.42
2 to 3 months	1.66	0.41	0.38	0.73	0.50
Number of Thrifts in Sample	236	240	224	197	171
Relativized Equity	.963	.918	.834	.828	.843
Proportion of Insured Deposits	.867	.866	.880	.896	.901

Table 2: Results of Maximum Likelihood Estimation of Model for Default Payouts

Model:

$$\frac{(V_S(t,t+\tau) - V_J(t,t+\tau))}{1 - V_J(t,t+\tau)} = \left\{ \frac{m_i - \Phi_J(m_i,\lambda_i)}{p_S(1 - \Phi_J(m_i,\lambda_i))} + \varepsilon^i_{t\tau} \right\} \text{ with probability } p_i \text{ for } i=1,2,3,$$

where

$p_1=(1 - N((t-\eta_1)/\sigma_1))$, $p_2=N((t-\eta_1)/\zeta_1)(1 - N((t-\eta_2)/\zeta_2))$ and $p_3=N((t-\eta_1)/\zeta_1)N((t-\eta_2)/\zeta_2)$,

$$\Phi_J(m,\lambda) = \frac{(m - p_S)}{1-p_S} + \frac{p_S}{1-p_S}\mathbb{B}(p_S;m\lambda,(1-m)\lambda) - \frac{m}{1-p_S}\mathbb{B}(p_S;m\lambda+1,(1-m)\lambda)$$

$\mathbb{B}(u;\alpha,\beta)$ is the incomplete Beta function,

$N(x)$ is the standard normal distribution function,

ε^i_τ, are i.i.d. zero mean normal variates with standard deviations σ^i_τ,

for τ equals 1, 2 and 3 months and i=1,2, and 3 periods,

and the time period is March 1987 to December 1991.

Parameter (t-value)	m	λ	σ_1	σ_2	σ_3	η	ζ
First Regime	.3628 (33.1)	22.98 (55.33)	.0996 (5.57)	.1065 (5.51)	.1798 (5.20)		
Second Regime	.2878 (23.35)	26.37 (63.52)	.0374 (2.53)	.0520 (2.15)	.1235 (2.46)	March 1990 37.35 (89.96)	0.006 (.015)
Third Regime	.1567 (10.36)	7.3705 (17.75)	.1109 (3.31)	.0866 (2.99)	.0770 (3.20)	Nov. 1990 45.35 (109.21)	.001 (.002)

Table 3: Results of GMM Estimation of Model for Default Arrival Process

Model: $y_U(t,T) = c_U[\ 1/(\ln(x_t + \gamma_U - \sigma_U^2(T-t)/2)^2] + \varepsilon^U_{t,(T-t)}$

for U=S,J for the insured and uninsured cases.

Coefficients	Insured CD's Coefficient estimate (t-stat)	Uninsured CD's Coefficient estimate (t-stat)
Subsample 1		
c	.003357 (7.45)	.003419 (5.36)
γ	.2585 (1477.00)	.2298 (124.75)
Subsample 2		
c	.000039 (3.10)	.000016 (2.66)
γ	.2515 (254.07)	.2242 (1028.61)
σ^2	.4535 (59.29)	.1342 (41.06)
χ^2_{22} (p-value)	17.92 (.81)	17.96 (.78)

Figure 1

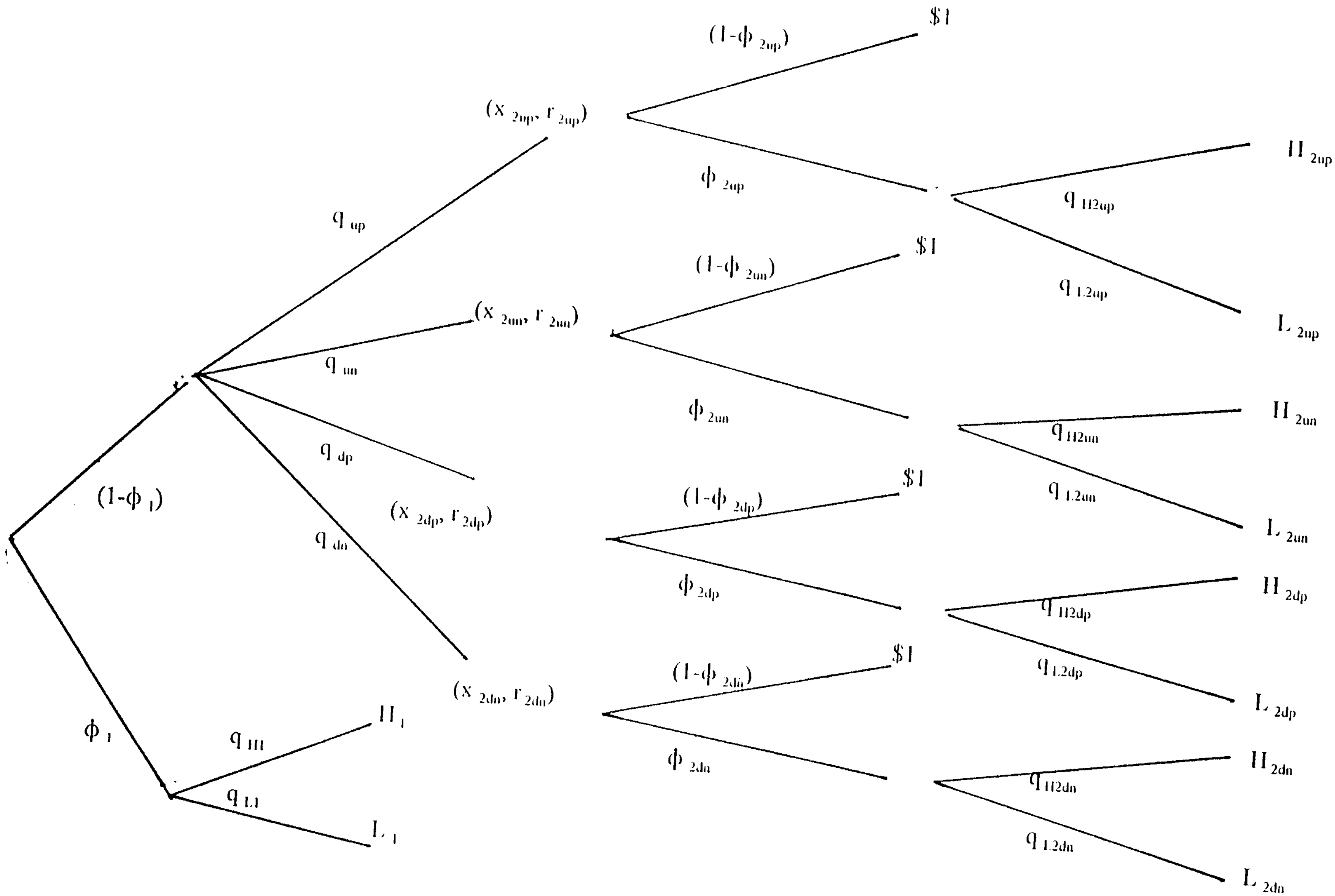

(1-ϕ 1)
ϕ 1
q H1
q L1
H 1
L 1
q up
q un
q dp
q dn
(x 2up, r 2up)
(x 2un, r 2un)
(x 2dp, r 2dp)
(x 2dn, r 2dn)
(1-ϕ 2up)
ϕ 2up
$1
(1-ϕ 2un)
ϕ 2un
$1
(1-ϕ 2dp)
ϕ 2dp
$1
(1-ϕ 2dn)
ϕ 2dn
$1
q H2up
q L2up
H 2up
L 2up
q H2un
q L2un
H 2un
L 2un
q H2dp
q L2dp
H 2dp
L 2dp
q H2dn
q L2dn
H 2dn
L 2dn

Figure 1

Graph of the instantaneous arrival rate of default, $\phi(x)$,

as a function of excess equity returns, x.

$$\phi(x) = \frac{c}{(\ln(x/\delta))^2}$$

for c=.0003; and δ=1.0.

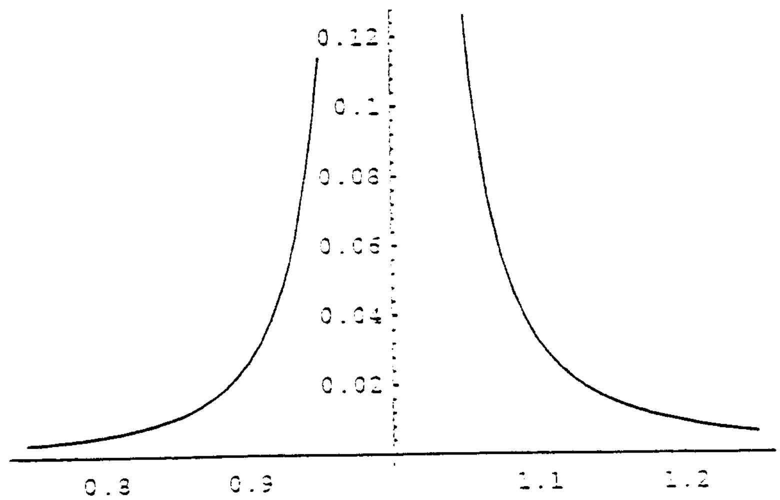


Figure 3

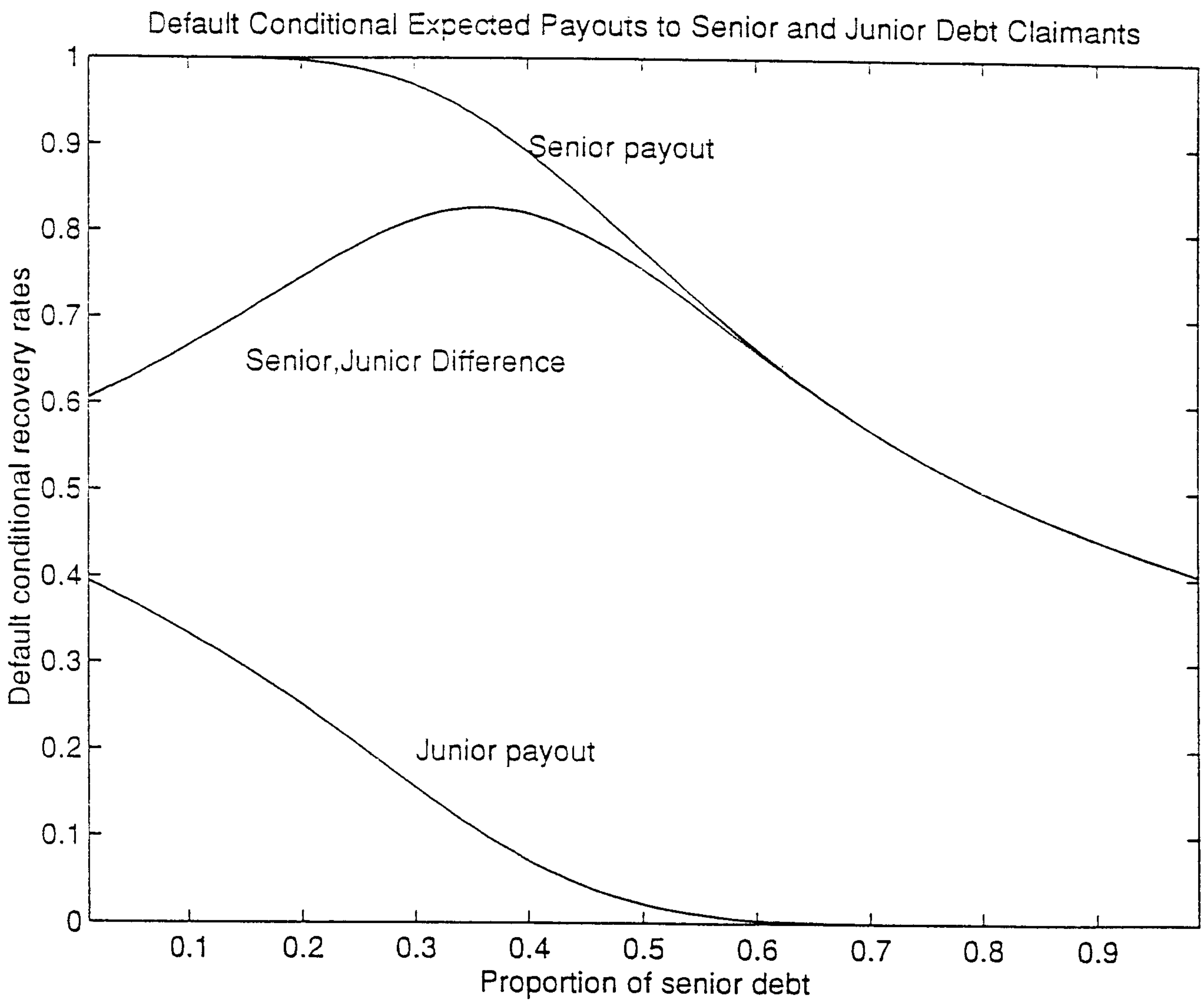

Default Conditional Expected Payouts to Senior and Junior Debt Claimants
Senior payout
Senior,Junior Difference
Junior payout
Default conditional recovery rates
Proportion of senior debt
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Figure 4

The effect of mean payout rates on relative spreads

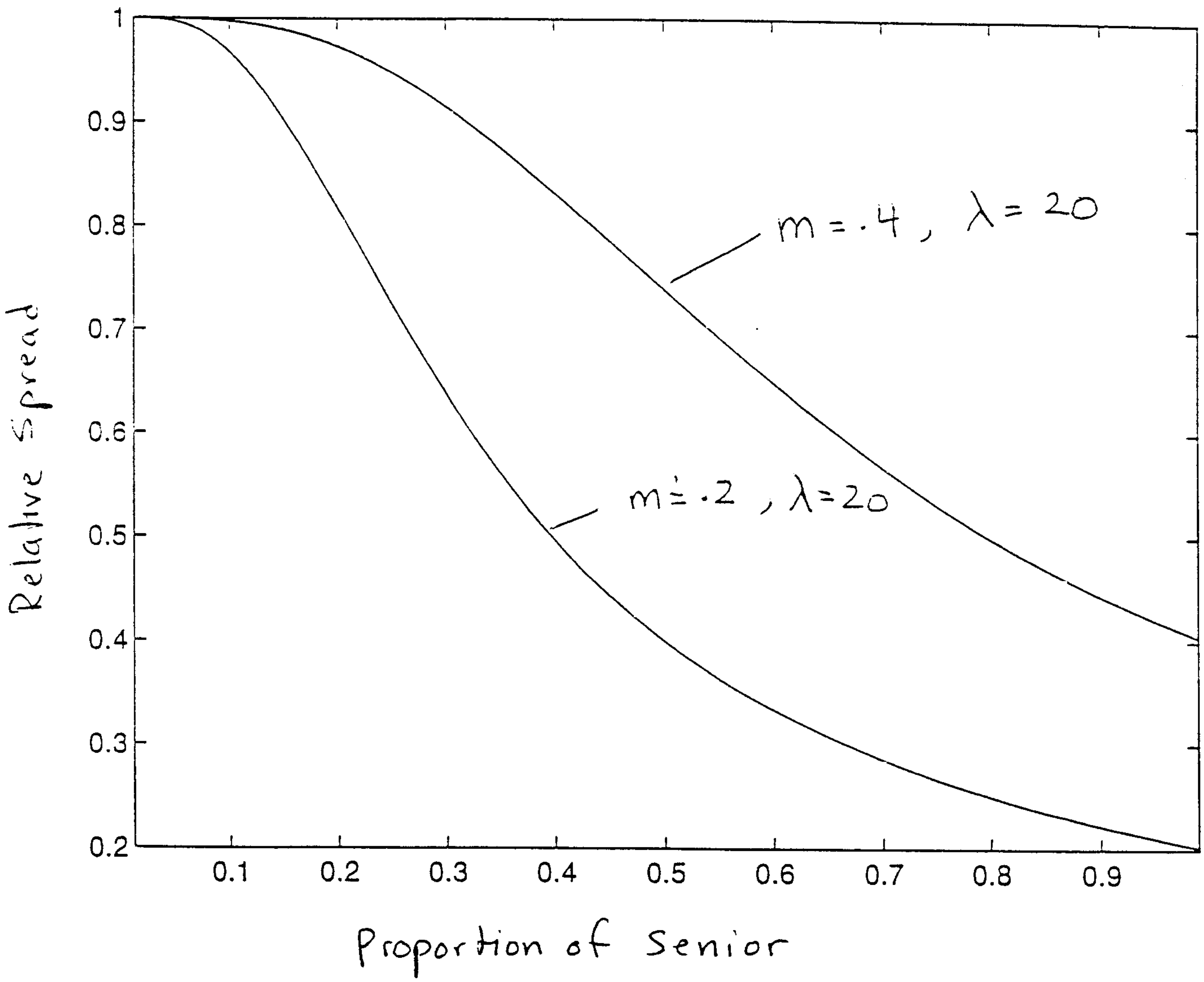


Figure 5

The effect of payout rate volatility on relativ

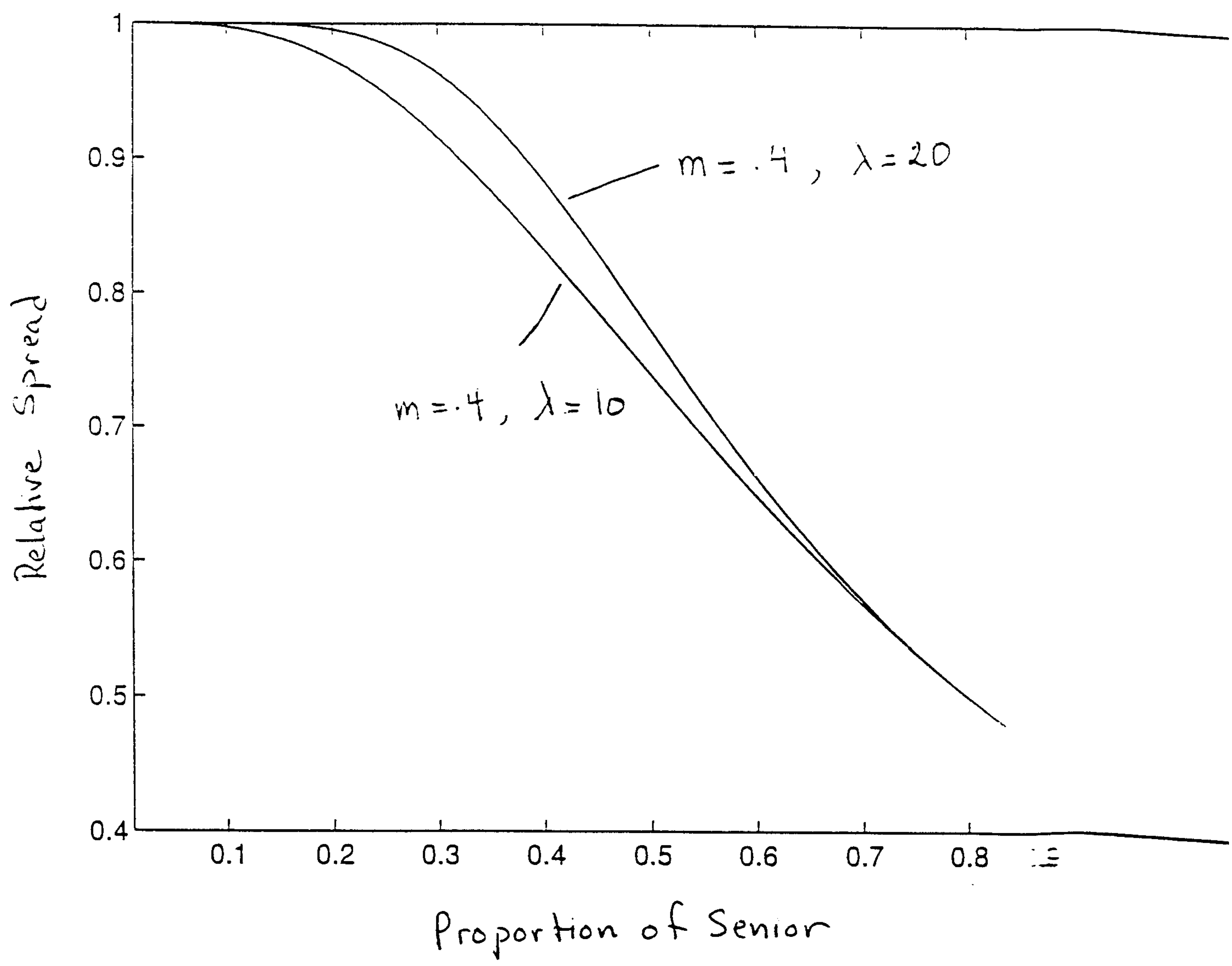
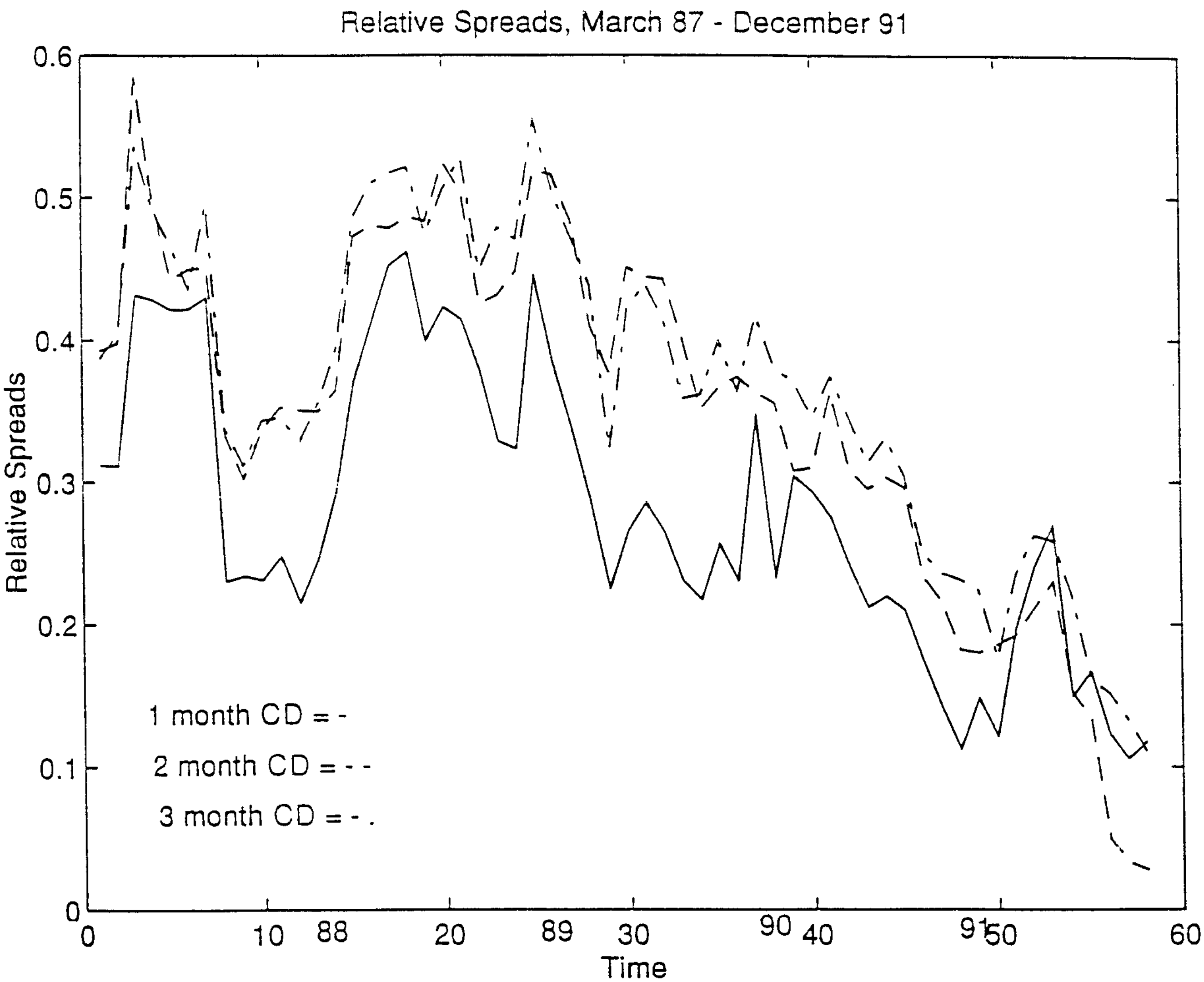
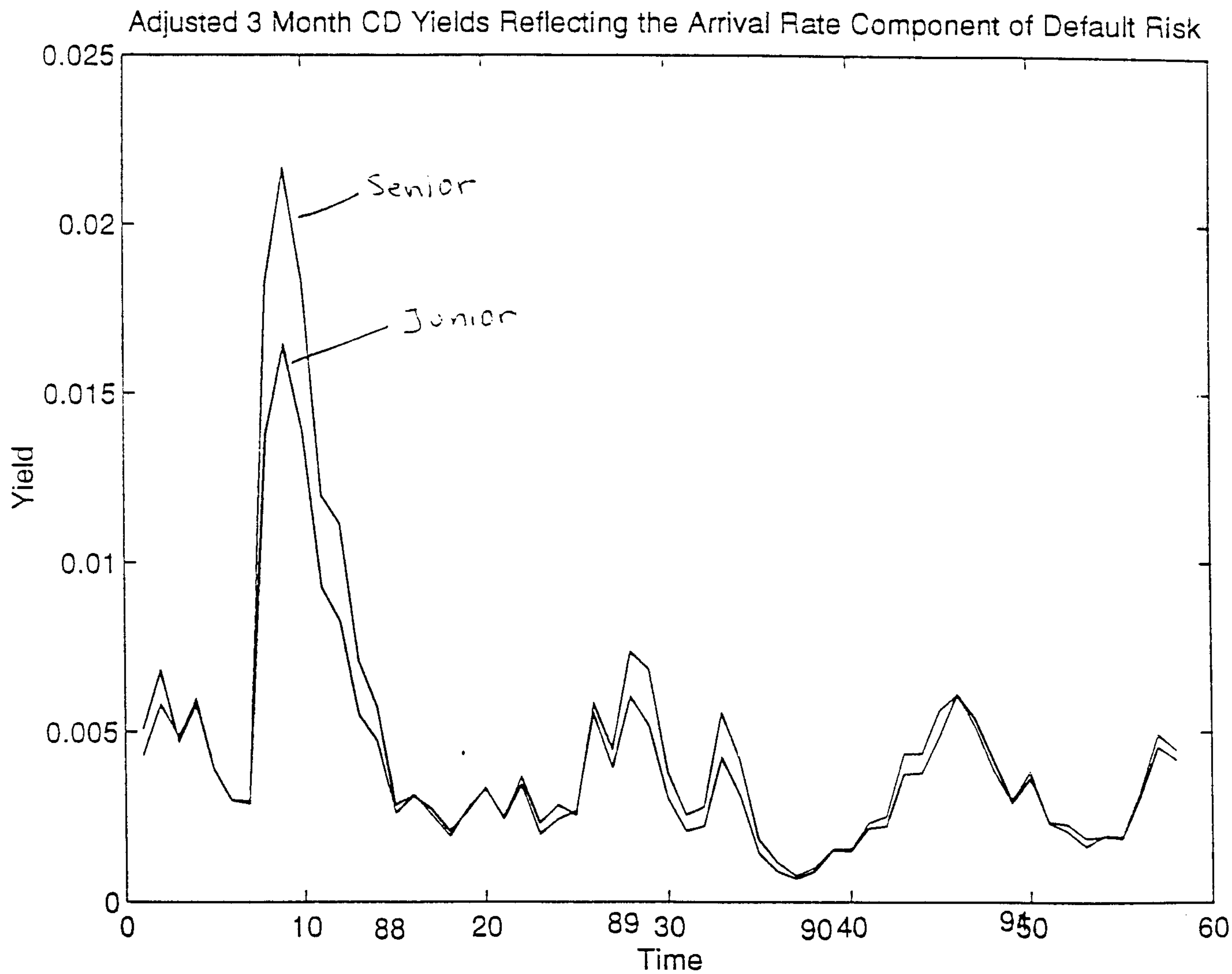


Figure 6



Relative Spreads, March 87 - December 91
Relative Spreads
0.6
0.5
0.4
0.3
0.2
0.1
0
10
88
20
89
30
90
40
91
50
60
Time
1 month CD = -
2 month CD = - -
3 month CD = - .

Figure 7



Adjusted 3 Month CD Yields Reflecting the Arrival Rate Component of Default Risk
0.025
0.02
0.015
0.01
0.005
0
Yield
Senior
Junior
0
10
88
20
89
30
90
40
91
50
60
Time