MORDELL EXCEPTIONAL LOCUS FOR SUBVARIETIES OF THE ADDITIVE GROUP

DRAGOS GHIOCA

Abstract. We define the Mordell exceptional locus $Z(V)$ for affine varieties $V \subset \mathbb{G}_a^g$ with respect to the action of a product of Drinfeld modules on the coordinates of $\mathbb{G}_a^g$. We show that $Z(V)$ is a closed subset of V. We also show that there are finitely many maximal algebraic ϕ-modules whose translates lie in V. Our results are motivated by Denis-Mordell-Lang conjecture for Drinfeld modules.

1. Introduction

Faltings proved the Mordell-Lang conjecture in the following form (see [Fal94]).

Theorem 1.1 (Faltings). *Let G be an abelian variety defined over the field of complex numbers $\mathbb{C}$. Let $X \subset G$ be a closed subvariety and $\Gamma \subset G(\mathbb{C})$ a finitely generated subgroup of $G(\mathbb{C})$. Then $X(\mathbb{C}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.*

In particular, Theorem 1.1 says that an irreducible subvariety X of an abelian variety G has a Zariski dense intersection with a finitely generated subgroup of $G(\mathbb{C})$ only if X is a translate of an algebraic subgroup of G.

We define the Mordell exceptional locus of $X \subset G$ as the set (see also [Abr94])

$$Z(X) = \{x \in X \mid \exists B,\ \dim B > 0,\ B \text{ an algebraic subgroup, } x + B \subset X\}.$$

Thus, Theorem 1.1 says that for each finitely generated subgroup Γ, we have that $(X \setminus Z(X)) \cap \Gamma$ is finite. The Mordell exceptional locus of subvarieties of abelian varieties was shown to be closed (see [Kaw80], [Bog81] and [Abr94]). This last paper served as inspiration for our work.

If we try to formulate the Mordell-Lang conjecture in the context of algebraic subvarieties contained in a power of the additive group scheme $\mathbb{G}_a$, the conclusion is either false (in the characteristic 0 case, as shown by the curve $y = x^2$ which has an infinite intersection with the finitely generated subgroup $\mathbb{Z} \times \mathbb{Z}$, without being itself a translate of an algebraic subgroup of $\mathbb{G}_a^2$) or it is trivially true (in the characteristic $p > 0$ case, because every finitely generated subgroup of a power of $\mathbb{G}_a$ is finite). Denis [Den92]

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formulated a Mordell-Lang conjecture for powers of $\mathbb{G}_a$ in characteristic p in the context of Drinfeld modules. Denis replaced the *finitely generated subgroup* from the usual Mordell-Lang statement with a *finitely generated ϕ-submodule*, where ϕ is a Drinfeld module. He also strengthened the conclusion of the Mordell-Lang statement by asking that the *subgroups* whose cosets are contained in the intersection of the algebraic variety with the finitely generated ϕ-submodule be actually *ϕ-submodules*. Several cases of Denis-Mordell-Lang conjecture were proved by the author (see [Ghi05] and [Ghi06]), and by Thomas Tucker and the author (see [GT07]).

Similar with the case of abelian varieties, Denis-Mordell-Lang conjecture suggests that the intersection of a variety V with a finitely generated ϕ-module should be finite outside the Mordell exceptional locus $Z(V)$ of V (see our Definition 2.2 for $Z(V)$). In the present paper we prove that $Z(V)$ is closed (see our Theorem 2.4). In addition, we show that there exists no infinite family of algebraic ϕ-modules (see Corollary 3.9), and that for every affine variety $V \subset \mathbb{G}_a^g$ there are finitely many maximal algebraic ϕ-modules whose translates lie in V (see our Theorem 2.5). Both of these statements are further indications that Denis-Mordell-Lang conjecture should be true.

We briefly sketch the plan of our paper. In Section 2 we set the notation, describe the Denis-Mordell-Lang conjecture and then state our main results. In Section 3 we prove these main results (Theorems 2.4 and 2.5).

2. Notation and statement of our main results

All subvarieties appearing in this paper are *closed*. We define next the notion of a Drinfeld module.

Let p be a prime and let q be a power of p. Let C be a projective non-singular curve defined over $\mathbb{F}_q$. Let A be the ring of $\mathbb{F}_q$-valued functions defined on C, regular away from a fixed closed point $\infty \in C$. Let K be a finite field extension of the fraction field $\mathrm{Frac}(A)$ of A. We let K^{alg} be a fixed algebraic closure of K, and let K^{sep} be the separable closure of K inside K^{alg}.

We define the operator τ as the Frobenius on $\mathbb{F}_q$, extended so that for every $x \in K^{\mathrm{alg}}$, we have $\tau(x) = x^q$. Then for every subfield $L \subset K^{\mathrm{alg}}$, we let $L\{\tau\}$ be the ring of polynomials in τ with coefficients from L (the addition is the usual addition, while the multiplication is given by the usual composition of functions).

Following Goss [Gos96], we call a Drinfeld module of generic characteristic defined over K a morphism $\phi : A \to K\{\tau\}$ for which the coefficient of τ^0 in ϕ_a is a for every $a \in A$, and there exists $a \in A$ such that $\phi_a \neq a\tau^0$. All Drinfeld modules appearing in this paper are of generic characteristic.

For every field extension $K \subset L$, any Drinfeld module ϕ induces an action on $\mathbb{G}_a(L)$ by $a * x := \phi_a(x)$, for each $a \in A$.

Let g be a fixed positive integer. Let $\phi_1 : A \to K\{\tau\}, \dots, \phi_g : A \to K\{\tau\}$ be Drinfeld modules. From now on, we denote by ϕ the $(\phi_1, \dots, \phi_g)$-action on $\mathbb{G}_a^g$ (where each Drinfeld module ϕ_i acts on the corresponding coordinate of the affine space).

A point $x \in \mathbb{G}_a^g(K^{\text{alg}})$ is torsion for the ϕ-action if there exists $a \in A \setminus \{0\}$ such that $\phi_a(x) = 0$. We denote by $\phi[a]$ the finite set of all torsion points x killed by ϕ_a. We denote by ϕ_{tor} the set of all torsion points in $\mathbb{G}_a^g(K^{\text{alg}})$.

The subgroups of $\mathbb{G}_a^g(K^{\text{alg}})$ invariant under the action of ϕ are called *ϕ-submodules.*

Definition 2.1. *An algebraic ϕ-(sub)module of $\mathbb{G}_a^g$ is an irreducible algebraic subgroup of $\mathbb{G}_a^g$ invariant under ϕ.*

Now we can define the Mordell exceptional locus of an affine subvariety $V \subset \mathbb{G}_a^g$.

Definition 2.2. *Let $V \subset \mathbb{G}_a^g$ be an affine subvariety. We let $Z(V)$ be the set of all points $y \in V$ with the property that there exists a positive dimensional algebraic ϕ-submodule $Y \subset \mathbb{G}_a^g$ such that $(y + Y) \subset V$.*

Denis proposed in [Den92] the following conjecture.

Conjecture 2.3. *Let $V \subset \mathbb{G}_a^g$ be an affine variety defined over $\overline{K}$. Let Γ be a finitely generated ϕ-submodule of $\mathbb{G}_a^g(\overline{K})$. Then there exist algebraic ϕ-submodules $B_1, \dots, B_l$ of $\mathbb{G}_a^g$ and there exist $\gamma_1, \dots, \gamma_l \in \Gamma$ such that*

$$V(\overline{K}) \cap \Gamma = \bigcup_{i=1}^{l} (\gamma_i + B_i(\overline{K})) \cap \Gamma.$$

As explained in Introduction, the results of our paper were motivated by Conjecture 2.3. Our main result is the following.

Theorem 2.4. *With the above notation for ϕ and V, the Mordell exceptional locus $Z(V)$ is Zariski closed.*

The following important result is a consequence of Theorem 2.4.

Theorem 2.5. *Let $V \subset \mathbb{G}_a^g$ be an affine subvariety. There are finitely many maximal algebraic ϕ-submodules Y such that a translate of Y lies in V (where Y is maximal in the sense that there is no larger algebraic ϕ-module whose translate lies in V).*

3. Proofs of our main results

We continue with the notation from Section 2. Hence $\phi_1, \dots, \phi_g$ are Drinfeld modules, and we denote by ϕ the action of $(\phi_1, \dots, \phi_g)$ on $\mathbb{G}_a^g$. Unless otherwise stated, $V \subset \mathbb{G}_a^g$ is an affine subvariety, and $Z(V)$ is its Mordell exceptional locus (as defined in Definition 2.2).

We first state a result which we will use later (Lemme 4 of [Den92]).

Lemma 3.1 (Denis). *Let $Y \subset \mathbb{G}_a^g$ be an irreducible subvariety, and let $t \in A$ be a non-constant function. If $\phi_t(Y) = Y$, then Y is a translate of an algebraic ϕ-module.*

The following corollary follows easily from Lemma 3.1.

Corollary 3.2. *Let $0 \in Y \subset \mathbb{G}_a^g$ be an irreducible subvariety, and let $t \in A$ be a non-constant function. If $\phi_t(Y) = Y$, then Y is an algebraic ϕ-module.*

Proof. According to Lemma 3.1, $Y = y + Z$ is a translate of an algebraic ϕ-module Z. Because $0 \in Y$, then $-y \in Z$, and so, because Z is an algebraic group, we conclude that $y + Z = Z$. Therefore $Y = Z$ is an algebraic ϕ-module. ☐

The following Fact is a consequence of Lemma 3.1.

Fact 3.3. *Let $t \in A$ be a non-constant function. Let $0 \in Y \subset \mathbb{G}_a^g$ be a variety such that $\phi_t(Y) \subset Y$. Let Z be an irreducible component of Y containing 0. Then Z is an algebraic ϕ-module.*

Proof. First we prove the following Claim.

Claim 3.4. *Let Z be an irreducible subvariety of $\mathbb{G}_a^g$ containing 0. Suppose that for some positive integers $m < n$, we have $\phi_{t^m}(Z) = \phi_{t^n}(Z)$. Then Z is an algebraic ϕ-module.*

Proof of Claim 3.4. By our assumption, the irreducible subvariety $\phi_{t^m}(Z)$ is invariant under $\phi_{t^{n-m}}$. Hence, using Corollary 3.2, we conclude that $Z_0 := \phi_{t^m}(Z)$ is an algebraic ϕ-module. In particular, $\phi_{t^m}(Z_0) = Z_0$. Thus, using that $\phi_{t^m}(Z) = \phi_{t^m}(Z_0)$, we get that

$$Z \subset \bigcup_{z \in \phi[t^m]} (z + Z_0). \tag{3.4.1}$$

Because Z is irreducible, then (3.4.1) yields that there exists $z \in \phi[t^m]$ such that $Z \subset (z + Z_0)$. Because $\dim(Z) = \dim(\phi_{t^m}(Z)) = \dim(Z_0)$, and because Z_0 is irreducible, we conclude that $Z = z + Z_0$. Because $0 \in Z = z + Z_0$, we obtain that $-z \in Z_0$, and so, $Z = z + Z_0 = Z_0$ is an algebraic ϕ-module, as desired. ☐

The following result is an easy corollary of Claim 3.4.

Corollary 3.5. *Let ℓ be a positive integer, and let $S := \{Y_i\}_{i=1}^{\ell}$ be a finite set of irreducible subvarieties of $\mathbb{G}_a^g$, each containing 0, such that ϕ_t acts on S (by permuting the varieties). Then each Y_i is an algebraic ϕ-module.*

Proof of Corollary 3.5. Because ϕ_t acts on the finite set S, then there exist positive integers $m < n$ such that for each $i \in \{1, \dots, \ell\}$, we have $\phi_{t^m}(Y_i) = \phi_{t^n}(Y_i)$. Then Claim 3.4 yields the conclusion of Corollary 3.5. ☐

Let d be the maximal dimension of the irreducible components of Y passing through 0. Let Z be an irreducible component of Y, passing through 0. We will prove Fact 3.3 by induction on $s := d - \dim(Z)$.

First we prove the case $s = 0$. So, let $\{Z_i\}_{i=1}^{\ell}$ be all the irreducible components of Y of dimension d, which contain 0. Because $0 \in \phi_t(Z_i) \subset \phi_t(Y) \subset Y$, then $\phi_t(Z_i)$ is contained in an irreducible component Z_j of Y (of maximal dimension d, because $\dim(\phi_t(Z_i)) = \dim(Z_i) = d$), which passes through 0. Because $\dim(\phi_t(Z_i)) = \dim(Z_j)$ and both $\phi_t(Z_i)$ and Z_j are irreducible, we conclude that $\phi_t(Z_i) = Z_j$. Hence, ϕ_t acts on the finite set $\{Z_i\}_{i=1}^{\ell}$. Thus, Corollary 3.5 yields that each Z_i is an algebraic ϕ-module.

Let $s \geq 1$. We assume that we proved Fact 3.3 for all irreducible components of dimension greater than $(d-s)$, and we will prove next that Fact 3.3 holds also for the irreducible components of dimension $d-s$.

Let $T := \{W_i\}_{i=1}^{k}$ be all the irreducible components of Y of dimension $(d-s)$, which contain 0. If ϕ_t acts on the finite set T, then Corollary 3.5 yields that each W_i is an algebraic ϕ-module, as desired. Therefore, assume from now on that ϕ_t does not act on T. However, for each $W := W_i$, there exists another irreducible component Z of Y passing through 0, such that $\phi_t(W) \subset Z$. Assume $Z \notin T$. Then $\dim(Z) > d-s$. By the induction hypothesis, Z is an algebraic ϕ-module. Hence, because $Z = \phi_t(Z)$ contains $\phi_t(W)$, then

$$W \subset \bigcup_{y \in \phi[t]} y + Z. \tag{3.5.1}$$

Because W is irreducible, then there exists $y \in \phi[t]$ such that $W \subset y + Z$. But $0 \in W$, and so, $0 \in y + Z$. Therefore $-y \in Z$, and because Z is an algebraic group, we conclude that $y + Z = Z$. Hence $W \subset Z$, which contradicts the fact that W is an irreducible component of Y, different from Z. This contradiction shows that actually ϕ_t acts on the finite set T, and so, it concludes our inductive proof. ☐

We are ready to prove that $Z(V)$ is a (closed) subvariety of V.

Proof of Theorem 2.4. Our proof follows the second proof of Theorem 1 from [Abr94]. Let $t \in A$ be a non-constant function.

For each $m \geq 2$ we define the map $F_m : (\mathbb{G}_a^g)^m \to (\mathbb{G}_a^g)^{m-1}$ by

$$(y_1, \dots, y_m) \to (\phi_t(y_1) - y_2, \phi_t(y_2) - y_3, \dots, \phi_t(y_{m-1}) - y_m).$$

Clearly, the map $F'_m : \mathbb{G}_a^{gm} \to \mathbb{G}_a^{gm}$ given by

$$F'_m(y_1, \dots, y_m) := (y_1, \phi_t(y_1) - y_2, \phi_t(y_2) - y_3, \dots, \phi_t(y_{m-1}) - y_m)$$

is an isomorphism. We let F_m^V be the map F_m restricted to V^m.

We let $D_m : V \to \mathbb{G}_a^{g(m-1)}$ defined by $D_m(y) = \phi_{t-1} \cdot (y, y, \dots, y)$. We let $Y_m \subset V^{m+1}$ be defined as

$$\{(y_1, \dots, y_m, y) \in V^m \times V \mid F_m^V(y_1, \dots, y_m) = D_m(y)\}. \tag{3.5.2}$$

Using the fact that F'_m is an isomorphism, we obtain that Y_m embeds into $V \times V$ via the map

$$(y_1, \dots, y_m, y) \to (y_1, y). \tag{3.5.3}$$

Let $Y'_m \subset V \times V$ be the image of Y_m through the map in (3.5.3). We claim that for $n > m$, we have $Y'_n \subset Y'_m$.

Indeed, if $(y_1, \dots, y_n, y) \in Y_n$, then $(y_1, \dots, y_m, y) \in Y_m$. Therefore $\{Y'_m\}_{m \geq 2}$ is a descending chain of subvarieties of $V \times V$, which has to stabilize. Hence, for some positive integer n, we have $Y'_m = Y'_n$ for each $m \geq n$.

We note that each Y_m contains the diagonal of V^{m+1}. Hence, each Y'_m contains the diagonal Δ of $V \times V$.

We have the natural projection π_2 of $Y'_n \subset V \times V$ on the second coordinate. The following Claim is the key to our proof.

Claim 3.6. *For each $y \in V$, and for each irreducible component $Z \times \{y\}$ of the fiber $\pi_2^{-1}(y)$, which passes through (y, y), the translate $-y + Z$ is an algebraic ϕ-module. Moreover, $\pi_2^{-1}(y)$ contains a positive dimensional irreducible component passing through (y, y) if and only if there exists a positive dimensional algebraic ϕ-module Z such that $(y + Z) \subset V$, if and only if $y \in Z(V)$.*

Proof of Claim 3.6. Let $y \in V$, and let $Y' \times \{y\} = \pi_2^{-1}(y) \subset Y'_n$ be the fiber above y. Hence $Y' \subset V$, and we let $Y := (Y' - y)$. Then $0 \in Y$ (because $\Delta \subset Y'_n$, and so, $y \in Y'$). We claim that $\phi_t(Y) \subset Y$.

Indeed, every point $y_1 \in Y'$ lies below a point $(y_1, y) \in Y'_n$, and in addition because $\{Y'_m\}_m$ stabilizes for $m \geq n$, we obtain that $(y_1, y) \in Y'_m$ for all $m \geq n$. In particular, using (3.5.2), we conclude that there exists an infinite sequence $\{y_i\}_{i \geq 1} \subset V$ such that

$$\phi_t(y_i) - y_{i+1} = \phi_{t-1}(y) \text{ for every } i \geq 1. \tag{3.6.1}$$

Therefore,

$$\phi_t(y_i - y) = y_{i+1} - y \text{ for all } i \geq 1. \tag{3.6.2}$$

Moreover, (3.6.1) yields that also $(y_2, y) \in Y'_n$, and so $(y_2 - y) \in Y$. Hence, (3.6.2) yields that $\phi_t(Y) \subset Y$. Therefore, using Fact 3.3, we conclude that each irreducible component of Y containing 0 is an algebraic ϕ-module.

Now, conversely, assume $(y + Y) \subset V$ and Y is maximal in the sense that there exists no larger algebraic ϕ-module whose translate by y lies in V. Then, because Y is invariant under ϕ_t, for each $y_1 \in (y + Y)$ there exists an infinite sequence

$$\{y_i\}_{i \geq 1} \subset (y + Y) \subset V$$

such that (3.6.2) holds, and so, (3.6.1) holds. Therefore $(y_1, y) \in Y'_n$, and so, $(y + Y) \times \{y\}$ lies in an irreducible component $Z \times \{y\}$ of the fiber $\pi_2^{-1}(y)$. We note that $Z \times \{y\}$ passes through (y, y) because $y \in (y + Y)$. Moreover, as shown in the above paragraph, $Z \subset V$ is a translate by y of an algebraic ϕ-module. Because Y is maximal, then $(y + Y) = Z$. Hence $\dim(Z) > 0$ if and only if $\dim(Y) > 0$, if and only if $y \in Z(V)$. ☐

We define the subset U of points $x \in Y'_n$ such that if $y := \pi_2(x) \in V$, then there exists a positive dimensional irreducible component of the fiber $\pi_2^{-1}(y)$, containing x. According to part (d) of 3.22 (page 95) in [Har77], the subset U is Zariski closed. We let $\tilde{Z} := U \cap \Delta$. Then $\tilde{Z}$ is Zariski closed, and we claim that

$$\tilde{Z} = \{(y, y) \mid y \in Z(V)\}. \tag{3.6.3}$$

Indeed, if $y \in Z(V)$, then there exists a positive dimensional algebraic ϕ-module Y such that $(y+Y) \subset V$. We may assume Y is a maximal algebraic ϕ-module with the property that its translate by y lies in V. Then $(y+Y) \times \{y\}$ is an irreducible component of the fiber $\pi_2^{-1}(y)$, which contains (y, y) (as shown in Claim 3.6). Therefore, $(y, y) \in U \cap \Delta = \tilde{Z}$. Now, conversely, if $(y, y) \in U$, then there exists a positive dimensional irreducible component Y' of $\pi_2^{-1}(y)$ passing through (y, y). Then $Y' = y + Y$ for some positive dimensional algebraic ϕ-module Y (see Claim 3.6). Thus $y \in Z(V)$, as desired.

Because $\tilde{Z}$ is Zariski closed, then (3.6.3) yields that also $Z(V)$ is a closed subvariety. This concludes the proof of Theorem 2.4. □

Before proceeding to the proof of Theorem 2.5, we will prove several preliminary results.

Lemma 3.7. *Let $Y \subset \mathbb{G}_a^g$ be an algebraic ϕ-submodule. Then $\phi_{\text{tor}} \cap Y$ is Zariski dense in Y.*

Proof. Let $m := \dim(Y)$. Then there exists a suitable finite-to-one, dominant projection π of Y on m coordinates of $\mathbb{G}_a^g$. At the expense of relabelling the coordinates, we may assume the projection is on the first m coordinates. Because $\pi(Y)$ is actually an algebraic group, and π is a group homomorphism, we conclude that $\pi(Y) = \mathbb{G}_a^m$. Moreover, π has finite fibers. By abuse of notation, we also denote by ϕ the induced action of $(\phi_1, \dots, \phi_m)$ on $\mathbb{G}_a^m$.

Claim 3.8. *The preimage of a torsion point of $\mathbb{G}_a^m$ through π^{-1} is a finite set of torsion points in Y.*

Proof of Claim 3.8. Let x be a torsion point of $\mathbb{G}_a^m$. Let S_0 be the orbit of x under the action of ϕ; hence S_0 is a finite ϕ-submodule of $\mathbb{G}_a^m$. Moreover S_0 is a finite set of torsion points. Because π has finite fibers, $S := \pi^{-1}(S_0)$ is a finite subset of Y. Moreover, because π commutes with the ϕ-action, we conclude that S is also a ϕ-module. Hence, S consists of finitely many torsion points (if S would contain a non-torsion point z, then the infinite ϕ-orbit of z would be contained in S, contradicting the fact that S is finite). Therefore, the preimage of x is indeed a finite set of torsion points in Y. □

Because $\phi_{\text{tor}}(\mathbb{G}_a^m)$ is a cartesian product of infinite subsets of the affine line, then $\phi_{\text{tor}}(\mathbb{G}_a^m)$ is Zariski dense in $\mathbb{G}_a^m$. We conclude that the Zariski closure of $\pi^{-1}(\phi_{\text{tor}}(\mathbb{G}_a^m)) \subset Y$ has dimension m. Hence, it equals Y (because Y is irreducible). Thus $\pi^{-1}(\phi_{\text{tor}}(\mathbb{G}_a^m))$ is a Zariski dense set of torsion points in Y (see Claim 3.8). This concludes the proof of Lemma 3.7. □

The following key result is an immediate corollary of Lemma 3.7.

Corollary 3.9. *There are no infinite algebraic families of algebraic ϕ-submodules of $\mathbb{G}_a^g$.*

Proof. Using Lemma 3.7, every algebraic ϕ-submodule of $\mathbb{G}_a^g$ contains a Zariski dense set of torsion points. Hence each algebraic ϕ-submodule of $\mathbb{G}_a^g$ is defined over K^{sep} (because every torsion point of ϕ is defined over K^{sep}). Therefore, there are no infinite algebraic families of algebraic ϕ-submodules of $\mathbb{G}_a^g$. □

Definition 3.10. *For an irreducible subvariety $V \subset \mathbb{G}_a^g$, we call the ϕ-*stabilizer *of V (denoted by $\text{Stab}_\phi(V)$) the largest algebraic ϕ-submodule Y such that $Y + V = V$.*

The ϕ-stabilizer of V is well-defined because if the algebraic ϕ-modules Y_1 and Y_2 have the property that $Y_1 + V = V$ and $Y_2 + V = V$, then the connected component Y_0 of $(Y_1 + Y_2)$ is also an algebraic ϕ-module such that $Y_0 + V = V$. Moreover, Y_1 and Y_2 are contained in Y_0.

The following result is a corollary of Theorem 2.4.

Corollary 3.11. *Let $V \subset \mathbb{G}_a^g$ be a positive dimensional irreducible affine variety. If $Z(V) = V$, then $\dim \text{Stab}_\phi(V) > 0$. More precisely, $\text{Stab}_\phi(V)$ is the unique maximal algebraic ϕ-submodule whose translate lies in V.*

Proof. Using the notation as in the proof of Theorem 2.4, the fact that $Z(V) = V$ yields that $\tilde{Z} = \Delta$. In particular, $\Delta \subset U$. Because V is irreducible, then Δ is irreducible. Let U_0 be an irreducible component of U which contains Δ. Then the restriction of $\pi_2 : Y_n' \to V$ to U_0 is a dominant morphism. By abuse of notation, this restriction will also be called π_2.

The fibers of $\pi_2 : U_0 \to V$ form an algebraic family of algebraic ϕ-submodules (there is only one family because both V and U_0 are irreducible). Thus, they are translates of the *same* positive dimensional algebraic ϕ-submodule Y, since there is no non-constant algebraic family of algebraic ϕ-modules (see Corollary 3.9). Therefore for each $y \in V$, we have $y + Y \subset V$. Hence $Y \subset \text{Stab}_\phi(V)$, which shows that $\text{Stab}_\phi(V)$ is positive dimensional. Moreover, no larger algebraic ϕ-submodule Y' has a translate which lies in V (because all fibers of U_0 are translates of the same algebraic ϕ-module). □

We are ready to prove Theorem 2.5.

Proof of Theorem 2.5. Clearly, there is only one algebraic ϕ-module of dimension 0. So, let Y be a maximal algebraic ϕ-module (of positive dimension) whose coset lies in V. Therefore, a coset $(y+Y)$ lies in $Z(V)$. Because $Z(V)$ is a closed subset of V (as shown by Theorem 2.4), then $(y + Y)$ lies in one of the *finitely* many irreducible components V_1 of $Z(V)$. Because V_1 is irreducible and $Z(V_1) = V_1$ (because $Z(Z(V)) = Z(V)$), then Corollary 3.11 shows that there exists a unique maximal algebraic ϕ-submodule whose coset lies in V_1. Therefore Y is one of the finitely many ϕ-stabilizers of the irreducible components of $Z(V)$, which concludes the proof of Theorem 2.5. ☐

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Dragos Ghioca, Department of Mathematics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1,

E-mail address: dghioca@math.mcmaster.ca