RANDOM ASSIGNMENT OF MULTIPLE INDIVISIBLE OBJECTS

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Abstract. We consider random assignment of multiple indivisible objects. When each agent receives one object, Bogomolnaia and Moulin (2001) show that the probabilistic serial mechanism is ordinally efficient, envy-free and weakly strategy-proof. When each agent receives more than one object, we propose a generalized probabilistic serial mechanism that is ordinally efficient and envy-free but not weakly strategy-proof. Our main result shows that, if each agent receives more than one object, there exists no mechanism that is ordinally efficient, envy-free and weakly strategy-proof. *JEL Classification Numbers*: C70, D61, D63.

Keywords: random assignment, multiple object assignment, probabilistic serial mechanism, ordinal efficiency, envy-freeness, strategy-proofness.

1. Introduction

In an assignment problem, a number of indivisible objects that are collectively owned need to be assigned to a set of agents. Housing allocation in universities and student placement in public schools are important examples of assignment problems in real life.[1] The goal of the mechanism designer is to assign the objects in an efficient and fair way,

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[1]See Abdulkadiroğlu and Sönmez (1999) and Chen and Sönmez (2002) for application to house allocation, and Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003b) for student placement. For the latter application, Abdulkadiroğlu, Pathak, and Roth (2005) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) discuss practical considerations in designing student placement mechanisms in New York City and Boston.

while eliciting the true preferences of the agents. The mechanism may need to satisfy other constraints as well. For example, considerations about fairness preclude monetary transfers and random assignments are called for in many applications. Also, the assignment usually has to be based on agents' reports of ordinal preferences over objects rather than full cardinal preferences, as elicitation of cardinal preferences may be difficult.[2]

Popular assignment mechanisms such as the random priority (random serial dictatorship) often result in inefficiency from the ex ante point of view. Motivated by this observation, Bogomolnaia and Moulin (2001, BM henceforth) define a random assignment to be ordinally efficient if it is not first-order stochastically dominated for all agents by any other random assignment. Ordinal efficiency is probably the most compelling efficiency concept in the context of assignment mechanisms based solely on ordinal preferences.

In a random assignment problem in which each agent receives one object, BM further propose the probabilistic serial mechanism. The random assignment prescribed by the probabilistic serial mechanism is ordinally efficient. It is also envy-free, that is, every agent prefers her random assignment to the one of any other agent. While the probabilistic serial mechanism is not strategy-proof, BM show that it is weakly strategy-proof, that is, no agent who reports false preferences can obtain a random assignment that stochastically dominates the one she receives if she reports her true preferences. Ordinal efficiency and envy-freeness, together with weak strategy-proofness, make the probabilistic serial mechanism stand out as a promising assignment mechanism.

This paper considers random assignment of multiple objects, in which each agent receives more than one object and preferences of agents are additively separable across objects. Draft in professional sports such as baseball and basketball is a typical example, in which clubs take turns to select several players. We propose a natural generalization of the probabilistic serial mechanism for this setting. The generalized probabilistic serial mechanism is ordinally efficient and envy-free in this environment, but it is not weakly strategy-proof any more. Our main result shows that the difficulty we face in assignment

[2]The pseudo-market mechanism proposed by Hylland and Zeckhauser (1979) is one of the few solutions to the random assignment problem existent in the literature in which agents report their cardinal preferences over objects.

of multiple objects is not restricted to the probabilistic serial mechanism and is fundamental to the environment. More specifically, we show that, if there are at least two agents and each agent receives more than one object, then there exists no mechanism that is ordinally efficient, envy-free and weakly strategy-proof.[3]

Relation to the Literature

Our paper offers a new perspective to the literature on assignment of multiple objects, whose focus has mainly been on deterministic, rather than random, assignment mechanisms. Allowing for general preferences over sets of objects, Papai (2001) shows that sequential dictatorships are the only deterministic mechanisms that are nonbossy, strategy-proof and Pareto optimal. Ehlers and Klaus (2003) assume separable preferences across objects and show that the sequential dictatorships are the only deterministic mechanisms that are coalition-proof and Pareto optimal. Our impossibility result demonstrates that assignment of multiple objects is difficult even if one allows for random assignment.[4]

This paper also contributes to incentive issues in the random assignment problem. Although the probabilistic serial mechanism satisfies the three conditions in the original setting of BM, these conditions are incompatible when preferences may not be strict (Katta and Sethuraman 2006) or some agents have initial property rights on objects (Yilmaz 2006). By contrast, Kojima and Manea (2006) show that reporting true preferences is even a dominant strategy in a sufficiently large market with no initial property rights and strict preferences.

More broadly, the paper is part of the growing literature on ordinal efficiency. Abdulkadiroğlu and Sönmez (2003a) give a characterization of ordinal efficiency based on the idea of dominated sets of assignments. McLennan (2002) proves that any ordinally efficient random assignment with respect to some ordinal preferences is welfare-maximizing with respect to some expected utility function consistent with the ordinal preferences.[5] Kesten

[3]Since the impossibility result holds when preferences are additively separable, it generalizes to cases in which more general preferences are allowed.

[4]In a closely related context of multiple objects assignment problem with initial property rights, Konishi, Quint, and Wako (2001) show that there is no Pareto efficient, individually rational and strategy-proof deterministic mechanism.

[5]A short constructive proof is given by Manea (2006).

(2006) introduces the top trading cycles from equal division mechanism, and shows that it is equivalent to the probabilistic serial mechanism. On the restrictive domain of the scheduling problem, Crès and Moulin (2001) show that the probabilistic serial mechanism is group strategy-proof and stochastically dominates the random priority, and Bogomolnaia and Moulin (2002) give two characterizations of the probabilistic serial mechanism.

The remainder of the paper is organized as follows. Section 2 presents the model, Section 3 presents the main result, and Section 4 concludes.

2. Model

A **random assignment problem** is a quadruple $\Gamma = (N, (\succ_i)_{i \in N}, O, q)$. N represents the set of **agents**, and O represents the set of **objects**; both N and O are finite. Each agent can consume q objects, where $q \geq 2$ is an integer called the **quota**.[6] We assume $|O| = q|N|$ (for any set S, we denote by $|S|$ the cardinality of S). Each agent $i \in N$ has a **strict preference** $\succ_i$ over O. We assume that preferences over sets of objects are additively separable across objects, that is, there is a cardinal utility function $u_i : O \to \mathbb{R}$ such that utility of receiving $O' \subset O$ is $\sum_{a \in O'} u_i(a)$.[7] We write $a \succeq_i b$ if either $a \succ_i b$ or $a = b$ holds. When N and O are fixed, we write $\succ$ for $(\succ_i)_{i \in N}$ and $\succ_{N'}$ for $(\succ_i)_{i \in N'}$ where N' is a subset of N.

A **deterministic assignment** for the problem Γ is a correspondence α from N to O such that q objects are assigned to i for each $i \in N$, with $\alpha(i)$ denoting the set of objects that i receives at α. A **deterministic assignment matrix** is a matrix $X = [X_{ia}]_{i \in N, a \in O}$ with $X_{ia} \in \{0, 1\}$ for all i and a, $\sum_{a \in O} X_{ia} = q$ for all i, and $\sum_{i \in N} X_{ia} = 1$ for all a. For each deterministic assignment α, there exists a (one-to-one) **corresponding**

[6]The assumption that all agents receive the same number of objects is discussed in Conclusion.

[7]Additive separability is a very restrictive assumption on preferences. As our main result, Theorem 1, is a negative one, it generalizes to cases in which more general preferences are allowed. Perhaps the notions of ordinal efficiency, envy-freeness and weak strategy-proofness in this paper make the most sense with the interpretation that agents have preferences over sets of objects that is additively separable across objects, and preferences over lotteries satisfy the usual expected utility hypothesis with respect to the utility function u_i for each agent i. However, formally the interpretation is not necessary once we directly define these notions based on ordinal preferences over individual objects as in this paper.

deterministic assignment matrix X^α such that $X^\alpha_{ia} = 1$ if and only if $a \in \alpha(i)$. Denote by $\mathcal{A}$ the set of all deterministic assignments.

A **lottery assignment** is a probability distribution w over $\mathcal{A}$, with $w(\alpha)$ denoting the probability of assignment α. A **random assignment** is a matrix $P = [P_{ia}]_{i\in N, a\in O}$, where $P_{ia} \geq 0$ for all i and a, $\sum_{a\in O} P_{ia} = q$ for all i, and $\sum_{i\in N} P_{ia} = 1$ for all a; P_{ia} stands for the probability that agent i receives object a. For each lottery assignment w, there exists a **corresponding random assignment** P^w, with P^w_{ia} equal to the probability that agent i is assigned object a under w, i.e., $P^w_{ia} = \sum_{\alpha\in\mathcal{A}, a\in\alpha(i)} w(\alpha)$. We say that the lottery assignment w **induces** the random assignment P^w.

The following proposition is a straightforward generalization of the Birkhoff-von Neumann theorem (see Birkhoff (1946) and von Neumann (1953)), showing that the correspondence $w \to P^w$ from lottery assignments to random assignments is surjective on the set of all random assignments. See Kojima and Manea (2006) for a proof.

Proposition 1. Every random assignment can be written as a convex combination of deterministic assignment matrices, hence any random assignment is induced by a lottery assignment.[8]

By Proposition 1, for any random assignment P, there exists a lottery over deterministic assignments that induces P. Henceforth, we identify a lottery assignment with a random assignment, and use these terminologies interchangeably. A **mechanism** is a function φ from the set of all possible preference profiles to random assignments.

A random assignment $P_i = (P_{ia})_{a\in O}$ for $i \in N$ **(first-order) stochastically dominates** another random assignment P'_i for i **at** $\succ_i$, denoted $P_i sd(\succ_i) P'_i$, if

$$\sum_{b\succeq_i a} P_{ib} \geq \sum_{b\succeq_i a} P'_{ib} \text{ for all } a \in O, \tag{2.1}$$

with strict inequality for some a. A random assignment $P = (P_{ia})_{i\in N, a\in O}$ **ordinally dominates** another random assignment P' **at** $\succ$ if $P_i = P'_i$ or $P_i sd(\succ_i) P'_i$ for all $i \in N$ and $P_i sd(\succ_i) P'_i$ for some $i \in N$. A random assignment P is **ordinally efficient at** $\succ$ if it is not ordinally dominated at $\succ$ by any other random assignment. If P ordinally dominates P' at $\succ$, then every agent i prefers P_i to P'_i according to any expected utility

[8]The convex combination is not unique in general.

function consistent with $\succ_i$. A mechanism φ is ordinally efficient if random assignment $\varphi(\succ)$ is ordinally efficient at $\succ$ for any preference profile $\succ$.

A random assignment P is **envy-free at** $\succ$ if, for any $i, j \in N$, we have $P_i = P_j$ or $P_i sd(\succ_i) P_j$. In words, an assignment is envy-free if every agent weakly prefers her own random assignment to a random assignment of any other agent, irrespective of her risk attitude. A mechanism φ is envy-free if $\varphi(\succ)$ is envy-free at $\succ$ for any preference profile $\succ$.

A mechanism φ is **strategy-proof** if for any preference profile $\succ$, agent $i \in N$ and her preference $\succ'_i$, we have $\varphi_i(\succ) sd(\succ_i) \varphi_i(\succ'_i, \succ_{-i})$ or $\varphi_i(\succ) = \varphi_i(\succ'_i, \succ_{-i})$. In words, a mechanism is strategy-proof if a random assignment under truthtelling (weakly) stochastically dominates the one under any misreported preferences.

Strategy-proofness is a strong requirement, and it turns out to be violated by many mechanisms even when $q = 1$. This observation motivates the following weakening of the property. A mechanism φ is **weakly strategy-proof** if there exists no preference profile $\succ$, agent $i \in N$ and her preference $\succ'_i$ such that $\varphi_i(\succ'_i, \succ_{-i}) sd(\succ_i) \varphi_i(\succ)$. In words, a mechanism is weakly strategy-proof if an agent cannot misstate her preferences and obtain a random assignment that stochastically dominates the one corresponding to truth-telling. Clearly weak strategy-proofness is a mild requirement, and they are satisfied by various mechanisms.[9]

Now we introduce the **probabilistic serial** mechanism (PS), which is an adaptation of the mechanism proposed by BM to our setting. The idea is to regard each object as a divisible object of probability shares. Each agent "eats" the best available object with speed one at every moment of time $t \in [0, q]$ (object a is available at time t if less than one share of a has been eaten away by time t). The resulting profile of shares of objects eaten by agents by time q corresponds to a random assignment, which we call the **probabilistic serial random assignment**.

Formally, the probabilistic serial random assignment is defined as follows. For any $a \in O' \subset O$, let $N(a, O') = \{i \in N | a \succeq_i b \text{ for every } b \in O'\}$ be the set of agents whose most

[9]BM show that the probabilistic serial mechanism is weakly strategy-proof when each agent receives one object, while the random priority satisfies strategy-proofness.

preferred object in O' is a. For a preference profile $\succ$, the assignment under the probabilistic serial mechanism is defined by the following sequence of steps. Let $O^0 = O$, $t^0 = 0$, and $P^0_{ia} = 0$ for every $i \in N$ and $a \in O$. Given $O^0, t^0, [P^0_{ia}]_{i\in N, a\in O}, \ldots, O^{v-1}, t^{v-1}, [P^{v-1}_{ia}]_{i\in N, a\in O}$, for any $a \in O^{v-1}$ define

$$t^v(a) = \sup\left\{ t \in [0,q] : \sum_{i\in N} P^{v-1}_{ia} + |N(a, O^{v-1})|(t - t^{v-1}) < 1 \right\}. \tag{2.2}$$

Define

$$t^v = \min_{a\in O^{v-1}} t^v(a), \tag{2.3}$$

$$O^v = O^{v-1} \setminus \{a \in O^{v-1} | t(a) = t^v\}, \tag{2.4}$$

$$P^v_{ia} = \begin{cases} P^{v-1}_{ia} + t^v - t^{v-1}, & i \in N(a, O^{v-1}), \\ P^{v-1}_{ia}, & \text{otherwise.} \end{cases} \tag{2.5}$$

Since O is a finite set, there exists $\bar{v}$ such that $t^{\bar{v}} = q$. We define $\mathrm{PS}(\succ) := \mathrm{P}^{\bar{\mathrm{v}}}$ to be the probabilistic serial random assignment for the preference profile $\succ$.

Example 1. Let $N = \{1,2\}$, $O = \{a,b,c,d\}$ with $q = 2$. Let

$$\succ_1 : a, b, c, d, \tag{2.6}$$

$$\succ_2 : b, c, a, d, \tag{2.7}$$

where notation (2.6) is read as "according to preference $\succ_1$, agent 1 prefers a most, b second, c third, and d least," and so forth.[10] Let PS be the probabilistic serial mechanism. Following the definition of PS,

$$\mathrm{PS}(\succ_1, \succ_2) = \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \end{pmatrix} \tag{2.8}$$

is obtained.[11]

[10] We will use analogous notation throughout the paper.

[11] Given $N = \{1,2\}$, $O = \{a,b,c,d\}$ and mechanism φ, we write

$$\varphi(\succ) = \begin{pmatrix} \varphi_{1a}(\succ) & \varphi_{1b}(\succ) & \varphi_{1c}(\succ) & \varphi_{1d}(\succ) \\ \varphi_{2a}(\succ) & \varphi_{2b}(\succ) & \varphi_{2c}(\succ) & \varphi_{2d}(\succ) \end{pmatrix}.$$

Similar notation will be used elsewhere in this paper as well.

3. Result

BM consider the case in which each agent receives one object. In that environment, they show that the probabilistic serial mechanism is ordinally efficient, envy-free and weakly strategy-proof. Ordinal efficiency and envy-freeness are violated by another popular mechanism called the **random priority** (BM) or the **random serial dictatorship** (Abdulkadiroğlu and Sönmez 1998), while random priority is strategy-proof.[12]

BM's proof of ordinal efficiency and envy-freeness can be adapted easily to our setting with multiple objects. We state it as a proposition without a proof.

Proposition 2. For any problem Γ, the probabilistic serial mechanism is ordinally efficient and envy-free.

As we demonstrate below, the probabilistic serial mechanism is not weakly strategy-proof if each agent receives more than one object.

Example 2 (Probabilistic serial mechanism is not weakly strategy-proof)**.** Consider the problem described in Example 1, and now introduce a new preference relation

$$\succ_1' : b, a, c, d.$$

We obtain

$$\mathrm{PS}(\succ_1', \succ_2) = \begin{pmatrix} 1 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \end{pmatrix}. \tag{3.1}$$

Equations (2.8) and (3.1) imply $\mathrm{PS}_1(\succ_1', \succ_2)\mathrm{sd}(\succ_1)\mathrm{PS}_1(\succ_1, \succ_2)$. Therefore PS is not weakly strategy-proof.

The violation of weak strategy-proofness is not an isolated deficiency of the probabilistic serial mechanism. The following theorem shows fundamental difficulty to find a mechanism with all the above-mentioned desirable properties.

Theorem 1. Fix N, O and q. Suppose that $|N| \geq 2$, that is, there are at least two agents. Then there exists no mechanism that satisfies ordinal efficiency, envy-freeness and weak strategy-proofness.

[12] The probabilistic serial mechanism does not satisfy strategy-proofness even if $q = 1$.

Proof. First, note that we can restrict attention to a case in which $|N| = 2$ and $q = 2$, as our construction for such a case can be embedded to a problem with more than two agents and larger quotas. Let $N = \{1, 2\}$ and $O = \{a, b, c, d\}$, and suppose on the contrary that there exists a mechanism φ that satisfies ordinal efficiency, envy-freeness and weak strategy-proofness. Consider preferences given by

$$\succ_1: a, b, c, d, \tag{3.2}$$

$$\succ_1': b, a, c, d, \tag{3.3}$$

$$\succ_2: b, c, a, d, \tag{3.4}$$

$$\succ_2': b, a, c, d. \tag{3.5}$$

Step 1. Consider preference profile $(\succ_1, \succ_2')$. By assumption we have

$$\sum_{o\in O} \varphi_{1o}(\succ_1, \succ_2') = \sum_{o\in O} \varphi_{2o}(\succ_1, \succ_2') = 2. \tag{3.6}$$

Since φ is envy-free, we have

$$\sum_{o\in\{a,b,c\}} \varphi_{1o}(\succ_1, \succ_2') = \sum_{o\in\{a,b,c\}} \varphi_{2o}(\succ_1, \succ_2'), \tag{3.7}$$

$$\sum_{o\in\{a,b\}} \varphi_{1o}(\succ_1, \succ_2') = \sum_{o\in\{a,b\}} \varphi_{2o}(\succ_1, \succ_2'). \tag{3.8}$$

Since $\varphi_{1d}(\succ_1, \succ_2') + \varphi_{2d}(\succ_1, \succ_2') = \varphi_{1c}(\succ_1, \succ_2') + \varphi_{2c}(\succ_1, \succ_2') = 1$, equations (3.6)-(3.8) imply

$$\varphi_{1c}(\succ_1, \succ_2') = \varphi_{1d}(\succ_1, \succ_2') = \varphi_{2c}(\succ_1, \succ_2') = \varphi_{2d}(\succ_1, \succ_2') = 1/2. \tag{3.9}$$

In order to show $\varphi_{1a}(\succ_1, \succ_2') = 1$, suppose on the contrary that $\varphi_{1a}(\succ_1, \succ_2') < 1$. Then $\varphi_{1b}(\succ_1, \succ_2') > 0$ by equations (3.6) and (3.9). Since $\varphi_{1a}(\succ_1, \succ_2') + \varphi_{2a}(\succ_1, \succ_2') = 1$ and $\varphi_{1b}(\succ_1, \succ_2') + \varphi_{2b}(\succ_1, \succ_2') = 1$, we have $\varphi_{2a}(\succ_1, \succ_2') > 0$ and $\varphi_{2b}(\succ_1, \succ_2') < 1$. Then the random assignment $\varphi(\succ_1, \succ_2')$ is ordinally dominated by

$$\begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \end{pmatrix}, \tag{3.10}$$

contradicting ordinal efficiency of φ. Hence we have $\varphi_{1a}(\succ_1, \succ_2') = 1$. This implies that $\varphi_{1b}(\succ_1, \succ_2') = 0$, $\varphi_{2a}(\succ_1, \succ_2') = 0$ and $\varphi_{2b}(\succ_1, \succ_2') = 1$. Therefore

$$\varphi(\succ_1, \succ_2') = \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \end{pmatrix}. \tag{3.11}$$

Step 2. Consider preference profile $(\succ_1, \succ_2)$. Since φ is envy-free, by an argument similar to the first part of Step 1 we have

$$\varphi_{1d}(\succ_1, \succ_2) = \varphi_{2d}(\succ_1, \succ_2) = 1/2. \tag{3.12}$$

Since φ is ordinally efficient, by an argument similar to Step 1 we obtain

$$\varphi_{1a}(\succ_1, \succ_2) = 1, \varphi_{2a}(\succ_1, \succ_2) = 0. \tag{3.13}$$

Hence there exists $p \in [0, 1/2]$ such that

$$\varphi(\succ_1, \succ_2) = \begin{pmatrix} 1 & p & 1/2 - p & 1/2 \\ 0 & 1-p & 1/2 + p & 1/2 \end{pmatrix}. \tag{3.14}$$

If $p > 0$, then (3.11) and (3.14) imply $\varphi_2(\succ_1, \succ_2') sd(\succ_2) \varphi_2(\succ_1, \succ_2)$, that is, the random assignment for player 2 under reported preference $\succ_2'$ stochastically dominates the one under reported preference $\succ_2$ when her true preference is $\succ_2$. This is a contradiction to weak strategy-proofness of φ. Therefore we conclude $p = 0$, resulting in

$$\varphi(\succ_1, \succ_2) = \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \end{pmatrix}. \tag{3.15}$$

Step 3. Consider preference profile $(\succ_1', \succ_2)$. Since φ is envy-free,

$$\varphi_{1b}(\succ_1', \succ_2) = \varphi_{1d}(\succ_1', \succ_2) = \varphi_{2b}(\succ_1', \succ_2) = \varphi_{2d}(\succ_1', \succ_2) = 1/2. \tag{3.16}$$

Since φ is ordinally efficient,

$$\varphi_{1a}(\succ_1', \succ_2) = \varphi_{2c}(\succ_1', \succ_2) = 1, \varphi_{1c}(\succ_1', \succ_2) = \varphi_{2a}(\succ_1', \succ_2) = 0. \tag{3.17}$$

Hence

$$\varphi(\succ_1', \succ_2) = \begin{pmatrix} 1 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1 & 1/2 \end{pmatrix}. \tag{3.18}$$

Then (3.15) and (3.18) imply $\varphi_1(\succ_1', \succ_2) sd(\succ_1) \varphi_1(\succ_1, \succ_2)$, that is, the random assignment for player 1 under reported preference $\succ_1'$ stochastically dominates the one under reported preference $\succ_1$ when her true preference is $\succ_1$. This is a contradiction to weak strategy-proofness of φ, completing the proof. □

3.1. **Tightness of Theorem 1.** The following examples show that Theorem 1 is tight. More specifically, if any one of the above three conditions of the Theorem is relaxed, then there exists a mechanism that satisfies the remaining two conditions.

Example 3 (Without ordinal efficiency)**.** Consider the mechanism such that, for any problem with n agents, $\varphi_{ia}(\succ) = 1/n$ for any preference profile $\succ$, agent i and object a. Clearly, φ is envy-free and weakly strategy-proof, but not ordinally efficient.

Example 4 (Without envy-freeness)**.** Consider the deterministic serial dictatorship: agent 1 is assigned her most desirable q objects from O, and agent $i \geq 2$ is assigned her most desirable q objects from O that have not been assigned to agents $1, \dots, i-1$. Clearly, φ is ordinally efficient and weakly strategy-proof, but not envy-free.

Example 5 (Without weak strategy-proofness)**.** Consider the probabilistic serial mechanism PS. As stated in Proposition 2, PS is ordinally efficient and envy-free. As shown in Example 2, PS is not weakly strategy-proof.

3.2. **Random Priority.** Given that PS loses one of its desirable properties when generalized to multiple objects, it is interesting to consider random priority (denoted RP).[13] In the current setup, one natural extension of RP is as follows: (i) randomly order agents with equal probability, and (ii) the first agent obtains her q favorite objects, the second agent obtains her q favorite objects among the remaining objects, and so on. It is easy to see that the mechanism is strategy-proof. Furthermore the mechanism satisfies **equal treatment of equals**, that is, $\mathrm{RP_i}(\succ) = \mathrm{RP_j}(\succ)$ if $\succ_i = \succ_j$.

However, RP violates ordinal efficiency, as in the case with $q = 1$. To see this point, consider the following example, which is an adaptation of an example in BM.

[13] I am grateful to an anonymous referee for encouraging me to discuss RP and for suggesting Example 6.

Example 6 (Ordinal inefficiency of the random priority mechanism)**.** Let $N = \{1, 2\}$, $O = \{a, b, c, d\}$ with $q = 2$. Let

$$\succ_1 : a, b, c, d, \tag{3.19}$$

$$\succ_2 : b, a, d, c. \tag{3.20}$$

Following the definition of RP,

$$\mathrm{RP}(\succ_1, \succ_2) = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix} \tag{3.21}$$

is obtained. This random assignment is ordinally inefficient, since it is ordinally dominated by

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{3.22}$$

Interestingly, the ordinal inefficiency of RP seems even severer when each agent receives multiple objects than when each agent obtains a single object. Proposition 2 of BM shows that if there are only three agents and each agent receives one object, then RP is ordinally efficient. By contrast, Example 6 demonstrates that RP is ordinally inefficient even if there are only two agents.

4. Conclusion

We showed that ordinal efficiency, envy-freeness and weak strategy-proofness are incompatible in random assignment of multiple objects. The result contributes to the literatures on the assignment problem of multiple objects and the random assignment problem by presenting a new challenge in the current general setting.

This paper assumed that all agents receive the same number of objects. The PS mechanism is still ordinally efficient even when different agents are allowed to receive different numbers of objects (the proof follows Theorem 1 of BM). However, extending envy-freeness is nontrivial, since one needs to make a modeling decision on how agents with different quotas compare random assignments of one another. On the other hand, clearly our negative result, Theorem 1, is unchanged in this environment.

We also assumed that all objects are distinct. If we allow the possibility that some objects are identical to one another, Theorem 1 can be extended with suitable modification since it is a negative result.

In extending the PS mechanism, we only considered the symmetric eating speed. BM consider the entire class of simultaneous eating algorithms, where different agents may eat with different speeds. An analogous extension can be done in our environment, and such a mechanism remains ordinally efficient. Indeed, any ordinally efficient random assignment can be obtained as a result of a certain (possibly asymmetric) simultaneous eating algorithm, as in BM's case. As in BM's case, the mechanism may not be envy-free if the eating speed is asymmetric.

Since the three conditions we consider are incompatible, the probabilistic serial mechanism is perhaps a sensible solution even if each agent receives more than one object. In that sense, one could read the current paper as a constructive proposal for mechanism design for assigning multiple objects. However, the lack of weak strategy-proofness of the probabilistic serial mechanism is without doubt a problem. It may be interesting to relax some of our requirements and obtain a sensible mechanism. While Examples 3-5 study the issue in such a direction, exploring mechanisms with other desirable properties may be an interesting direction of future research.

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