ERROR BOUNDS FOR DEGENERATE CONE INCLUSION PROBLEMS

YIRAN HE* AND JIE SUN†

Abstract. Error bounds for cone inclusion problems in Banach spaces are established under conditions weaker than Robinson's constraint qualification. The results allow the cone to be more general than the origin, therefore also generalize a classical error bound result concerning equality-constrained sets in optimization.

Key words. cone inclusion problems, error bounds, Robinson's constraint qualification, tangent cones.

1. Introduction. This paper studies various error bounds for the following inclusion problem in mathematical optimization.

$$\text{Given } g \text{ and } K, \text{ find } x \text{ such that } g(x) \in K, \tag{1.1}$$

where $g : X \to \mathbb{E}$ is continuously differentiable in a neighborhood of $\bar{x} \in X$, X and $\mathbb{E}$ are Banach spaces, and K is a closed convex cone in $\mathbb{E}$. We denote M the solution set of (1.1), and assume that $\bar{x} \in M$. We will prove that under suitable conditions there exist a neighborhood U of $\bar{x}$ and a constant $c > 0$ such that

$$\operatorname{dist}(x, M) \le c \cdot e(x), \quad \text{for all } x \in U, \tag{1.2}$$

where $\operatorname{dist}(x, M) := \inf_{y \in M} \|x - y\|$ denotes the distance from x to M, and $e(x)$ is a residual function measuring the violation of the relationship $x \in M$.

In the case where $K = \{0\}$, a classic result due to Graves [6] and Lyusternik [13] says that if the derivative $g'(\bar{x})$ is surjective, then the residual function $e(x)$ in (1.2) can be taken as $\|g(x)\|$; when $g'(\bar{x})$ is not necessarily surjective and $\mathbb{E} = \mathbb{R}^m$, Izmailov and Solodov [9] proved under the so-called 2-regularity assumption that the residual function in (1.2) can be taken as

$$e(x) = \|Q(g(x))\| + \frac{\|P(g(x))\|}{\|x - \bar{x}\|},$$

where Q and P are respectively the projectors onto the range of $g'(\bar{x})$ and onto its orthogonal complement. As remarked in [9], this result covers the classic Graves-Lyusternik theorem, because if $g'(\bar{x})$ is surjective then $P \equiv 0$.

For the case where K is a closed convex cone, Robinson [22] proved that if the so-called Robinson's constraint qualification at $\bar{x}$ holds, equivalently [2], if the Minkowski sum of the range of $g'(\bar{x})$ and the radial cone $\mathcal{R}_K(g(\bar{x}))$ of K at $g(\bar{x})$ (the latter is by definition the cone generated by $K - g(\bar{x})$) is equal to the whole space $\mathbb{E}$, then error bound (1.2) holds with $e(x) = \operatorname{dist}(g(x), K)$. Robinson's result also generalizes the Graves-Lyusternik theorem because if $K = \{0\}$ then $\mathcal{R}_K(g(\bar{x})) = \{0\}$ and Robinson's constraint qualification reduces to the surjectivity of $g'(\bar{x})$.

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*Department of Decision Sciences, National University of Singapore and Department of Mathematics, Sichuan Normal University, PRC. His research is partially supported by Grants RP314000-028/042-112 of National University of Singapore. email: dschy@nus.edu.sg.

†Department of Decision Sciences and Singapore-MIT Alliance, National University of Singapore. His research is partially supported by Grants RP314000-028/042-112 of National University of Singapore and Singapore-MIT Alliance. email: jsun@nus.edu.sg.

Our goal is to extend the results of Robinson and Izmailov-Solodov to the cases where $K \neq \{0\}$ and Robinson's constraint qualification may not hold, which we call the degenerate cases. As a natural development of the error bound results, we also study certain tangency formulas that are of importance for perturbation analysis in optimization.

In an effort to extend the classical results of error bound theory but along a somewhat different direction, Ng, Yang and Zheng [16, 17, 18, 28, 29] and Wu and Ye [25, 26, 27] have made important progress recently by developing error bounds for nonsmooth problems, where the functions involved are only lower semicontinuous or locally Lipschitzian, under various constraint qualifications. For a comprehensive survey on these topics and error bound theory in general, see the eminent paper of Pang [19] and the book of Facchinei and Pang [7]. Specifically, the latter contains an extensive list of literature up to 2002 in this field.

We organize this paper as follows. After present some preliminaries in the next section, we discuss error bounds for the general case in Section 3 and for some refined cases in Section 4, respectively. Then we establish tangency formulas in Section 5, which are based on the error bound results in the previous sections.

2. Preliminaries. Here and below, when we consider a projector onto a closed linear subspace, we always assume this closed linear subspace has an orthogonal complement subspace, which, in particular, is true for all Hilbert spaces. Let G be a convex process from Banach space X to Banach space $\mathbb{E}$; i.e., G is a set-valued mapping whose graph is a convex cone. Clearly the inverse or linear transformation of a convex process is a convex process. The norm of G^{-1} is defined as

$$\|G^{-1}\| := \sup\{\text{dist}(0, G^{-1}(y)) : \|y\| = 1\}.$$

LEMMA 2.1. *Let $G : X \to \mathbb{E}$ be a convex process. If G is surjective, then $\|G^{-1}\| < \infty$.*

Proof. See Corollary to Theorem 2 of [20]. □

Let A, B be two closed sets in a Banach space X. $H(A, B)$ denotes the Hausdorff distance between A and B, that is,

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}, \text{ where } \delta(A, B) = \sup_{x \in A} \text{dist}(x, B).$$

The following lemma is an immediate consequence of [20, Theorem 6].

LEMMA 2.2. *Let $G : X \to \mathbb{E}$ be a convex process with $G(X) = \mathbb{E}$ and let S_1 and S_2 be nonempty subsets of $\mathbb{E}$. Then there hold*

$$\delta(G^{-1}(S_1), G^{-1}(S_2)) \leq \|G^{-1}\| \delta(S_1, S_2) \text{ and}$$
$$H(G^{-1}(S_1), G^{-1}(S_2)) \leq \|G^{-1}\| H(S_1, S_2).$$

The following theorem is well known as the set-valued contraction mapping principle due to Nadler [15] whose proof can be found in [8, p.31.].

THEOREM 2.3. *Let $\mathbb{B}(\bar{x}; r) = \{x : \|x - \bar{x}\| \leq r\}$ and $\mathbb{B}(\bar{x}; r) \subset X$. If $\Phi : \mathbb{B}(\bar{x}; r) \to 2^X$ is a set-valued mapping with $\Phi(x)$ being nonempty and closed and if there exists a constant $\theta \in (0, 1)$ such that*

$$H(\Phi(x_1), \Phi(x_2)) \leq \theta \|x_1 - x_2\|, \quad \text{for all } x_1, x_2 \in \mathbb{B}(\bar{x}; r),$$

and $\operatorname{dist}(\bar{x}, \Phi(\bar{x})) < (1-\theta)r$*, then there exists some* $\xi \in \mathbb{B}(\bar{x}; r)$ *such that*

$$\xi \in \Phi(\xi) \quad \textit{and} \quad \|\xi - \bar{x}\| \leq \frac{2}{1-\theta} \operatorname{dist}(\bar{x}, \Phi(\bar{x})).$$

The proofs of the following two results are similar to that of Lemmas 1 and 2 in [9].

LEMMA 2.4. *Let* $\bar{x} \in X$ *and* $g : \mathbb{B}(\bar{x}; \rho) \to \mathbb{E}$ *be continuously differentiable with its derivative* g' *being Lipschitzian on* $\mathbb{B}(\bar{x}; \rho)$*. Suppose that* $Y \subseteq \mathbb{E}$ *is a closed linear subspace containing the range of* $g'(\bar{x})$*, and* Q *and* P *are respectively the projectors onto* Y *and onto the orthogonal complement* $Y^{\perp}$ *of* Y*. If* Pg' *is directionally differentiable at* $\bar{x}$*, then for any* $\varepsilon > 0$ *there exists* $\delta \in (0, \rho)$ *such that for all* $x \in \mathbb{B}(\bar{x}, \delta)$*, there hold*

$$\|Q(g(x) - g(\bar{x})) - g'(\bar{x})(x - \bar{x})\| \leq \varepsilon \|x - \bar{x}\| \ \textit{and}$$
$$\|P(g(x) - g(\bar{x})) - \frac{1}{2}(Pg')'(\bar{x}; x - \bar{x})(x - \bar{x})\| \leq \varepsilon \|x - \bar{x}\|^2.$$

LEMMA 2.5. *Under the conditions of Lemma 2.4, for any* $\varepsilon > 0$ *there exists* $\delta \in (0, \rho)$ *and* $r \in (0, 1)$ *such that for all* $x \in \mathbb{B}(\bar{x}, \delta)$ *and* $\xi_1, \xi_2 \in \mathbb{B}(0, r\|x - \bar{x}\|)$*, there hold*

$$\|Q(g(x + \xi_1) - g(x + \xi_2)) - g'(\bar{x})(\xi_1 - \xi_2)\| \leq \varepsilon \|\xi_1 - \xi_2\| \ \textit{and}$$
$$\|x - \bar{x}\|^{-1} \|P(g(x + \xi_1) - g(x + \xi_2)) - (Pg')'(\bar{x}; x - \bar{x})(\xi_1 - \xi_2)\| \leq \varepsilon \|\xi_1 - \xi_2\|.$$

DEFINITION 2.6. *We say that a closed convex set* K *is* localizable *at* $x^* \in K$ *if there exists a neighborhood* V *of the original point such that*

$$\operatorname{dist}(x, T_K(x^*)) = \operatorname{dist}(x, K - x^*) \quad \forall\, x \in V,$$

where $T_K(x^*)$ *denotes the tangent cone of* K *at* $x^* \in K$*.*

DEFINITION 2.7. *[2, Definition 2.195] A convex set* K *is said to be* generalized polyhedral *if there exist a continuous linear mapping* $A : X \to \mathbb{E}$ *with closed range,* $\rho \in \mathbb{E}$*, and* $a_i \in X^*$*,* $b_i \in \mathbb{R}$ *for* $i = 1, 2, \ldots, m$*, such that*

$$K = \{y \in X : Ay = \rho, \ \textit{and} \ \langle a_i, y \rangle \leq b_i, i = 1, 2, \ldots, m\}.$$

LEMMA 2.8. *A generalized polyhedral set* K *is localizable at every* $x^* \in K$*.*

Proof. Let K be a nonempty generalized polyhedral set and $x^* \in K$. Then K has the representation of

$$\{x : Ax = \rho, \ \text{and} \ \langle a_i, x \rangle \leq b_i, i = 1, 2, \ldots, m\},$$

where A is a linear continuous operator with closed range. Without loss of generality, we assume that each a_i is of norm one. For $x^* \in K$, we let I denote its active indices, and J the inactive indices:

$$I := \{i : \langle a_i, x^* \rangle = b_i\}, \quad \text{and } J := \{i : \langle a_i, x^* \rangle < b_i\}.$$

Take $\delta = \min\{b_i - \langle a_i, x^*\rangle : i \in J\}$. Then $\delta > 0$. (Here we adopt the convention that if J is empty, then δ takes $+\infty$.) Noting that as a consequence of Hoffman's lemma [2, Theorem 2.200],

$$T_K(x^*) = \{x : Ax = 0, \text{ and } \langle a_i, x\rangle \le 0, \forall\, i \in I\}$$

We have for each $x \in T_K(x^*)$ with $\|x\| \le \delta$,

$$\begin{aligned}
&\langle a_i, x + x^*\rangle \le b_i, \quad \forall\, i \in I;\\
&\langle a_i, x + x^*\rangle \le \langle a_i, x\rangle + b_i - \delta \le \|x\| + b_i - \delta \le b_i, \quad \forall\, i \in J.
\end{aligned}$$

This shows that $x + x^* \in K$, and hence $T_K(x^*) \cap \delta\mathbb{B} = (K - x^*) \cap \delta\mathbb{B}$.

It remains to prove that $\mathrm{dist}(x, T_K(x^*)) = \mathrm{dist}(x, K - x^*)$. Let u be the projection of x onto $T_K(x^*)$. Then

$$u \in T_K(x^*), x - u \in N_K(x^*), \text{ and } \langle x - u, u\rangle = 0.$$

In view of the last equality, $\|x\| \le \delta$ implies that $\|u\| \le \delta$. Thereby $u \in T_K(x^*) \cap \delta\mathbb{B}$, and hence it follows that $u \in K - x^*$. Since $K - x^* \subset T_K(x^*)$, we have therefore $\mathrm{dist}(x, T_K(x^*)) = \mathrm{dist}(x, K - x^*)$. □

3. The general case. Let $X, \mathbb{E}$ be two Banach spaces, $K \subset \mathbb{E}$ be a closed convex cone, and let $g : X \to \mathbb{E}$ be continuously differentiable with g' being locally Lipschitzian. $\bar{x} \in X$, and Y is the closed linear subspace spanned by $g'(\bar{x})X - \mathcal{R}_K(g(\bar{x}))$, Q and P are respectively the projectors in $\mathbb{E}$ onto Y and $Y^\perp$.

We will often use in the following the relationship that $K \subset \mathcal{R}_K(g(\bar{x}))$, which is true as long as K is a closed convex cone.

Given $h \in X$, let

$$G(h)z = g'(\bar{x})z + (Pg')'(\bar{x}; h)z, \tag{3.1}$$

$$G(h, K)z = G(h)z - K. \tag{3.2}$$

LEMMA 3.1. *Given $x \in X$, let $\Phi_x(\xi) = G(x - \bar{x}, K)^{-1}(G(x - \bar{x})\xi - g(x + \xi))$. Then*

$$\Phi_x(\xi) = G\left(\frac{x - \bar{x}}{\|x - \bar{x}\|}, K\right)^{-1}\left[G\left(\frac{x - \bar{x}}{\|x - \bar{x}\|}\right)\xi - Q(g(x + \xi)) - \frac{P(g(x + \xi))}{\|x - \bar{x}\|}\right] \tag{3.3}$$

Proof. Let $\eta \in \Phi_x(\xi)$. Then

$$G(x - \bar{x})\xi - g(x + \xi) \in G(x - \bar{x}, K)\eta = G(x - \bar{x})\eta - K,$$

that is, $G(x - \bar{x})(\eta - \xi) + g(x + \xi) \in K$. Since $g'(\bar{x})X \subset Y$ and $K \subset Y$, it follows that

$$g'(\bar{x})(\eta - \xi) + Q(g(x + \xi)) \in K$$

and

$$(Pg')'(\bar{x}; x - \bar{x})(\eta - \xi) + P(g(x + \xi)) = 0.$$

The last expression implies that

$$(Pg')'\left(\bar{x}; \frac{x - \bar{x}}{\|x - \bar{x}\|}\right)(\eta - \xi) + \frac{P(g(x + \xi))}{\|x - \bar{x}\|} = 0$$

This shows that

$$G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)(\eta-\xi)+Q(g(x+\xi))+\frac{P(g(x+\xi))}{\|x-\bar{x}\|}\in K,$$

and hence

$$G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)\xi-Q(g(x+\xi))-\frac{P(g(x+\xi))}{\|x-\bar{x}\|}\in G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)\eta-K,$$

this shows that η belongs to the right hand side of (3.3). Thus, $\Phi_x(\xi)\subseteq$ the right hand side of (3.3). Clearly, the above argument can be inverted to get the reverse inclusion relationship. □

Lemma 3.2.

$$\begin{aligned}\Phi_x(0)&=G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|},K\right)^{-1}\left[Q(-g(x))+\frac{P(-g(x))}{\|x-\bar{x}\|}\right]\\&=G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|},K\right)^{-1}\left[Q(-g(x))+\frac{P(-g(x))}{\|x-\bar{x}\|}+K\right]\end{aligned}$$

Proof. In view of Lemma 3.1, it suffices to prove that the third set is contained in the second one. Let η belong to the third one. Then there exists

$$\xi\in Q(-g(x))+\frac{P(-g(x))}{\|x-\bar{x}\|}+K$$

such that $\xi\in G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)\eta-K$. Since K is a closed convex cone, it follows that

$$G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)\eta\in\xi+K\subset Q(-g(x))+\frac{P(-g(x))}{\|x-\bar{x}\|}+K,$$

which implies that η belongs to the second set. □

Theorem 3.3. *Let X and $\mathbb{E}$ be two Banach spaces, and K be a closed convex cone in $\mathbb{E}$. Let $\bar{x}\in X$, $\rho>0$ and $g:\mathbb{B}(\bar{x};\rho)\to\mathbb{E}$ be continuously differentiable with its derivative g' being Lipschitzian on $\mathbb{B}(\bar{x};\rho)$. Suppose that Y is the closed subspace spanned by $g'(\bar{x})X-\mathcal{R}_K(g(\bar{x}))$ and P and Q are respectively the projectors in $\mathbb{E}$ onto $Y^{\perp}$ and onto Y. If K is localizable at $g(\bar{x})$, Pg' is directionally differentiable at $\bar{x}$ and if there exists $\nu>0$ such that*

$$C:=\sup\{\|G(h,K)^{-1}\|:h\in\mathbb{T}_\nu\ \text{with}\ \|h\|=1\}<\infty,\tag{3.4}$$

where $\mathbb{T}_\nu=\{h\in X:\operatorname{dist}(g'(\bar{x})h,T_K(g(\bar{x})))\le\nu,\|(Pg')'(\bar{x};h)h\|\le\nu\}$, then there exist $\delta\in(0,\rho)$ and a positive scalar c such that

$$\operatorname{dist}(x,M)\le c\left[\operatorname{dist}(Q(g(x)),K)+\frac{\|P(g(x))\|}{\|x-\bar{x}\|}\right],\quad\forall\,x\in\mathbb{B}(\bar{x};\delta).$$

Proof. Let $\theta\in(0,1)$. In view of Lemma 2.5, there exist $\delta_1\in(0,\rho/2)$ and $r\in(0,1)$ such that for all $x\in\mathbb{B}(\bar{x},\delta_1)$ and for all $\xi_1,\xi_2\in\mathbb{B}(0,r\|x-\bar{x}\|)$,

$$\|Q(g(x+\xi_1)-g(x+\xi_2))-g'(\bar{x})(\xi_1-\xi_2)\|\le\frac{\theta}{2C}\|\xi_1-\xi_2\|,\tag{3.5}$$

$$\|x-\bar{x}\|^{-1}\|P(g(x+\xi_1)-g(x+\xi_2))-(Pg')'(\bar{x};x-\bar{x})(\xi_1-\xi_2)\|\le\frac{\theta}{2C}\|\xi_1-\xi_2\|.\tag{3.6}$$

Take $\bar{\nu} \in (0, \min\{\nu, (1-\theta)r/3C\})$. In view of Lemma 2.4, there exists $\delta_2 \in (0, \delta_1)$ such that for all x in $\mathbb{B}(\bar{x}, \delta)$,

$$\|Q(g(x) - g(\bar{x})) - g'(\bar{x})(x - \bar{x})\| \leq \frac{\bar{\nu}}{2}\|x - \bar{x}\|, \tag{3.7}$$

$$\|P(g(x) - g(\bar{x})) - \frac{1}{2}(Pg')'(\bar{x}; x - \bar{x})(x - \bar{x})\| \leq \frac{\bar{\nu}}{4}\|x - \bar{x}\|^2. \tag{3.8}$$

Since K is localizable at $g(\bar{x})$, by Definition 2.6, there exists $\delta \in (0, \delta_2)$ such that

$$\operatorname{dist}(g'(\bar{x})(x - \bar{x}), T_K(g(\bar{x}))) = \operatorname{dist}(g'(\bar{x})(x - \bar{x}), K - g(\bar{x})), \quad \forall\, x \in \mathbb{B}(\bar{x}, \delta). \tag{3.9}$$

Let us first consider the case when $x \in \mathbb{B}(\bar{x}, \delta) \setminus \{\bar{x}\}$ and

$$\frac{x - \bar{x}}{\|x - \bar{x}\|} \in \mathbb{T}_{\bar{\nu}}. \tag{3.10}$$

Let

$$e(x) := \operatorname{dist}(Qg(x), K) + \|x - \bar{x}\|^{-1}\|P(g(x))\|,$$

and

$$r(x) := r\|x - \bar{x}\|.$$

It follows that for x belonging to $\mathbb{B}(\bar{x}; \delta)$ and satisfying (3.10),

$$\begin{aligned}
\operatorname{dist}(Qg(x), K) &\leq \|Q(g(x) - g(\bar{x})) - g'(\bar{x})(x - \bar{x})\| + \operatorname{dist}(g(\bar{x}) + g'(\bar{x})(x - \bar{x}), K) \\
&= \|Q(g(x) - g(\bar{x})) - g'(\bar{x})(x - \bar{x})\| + \operatorname{dist}(g'(\bar{x})(x - \bar{x}), K - g(\bar{x})) \\
&= \|Q(g(x) - g(\bar{x})) - g'(\bar{x})(x - \bar{x})\| + \operatorname{dist}(g'(\bar{x})(x - \bar{x}), T_K(g(\bar{x}))) \\
&= \|Q(g(x) - g(\bar{x})) - g'(\bar{x})(x - \bar{x})\| + \\
&\quad + \|x - \bar{x}\| \operatorname{dist}\left(g'(\bar{x})\left(\frac{x - \bar{x}}{\|x - \bar{x}\|}\right), T_K(g(\bar{x}))\right) \\
&\leq \frac{\bar{\nu}}{2}\|x - \bar{x}\| + \bar{\nu}\|x - \bar{x}\|,
\end{aligned}$$

where the first inequality follows from the triangular inequality, the second equality from (3.9), and the last equality is due to $T_K(g(\bar{x}))$ being a cone.

Since $P(g(\bar{x})) = 0$, we have for x belonging to $\mathbb{B}(\bar{x}; \delta)$ and satisfying (3.10)

$$\begin{aligned}
\|P(g(x))\| &\leq \|P(g(x) - g(\bar{x})) - (Pg')'(\bar{x}; x - \bar{x})(x - \bar{x})\| \\
&\quad + \|(Pg')'(\bar{x}; x - \bar{x})(x - \bar{x})\| \\
&\leq \frac{\bar{\nu}}{4}\|x - \bar{x}\|^2 + \bar{\nu}\|x - \bar{x}\|^2.
\end{aligned}$$

Therefore

$$e(x) < 3\bar{\nu}\|x - \bar{x}\| \leq \frac{(1-\theta)}{C} r(x), \quad \text{for all } x \in \mathbb{B}(\bar{x}; \delta) \setminus \{\bar{x}\}.$$

For any given $x \in \mathbb{B}(\bar{x}; \delta) \setminus \{\bar{x}\}$, define $\Phi_x : \mathbb{B}(0; r(x)) \to 2^X$ by

$$\Phi_x(\xi) = G(x - \bar{x}, K)^{-1}(G(x - \bar{x})\xi - g(x + \xi)).$$

By Lemma 2.2 and Lemma 3.1, it follows from (3.5) and (3.6) that

$$
\begin{aligned}
&H(\Phi_x(\xi_1), \Phi_x(\xi_2)) \\
&\leq \left\| G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \right\| \left\| G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)(\xi_1 - \xi_2) - Q(g(x+\xi_1) - g(x+\xi_2)) \right. \\
&\qquad\qquad\qquad\qquad \left. - \|x-\bar{x}\|^{-1} P(g(x+\xi_1) - g(x+\xi_2)) \right\| \\
&\leq \left\| G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \right\| \cdot \big\{ \|Q(g(x+\xi_1) - g(x+\xi_2)) - g'(\bar{x})(\xi_1 - \xi_2)\| \\
&\qquad + \|x-\bar{x}\|^{-1} \|P(g(x+\xi_1) - g(x+\xi_2)) - (Pg')'(\bar{x}; x-\bar{x})(\xi_1 - \xi_2)\| \big\} \\
&\leq \theta \|\xi_1 - \xi_2\|.
\end{aligned}
$$

By Lemma 3.2,

$$
\begin{aligned}
\Phi_x(0) &= G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \left(Q(-g(x)) + \frac{P(-g(x))}{\|x-\bar{x}\|} + K \right) \\
&\supset G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \big(Q(-g(x) + K) \big) \\
&\quad + G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \left(\frac{P(-g(x))}{\|x-\bar{x}\|} \right),
\end{aligned}
$$

where the inclusion relation follows from the fact that $G\big(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\big)^{-1}$ is the inverse of a convex process, so itself is a convex process. It follows from Lemma 2.2 that

$$
\begin{aligned}
&\operatorname{dist}(0, \Phi_x(0)) \\
&\leq \operatorname{dist}\left(0, G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \big(Q(-g(x)) + K\big)\right) \\
&\quad + \operatorname{dist}\left(0, G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \left(\frac{P(-g(x))}{\|x-\bar{x}\|}\right)\right) \\
&\leq \delta\left(G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1}(0), G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \big(Q(-g(x)) + K\big)\right) \\
&\quad + \delta\left(G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1}(0), G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \left(\frac{P(-g(x))}{\|x-\bar{x}\|}\right)\right) \\
&\leq \left\| G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}, K\right)^{-1} \right\| \cdot \left\{ \operatorname{dist}(Q(g(x), K)) + \frac{\|P(g(x)\|}{\|x-\bar{x}\|} \right\} \\
&\leq C \cdot e(x) \\
&< (1-\theta) r(x), \qquad \text{for all } x \in \mathbb{B}(\bar{x}; \delta) \setminus \{\bar{x}\},
\end{aligned}
$$

where the first inequality follows from the metric properties of norm.

Hence $\Phi_x(\cdot)$ satisfies the assumptions of Theorem 2.3. Applying Theorem 2.3, we obtain that for every x in $\mathbb{B}(\bar{x}; \delta) \setminus \{\bar{x}\}$ there exists a $\xi(x) \in \mathbb{B}(0, r(x))$ such that

$$
\xi(x) \in \Phi_x(\xi(x)) \quad \text{and } \|\xi(x)\| \leq \frac{2}{1-\theta} \operatorname{dist}(0, \Phi_x(0)).
$$

Therefore

$$\operatorname{dist}(x,M) \le \|\xi(x)\| \le \frac{2}{1-\theta}\operatorname{dist}(0,\Phi_x(0)) \le \frac{2C}{1-\theta}e(x).$$

It remains to consider the case when $x \in \mathbb{B}(\bar{x},\delta)\setminus\{\bar{x}\}$ and

$$\frac{x-\bar{x}}{\|x-\bar{x}\|} \notin \mathbb{T}_{\bar{\nu}}.$$

In this case, applying triangular inequality and (3.9), we have either

$$\begin{aligned}\operatorname{dist}(Q(g(x)),K) &\ge \operatorname{dist}(g(\bar{x})+g'(\bar{x})(x-\bar{x}),K) - \|Q(g(x)-g(\bar{x}))-g'(\bar{x})(x-\bar{x})\|\\ &= \operatorname{dist}(g'(\bar{x})(x-\bar{x}),T_K(g(\bar{x}))) - \|Q(g(x)-g(\bar{x}))-g'(\bar{x})(x-\bar{x})\|\\ &> \bar{\nu}\|x-\bar{x}\| - \frac{\bar{\nu}}{2}\|x-\bar{x}\| = \frac{\bar{\nu}}{2}\|x-\bar{x}\|\end{aligned}$$

or

$$\begin{aligned}\|P(g(x))\| \ge{}& \frac{1}{2}\|(Pg')'(\bar{x};x-\bar{x})(x-\bar{x})\|\\ &- \|P(g(x)-g(\bar{x})) - \frac{1}{2}(Pg')'(\bar{x};x-\bar{x})(x-\bar{x})\|\\ \ge{}& \frac{\bar{\nu}}{2}\|x-\bar{x}\|^2 - \frac{\bar{\nu}}{4}\|x-\bar{x}\|^2 = \frac{\bar{\nu}}{4}\|x-\bar{x}\|^2.\end{aligned}$$

Therefore

$$\operatorname{dist}(x,M) \le \|x-\bar{x}\| \le \frac{2}{\bar{\nu}}e(x).$$

Take $c = \max\{\frac{2C}{1-\theta},\frac{2}{\bar{\nu}}\}$, then

$$\operatorname{dist}(x,M) \le c\cdot e(x), \quad \text{for all } x \in \mathbb{B}(\bar{x};\delta).$$

This completes the proof. □

If the Banach space X is of finite dimension, we have the following result implied by Theorem 3.3.

THEOREM 3.4. *Let K be a closed convex cone in a Banach space $\mathbb{E}$. Let $\bar{x} \in \mathbb{R}^n$, $\rho > 0$ and $g : \mathbb{B}(\bar{x};\rho) \to \mathbb{E}$ be continuously differentiable with its derivative g' being Lipschitzian on $\mathbb{B}(\bar{x};\rho)$. Suppose that Y is the closed subspace spanned by $g'(\bar{x})X - \mathcal{R}_K(g(\bar{x}))$, and that P and Q are respectively the projectors in $\mathbb{E}$ onto $Y^\perp$ and onto Y. If K is localizable at $g(\bar{x})$, Pg' is directionally differentiable at $\bar{x}$ and if for every $h \in \mathbb{R}^n$ with $\|h\| = 1$,*

$$\begin{gathered} g'(\bar{x})h \in T_K(g(\bar{x})),\ \text{and}\ (Pg')'(\bar{x};h)h = 0\\ \Downarrow\\ \xi \mapsto g'(\bar{x})\xi + (Pg')'(\bar{x};h)\xi - K\ \text{is surjective,}\end{gathered} \tag{3.11}$$

then there exist $\delta > 0$ and a positive scalar c such that,

$$\operatorname{dist}(x,M) \le c\left\{\operatorname{dist}(Q(g(x)),K) + \frac{\|P(g(x))\|}{\|x-\bar{x}\|}\right\}, \quad \forall\, x \in \mathbb{B}(\bar{x};\delta).$$

Proof. It suffices to prove that the assumption (3.4) holds. If (3.4) does not hold, then there exists $h_n \in X$ with $\|h_n\| = 1$ such that $h_n \in \mathbb{T}_{1/n}$ and

$$\|G(h_n, K)^{-1}\| \geq n. \tag{3.12}$$

Without loss of generality we assume that h_n converges to $\bar{h}$. Then $\|\bar{h}\| = 1$. Since $h_n \in \mathbb{T}_{1/n}$, that is, $\text{dist}(g'(\bar{x})h_n, T_K(g(\bar{x}))) < 1/n$ and $\|(Pg')'(\bar{x}; h_n)h_n\| < 1/n$, it follows that

$$g'(\bar{x})\bar{h} \in T_K(g(\bar{x})), \text{ and } (Pg')'(\bar{x}; \bar{h})\bar{h} = 0.$$

In view of (3.11) and Lemma 2.1,

$$\|G(\bar{h}, K)^{-1}\| < \infty.$$

Since (Pg') is locally Lipschitzian around $\bar{x}$ and (Pg') is directionally differentiable at $\bar{x}$, it follows from [2, Proposition 2.49] that

$$\lim_{n\to\infty} \|(Pg')'(\bar{x}; h_n) - (Pg')'(\bar{x}; \bar{h})\| = 0.$$

Note that $G(h_n, K) = G(\bar{h}, K) + (Pg')'(\bar{x}; h_n) - (Pg')'(\bar{x}; \bar{h})$, as a consequence of [20, Theorem 5], there exists some $\beta \in (0, \infty)$ such that $\|G(h_n, K)^{-1}\| < \beta$ for sufficiently large n, which is a contradiction to (3.12). □

REMARK 3.1. *When considering error bound for the inclusion problem* (1.1), *Robinson [22] proved that if $g'(\bar{x})X - \mathcal{R}_K(g(\bar{x}))$ is equal to the whole space $\mathbb{E}$ then*

$$\text{dist}(x, M) \leq c\,\text{dist}(g(x), K), \quad \textit{for some } c > 0 \textit{ and for all } x \textit{ around } \bar{x}.$$

In the notations of Theorem 3.3, if $g'(\bar{x})X - \mathcal{R}_K(g(\bar{x})) = \mathbb{E}$, then $Y = \mathbb{E}$. Hence $P = 0$ and the assumption (3.4) *is satisfied which is due to Lemma 2.1. Thus Theorem 3.3 covers the above (nondegenerate) case.*

REMARK 3.2. *If $K = \{0\}$, then Theorem 3.3 reduces to Izmailov and Solodov [9, Theorem 4].*

4. The refined cases. Although Theorem 3.3 covers several important known results, there are still some cases beyond its scope. For example, it is possible that $g'(\bar{x})X - \mathcal{R}_K(g(\bar{x}))$ is not equal to the whole space $\mathbb{E}$, but its span is equal to $\mathbb{E}$: if $g'(\bar{x}) = 0$ and the interior of K is nonempty, then the linear subspace spanned by the set $g'(\bar{x})X - \mathcal{R}_K(g(\bar{x}))$ is $\mathbb{E}$ itself. In this case, assumption (3.4) may not be satisfied as $g'(\bar{x})X - \mathcal{R}_K(g(\bar{x})) = -\mathcal{R}_K(g(\bar{x})) \neq \mathbb{E}$ in general. Thus the error bound result in the previous section will not apply.

This section aims to refine the discussion of the previous section to cover some more delicate cases of degenerate inclusion problems. *Throughout this section, we assume that $\mathbb{E} = \mathbb{R}^m$ and $K \subset \mathbb{E}$ with K being a nonempty closed convex polyhedral cone.*

THEOREM 4.1. *Let $\bar{x} \in \mathbb{R}^n$, $\rho > 0$ and $g : \mathbb{B}(\bar{x}; \rho) \to \mathbb{R}^m$ be continuously differentiable with its derivative g' being Lipschitzian on $\mathbb{B}(\bar{x}; \rho)$. Suppose that $g'(\bar{x}) = 0$ and g' is directionally differentiable at $\bar{x}$. If for every h with $\|h\| = 1$,*

$$g'(\bar{x})h = 0 \textit{ and } (g')'(\bar{x}; h)h \in T_K(g(\bar{x})) \Longrightarrow \xi \mapsto (g')'(\bar{x}; h)\xi \textit{ is surjective,}$$

then there exist $\delta > 0$ and $c > 0$ such that

$$\text{dist}(x, M) \leq c \left\{ \frac{\text{dist}(g(x), K)}{\|x - \bar{x}\|} \right\}, \quad \forall x \in \mathbb{B}(\bar{x}; \delta).$$

In fact, we will prove a more general result stated below. To simplify notation, we will use $\Im$ to stand for the tangent cone $T_K(g(\bar{x}))$ in the following.

LEMMA 4.2. *Let $a, b \in \mathbb{R}^m$ and $\alpha, \beta \in X$. If $A : X \to \mathbb{R}^m$ is a continuous linear operator with $A(X) = \mathbb{R}^m$, then*

$$H(M(a,\alpha), M(b,\beta)) = \sup_{y^* \in K^- \cup K^+} \{\langle y^*, A(\alpha-\beta) + a - b\rangle - \|A^* y^*\|\}$$
$$\leq \inf\{\|y\| : Ay = A(\alpha-\beta) + a - b\} < \infty,$$

where $M(a,\alpha) = \alpha + A^{-1}(a + K)$.

Proof. By virtue of [3, Lemma 4],

$$\inf_{A\xi - a \in K} \|x - \xi\| = \sup\left\{\langle y^*, Ax - a\rangle : y^* \in K^-, \|A^* y^*\| \leq 1\right\}.$$

$$\begin{aligned}
\delta(M(a,\alpha), M(b,\beta)) &= \sup_{z \in M(a,\alpha)} \inf_{A(y) - A(\beta) - b \in K} \|z - y\| \\
&= \sup_{z \in M(a,\alpha)} \sup\left\{\langle y^*, Az - A(\beta) - b\rangle : y^* \in K^-, \|A^* y^*\| \leq 1\right\} \\
&= \sup_{\substack{y^* \in K^- \\ \|A^* y^*\| \leq 1}} \sup_{z \in M(a,\alpha)} \langle y^*, Az - A(\beta) - b\rangle \\
&= \sup_{\substack{y^* \in K^- \\ \|A^* y^*\| \leq 1}} \sup_{A(z) - A(\alpha) - a \in K} \langle y^*, Az - A(\beta) - b\rangle \\
&= \sup_{\substack{y^* \in K^- \\ \|A^* y^*\| \leq 1}} \left\{\sup_{x \in K} \langle y^*, x\rangle + \langle y^*, A(\alpha) + a - A(\beta) - b\rangle\right\} \\
&= \sup_{\substack{y^* \in K^- \\ \|A^* y^*\| \leq 1}} \{\langle y^*, A(\alpha) + a - A(\beta) - b\rangle\}.
\end{aligned}$$

Similarly,

$$\delta(M(b,\beta), M(a,\alpha)) = \sup_{\substack{y^* \in K^- \\ \|A^* y^*\| \leq 1}} \{\langle y^*, A(\beta) + b - A(\alpha) - a\rangle\}.$$

Thus we have verified the first part. For the second part,

$$\begin{aligned}
&\sup_{\substack{y^* \in K^- \cup K^+ \\ \|A^* y^*\| \leq 1}} \{\langle y^*, A(\alpha-\beta) + a - b\rangle\} \\
&\quad \leq \sup_{\|A^* y^*\| \leq 1} \{\langle y^*, A(\alpha-\beta) + a - b\rangle\} \\
&\quad = \inf\{\|y\| : Ay = A(\alpha-\beta) + a - b\} < \infty,
\end{aligned}$$

the equality is due to [23, Theorem 16.3], and the last inequality is due to A being surjective. In fact, for given $\alpha, \beta \in X$ and $a, b \in \mathbb{R}^m$, take $x_0 \in A^{-1}(a - b) \neq \emptyset$. We have $A(x_0 + \alpha - \beta) = A(\alpha - \beta) + a - b$. ∎

Let $g : X \to \mathbb{R}^m$ be continuously differentiable with g' being locally Lipschitzian, $\bar{x} \in X$, $g'(\bar{x})X = Y$ and P and Q are respectively the projectors in $\mathbb{R}^m$ onto $Y^\perp$ and onto Y. Let

$$\Phi_x(\xi) = \xi - G(x - \bar{x})^{-1}(g(x + \xi) - K).$$

We have

LEMMA 4.3. *If $P(K)+Q(K)=K$, then*

$$\Phi_x(\xi) = \xi - G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\left(Q(-g(x+\xi)+K) + \frac{P(-g(x+\xi)+K)}{\|x-\bar{x}\|}\right)$$
$$= \xi - G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\left(Q(-g(x+\xi)) + \frac{P(-g(x+\xi))}{\|x-\bar{x}\|} + K\right).$$

Proof. Let $\eta \in \Phi_x(\xi)$. Then $G(x-\bar{x})(\eta-\xi) \in -g(x+\xi)+K$, and hence

$$g'(\bar{x})(\eta-\xi) \in Q(-g(x+\xi)+K)$$

and

$$(Pg')'(\bar{x}; x-\bar{x})(\eta-\xi) \in P(-g(x+\xi)+K).$$

The last expression implies that

$$(Pg')'\left(\bar{x}; \frac{x-\bar{x}}{\|x-\bar{x}\|}\right)(\eta-\xi) \in \frac{P(-g(x+\xi))}{\|x-\bar{x}\|} + P(K).$$

Therefore

$$G\left(\bar{x}; \frac{x-\bar{x}}{\|x-\bar{x}\|}\right)(\eta-\xi) \in Q(-g(x+\xi)+K) + \frac{P(-g(x+\xi)+K)}{\|x-\bar{x}\|}.$$

Since $Q(K)+P(K)=K$, the above deduction can be inverted. Thus the proof is complete. □

LEMMA 4.4. *Let Y be a subspace $\mathbb{R}^\ell$ of $\mathbb{R}^m$. If K is the Cartesian product of $\{0\}^q$ and $\mathbb{R}^p_+$ with $q+p=m$, and P and Q are respectively projectors in $\mathbb{R}^m$ onto Y and onto $Y^\perp$, then $Q(K)+P(K)=K$.*

Proof. It suffices to prove $Q(K)+P(K)\subset K$. If $K=\{0\}^q\times\mathbb{R}^p_+$ $(q+p=m)$, then for $u,v\in K$, $P(u)=(u^Y,0)$ and $Q(v)=(0,v^{Y^\perp})$, where u^Y denotes the components of u corresponding to the subspace Y. The notation $v^{Y^\perp}$ is similarly defined. Since $u\in K$, every component of u^Y is nonnegative, so does $v^{Y^\perp}$. Therefore $P(u)+Q(v)=(u^Y,v^{Y^\perp})\in\mathbb{R}^m_+$. One can also check the $\mathbb{R}^q$-components of the vector $(u^Y,v^{Y^\perp})$ is zero, that is, $P(u)+Q(v)\in K$. □

Now we prove a main result of this section.

THEOREM 4.5. *Let X be a Banach space, $\bar{x}\in X$, $\rho>0$ and $g:\mathbb{B}(\bar{x};\rho)\to\mathbb{R}^m$ be continuously differentiable with its derivative g' being Lipschitzian on $\mathbb{B}(\bar{x};\rho)$. Suppose that there exists a subspace Y of $\mathbb{R}^m$ such that*

(i) *$g'(\bar{x})X\subset Y$,*
(ii) *$P(K)+Q(K)=K$, where P and Q are respectively the projectors in $\mathbb{R}^m$ onto $Y^\perp$ and onto Y,*
(iii) *Pg' is directionally differentiable at $\bar{x}$,*
(iv) *there exists $\nu>0$ such that*

$$C := \sup\{\|G(h)^{-1}\| : h\in T_\nu \text{ with } \|h\|=1\} < \infty, \tag{4.1}$$

where $T_\nu=\{h\in X : \operatorname{dist}(g'(\bar{x})h, Q(\Im))\le\nu, \operatorname{dist}((Pg')'(\bar{x};h)h, P(\Im))\le\nu\}$.

Then there exist $\delta > 0$ and a positive scalar c such that

$$\operatorname{dist}(x, M) \leq c\left\{ \operatorname{dist}(Q(g(x) - g(\bar{x})), Q(K)) + \frac{\operatorname{dist}(P(g(x) - g(\bar{x})), P(K))}{\|x - \bar{x}\|} \right\}, \quad \forall\, x \in \mathbb{B}(\bar{x}; \delta).$$

Proof. Let $\theta \in (0, 1)$. In view of Lemma 2.5, there exist $\delta_1 \in (0, \rho/2)$ and $r \in (0, 1)$ such that for all $x \in \mathbb{B}(\bar{x}, \delta_1)$ and for all ξ_1, ξ_2 in $\mathbb{B}(0, r\|x - \bar{x}\|)$,

$$\|Q(g(x + \xi_1) - g(x + \xi_2)) - g'(\bar{x})(\xi_1 - \xi_2)\| \leq \frac{\theta}{2C}\|\xi_1 - \xi_2\|, \tag{4.2}$$

$$\frac{\|P(g(x + \xi_1) - g(x + \xi_2)) - (Pg')'(\bar{x}; x - \bar{x})(\xi_1 - \xi_2)\|}{\|x - \bar{x}\|} \leq \frac{\theta}{2C}\|\xi_1 - \xi_2\|. \tag{4.3}$$

Take $\bar{\nu} \in (0, \min\{\nu, (1 - \theta)r/3C\})$. In view of Lemma 2.4, there exists $\delta_2 \in (0, \delta_1)$ such that for all $x \in \mathbb{B}(\bar{x}; \delta)$,

$$\|Q(g(x) - g(\bar{x})) - g'(\bar{x})(x - \bar{x})\| \leq \frac{\bar{\nu}}{2}\|x - \bar{x}\|,$$

$$\|P(g(x) - g(\bar{x})) - \frac{1}{2}(Pg')'(\bar{x}; x - \bar{x})(x - \bar{x})\| \leq \frac{\bar{\nu}}{4}\|x - \bar{x}\|^2.$$

Since K is polyhedral cone, and hence $Q(K)$ and $P(K)$ are also polyhedral cone, it is easy to check that $P(\Im) = T_{P(K)}(Pg(\bar{x}))$ and $Q(\Im) = T_{Q(K)}(Qg(\bar{x}))$. In view of Lemma 2.8, $Q(K)$ and $P(K)$ are localizable at $g(\bar{x})$, thereby there exists $\delta \in (0, \delta_2)$ such that

$$\operatorname{dist}(g'(\bar{x})(x - \bar{x}), Q(\Im)) = \operatorname{dist}(g'(\bar{x})(x - \bar{x}), Q(K - g(\bar{x}))), \quad \forall\, x \in \mathbb{B}(\bar{x}, \delta) \tag{4.4}$$

$$\operatorname{dist}(\frac{1}{2}(Pg')'(\bar{x}; x - \bar{x})(x - \bar{x}), P(\Im)) = \operatorname{dist}(\frac{1}{2}(Pg')'(\bar{x}; x - \bar{x})(x - \bar{x}), P(K - g(\bar{x}))), \quad \forall\, x \in \mathbb{B}(\bar{x}, \delta); \tag{4.5}$$

where the second expression makes use of the continuity of $(Pg')'(\bar{x}; \xi)(\xi)$ with respect to ξ.

Let us first consider the case when

$$\frac{x - \bar{x}}{\|x - \bar{x}\|} \in T_{\bar{\nu}}. \tag{4.6}$$

Let

$$e(x) := \operatorname{dist}(Qg(x), Q(K)) + \|x - \bar{x}\|^{-1} \operatorname{dist}\left(P(g(x)), P(K)\right),$$

and

$$r(x) := r\|x - \bar{x}\|.$$

Then for x belonging to $\mathbb{B}(\bar{x};\delta)$ and satisfying (4.6),

$$\begin{aligned}
&\operatorname{dist}(Qg(x), Q(K))\\
&\le \|Q(g(x)-g(\bar{x}))-g'(\bar{x})(x-\bar{x})\| + \operatorname{dist}(Q(g(\bar{x}))+g'(\bar{x})(x-\bar{x}), Q(K))\\
&= \|Q(g(x)-g(\bar{x}))-g'(\bar{x})(x-\bar{x})\| + \operatorname{dist}(g'(\bar{x})(x-\bar{x}), Q(K)-Q(g(\bar{x})))\\
&= \|Q(g(x)-g(\bar{x}))-g'(\bar{x})(x-\bar{x})\| + \operatorname{dist}(g'(\bar{x})(x-\bar{x}), Q(\Im))\\
&= \|Q(g(x)-g(\bar{x}))-g'(\bar{x})(x-\bar{x})\| + \|x-\bar{x}\| \operatorname{dist}\left(g'(\bar{x})\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right), Q(\Im)\right)\\
&\le \frac{\bar{\nu}}{2}\|x-\bar{x}\| + \bar{\nu}\|x-\bar{x}\|,
\end{aligned}$$

where the second equality follows from (4.4); and

$$\begin{aligned}
\operatorname{dist}(P(g(x)), P(K)) &\le \|P(g(x)-g(\bar{x})) - (Pg')'(\bar{x}; x-\bar{x})(x-\bar{x})\|\\
&\quad + \operatorname{dist}((Pg')'(\bar{x}; x-\bar{x})(x-\bar{x}), P(K)-Pg(\bar{x}))\\
&= \|P(g(x)-g(\bar{x})) - (Pg')'(\bar{x}; x-\bar{x})(x-\bar{x})\|\\
&\quad + \operatorname{dist}((Pg')'(\bar{x}; x-\bar{x})(x-\bar{x}), P(\Im))\\
&\le \frac{\bar{\nu}}{4}\|x-\bar{x}\|^2 + \bar{\nu}\|x-\bar{x}\|^2,
\end{aligned}$$

where the equality follows from (4.5). It follows that

$$e(x) < 3\bar{\nu}\|x-\bar{x}\| \le \frac{(1-\theta)}{C} r(x), \quad \text{for all } x \in \mathbb{B}(\bar{x};\delta)\setminus\{\bar{x}\}. \tag{4.7}$$

Define $\Phi_x : \mathbb{B}(0; r(x)) \to 2^X$ by

$$\Phi_x(\xi) = \xi - G(x-\bar{x})^{-1}(g(x+\xi)-K).$$

By Lemma 4.3 and Lemma 4.2, it follows that

$$\begin{aligned}
&H(\Phi_x(\xi_1), \Phi_x(\xi_2))\\
&\quad\le \inf\left\{\|y\| : G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right) y = G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)(\xi_1-\xi_2)\right.\\
&\quad\quad \left. - (Q)(g(x+\xi_1)-g(x+\xi_2)) - \frac{P(g(x+\xi_1)-g(x+\xi_2))}{\|x-\bar{x}\|}\right\}\\
&\quad\le \left\|G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\right\| \cdot \big\{ \|Q(g(x+\xi_1)-g(x+\xi_2)) - g'(\bar{x})(\xi_1-\xi_2)\|\\
&\quad\quad + \|x-\bar{x}\|^{-1}\|P(g(x+\xi_1)-g(x+\xi_2)) - (Pg')'(\bar{x}; x-\bar{x})(\xi_1-\xi_2)\|\big\}\\
&\quad\le \theta\|\xi_1-\xi_2\|,
\end{aligned}$$

where the second inequality follows from the definition of right inverse of linear operator, and the last one from (4.1), (4.2) and (4.3). By Lemma 4.3,

$$\begin{aligned}
\Phi_x(0) &= G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\left(Q(-g(x)+K) + \frac{P(-g(x)+K)}{\|x-\bar{x}\|}\right)\\
&\supset G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\Big(Q(-g(x)+K)\Big)\\
&\quad + G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\left(\frac{P(-g(x)+K)}{\|x-\bar{x}\|}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\operatorname{dist}(0,\Phi_x(0))\\
&\le \operatorname{dist}\left(0, G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\big(Q(-g(x)+K)\big)\right)\\
&\qquad + \operatorname{dist}\left(0, G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\left(\frac{P(-g(x)+K)}{\|x-\bar{x}\|}\right)\right)\\
&\le \delta\left(G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}(0), G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\big(Q(-g(x)+K)\big)\right)\\
&\qquad + \delta\left(G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}(0), G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\left(\frac{P(-g(x)+K)}{\|x-\bar{x}\|}\right)\right)\\
&\le \left\|G\left(\frac{x-\bar{x}}{\|x-\bar{x}\|}\right)^{-1}\right\|\cdot\{\operatorname{dist}(Q(g(x)),Q(K)) + \|x-\bar{x}\|^{-1}\operatorname{dist}(P(g(x)),P(K))\}\\
&\le C\cdot e(x)\\
&< (1-\theta)r(x), \qquad \text{for all } x\in\mathbb{B}(\bar{x};\delta)\setminus\{\bar{x}\},
\end{aligned}$$

where the third inequality follows from Lemma 2.2, and the last one follows from (4.7). Hence $\Phi_x(\cdot)$ satisfies the assumptions of Theorem 2.3. Applying Theorem 2.3, we obtain that for every $x\in\mathbb{B}(\bar{x};\delta)\setminus\{\bar{x}\}$ there exists a $\xi(x)\in\mathbb{B}(0,r(x))$ such that

$$\xi(x)\in\Phi_x(\xi(x)) \quad \text{and } \|\xi(x)\|\le\frac{2}{1-\theta}\operatorname{dist}(0,\Phi_x(0)).$$

Therefore

$$\begin{aligned}
\operatorname{dist}(x,M) &\le \|\xi(x)\|\\
&\le \frac{2C}{1-\theta}\left[\operatorname{dist}(Q(g(x)),Q(K)) + \frac{\operatorname{dist}(P(g(x)),P(K))}{\|x-\bar{x}\|}\right]\\
&= \frac{2C}{1-\theta}e(x).
\end{aligned}$$

It remains to consider the case when

$$\frac{x-\bar{x}}{\|x-\bar{x}\|}\notin T_{\bar{\nu}}.$$

In this case, we have either

$$\begin{aligned}
\operatorname{dist}(Q(g(x)),Q(K)) &\ge \operatorname{dist}(Qg(\bar{x})+g'(\bar{x})(x-\bar{x}),Q(K))\\
&\quad - \|g'(\bar{x})(x-\bar{x})-Q(g(x)-g(\bar{x}))\|\\
&= \operatorname{dist}(g'(\bar{x})(x-\bar{x}),Q(\Im)) - \|g'(\bar{x})(x-\bar{x})-Q(g(x)-g(\bar{x}))\|\\
&> \bar{\nu}\|x-\bar{x}\| - \frac{\bar{\nu}}{2}\|x-\bar{x}\| = \frac{\bar{\nu}}{2}\|x-\bar{x}\|
\end{aligned}$$

or

$$
\begin{aligned}
\operatorname{dist}(P(g(x)),P(K)) &\geq \operatorname{dist}\left(\frac{1}{2}(Pg')'(\bar{x};x-\bar{x})(x-\bar{x}),P(K)-Pg(\bar{x})\right)\\
&\quad-\left\|\frac{1}{2}(Pg')'(\bar{x};x-\bar{x})(x-\bar{x})-P(g(x)-g(\bar{x}))\right\|\\
&=\operatorname{dist}\left(\frac{1}{2}(Pg')'(\bar{x};x-\bar{x})(x-\bar{x}),P(\Im)\right)\\
&\quad-\left\|\frac{1}{2}(Pg')'(\bar{x};x-\bar{x})(x-\bar{x})-P(g(x)-g(\bar{x}))\right\|\\
&=\frac{1}{2}\operatorname{dist}\left((Pg')'(\bar{x};x-\bar{x})(x-\bar{x}),P(\Im)\right)\\
&\quad-\left\|\frac{1}{2}(Pg')'(\bar{x};x-\bar{x})(x-\bar{x})-P(g(x)-g(\bar{x}))\right\|\\
&\geq\frac{\bar{\nu}}{2}\|x-\bar{x}\|^2-\frac{\bar{\nu}}{4}\|x-\bar{x}\|^2=\frac{\bar{\nu}}{4}\|x-\bar{x}\|^2.
\end{aligned}
$$

Therefore

$$\operatorname{dist}(x,M)\leq\|x-\bar{x}\|\leq\frac{4}{\bar{\nu}}e(x).$$

This completes the proof. □

COROLLARY 4.6. *Let $\bar{x}\in\mathbb{R}^n$ and $g:\mathbb{B}(\bar{x};\rho)\to\mathbb{R}^m$ be continuously differentiable with its derivative g' being Lipschitzian on $\mathbb{B}(\bar{x};\rho)$. Suppose that Y is a subspace of $\mathbb{R}^m$ such that*

(i) *$g'(\bar{x})X\subset Y$,*
(ii) *$P(K)+Q(K)=K$, where P and Q are respectively the projectors in $\mathbb{R}^m$ onto $Y^\perp$ and onto Y,*
(iii) *Pg' is directionally differentiable at $\bar{x}$,*
(iv) *for $h\in\mathbb{R}^n$ with $\|h\|=1$,*

$$
\begin{gathered}
g'(\bar{x})h\in Q(\Im),\ \text{ and } (Pg')'(\bar{x};h)h\in P(\Im)\\
\Downarrow\\
\xi\to g'(\bar{x})\xi+(Pg')'(\bar{x};h)\xi \text{ is surjective.}
\end{gathered}
\tag{4.8}
$$

Then there exist $\delta>0$ and a positive scalar c such that for all $x\in\mathbb{B}(\bar{x};\delta)$,

$$\operatorname{dist}(x,M)\leq c\left[\operatorname{dist}(Q(g(x)-g(\bar{x})),Q(K))+\frac{\operatorname{dist}(P(g(x)-g(\bar{x})),P(K))}{\|x-\bar{x}\|}\right].$$

Proof. It suffices to prove that (4.8) implies (4.1). If not, then there exists $h_n\in\mathbb{R}^n$ with $\|h_n\|=1$ such that $h_n\in T_{1/n}$ and $\|G(h_n)^{-1}\|\geq n$. Without loss of generality, we assume that h_n converges to $\bar{h}$. Then $\|\bar{h}\|=1$. Since $h_n\in T_{1/n}$, that is,

$$\operatorname{dist}(g'(\bar{x})h_n,Q(\Im))<1/n,\ \text{ and }\ \operatorname{dist}((Pg')'(\bar{x};h_n)h_n,P(\Im))<1/n,$$

it follows that

$$g'(\bar{x})\bar{h}\in Q(\Im),\ \text{ and }\ (Pg')'(\bar{x};\bar{h})\bar{h}\in P(\Im).$$

In view of (4.8), $\|G(\bar h)^{-1}\| < \infty$. Since Pg' is directionally differentiable and locally Lipschitzian, it follows that $\lim_{n\to\infty} \|G(h_n) - G(\bar h)\| = 0$. This together with [20, Theorem 5] implies that

$$\|G(h_n)^{-1}\| \le \frac{\|G(\bar h)^{-1}\|}{1 - \|G(\bar h)^{-1}\|\|G(\bar h) - G(h_n)\|} < \infty \text{ for sufficiently large } n,$$

which is a contradiction. ∎

REMARK 4.1. *If $K = \{0\}$, then Theorem 4.5 recovers Theorem 4 in [9].*

Now let us consider the following differentiable inequalities:

$$g_1(x) \le 0, \dots, g_m(x) \le 0. \tag{4.9}$$

Let $g(x) := (g_1(x), \dots, g_m(x))$, and M be the feasible set of (4.9) with $\bar x \in M$. Applying Lemma 4.4 and Theorem 4.5, we have the following error bound result for (4.9). Note that when $K = \mathbb{R}^m_-$,

$$T_K(g(\bar x)) = \{h \in \mathbb{R}^m : h_i \le 0, \forall\, i \in I(\bar x)\},$$

where $I(\bar x)$ denotes the active indices of g at $\bar x$.

Let ℓ be a subset of $\{1, 2, \dots, m\}$, we use $\mathbb{R}^\ell$ to denote the following linear subspace:

$$\mathbb{R}^\ell := \{x \in \mathbb{R}^m : x_i = 0, \forall\, i \notin \ell\}.$$

COROLLARY 4.7. *Let $g_i : \mathbb{B}(\bar x; \rho) \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable with its derivative g_i' being Lipschitzian on $\mathbb{B}(\bar x; \rho)$. Suppose that there exists a subspace $\mathbb{R}^\ell$ such that*

(i) *$\mathbb{R}^\ell$ contains the subspace spanned by $\{g_i'(\bar x), i = 1, \dots, m\}$,*
(ii) *for $i = 1, \dots, m$, Pg_i' is directionally differentiable at $\bar x$, where P is the projector in $\mathbb{R}^m$ onto the subspace $\mathbb{R}^{\{1,\dots,m\}\setminus\ell}$,*
(iii) *for every $h \in \mathbb{R}^n$ with $\|h\| = 1$,*

$$\left.\begin{array}{r} g_i'(\bar x)h = 0, \forall\, i \notin \ell \\ g_i'(\bar x)h \le 0, \forall\, i \in I(\bar x) \cap \ell \\ (Pg_i')'(\bar x; h)h = 0, \forall\, i \in \ell \\ (Pg_i')'(\bar x; h)h \le 0, \forall\, i \in I(\bar x) \setminus \ell \end{array}\right\} \Rightarrow \begin{array}{l} \{g_i'(\bar x) + (Pg_i')'(\bar x; h), i = 1, \dots, m\} \\ \textit{are linearly independent.} \end{array} \tag{4.10}$$

Then there exist $\delta > 0$ and a positive scalar c such that for all $x \in \mathbb{B}(\bar x; \delta)$,

$$\operatorname{dist}(x, M) \le c\left[\max_{i\in\ell} g_i(x)_+ + \frac{\max_{i\notin\ell} g_i(x)_+}{\|x - \bar x\|}\right].$$

As a consequence of Theorem 4.1, we have

COROLLARY 4.8. *Let $g_i : \mathbb{B}(\bar x; \rho) \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable with its derivative g_i' being Lipschitzian on $\mathbb{B}(\bar x; \rho)$. Suppose that $g_i'(\bar x) = 0$ and g_i' is directionally differentiable at $\bar x$ for $i = 1, \dots, m$. If for every $h \in \mathbb{R}^n$ with $\|h\| = 1$,*

$$\begin{array}{c} (g_i')'(\bar x; h)h \le 0 \textit{ for all } i \in I(\bar x) \\ \Downarrow \\ \{(g_i')'(\bar x; h) : i = 1, \dots, m\} \textit{ are linearly independent,} \end{array} \tag{4.11}$$

then there exist $\delta > 0$ and a positive scalar c such that

$$\operatorname{dist}(x, M) \le c \frac{\max_{1\le i\le m} g_i(x)_+}{\|x-\bar{x}\|}, \quad \forall\, x \in \mathbb{B}(\bar{x};\delta);$$

in particular,

$$\operatorname{dist}(x, M) \le c \left(\max_{1\le i\le m} g_i(x)_+ \right)^{1/2}, \quad \forall\, x \in \mathbb{B}(\bar{x};\delta).$$

Corollary 4.7 establishes error bound for differentiable inequalities. Error bounds for convex inequalities are well established in the literature [10, 11, 19, 21]. But the study of error bound for nonconvex inequalities and for more general nonconvex inclusion problem is far from completeness. Considering analytic inequalities, [12] proved that a local error bound is available without assuming any nondegeneracy conditions, but the exponent in the residual function is in general unknown. For differentiable inequalities, [22] proved the existence of local error bounds under Robinson's constraint qualification in [2]. Without Robinson's constraint qualification, our Corollaries 4.7 and 4.8 show that local error bounds with an exponent at least $1/2$ exist for differentiable inequalities.

The assumptions in Corollary 4.8 cover some cases where Robinson's constraint qualification may not hold. In fact, for system (4.9), Robinson's constraint qualification is equivalent to the Mangasarian-Fromovitz constraint qualification ([14], [2, p.71.]):

There exists $h \in \mathbb{R}^n$ such that $g_i'(\bar{x})h < 0$, for every active index i.

On the other hand, since $g_i'(\bar{x}) = 0$ for every i, the Mangasarian-Fromovitz constraint qualification cannot be satisfied in this case. In other words, our results can apply to some degenerate inequalities that do not satisfy the classical nondegeneracy conditions.

Example 4.1. *Let $g_1(x) = x_1^2 - x_2^2$, $g_2(x) = x_2^2/2$ and $\bar{x} = (0,0)$. Consider the following inequalities:*

$$g_1(x) \le 0, g_2(x) \le 0.$$

Let M denote the feasible set of the above inequalities. Clearly $g_1'(\bar{x})$ and $g_2'(\bar{x})$ are both equal to zero. Therefore, Robinson's constraint qualification does not hold. Since $(g_1')'(\bar{x};h)h = 2h_1^2 - 2h_2^2$ and $(g_2')'(\bar{x};h)h = h_2^2$, it can be seen that $h = 0$ is the only element such that $(g_1')'(\bar{x};h)h \le 0$ and $(g_2')'(\bar{x};h)h \le 0$. Thus condition (4.11) is satisfied. Applying Corollary 4.8, we can obtain an error bound for the system $g_1(x) \le 0$ and $g_2(x) \le 0$; namely, there exists a neighborhood U of $\bar{x}$ and a constant $c > 0$ such that

$$\operatorname{dist}(x, M) \le c \cdot \max\{g_1(x), g_2(x), 0\}^{1/2}, \quad \textit{for all } x \in U.$$

An interesting special case is that the system (4.9) consists of a single inequality.

Corollary 4.9. *Let a function $g : \mathbb{B}(\bar{x};\rho) \subset \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable at $\bar{x} \in M := \{x : g(x) \le 0\}$, and let $g(\bar{x}) = 0$.*

(i) *If $\nabla g(\bar{x}) \neq 0$, then there exist $\delta > 0$ and a positive scalar c such that*

$$\operatorname{dist}(x, M) \le c \cdot \max\{g(x), 0\}, \quad \forall\, x \in \mathbb{B}(\bar{x};\delta).$$

(ii) *If* $\nabla g(\bar{x}) = 0$ *and if* $\nabla^2 g(\bar{x})$ *is nonsingular, then there exist* $\delta > 0$ *and a positive scalar* c *such that*

$$\operatorname{dist}(x, M) \le c \frac{\max\{g(x), 0\}}{\|x - \bar{x}\|}, \quad \forall\, x \in \mathbb{B}(\bar{x}; \delta);$$

in particular, $\operatorname{dist}(x, M) \le c \cdot \max\{g(x), 0\}^{1/2}, \quad \forall\, x \in \mathbb{B}(\bar{x}; \delta)$.

Proof. Because $\nabla g(\bar{x}) \neq 0$ implies that $\nabla g(\bar{x})$ is surjective, the first assertion follows from Theorem 3.3. The second assertion is an immediate consequence of Corollary 4.8. Indeed, it is not difficult to prove that the assumption that $\nabla^2 g(\bar{x})$ is nonsingular is equivalent to the condition (4.11) when $m = 1$:

$$\forall\, h \in \mathbb{R}^n \text{ with } \|h\| = 1, h'\nabla^2 g(\bar{x})h \le 0 \Longrightarrow \nabla^2 g(\bar{x})h \neq 0.$$

This completes the proof. ∎

5. Tangency formulas. Tangency formulas play important roles in variational analysis, especially in the establishment of optimality conditions (see [24]). As an application of error bounds established in the previous section, tangency formulas are established in this section.

For a nonempty closed set M, let $T_M^b(x)$ and $T_M^i(\bar{x})$ denote the *(Bouligand) contingent cone* and *intermediate cone* of M at $\bar{x} \in M$, respectively. That is, $h \in T_M^b(\bar{x})$ if and only if there exists $t_n \to 0+$ such that $\operatorname{dist}(x + t_n h, M) = o(t_n)$, and $h \in T_M^i(\bar{x})$ if and only if $\operatorname{dist}(x + th, M) = o(t)$. It can also be proved that $h \in T_M^b(\bar{x})$ if and only if there exist $t_n \to 0+$ and $x_n \in M$ such that $(x_n - \bar{x})/t_n$ converges to h. Furthermore, It holds that $T_M^i(\bar{x}) \subset T_M^b(\bar{x})$. If M is convex, then $T_M^b(x)$ and $T_M^i(\bar{x})$ coincide. One can refer to [1] and [2] for more details on tangent cones of different types.

PROPOSITION 5.1. *Let* X *and* $\mathbb{E}$ *be Banach spaces, and let* V *be an open neighborhood of* $\bar{x} \in X$, $g : X \to \mathbb{E}$ *be continuously differentiable on* V *and* K *a closed convex set in* $\mathbb{E}$. *Let* P *be the projector in* $\mathbb{E}$ *onto a closed subspace whose orthogonal complement contains the range of* $g'(\bar{x})$, $Q \equiv I - P$, $M := \{x \in X : g(x) \in K\}$ *and* $\bar{x} \in M$. *If there exist a neighborhood* $U \subset V$ *of* $\bar{x}$ *and a constant* $c > 0$ *such that*

$$\operatorname{dist}(x, M) \le c \cdot \left\{ \operatorname{dist}(Q(g(x)), Q(K)) + \frac{\operatorname{dist}(P(g(x)), P(K))}{\|x - \bar{x}\|} \right\}, \quad \forall\, x \in U,$$

then

$$\begin{aligned} T_M^b(\bar{x}) &= T_M^i(\bar{x}) \\ &= \{h : g'(\bar{x})h \in \Im \text{ and } (Pg')'(\bar{x}; h)h \in P(\Im)\}. \end{aligned}$$

Recall that $\Im := T_K(g(\bar{x}))$.

Proof. Let $h \in T_M^b(\bar{x})$. Then there exist $t_n \to 0+$ and $x_n \in M$ such that

$$\lim_{n\to\infty} \frac{x_n - \bar{x}}{t_n} = h.$$

By the mean value theorem,

$$\begin{aligned} \left\| \frac{g(x_n) - g(\bar{x} + t_n h)}{t_n} \right\| &= {t_n}^{-1} \left\| \int_0^1 g'(\bar{x} + t_n h + \lambda(x_n - \bar{x} - t_n h))(x_n - \bar{x} - t_n h) d\lambda \right\| \\ &\le \left\| \frac{x_n - \bar{x}}{t_n} - h \right\| \int_0^1 \|g'(\bar{x} + t_n h + \lambda(x_n - \bar{x} - t_n h))\| d\lambda. \end{aligned}$$

Since for every $\lambda \in [0,1]$, $\lim_{n\to\infty} g'(\bar{x} + t_n h + \lambda(x_n - \bar{x} - t_n h)) = g'(\bar{x})$ which is a constant mapping with respect to λ, it follows from the bounded convergence theorem that the right hand side of the last inequalities converges to zero. Note that

$$g'(\bar{x})h = \lim_{n\to\infty} \frac{g(\bar{x} + t_n h) - g(\bar{x})}{t_n},$$

it follows that

$$\frac{g(x_n) - g(\bar{x})}{t_n} = \frac{g(x_n) - g(\bar{x} + t_n h)}{t_n} + \frac{g(\bar{x} + t_n h) - g(\bar{x})}{t_n} \to g'(\bar{x})h.$$

This implies that $g'(\bar{x})h \in T_K(g(\bar{x}))$.

Now we prove that $(Pg')'(\bar{x}; h)h \in P(\Im)$. By the mean value theorem,

$$\begin{aligned}
Pg(x_n) - Pg(\bar{x}) &= \int_0^1 Pg'(\bar{x} + \lambda(x_n - \bar{x}))(x_n - \bar{x})d\lambda \\
&= \int_0^1 \Big(Pg'(\bar{x} + \lambda(x_n - \bar{x})) - Pg'(\bar{x}) \Big)(x_n - \bar{x})d\lambda \\
&= \int_0^1 (Pg')'(\bar{x}; \lambda(x_n - \bar{x}))(x_n - \bar{x})d\lambda + o(\|x_n - \bar{x}\|^2) \\
&= \frac{1}{2}(Pg')'(\bar{x}; x_n - \bar{x})(x_n - \bar{x}) + o(\|x_n - \bar{x}\|^2).
\end{aligned}$$

The third equality follows from the directional differentiability of Pg'. Since $g(x_n) - g(\bar{x})$ belongs to $\Im \equiv T_K(g(\bar{x}))$, dividing both sides of the above expression by t_n^2, we obtain that

$$\begin{aligned}
P(\Im) \ni \frac{Pg(x_n) - Pg(\bar{x})}{t_n^2} &= \frac{1}{2}(Pg')'\left(\bar{x}; \frac{x_n - \bar{x}}{t_n}\right)\left(\frac{x_n - \bar{x}}{t_n}\right) + t_n^{-1} o(\|x_n - \bar{x}\|^2) \\
&\to \frac{1}{2}(Pg')'(\bar{x}; h)h \quad (\text{as } n \to \infty).
\end{aligned}$$

Therefore $(Pg')'(\bar{x}; h)h \in P(\Im)$. Thus we have shown that

$$T_M^b(\bar{x}) \subset \{h : g'(\bar{x})h \in \Im \text{ and } (Pg')'(\bar{x}; h)h \in P(\Im)\}.$$

Now let h be such that $g'(\bar{x})h \in \Im$ and $(Pg')'(\bar{x}; h)h \in P(\Im)$. Since $g'(\bar{x})h \in Y$, $g'(\bar{x})h \in Q(\Im)$. Since K is convex and since P and Q are linear mappings, by virtue of [1, Table 4.3],

$$Q(\Im) \subset T_{Q(K)}(Q(g(\bar{x}))) \text{ and } P(\Im) \subset T_{P(K)}(P(g(\bar{x}))).$$

Therefore $g'(\bar{x})h \in T_{Q(K)}(Q(g(\bar{x})))$, which implies by the equivalent characterizations of tangent cones that

$$\operatorname{dist}(Q(g(\bar{x}) + tg'(\bar{x})h), Q(K)) = o(t). \tag{5.1}$$

Similarly, since $(Pg')'(\bar{x}; h)h \in P(\Im) \subset T_{P(K)}(P(g(\bar{x})))$, we have

$$\operatorname{dist}(P(g(\bar{x})) + \frac{t}{2}(Pg')'(\bar{x}; h)h, P(K)) = o(t),$$

and hence for sufficiently small $t > 0$,

$$\begin{aligned}\operatorname{dist}(P(g(\bar{x})) + \frac{t^2}{2}(Pg')'(\bar{x};h)h, P(K)) &= \operatorname{dist}(\frac{t^2}{2}(Pg')'(\bar{x};h)h, P(K) - P(g(\bar{x}))) \\ &\leq t\operatorname{dist}(\frac{t}{2}(Pg')'(\bar{x};h)h, P(K) - P(g(\bar{x}))) \\ &= t\operatorname{dist}(P(g(\bar{x})) + \frac{t}{2}(Pg')'(\bar{x};h)h, P(K)) \\ &= o(t^2), \qquad (5.2)\end{aligned}$$

where the inequality follows from the facts that $0 \in P(K) - P(g(\bar{x}))$ and hence the set $t\{P(K) - P(g(\bar{x}))\}$ is contained in the convex set $P(K) - P(g(\bar{x}))$ as $t > 0$ is small enough.

The expressions (5.1) and (5.2) together with Lemma 2.4 respectively yield that

$$\begin{aligned}\operatorname{dist}(Q(g(\bar{x}+th)), Q(K)) &\leq \|Q(g(\bar{x}+th) - g(\bar{x}) - tg'(\bar{x})h)\| \\ &\quad + \operatorname{dist}(Q(g(\bar{x}) + tg'(\bar{x})h), Q(K)) \\ &= o(t),\end{aligned}$$

and

$$\begin{aligned}\operatorname{dist}(P(g(\bar{x}+th)), P(K)) &\leq \|P(g(\bar{x}+th) - g(\bar{x})) - \frac{t^2}{2}(Pg')'(\bar{x};h)h\| \\ &\quad + \operatorname{dist}(P(g(\bar{x})) + \frac{t^2}{2}(Pg')'(\bar{x};h)h, P(K)) \\ &= o(t^2).\end{aligned}$$

In view of the assumption of error bounds, it follows that for sufficiently small $t > 0$,

$$\begin{aligned}\operatorname{dist}(\bar{x}+th, M) &\leq c\left\{\operatorname{dist}(Q(g(\bar{x}+th)), Q(K)) + \frac{\operatorname{dist}(P(g(\bar{x}+th)), P(K))}{\|th\|}\right\} \\ &= o(t).\end{aligned}$$

This shows that $h \in T_M^i(\bar{x})$.

Note that $T_M^i(\bar{x}) \subset T_M^b(\bar{x})$, hence we have $T_M^i(\bar{x}) = T_M^b(\bar{x})$ as desired. ☐

By virtue of Proposition 5.1, each error bound result corresponds a result of tangent cone calculus. To save space, we will not state these corresponding tangency formulas one by one. Instead, we would like to state a result corresponding to Corollary 4.9 which does not require a proof.

Proposition 5.2. *Let a function $g : \mathbb{B}(\bar{x};\rho) \subset \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable at $\bar{x} \in M := \{x : g(x) \leq 0\}$, and let $g(\bar{x}) = 0$.*

(i) *If $\nabla g(\bar{x}) \neq 0$, then*

$$T_M^b(\bar{x}) = T_M^i(\bar{x}) = \{h \in \mathbb{R}^n : \nabla g(\bar{x})h \leq 0\}.$$

(ii) *If $\nabla g(\bar{x}) = 0$ and if $\nabla^2 g(\bar{x})$ is nonsingular, then*

$$T_M^b(\bar{x}) = T_M^i(\bar{x}) = \{h \in \mathbb{R}^n : h'\nabla^2 g(\bar{x})h \leq 0\}.$$

Generally speaking, under the conditions of Proposition 5.2, one cannot obtain similar expressions for the Clarke tangent cone (see [4, 5] for the definition and properties of Clarke tangent cone). For example, consider $g(x,y) = x^2 - y^2$ for $(x,y) \in \mathbb{R}^2$.

Then g is continuously differentiable around the original point $(0,0)$, and $\nabla^2 g(0,0)$ is nonsingular. By Proposition 5.2 (ii), the contingent cone and the intermediate tangent cone of the set M coincide and are equal to M itself; however, it can be checked that the Clarke tangent cone of M at the origin is only a singleton $\{(0,0)\}$.

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