

CHAPTER 4

Problems I wish I could Solve

Stephen Watson

Department of Mathematics
York University
North York, Ontario, Canada
watson@yorkvm1.bitnet

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1. Introduction

This assortment is a list of problems that I worked on between 1979 and 1989 which I failed to solve. Some of the problems are due to other topologists and set theorists and I have attributed them when references are available in the literature. An earlier list appeared in 1984 in the Italian journal, *Rend. Circ. Mat. Palermo*. This list was entitled “Sixty questions on regular non-paracompact spaces”. Thirteen of these questions have since been answered (we shall give the numbering in that earlier paper in each case).

- CARLSON, in [1984], used Nyikos’ solution from NYIKOS [1980] of the normal Moore space problem to show that if it is consistent that there is a weakly compact cardinal then it is consistent that normal Moore spaces of cardinality at most 2^{\aleph_0} are metrizable. This solved Palermo #11.
- In [19∞a] BALOGH showed at the STACY conference at York University that assuming the consistency of the existence of a supercompact cardinal, it is consistent that normal locally compact spaces are collectionwise normal thus solving Palermo #16. Tall had earlier established this result for spaces of cardinality less than \beth_ω . See Tall’s B1.
- In [1985] DANIELS and GRUENHAGE constructed a perfectly normal locally compact collectionwise Hausdorff space under \diamond^* which is not collectionwise normal, thus answering Palermo #22, Palermo #23 and Palermo #24 in one blow.
- Balogh showed that under $V = L$ countably paracompact locally compact spaces are collectionwise Hausdorff and that under $V = L$ countably paracompact locally compact metacompact spaces are paracompact thus answering Palermo #26 and giving a partial solution to Palermo #28.
- Burke showed that under **PMEA**, countably paracompact Moore spaces are metrizable thus solving a famous old problem and incidentally answering Palermo #30, Palermo #33 and Palermo #34.
- It turned out that Palermo #37 and Palermo #47 were somewhat ill-posed since Fleissner’s **CH** space already in existence at that time answered both in its **ZFC** version by being a para-Lindelöf metacompact normal space of character 2^{\aleph_0} which is not collectionwise normal.
- Daniels solved Palermo #56 by showing that in **ZFC** the Pixley-Roy space of the co-countable topology on ω_1 is collectionwise Hausdorff (it was her question to begin with).

2. Normal not Collectionwise Hausdorff Spaces

- ? **69. Problem 1.** (Palermo #2) *Does CH imply that there is a normal not collectionwise Hausdorff space of character \aleph_2 ?*
- ? **70. Problem 2.** (Palermo #4) *Does \neg CH imply that there is a normal not collectionwise Hausdorff space of character 2^{\aleph_0} ?*

A natural question which has never been asked explicitly but has occupied a huge amount of thought is “What is the least character of a normal space which is not collectionwise Hausdorff?”. There are many independence results here so we ask only that this number be calculated under each possible cardinal arithmetic. The above are the two simplest cases.

These questions point out the two kinds of consistent theorems which we have for getting normal spaces of small character to be collectionwise Hausdorff. One is the $V = L$ argument which requires character $\leq \aleph_1$. The other is the Cohen real or **PMEA** argument which requires character less than 2^{\aleph_0} . To answer either of the above questions negatively would thus require, I think, a new kind of consistency proof and that would certainly be interesting—and challenging. To answer either of the above questions positively would be simply astounding.

- ? **71. Problem 3.** *Does $2^{\aleph_0} < 2^{\aleph_1}$ imply that there is no family of size less than 2^{\aleph_1} which generates the power set of ω_1 under countable unions?*
- ? **72. Problem 4.** *Is $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$ consistent with the existence of a family of size \aleph_2 which generates the power set of ω_1 under countable unions?*

These questions arose in a paper (WATSON [1988a]) which studied possible methods of lowering the character of Bing’s space directly. That paper showed that the existence of such families of cardinality less than 2^{\aleph_1} implies the existence of a normal space of character less than 2^{\aleph_1} which is not collectionwise Hausdorff. I am most interested in these questions however purely as problems in combinatorial set theory.

JURIS STEPRĀNS in [1982], and independently JECH and PRIKRY in [1984] showed that the answer to the first question is yes so long as either $2^{\aleph_0} < \aleph_{\omega_1}$ holds or else $2^{\aleph_0} = \kappa, (\kappa^+)^L < 2^{\aleph_1}$ and the covering lemma over L is true (note the connection with problem 26). Thus a solution in either direction would be quite startling.

The other case of $2^{\aleph_0} = 2^{\aleph_1}$ brings to mind **MA** $_{\aleph_1}$ which, however, implies that the minimal size of such a family is 2^{\aleph_1} . The model where \aleph_ω Cohen reals are added to a model of **GCH** has such a subfamily of cardinality $\aleph_\omega < 2^{\aleph_0}$. This is a rather unsatisfying result and the second question is designed

to exploit this. This question probably has something to do with Kurepa's hypothesis.

Problem 5. (Palermo #7) *Is there an axiom which implies that first countable normal spaces are \aleph_1 -collectionwise Hausdorff, which follows from the product measure extension axiom, from \diamond^* , from \diamond for stationary systems and which holds in the reverse Easton model?* **73. ?**

Problem 6. (Palermo #5) *If \diamond_S holds for each stationary $S \subset \omega_1$, then are normal first countable spaces \aleph_1 -collectionwise Hausdorff?* **74. ?**

The first problem was proposed to the author by Frank Tall. He wanted to unify the various proofs (TALL [1977], FLEISSNER [1974], SHELAH [1979], NYIKOS[1980]) of the consistency of the statement that first countable normal spaces are collectionwise Hausdorff. I was unable to solve this question but wrote a paper (WATSON [1984]) in which I introduced an axiom Φ which implies that first countable normal spaces are \aleph_1 -collectionwise Hausdorff, which follows from \diamond^* and from \diamond for stationary systems. It thus provided a unified (and simple) proof for both Fleisser's result and Shelah's result. The referee noted that Φ held in a forcing model similar to the reverse Easton model but I never checked whether it held in that reverse Easton model used by Tall in 1968 to show the consistency of normal Moore spaces of cardinality less than \aleph_{ω_1} being metrizable. I was also never able to determine whether the product measure extension axiom used by Nyikos implies Φ . I am still interested in knowing whether this (or another) axiom can unify these four proofs. An observer from outside the normal Moore space fraternity might feel that this is a somewhat esoteric question but the fact of the matter is that the consistency of normal first countable spaces being \aleph_1 -collectionwise Hausdorff will remain of interest in the decades to come and a single proof would enhance our understanding of the set theoretic nature of this property.

The second problem is an attempt to ascertain why \diamond is not enough. Shelah's model in DEVLIN and SHELAH [1979] in which there is a non-metrizable normal Moore space satisfies \diamond but exploits a stationary set on which \diamond does not hold. It is that result together with the two consistent theorems of FLEISSNER [1974] and SHELAH [1979] which give rise to this desperate attempt to figure out what is going on with \diamond . After all, it is Fleissner who created \diamond for stationary systems so this is a question about the nature of \diamond -principles, not really a question about general topology at all. The Easton model can also be included in problem 5 (see TALL [1988]).

Problem 7. (Fleissner; Palermo #57; Tall's C1) *Does ZFC imply that there is a first countable \aleph_1 -collectionwise Hausdorff space which fails to be collectionwise Hausdorff?* **75. ?**

This is a central question on reflection. It has been much worked on. In [1977] SHELAH showed that the answer is yes for locally countable spaces if a supercompact cardinal is Lévy-collapsed to \aleph_2 . On the other hand $E(\omega_2)$ is enough to construct a counterexample. Thus large cardinals are needed to establish a consistency result if indeed it is consistent. In [1977b] Fleissner has conjectured that Lévy-collapsing a compact cardinal to \aleph_2 will yield a model in which first countable \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff. As in the pursuit of problem 11, far more effort has gone into obtaining a consistency result than has gone into trying to construct a counterexample. The conventional wisdom is that the set-theoretic technology is simply not ready yet and that we just have to wait. I conjecture that there is a **ZFC** example and that we really have to look somewhere other than ordinals for large first countable spaces which are not paracompact.

- ? 76. **Problem 8.** (Palermo #1) *Does **GCH** imply that normal first countable \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff?*

In [1977], SHELAH constructed two consistent examples of a normal Moore space which is \aleph_1 -collectionwise Hausdorff but which fails to be \aleph_2 -collectionwise Hausdorff. The first satisfied $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_3$ and the second satisfied $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$. These cardinal arithmetics make the question quite natural. FLEISSNER asked this question in [1977b].

- ? 77. **Problem 9.** (Palermo #6; Tall's # C2) *If κ is a strong limit cardinal, then are normal first countable spaces which are $< \kappa$ -collectionwise Hausdorff, κ -collectionwise Hausdorff?*

In WATSON [19∞a], we showed that the answer is yes for strong limit cardinals of countable cofinality even without normality (answering a question from FLEISSNER [1977c]; in that paper Fleissner had proved the same result assuming **GCH**). Aside from that nothing seems to be known about getting *normal* first countable spaces where collectionwise Hausdorff first fails at a limit cardinal. In SHELAH [1977], a consistent counterexample which is not normal was obtained.

- ? 78. **Problem 10.** (Palermo #3) *Does **CH** imply that normal first countable spaces are weakly \aleph_2 -collectionwise Hausdorff?*

The motive for this question is that the only examples of spaces which fail to be weakly collectionwise Hausdorff are those which do not use pressing-down arguments but rather simply Δ -system arguments. Such examples always seem to use an identification of the unseparated set with a subset of the real line. Of course, there is no reason why it should be this way but I spent a lot of time trying to construct an example some other way and got nowhere.

On the other hand, a theorem would be quite surprising and would provide convincing evidence that the Moore plane is canonical.

3. Non-metrizable Normal Moore Spaces

Problem 11. (Palermo #8; Tall's A1) *Does $2^{\aleph_0} = \aleph_2$ imply the existence of a non-metrizable normal Moore space?* **79. ?**

I only add that it is dangerous to spend 95% of the effort on a question trying to prove it in one direction. Very little effort has gone into trying to modify FLEISSNER'S [1982b, 1982a] construction of a non-metrizable Moore space from the continuum hypothesis. In fact, there's probably only about two or three people who really understand his construction (I am not one of them). If the conventional wisdom that collapsing a large cardinal should get the consistency of the normal Moore space conjecture with $2^{\aleph_0} = \aleph_2$, then why hasn't it been done?

Problem 12. (Palermo #9; Tall's A3) *Does the existence of a non-metrizable normal Moore space imply the existence of a metacompact non-metrizable normal Moore space?* **80. ?**

Problem 13. (Palermo #12) *Does the existence of a non-metrizable normal Moore space imply the existence of a normal Moore space which is not collectionwise normal with respect to metrizable sets?* **81. ?**

The reason this question remains of interest is that a tremendous amount of effort has gone into obtaining partial positive results. RUDIN and STARBIRD [1977] and NYIKOS [1981] both obtained some technical results of great interest. Nyikos showed, in particular, that if there is a non-metrizable normal Moore space, then there is a metacompact Moore space with a family of closed sets which is normalized but not separated. In WATSON [19∞a], it is shown that if there is a non-metrizable normal Moore space which is non-metrizable because it has a nonseparated discrete family of closed metrizable sets then there is a metacompact non-metrizable normal Moore space. This means that if you believe in a counterexample you had better solve problem 32 first! If you believe in a theorem as Rudin and Starbird and Nyikos did, you have a lot of reading to do. I think there is a counterexample.

Problem 14. (Palermo #10; see Tall's A2 and A4) *Does the existence of a non-metrizable normal Moore space imply the existence of a para-Lindelöf non-metrizable normal Moore space?* **82. ?**

The normal Moore space problem enjoyed lots of consistent counterexamples long before para-Lindelöf raised it's head. However in [1981] CARYN

NAVY managed to show how to use $\mathbf{MA} + \neg\mathbf{CH}$ to get a para-Lindelöf example. FLEISSNER'S \mathbf{CH} example of a non-metrizable normal Moore space from [1982b] turns out to be para-Lindelöf (after all, he was modifying Navy's example). The only part that is not clear is whether the singular cardinals hypothesis can get you a para-Lindelöf counterexample to the normal Moore space conjecture. Probably the best positive result that can be hoped for is to show that Fleissner's \mathbf{SCH} counterexample can be modified to be para-Lindelöf. That would be a good result since a deep understanding of Fleissner's space would be required and that space does have to be digested. A negative result is more likely and would really illustrate the difference between FLEISSNER'S \mathbf{CH} example and his \mathbf{SCH} example from [1982b] and [1982a] and that would be valuable.

- ? **83. Problem 15.** (Palermo #14) *Are Čech-complete locally connected normal Moore spaces metrizable?*
- ? **84. Problem 16.** (Palermo #13) *Are normal Moore spaces sub-metrizable?*

4. Locally Compact Normal Spaces

- ? **85. Problem 17.** (Palermo #15; Tall's B2) *Are locally compact normal metacompact spaces paracompact?*

I have spent a lot of time on this question. It was originally stated by TALL [1974] although it would be difficult to appreciate a result proved by ARHANGEL'SKIĬ in [1971] without thinking of it. In WATSON [1982], it was shown that, under $V = L$, locally compact normal metacompact spaces are paracompact. However, the techniques for constructing examples under $\mathbf{MA} + \neg\mathbf{CH}$ (the usual place to look) did not seem to provide any way of getting locally compact spaces and metacompact spaces at the same time. In [1983] PEG DANIELS showed in \mathbf{ZFC} that locally compact normal boundedly metacompact spaces are paracompact. This surprising result raised the hopes of everyone (except Frank Tall) that the statement in question might actually be a theorem in \mathbf{ZFC} . It hasn't worked out that way so far. Boundedly metacompact really is different from metacompact although most examples don't show this. If there is a theorem here in \mathbf{ZFC} it would be an astounding result. If there is an example in some model (as I think there is), it would require a deeper understanding of the Pixley-Roy space than has so far existed. Either way this is a central question. Another related question with a (much?) higher probability of a consistent counterexample is problem 18.

Problem 18. (Palermo #28) *Are countably paracompact locally compact metacompact spaces paracompact?* 86. ?

Problem 19. *Does \mathbf{MA}_{\aleph_1} imply that normal locally compact meta-Lindelöf spaces are paracompact?* 87. ?

This problem is related to problem 17. We mentioned that no consistent example of a locally compact normal metacompact space which is not paracompact is known. Actually, even constructing a consistent example of a locally compact normal meta-Lindelöf space which is not paracompact is non-trivial. We constructed such a space in WATSON [1986] but the proof makes essential use of a compact hereditarily Lindelöf space which is not hereditarily separable. These spaces do not exist under $\mathbf{MA} + \neg\mathbf{CH}$. Thus what this question does is up the ante. Anyone except Frank Tall working on a counterexample to problem 17 is probably using Martin's axiom. So what we are saying is "you'll never do it—bet you can't even get meta-Lindelöf". Of course, maybe there is an example but that would require a completely different approach to getting meta-Lindelöf together with locally compact and normal. That would be just as interesting to me because I tried for a long time to get the results of WATSON [1986] using Martin's axiom. Of course, such an example cannot be done in \mathbf{ZFC} because of BALOGH's result from [19∞b] that, under $V = L$, locally compact normal meta-Lindelöf spaces are paracompact.

Problem 20. (Palermo #17) *Does \mathbf{ZFC} imply that there is a perfectly normal locally compact space which is not paracompact?* 88. ?

This is my favorite question. If there is an example then what a strange creature it must be. A series of results running from MARY ELLEN RUDIN's result from [1979] that under $\mathbf{MA} + \neg\mathbf{CH}$ perfectly normal manifolds are metrizable runs through results of LANE [1980] and GRUENHAGE [1980] to culminate in a result of Balogh and Junnila that, under $\mathbf{MA} + \neg\mathbf{CH}$, perfectly normal locally compact collectionwise Hausdorff spaces are paracompact. On the other hand, under $V = L$, normal locally compact spaces are collectionwise Hausdorff. This means that, if there is in \mathbf{ZFC} a perfectly normal locally compact space which is not paracompact, then under $\mathbf{MA} + \neg\mathbf{CH}$ it is not collectionwise Hausdorff but that under $V = L$ it *is* collectionwise Hausdorff. Now there are two ways this can be done. First, by stating a set-theoretic condition, using it to construct one space and then using its negation to construct another space. $2^{\aleph_0} = 2^{\aleph_1}$ is the only worthwhile axiom I know whose negation is worth something (although see WEISS [1975] and [1977]). Second, by constructing a space whose collectionwise Hausdorffness happens to be independent. This is fine but the fragment of $V = L$ is small enough to force with countably closed forcing so the definition of such a space better depend pretty strongly on what the subsets of *both* ω and ω_1 are. There

are however examples in most models. Under **CH**, the Kunen line (JUHÁSZ, KUNEN and RUDIN [1976]) is an example of a perfectly normal locally compact S -space which is not paracompact. Under **MA** + \neg **CH**, the Cantor tree with \aleph_1 branches is an example.

On the other hand, a consistent theorem would be amazing. To get it by putting together the two consistency proofs would be quite hard. The **MA** + \neg **CH** result uses both $\mathfrak{p} = \mathfrak{c}$ and the non-existence of a Suslin line so that's a lot of set theory. The $V = L$ result can be done without **CH** but then you have to add weakly compact many Cohen reals or something like that (TALL [1984]). I haven't tried this direction at all, although set-theorists have. It looks impossible to me.

- ? 89. **Problem 21.** (Palermo #19) *Does the existence of a locally compact normal space which is not collectionwise Hausdorff imply the existence of a first countable normal space which is not collectionwise Hausdorff?*

I think the answer is yes. This belief stems from WATSON [1982] where an intimate relation between the two existence problems was shown. The only thing that is missing is the possibility that there might be a model in which normal first countable spaces are \aleph_1 -collectionwise Hausdorff and yet that in that model there is a normal first countable space which fails to be \aleph_2 -collectionwise Hausdorff. This seems unlikely, though it is open, and yet the question might be answered positively in any case (see SHELAH [1977]). A counterexample would have been more interesting before Balogh showed the consistency of locally compact normal spaces being collectionwise normal. However it may yet provide a clue to an answer to problem 17.

- ? 90. **Problem 22.** (Palermo #18; Tall's B5) *Are large cardinals needed to show that normal manifolds are collectionwise normal?*

In 1986, Mary Ellen Rudin built a normal manifold which is not collectionwise normal under the axiom \diamond^+ . Meanwhile, PETER NYIKOS [1989] has shown that if the existence of a weakly compact cardinal is consistent then it is consistent that normal manifolds are collectionwise normal. The most likely solution to this problem is doing it on the successor of a singular cardinal where \diamond -like principles tend to hold unless there are large cardinals (see FLEISSNER [1982a]). A consistent theorem that normal manifolds are collectionwise normal probably means starting from scratch, where so many have started before.

- ? 91. **Problem 23.** (Palermo #21) *Does ZFC imply that normal manifolds are collectionwise Hausdorff?*

Mary Ellen Rudin's example in problem 22 is collectionwise Hausdorff. This can be deduced from the fact that under $V = L$ normal first countable spaces

are collectionwise Hausdorff (FLEISSNER [1974]). Thus no example of any kind has yet been demonstrated to exist and all we have are a few consistent theorems.

Problem 24. (Palermo #20; Tall's B3) *Are normal locally compact locally connected spaces collectionwise normal?* **92. ?**

REED and ZENOR showed in [1976] that locally connected locally compact normal Moore spaces are metrizable in **ZFC**. ZOLTAN BALOGH showed in [19∞d] that connected locally compact normal submeta-Lindelöf spaces are paracompact under $2^\omega < 2^{\omega_1}$. BALOGH showed in [19∞c] that locally connected locally compact normal submeta-Lindelöf spaces are paracompact in **ZFC**. GRUENHAGE constructed in [1984] a connected locally compact non-metrizable normal Moore space under **MA** + \neg **CH**. Problem 24 attempts to determine whether covering properties have anything central to do with these phenomena.

Problem 25. (Palermo #25) *Does **ZFC** imply that there is a normal extremally disconnected locally compact space which is not paracompact?* **93. ?**

In [1978] KUNEN and PARSONS showed that if there is a weakly compact cardinal then there is a normal extremally disconnected locally compact space which is not paracompact. In [1977] KUNEN showed that there is an normal extremally disconnected space which is not paracompact. This is a great question. I suspect that useful ideas may be found in WATSON [19∞e] where normal spaces which are not collectionwise normal with respect to extremally disconnected spaces are constructed.

5. Countably Paracompact Spaces

Problem 26. (Palermo #27; Tall's D6) *Does $2^{\aleph_0} < 2^{\aleph_1}$ imply that separable first countable countably paracompact spaces are collectionwise Hausdorff?* **94. ?**

In [1937] JONES showed that, under $2^{\aleph_0} < 2^{\aleph_1}$ normal separable spaces have no uncountable closed discrete sets (and thus that separable normal Moore spaces are metrizable). In [1964] HEATH showed that, in fact, the existence of a normal separable space with an uncountable closed discrete set is equivalent to $2^{\aleph_0} = 2^{\aleph_1}$.

These results blend well with the ongoing problem of determining the relation between normality and countable paracompactness. The normal separable space with an uncountable closed discrete set is, in fact, countably paracompact and so FLEISSNER [1978], PRZYMUSIŃSKI [1977] and REED [1980] asked whether the existence of a countably paracompact separable space with

an uncountable closed discrete set is also equivalent to $2^{\aleph_0} = 2^{\aleph_1}$. FLEISSNER [1978] fueled this suspicion with a proof that the continuum hypothesis implies that countably paracompact separable spaces have no uncountable closed discrete set. In WATSON [1985], we showed that the existence of such a space is equivalent to the existence of a dominating family in ${}^{\omega_1}\omega$ of cardinality 2^{\aleph_0} . This was somewhat satisfying since the equivalence of the existence of such a family with $2^{\aleph_0} < 2^{\aleph_1}$ was known as an open problem in set theory. In 1983, Steprāns and Jech and Prikry independently showed that if the continuum is a regular cardinal and there are no measurable cardinals in an inner model, then the latter equivalence holds. The general set-theoretic problem remains open.

Back in general topology, what about first countable spaces? The examples that are used in all these results have character equal to the continuum. One expects first countability to be a big help but so far it seems useless in this context. The drawback to this question is that if the answer is no, one first has to solve the set-theoretic question and then figure out how to lower the character from the continuum to \aleph_0 . Getting the character down is always interesting. On the other hand, if there is a theorem, that might involve a hard look at the weak version of \diamond invented by Keith Devlin (DEVLIN and SHELAH [1978]) and lots of people would be interested in an essential use of that axiom.

? 95. **Problem 27.** (Palermo #31) *Does $2^{\aleph_0} < 2^{\aleph_1}$ imply that special Aronszajn trees are not countably paracompact?*

This question is quite attractive to some precisely because it is not a topological question. It is however a natural question about the structure of Aronszajn trees.

In [1980] FLEISSNER noted that the proof in FLEISSNER [1975] that under $\mathbf{MA} + \neg\mathbf{CH}$, special Aronszajn trees are normal could be modified to show that under $\mathbf{MA} + \neg\mathbf{CH}$ special Aronszajn trees are countably paracompact. We showed in WATSON [1985], that under $(\forall \text{ stationary } S \subset \omega_1) \diamond_S$, special Aronszajn trees are not countably paracompact. This proof, however, was implicit in FLEISSNER [1975, 1980]. Fleissner had shown that under $V = L$, special Aronszajn trees are not normal but the key result that gave rise to the present question is the proof by SHELAH and DEVLIN [1979], that $2^{\aleph_0} < 2^{\aleph_1}$ implies that special Aronszajn trees are not normal. FLEISSNER [1980] cites a result of Nyikos that normal implies countably paracompact in trees. This means that the present statement in question is weaker than the Devlin-Shelah result.

? 96. **Problem 28.** (Palermo #32) *If the continuum function is one-to-one and X is a countably paracompact first countable space, then is $e(X) \leq c(X)$?*

This is just an attempt to conjecture a form of Shapirovskii's improvement of Jones' lemma at each cardinal. Recall that JONES proved in [1937] that separable normal spaces have no uncountable closed discrete set under $2^{\aleph_0} < 2^{\aleph_1}$. That proof was sharpened by Shapirovskii to show that in normal first countable spaces $2^{\aleph_0} < 2^{\aleph_1}$ implies that closed discrete sets of cardinality \aleph_1 have a subset of cardinality \aleph_1 which can be separated by disjoint open sets. Thus if there are no disjoint families of more than \aleph_0 many open sets then there are no closed discrete sets of cardinality \aleph_1 . That proof was observed to extend to higher cardinals by FRANK TALL [1976] and neatly summarized in the form: if the continuum function is one-to-one and X is a normal first countable space then $e(X) \leq c(X)$. The question is just asking whether this nice statement about cardinal functions applies equally to countably paracompact first countable spaces. I believe that it does.

Problem 29. (Palermo #29) *Does \diamond^* imply that countably paracompact first countable spaces are \aleph_1 -collectionwise Hausdorff?* **97. ?**

This question is just something that I expected would have a positive answer but couldn't make any headway on. SHELAH [1979] showed that \diamond^* implies that normal first countable spaces are \aleph_1 -collectionwise Hausdorff and every other separation theorem which used normality eventually was extended to countable paracompactness (see WATSON [1985] and Burke's use of **PMEA**). The real question here is vague: "is there a distinction between the separation properties of normality and countable paracompactness". A negative answer to the specific question would answer the vague question quite clearly. A positive answer would get a little closer to the combinatorial essence underlying separation and that would be a worthy accomplishment.

Problem 30. *Does the existence of a countably paracompact non-normal Moore space imply the existence of a normal non-metrizable Moore space?* **98. ?**

This question was first asked by WAGE [1976] since, in that paper, he showed the converse to be true. All the available evidence indicates that the answer is yes. PETER NYIKOS [1980] showed that **PMEA** implies that normal Moore spaces are metrizable. This was later extended in a non-trivial way, by Dennis Burke, who showed that, under **PMEA**, countably paracompact Moore spaces are metrizable.

Problem 31. (Palermo #59) *Does **ZFC** imply that collectionwise Hausdorff ω_1 -trees are countably paracompact?* **99. ?**

6. Collectionwise Hausdorff Spaces

- ? 100. **Problem 32.** (Palermo #38) *Is there a normal not collectionwise normal space which is collectionwise normal with respect to collectionwise normal sets?*

This apparently frivolous main question is intended to be a specific version of a more serious question (first asked in WATSON [1988b]): Characterize those spaces Y and categories \mathcal{C} for which there exists a normal space which is collectionwise normal with respect to discrete families of sets in \mathcal{C} but not collectionwise normal with respect to copies of Y . This tries to get at the heart of many constructions like those in WATSON [19 ∞ e]. A less extreme question is: Characterize those spaces Y for which there exists a normal collectionwise Hausdorff space which is not collectionwise normal with respect to copies of Y . The main question is worth solving. If the answer is no, then I would be amazed and it would solve problem 12. If the answer is yes, then I think we would be getting at the heart of a topic on which I have spent a great deal of time (WATSON [19 ∞ e] is devoted to establishing partial results).

- ? 101. **Problem 33.** (Palermo #60) *Does ZFC imply that there is a collectionwise Hausdorff Moore space which is not collectionwise normal with respect to compact sets?*
- ? 102. **Problem 34.** (Palermo #39) *Is there a normal collectionwise Hausdorff space which is not collectionwise normal with respect to \aleph_1 many compact sets?*
- ? 103. **Problem 35.** (Palermo #40) *Is it consistent that there is a normal first countable collectionwise Hausdorff space which is not collectionwise normal with respect to compact sets?*

In an unpublished result from 1980, Fleissner and Reed constructed, by using a measurable cardinal, a regular collectionwise Hausdorff space which is not collectionwise normal with respect to compact sets. In [1983] MIKE REED constructed in ZFC a collectionwise Hausdorff first countable regular space which is not collectionwise normal with respect to compact metric sets. He also obtained a collectionwise Hausdorff Moore space which is not collectionwise normal with respect to compact metric sets under the continuum hypothesis or Martin's axiom and asked the first question. The answer could well turn out to be yes since normality is not required and of course that would be preferable to Reed's results. If the answer is no, that is more interesting because the proof would penetrate into the manner in which the closed unit interval can be embedded in a Moore space and that would be quite exciting. In WATSON [19 ∞ e], an example was constructed of a normal collectionwise

Hausdorff space which is not collectionwise normal with respect to copies of $[0, 1]$. In that example, the proof of not collectionwise normality is not a Δ -system argument but rather a measure-theoretic argument. As a result, we have no hope of using that method to get an example in which \aleph_1 many copies of $[0, 1]$ cannot be separated unless there is a subset of the reals of cardinality \aleph_1 which has positive measure. Thus an example answering the second question would have to be essentially different from that of WATSON [19 ∞ e] and I do not believe such an example exists. On the other hand, a theorem, under **MA** + \neg **CH** for example, would be quite interesting. The example of WATSON [19 ∞ e] is badly not first countable. Anyway, the only normal first countable collectionwise Hausdorff spaces which are not collectionwise normal are Fleissner's space of FLEISSNER [1976] and the ones based on Navy's space of NAVY [1981] and FLEISSNER [1982b, 1982a]. The first of these requires the unseparated sets to be badly non-compact. The second requires the unseparated sets to be non-separable metric sets. Neither of these constructions is going to be easily modified to a positive solution to the third question. I don't think such a consistent example exists—it's asking too much. On the other hand, a negative answer means that rare thing: a **ZFC** result!

Problem 36. (Palermo #41) *Does **ZFC** imply that normal first countable collectionwise Hausdorff spaces are collectionwise normal with respect to scattered sets?* **104. ?**

Problem 37. (Palermo #35) *Does $V = L$ imply that normal first countable spaces are collectionwise normal with respect to separable sets?* **105. ?**

Problem 38. (Palermo #36) *Does $V = L$ imply that normal first countable spaces are collectionwise normal with respect to copies of ω_1 ?* **106. ?**

The prototypes of normal collectionwise Hausdorff spaces which are not collectionwise normal are Fleissner's space of FLEISSNER [1976] and Navy's space of NAVY [1981]. The latter space has an unseparated discrete family of Baire spaces of weight \aleph_1 . These Baire spaces are very non-scattered. On the other hand Fleissner's space has a unseparated discrete family of copies of the ordinal space ω_1 . These sets are scattered. The latter type has been successfully modified to be first countable: that is Fleissner's solution to the normal Moore space conjecture (FLEISSNER [1982b, 1982a]). The first question is trying to ask whether the former type can be modified to be first countable. This is a question which has great intrinsic interest. A consistent method which succeeds in lowering the character of the former prototype would undoubtedly be quite useful in many other contexts. A positive answer would be unthinkable. This first question is however mostly a question about my own inability to follow a proof in the literature. A paper (FLEISSNER [1982c]) has appeared which gives a negative answer to this question. The idea of this paper is very

clever and introduces an axiom which has since been used (RUDIN [1983]) to construct what is possibly the most clever example in general topology. However I have spent a great deal of effort trying to understand the proof in FLEISSNER [1982c]. I have conversed with the author who has suggested several changes. For various reasons I have been unable to locate anyone who has checked all of the details. It is undoubtedly the case that the proof is simply over my head but I just cannot follow it. In my stubborn fashion, I still want to know the answer to this first question. If these comments succeed in provoking someone to read FLEISSNER [1982c] and then to explain it to me, then they will have done both of us a favor, for they will have read an inspired paper and, in addition, set my mind at ease (in August 1989 Bill Fleissner circulated a corrigendum to that paper).

The second and third questions are follow-ups in my tribute to Fleissner's George (FLEISSNER [1976]), a normal collectionwise Hausdorff space which fails to be collectionwise normal with respect to copies of ω_1 . This space has been modified (WATSON [19 ∞ a]) to fail to be collectionwise normal with respect to separable sets (S -spaces actually), under suitable set-theoretic assumptions. The example of FLEISSNER [1982c] fails to be collectionwise normal with respect to copies of a space something like ω_1 . The second question asks: "Can a S -space (like Ostaszewski's space [1976]) be used?". The third question asks "Was it really necessary to use something different from ω_1 ?". FLEISSNER showed in [1977a] that it is consistent that normal first countable spaces are collectionwise normal with respect to copies of ω_1 by collapsing an inaccessible in a model of the constructible universe (see also DOW, TALL and WEISS [19 ∞]). I think that the techniques that one would have to develop in order to solve these questions would be useful in many areas of general topology, and thus worth the effort.

7. Para-Lindelöf Spaces

? 107. **Problem 39.** (Palermo #43) *Are para-Lindelöf regular spaces countably paracompact?*

This is the main open problem on para-Lindelöf spaces. The original question was whether para-Lindelöf was equivalent to paracompact— one more feather in the cap of equivalences of paracompactness established by Stone and Michael in the 1950s (see Burke's article in the handbook of Set-Theoretic Topology BURKE [1984]). This question was finally solved by Caryn Navy, a student of Mary Ellen Rudin, in NAVY [1981]. Her construction was a rather general one that permitted quite a lot of latitude; she obtained first countable ones under **MA** + \neg **CH** using the Moore plane, she obtained a **ZFC** example using Bing's space. FLEISSNER [1982b, 1982a] later modified this example to be a Moore space under the continuum hypothesis, thus solving

the normal Moore space conjecture. Certain properties seemed hard to get however. These difficulties each gave rise to questions which were listed in Navy's thesis. The main open problem listed above is due to the fact that all the constructions are intrinsically countably paracompact. I tried for a long time to build in the failure of countable paracompactness but each time para-Lindelöf failed as well. It may be useful to note that the whole idea of Navy's construction was to take Fleissner's space of FLEISSNER [1979] which was σ -para-Lindelöf but not paracompact and build in a way to "separate" the countably-many locally countable families so that one locally countable refinement is obtained. This way was normality. No other way of getting para-Lindelöf is known. I don't think another way of getting para-Lindelöf is even possible— Navy's method looks quite canonical to me (although see WATSON [19∞e]). I think the easiest way of getting a para-Lindelöf space which is not countably paracompact (at least consistently) is to iterate a normal para-Lindelöf space which is not collectionwise normal in an ω -sequence (see WATSON [19∞c]) and solve problem 40. I tried to do this but got bowled over by the details:

Problem 40. (Palermo #44) *Is there a para-Lindelöf Dowker space?* 108. ?

Another question which has not really been looked at but which I think is extremely important is:

Problem 41. (Palermo #42) *Are para-Lindelöf collectionwise normal spaces paracompact?* 109. ?

This was first asked by FLEISSNER and REED [1977]. So far, there are no ideas at all on how to approach this. Even the much weaker property of meta-Lindelöf creates big problems here:

Problem 42. (Palermo #58) *Is it consistent that meta-Lindelöf collectionwise normal spaces are paracompact?* 110. ?

In [1983] RUDIN showed that under $V = L$, there is a screenable normal space which is not paracompact. This space is collectionwise normal and meta-Lindelöf reducing our search to a **ZFC** example (although to use such a difficult space to solve this question consistently seems overkill— but I don't know of a simpler one).

Problem 43. (Palermo #46) *Are para-Lindelöf screenable normal spaces paracompact?* 111. ?

This question just throws in all the hardest properties and asks whether a theorem pops out. I predict a **ZFC** example will not be seen in this century (at

least not from me). If para-Lindelöf does indeed imply countably paracompact then such an example does not exist in any case, since normal screenable countably paracompact spaces are paracompact (NAGAMI [1955]).

? 112. **Problem 44.** (Palermo #45) *Are para-Lindelöf screenable spaces normal?*

There is an example of a screenable space which is not normal in BING [1951] but a lot of work has to be done to make it para-Lindelöf. Maybe that is the place to start. Keep in mind that para-Lindelöf spaces are strongly collectiowise Hausdorff (FLEISSNER and REED [1977]).

8. Dowker Spaces

The next few questions are **ZFC** questions about Dowker spaces. It's fairly easy to come up with a question about Dowker spaces. Just find a property that Mary Ellen Rudin's Dowker example in **ZFC** (RUDIN [1971]) does not have and ask if there is a Dowker space with that property. A lot can be done in particular models of **ZFC** to obtain very nice, well-behaved examples of Dowker spaces (see RUDIN [1955], JUHÁSZ, KUNEN and RUDIN [1976], DE CAUX [1976], WEISS [1981], BELL [1981] and RUDIN [1984, 1983]) but, in **ZFC**, there is only that one example around. I tried to construct another one in 1982 but only succeeded in getting one from a compact cardinal (WATSON [19 ∞ c]). On the one hand, this is worse than using **CH** or **MA** + \neg **CH** but on the other hand, postulating the existence of a compact cardinal has a different flavour than the other axioms. Anyway that example was scattered of height ω and hereditarily normal thus giving rise to the next three questions:

? 113. **Problem 45.** (Palermo #48) *Does **ZFC** imply that there is a hereditarily normal Dowker space?*

? 114. **Problem 46.** (Palermo #55) *Does **ZFC** imply that there is a σ -discrete Dowker space?*

? 115. **Problem 47.** (Palermo #54) *Does **ZFC** imply that there is a scattered Dowker space?*

The next two questions have been around for a while and rest on the following pathological properties of Mary Ellen Rudin's example (RUDIN [1971]): It has cardinality and character $(\aleph_\omega)^\omega$.

? 116. **Problem 48.** (Palermo #50) *Does **ZFC** imply that there is a Dowker space of cardinality less than \aleph_ω ?*

Problem 49. (Palermo #51; Rudin [1971]) *Does ZFC imply that there is a first countable Dowker space?* 117. ?

Problem 50. (RUDIN [1971]) *Is there a separable Dowker space?* 118. ?

In [1983], RUDIN showed that under $V = L$, there is a screenable normal space which is not paracompact. This space was quite difficult to construct. A ZFC example seems a long, long way off (although \diamond has been known to hold at large enough cardinals. On the other hand a consistent theorem would finish off this nearly forty year old question implicit in BING [1951]:

Problem 51. (Palermo #49) *Does ZFC imply that there is a screenable normal space which is not paracompact?* 119. ?

An even stronger property than screenable is that of having a σ -disjoint base. It remains completely open whether a normal space with a σ -disjoint base must be paracompact. The next question is conjectured to have a positive answer. This would start to clear up the mystery surrounding screenability and having a σ -disjoint base. A negative answer would require a good hard study of Rudin's space RUDIN [1983] and that is worthwhile anyway.

Problem 52. (Palermo #52) *Does ZFC imply that normal spaces with a σ -disjoint base are collectionwise normal (or paracompact)?* 120. ?

In reply to a question of Frank Tall, RUDIN [1983] showed that the existence of a screenable normal non-paracompact space implies the existence of a screenable normal non-collectionwise normal space. The next question asks whether collectionwise normality really is quite irrelevant.

Problem 53. (Palermo #53) *Does the existence of a screenable normal space which is not paracompact imply the existence of a screenable collectionwise normal space which is not paracompact?* 121. ?

9. Extending Ideals

If I is an ideal on X then I measures A if and only if A is a subset of X and either $A \in I$ or $X - A \in I$. If an ideal I on X has the property that whenever \mathcal{A} is a family of κ many subsets of X there is a countably complete ideal which extends I and which measures each of the elements of \mathcal{A} , then we say that the ideal I is κ -extendible. We say that an ideal I is κ -completable if there is a proper ideal J which is κ -complete and which contains I . If an ideal I on X has the property that whenever \mathcal{A} is a family of κ many subsets of X there is a countably complete ideal which extends I and which measures at least λ many elements of \mathcal{A} , then we say that the ideal I is (κ, λ) -extendible.

The idea of investigating these questions is due to Frank Tall whose interest is responsible for all the questions in this section. In a paper with Steprāns (STEPRĀNS and WATSON [1986]), we investigated many problems on the κ -extendibility and the (κ, λ) -extendibility of ideals. Many of these questions have remained open.

We showed in STEPRĀNS and WATSON [1986] that if an ideal I is $(\kappa^\omega)^+$ -completable then I is κ -extendible. We showed that the converse is true unless κ is greater or equal to either a weakly compact cardinal or something called a Ξ -cardinal (in particular an ideal I is ω -extendible if and only if I is $(2^{\aleph_0})^+$ -completable). We also showed that, if there are no measurable cardinals in an inner model and κ is not a Ξ -cardinal, then the κ -extendibility of an ideal is directly dependent on the completability of the ideal. However, if κ is a Ξ -cardinal and there are measurable cardinals in an inner model, then the best we can say is that κ^+ -completable implies κ -extendible which implies κ -completable. We were able to show that adding ineffably-many Cohen reals produced a model in which there is a κ -extendible ideal which is not κ^+ -completable. Problem 54 tries to establish whether we can get a cardinal (in a model which uses a large cardinal consistent with $V = L$) which is not weakly compact but which acts like one with respect to extendibility. Problem 55 asks whether we need an ineffable cardinal or could get away with a weakly compact cardinal (which would be more satisfying).

? 122. **Problem 54.** *Does the consistency of the existence of an ineffable cardinal imply the consistency of the existence of a cardinal κ which is not weakly compact such that each κ -completable ideal is κ -extendible?*

? 123. **Problem 55.** *Does the consistency of the existence of a weakly compact cardinal imply the consistency of the existence of a cardinal κ which is not weakly compact and a κ -extendible ideal which is not κ^+ -completable?*

The case of measurable cardinals is a bit different. If κ is a measurable cardinal then there is a κ^+ -extendible ideal which is not κ^+ -completable. If κ is a compact cardinal then any κ -completable ideal is κ^+ -extendible. On the other hand, if it is consistent that there is a supercompact cardinal, then it is consistent that there is a cardinal κ which is not measurable and a κ -completable ideal on κ which is κ^+ -extendible. Problem 56 asks whether a supercompact cardinal is needed for the simplest κ -completable ideal.

? 124. **Problem 56.** *Does the consistency of the existence of a measurable cardinal imply the consistency of the existence of a cardinal κ which is not measurable and yet so that $[\kappa]^{<\kappa}$ is κ^+ -extendible?*

The set theory involved in (κ, λ) -extendibility is even more interesting.

In [1978] LAVER showed that if it is consistent that there is a huge cardinal then it is consistent that **GCH** holds and $[\omega_1]^{<\omega_1}$ is (ω_2, ω_2) -extendible. On the other hand if \aleph_3 Cohen subsets of ω_1 are added to a model of **GCH** then $[\omega_1]^{<\omega_1}$ is (ω_3, ω_3) -extendible. This latter argument really needs \aleph_3 thus provoking problem 57.

Problem 57. *Does the (ω_2, ω_2) -extendibility of $[\omega_1]^{<\omega_1}$ imply the consistency of large cardinals?* **125. ?**

Todorčević has shown that $[\omega_1]^{<\omega_1}$ is not (ω_1, ω_1) -extendible in **ZFC**. This takes away any idea that a right-hand coordinate \aleph_2 is any stronger than a right-hand coordinate of \aleph_1 thus giving rise to problem 58.

Problem 58. *Does the fact that $[\omega_1]^{<\omega_1}$ is (ω_2, ω_1) -extendible imply that $[\omega_1]^{<\omega_1}$ is (ω_2, ω_2) -extendible?* **126. ?**

We know that $[2^{\omega_1}]^{<2^{\omega_1}}$ is (μ, μ) -extendible whenever $\mu < 2^{\omega_1}$. On the other hand, $[2^{\omega_1}]^{<2^{\omega_1}}$ is not $(2^{\omega_1}, \omega)$ -extendible whenever $2^\omega = 2^{\omega_1}$. This is not the strongest conceivable negative consistency result thus raising problem 59.

Problem 59. *Does **ZFC** imply that $[2^{\omega_1}]^{<2^{\omega_1}}$ is $(2^{2^{\omega_1}}, \omega)$ extendible?* **127. ?**

We showed that it is consistent with $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ that $[2^{\omega_1}]^{<2^{\omega_1}}$ is not $(2^{\omega_1}, 2^{\omega_1})$ -extendible. There is no reason to think that this is dependent on a particular cardinal arithmetic which brings up problem 60.

Problem 60. *Is it consistent with every cardinal arithmetic that $[2^{\omega_1}]^{<2^{\omega_1}}$ is not $(2^{\omega_1}, 2^{\omega_1})$ -extendible?* **128. ?**

Problem 61 is an even broader question on which we have made no progress at all. On the other hand, if it is consistent that there is a weakly compact cardinal, then it is consistent that **CH** holds and that $[2^{\omega_1}]^{<2^{\omega_1}}$ is $(2^{\omega_1}, 2^{\omega_1})$ -extendible. It is not at all clear that a large cardinal is needed for this result and that is problem 62.

Problem 61. *Does **ZFC** imply that $[\kappa]^{<\kappa}$ is $(2^\kappa, 2^\kappa)$ -extendible whenever $2^\kappa > 2^{\omega_1}$?* **129. ?**

Problem 62. *Does the consistency of the $(2^{\omega_1}, 2^{\omega_1})$ -extendibility of $[2^{\omega_1}]^{<2^{\omega_1}}$ imply the consistency of the existence of an inaccessible cardinal?* **130. ?**

10. Homeomorphisms

In the 1895 volume of *Mathematische Annalen*, Georg Cantor wrote an article CANTOR [1895] in which a “back-and-forth” argument, which has become standard (see, for example EBERHART [1977]), was used to show, among other things, that, for any countable dense subsets A and B of the real line \mathbb{R} , there is a homeomorphism $h: \mathbb{R} \rightarrow \mathbb{R}$ which takes A onto B in a monotonic manner. The study of the existence of homeomorphisms taking one countable dense set to another was extended to more general spaces by FORT in [1962]. He showed that the product of countably many manifolds with boundary (for example, the Hilbert cube) admits such homeomorphisms. In 1972, the abstract study was begun by BENNETT [1972] who called such spaces “countable dense homogeneous” or “CDH”.

Bennett asked a series of questions including: “Are CDH continua locally connected?”. This question was answered by FITZPATRICK [1972] in the same year who showed that CDH connected locally compact metric spaces are locally connected. Fitzpatrick has now asked the natural question:

? **131. Problem 63.** *Are CDH connected complete metric spaces locally connected?*

Ungar raised a new question in 1978: Is each open subspace of a CDH space also a CDH space? He showed that the answer is yes for dense open subsets of locally compact separable metric spaces with no finite cut-set. In [1954] FORD defined a space X to be strongly locally homogeneous if there is a base of open sets U such that, for each $x, y \in U$ there is a homeomorphism of X which takes x to y but is the identity outside U . In 1982, van Mill also raised a question: Are connected CDH spaces strongly locally homogeneous? In [19∞a] FITZPATRICK and ZHOU answered these two questions negatively for the class of Hausdorff spaces. Unfortunately, their spaces were not regular.

In SIMON and WATSON [19∞], Petr Simon and the author exhibit a completely regular CDH space which fails to be strongly locally homogeneous and which contains an open subset which is not CDH. The construction uses the classical tradition (see Stackel’s 1895 article STACKEL [1895]) of constructing smooth functions on \mathbb{R}^2 which take one countable dense set to another as developed by DOBROWOLSKI in [1976].

These examples still leave some natural questions:

? **132. Problem 64.** *Is there a connected metric CDH space which has an open subspace which is not CDH? Is there a connected normal (compact) CDH space which has an open subspace which is not CDH?*

FITZPATRICK and ZHOU [19∞b] have raised an interesting question which nicely turns a strong property of our example against itself. The open subset of that space is *not* homogeneous and so:

Problem 65. *Is there a connected CDH space with an open subset which is connected and homogeneous but not CDH?* **133. ?**

Jan van Mill probably had in mind metric spaces (he does open the paper with the comment that all spaces are separable metric) and so the next problem is his:

Problem 66. *Are connected CDH metrizable (or compact Hausdorff) spaces strongly locally homogeneous?* **134. ?**

In [1972] BENNETT showed that strongly locally homogeneous locally compact separable metric spaces are CDH. Bennett's theorem proved to be influential. In [1969] DE GROOT weakened locally compact to complete (see proof on p.317 of ANDERSON, CURTIS and VAN MILL [1982]). In [1974] RAVDIN replaced strongly locally homogeneous by "locally homogeneous of variable type" (which implies "representable" which is apparently weaker than strongly locally homogeneous) for complete separable metric spaces. In [1974] FLETCHER and MCCOY showed that "representable" complete separable metric spaces are CDH while in [1976] BALES showed that "representable" is equivalent to strongly locally homogeneous in any case.

In [1982] VAN MILL put an end to the sequence of weakenings of Bennett's theorem by constructing a subset of \mathbb{R}^2 which is connected locally connected and strongly locally homogeneous but not CDH. In [19 ∞] STEPRANS and ZHOU contributed another lower bound by constructing a separable manifold (thus strongly locally homogeneous and locally compact) which is not CDH. Their manifold had weight 2^ω and they asked whether it is consistent that there is a separable manifold of weight less than the continuum which is not CDH. In WATSON [19 ∞ d], we constructed a consistent example of a separable manifold of weight less than the continuum which fails to be CDH. All of these results leave:

Problem 67. *Which cardinal invariant describes the minimum weight of a separable manifold which fails to be CDH?* **135. ?**

The cardinal invariant which is the least cardinality of a dominating family in ${}^\omega\omega$ was shown by Steprāns and Zhou to be a lower bound and I believe that there is a nice answer to this question. I was unable to find it myself but think that a connection between these cardinal invariants and manifolds would be quite interesting.

The next question uses the concept of n -homogeneity. A space X is said to be n -homogeneous if for any two subsets $\{a_i : i \leq n\}$ and $\{b_i : i \leq n\}$ of X there is a homeomorphism of X onto itself which takes each a_i to b_i . Thus homogeneity is the same as 1-homogeneity.

? 136. **Problem 68.** *Are CDH connected spaces 2-homogeneous?*

In [1972] BENNETT showed that CDH connected first countable spaces are homogeneous and asked “Are CDH continua n -homogeneous?”. UNGAR [1978] answered this question in 1978 by showing that, for compact metric spaces, CDH is equivalent to n -homogeneous. Meanwhile, in [19 ∞] FITZPATRICK and LAUER showed that the assumption of first countability in Bennett’s theorem was unnecessary.

In a paper with Steprāns (STEPRĀNS and WATSON [1987]), we investigated the problem of when there is an autohomeomorphism of Euclidean space which takes one uncountable dense set to another. A little thought makes it clear that to have a chance of finding such a autohomeomorphism we need to try to send one κ -dense set to another. A set $A \subset X$ is κ -dense if $|A \cap V| = \kappa$ for every non-empty open set V in X . We say that $BA(X, \kappa)$ holds if, for every two κ -dense subsets of X there is an autohomeomorphism H of X such that $H(A) = B$. The axiom $BA(\mathbb{R}, \aleph_1)$ was shown to be consistent with $\mathbf{MA} + \neg\mathbf{CH}$ by BAUMGARTNER in [1973] and then shown to be independent of $\mathbf{MA} + \neg\mathbf{CH}$ by ABRAHAM and SHELAH in [1981]. We showed in STEPRĀNS and WATSON [1987] that $\mathbf{MA} + \neg\mathbf{CH}$ implies that $BA(\mathbb{R}^n, \kappa)$ for each $n > 1$ and $\kappa < 2^\omega$. This showed that \mathbb{R} and \mathbb{R}^n are different insofar as these autohomeomorphisms are concerned when $n > 1$ but leaves the question of whether \mathbb{R}^2 is different from \mathbb{R}^3 in this context.

? 137. **Problem 69.** *Does $BA(\mathbb{R}^m, \aleph_1)$ imply $BA(\mathbb{R}^n, \aleph_1)$ when n and m are greater than 1 and unequal?*

We do think however that there is a positive answer for:

? 138. **Problem 70.** *Does $BA(\mathbb{R}, \aleph_1)$ imply $BA(\mathbb{R}^n, \aleph_1)$?*

There seems to be no apparent monotonicity on the second coordinate either:

? 139. **Problem 71.** *Does $BA(\mathbb{R}^n, \kappa)$ imply $BA(\mathbb{R}^n, \lambda)$ when $\kappa \neq \lambda$?*

... and the best problem of all is Baumgartner’s:

? 140. **Problem 72.** *Is $BA(\mathbb{R}, \aleph_2)$ consistent?*

We actually obtained under $\mathbf{MA} + \neg\mathbf{CH}$ a stronger result which we denote by $BA^+(\mathbb{R}^n, \kappa)$ which states that if $\{A_\alpha : \alpha \in \kappa\}$ and $\{B_\alpha : \alpha \in \kappa\}$ are families of disjoint countable dense subsets of \mathbb{R}^n , then there is an autohomeomorphism H of \mathbb{R}^n such that, for each $\alpha \in \kappa$, $H(A_\alpha) = B_\alpha$. The advantage of this axiom is that at least it is monotonic in the second coordinate. However

it is not clear that it is the same as the original question:

Problem 73. *Does $BA(\mathbb{R}^n, \kappa)$ imply $BA^+(\mathbb{R}^n, \kappa)$ when $n \geq 2$?* **141. ?**

11. Absoluteness

In the summer of 1986, Alan Dow, Bill Weiss, Juris Steprāns and I met for a few weeks to discuss problems of absoluteness in topology. We compiled a list of what we knew and what we did not know:

If one adds a Cohen subset of ω_1 , then the collectionwise normal space 2^{ω_1} gets a closed copy of the Tychonoff plank and so becomes not normal.

If one forces with a Suslin tree, then one can embed the square of the Alexandroff compactification of the discrete space of cardinality \aleph_1 minus the “corner point” in 2^{ω_1} , thus making that collectionwise normal space not normal.

If adds a new subset of ω_1 with finite conditions (that is, adds \aleph_1 many Cohen reals), then Bing’s space will cease to be normal.

Problem 74. *Can Cohen forcing make a collectionwise normal space not collectionwise normal? not normal?* **142. ?**

Problem 75. *Can one Cohen real kill normality?* **143. ?**

Problem 76. *Is there, in **ZFC**, a countable chain condition partial order which kills collectionwise normality?* **144. ?**

If you take $(\omega_2 + 1) \times (\omega_1 + 1) - \{\omega_2, \omega_1\}$ and then collapse ω_2 into an ordinal of cardinality ω_1 and uncountable cofinality, it becomes collectionwise normal.

If you take a (ω, \mathfrak{b}^*) gap and fill it, you make a non-normal space into one which is collectionwise normal. Cohen forcing does preserve non-normality and non-collectionwise normality as demonstrated by DOW, TALL and WEISS in [19∞].

Problem 77. *Can countably-closed cardinal-preserving forcing make a non-normal space normal?* **145. ?**

If you take the Alexandroff double of the reals but only use some of the isolated points, then this space is metrizable if and only if the isolated points are a relative F_σ . Thus countable chain condition forcing can make a non-metrizable space into a metrizable space. If you take a non-normal ladder system on a stationary costationary set and make it into a nonstationary set by forcing a club, then you have taken a non-normal space and made it into a metrizable space by a cardinal-preserving forcing.

Problem 78. *Can countable chain condition forcing make a non-normal space metrizable?* **146. ?**

? **147. Problem 79.** *Is there, in ZFC, a cardinal-preserving forcing which makes a non-normal space metrizable?*

? **148. Problem 80.** *Can countably-closed forcing make a non-metrizable space metrizable?*

? **149. Problem 81.** *Does countably-closed forcing preserve hereditary normality?*

We also looked at problems involving cardinal functions: If you take the Alexandroff double with a Bernstein set of isolated points and make that set into a relative F_σ by means of countable chain condition forcing then the Lindelöf number will have increased from ω up to 2^ω .

On the other hand, an elementary submodel argument shows that the Lindelöf number cannot increase up to $(2^\omega)^+$ under countable chain condition forcing.

The Lindelöf number can however increase arbitrarily under countably closed forcing—just add a Cohen subset of κ and look at 2^κ .

Density (or Lindelöf number) cannot be decreased under the covering lemma but the topology on κ in which the bounded sets are the only proper closed sets can be made separable by forcing κ to have cofinality ω and that forcing is cardinal-preserving but denies the covering lemma.

Tightness can be increased by countable chain condition forcing from ω to 2^ω but not past the continuum by using the quotient of ω_1 many convergent sequences.

Finally tightness can also be increased by countably-closed forcing by using the binary ω_1 -tree and defining a neighborhood to be the complement of finitely-many branches.

? **150. Problem 82.** *Can character be lowered by cofinality-preserving forcing?*

? **151. Problem 83.** *Does ZFC imply that cardinal-preserving forcing cannot decrease the density of Hausdorff spaces?*

? **152. Problem 84.** *Can countably-closed forcing lower density?*

? **153. Problem 85.** *Can cardinal-preserving forcing make a first countable non-Lindelöf space Lindelöf?*

Problem 86. Does **ZFC** imply that countably-closed forcing preserves Lindelöf for first countable spaces? **154. ?**

Problem 87. Does **ZFC** imply that countably-closed forcing preserves compactness (or the Lindelöf property) in sequential spaces? **155. ?**

Under **PFA**, the answer is yes for compactness.

12. Complementation

In 1936, Birkhoff published “On the Combination of Topologies” in *Fundamenta Mathematicae* (BIRKHOFF [1936]). In this paper, he ordered the family of all topologies on a set by letting $\tau_1 < \tau_2$ if and only if $\tau_1 \subset \tau_2$. He noted that the family of all topologies on a set is a lattice. That is to say, for any two topologies τ and σ on a set, there is a topology $\tau \wedge \sigma$ which is the greatest topology contained in both τ and σ (actually $\tau \wedge \sigma = \tau \cap \sigma$) and there is a topology $\tau \vee \sigma$ which is the least topology which contains both τ and σ . This lattice has a greatest element, the discrete topology and a smallest element, the indiscrete topology whose open sets are just the null set and the whole set. In fact, the lattice of all topologies on a set is a complete lattice; that is to say there is a greatest topology contained in each element of a family of topologies and there is a least topology which contains each element of a family of topologies.

The study of this lattice ought to be a basic pursuit both in combinatorial set theory and in general topology.

This section is concerned with the nature of complementation in this lattice. We say that topologies τ and σ are complementary if and only if $\tau \wedge \sigma = 0$ and $\tau \vee \sigma = 1$. For simplicity, we call any topology other than the discrete and the indiscrete a proper topology (both the discrete topology and the indiscrete topology are uniquely complemented). As a result in finite combinatorics, JURIS HARTMANIS showed, in [1958], that the lattice of all topologies on a finite set is complemented. He also asked whether the lattice of all topologies on an infinite set is complemented. He showed that, in fact, there are at least two complements for any proper topology on a set of size at least 3.

The next series of results were obtained by MANUEL BERRI [1966], Haim Gaifman and Anne Steiner. GAIFMAN [1961] brought some startling new methods to play that foreshadowed some of the arguments of Hajnal and Juhasz in their work on L -spaces and S -spaces (HAJNAL and JUHÁSZ [1968, 1969]) and showed in 1961 that the lattice of all topologies on a countable set is complemented. In fact, Gaifman showed that any proper topology on a countable set has at least two complements. In [1966] STEINER used a careful analysis of Gaifman’s argument to show that the lattice of all topologies on any set is complemented. A slightly modified proof of Steiner’s result was given by

VAN ROOIJ in [1968]. The question of the number of distinct complements a topology on a set must possess was first raised by BERRI [1966] before Steiner's theorem was obtained. He asked if every complemented proper topology on an infinite set must have at least two complements. SCHNARE [1968] showed that any proper topology (even not T_0) on a infinite set has indeed infinitely many complements (see also DACIC [1969]).

The last paper on this subject appeared in 1969 and was written by Paul Schnare as well (SCHNARE [1969]). In this paper, Schnare showed that any proper topology on an infinite set of cardinality κ has at least κ distinct complements. He also pointed out that there are at most 2^{2^κ} many complements on a set of cardinality κ . By exhibiting examples of topologies on a set of cardinality κ which possess exactly κ complements, exactly 2^κ complements and exactly 2^{2^κ} complements, Schnare showed under the generalized continuum hypothesis that three values are possible for the number of complements of a topology on an infinite set and that these three values are attained.

In 1989, I obtained some results in WATSON [1989] which solved the problem of establishing the exact number of complements of a topology on a fixed set of cardinality \aleph_n by showing that there are exactly $2n + 4$ possible values (although, depending on the cardinal arithmetic, some of these may coincide). This removed the assumption of the generalized continuum hypothesis in Schnare's theorem in the countable case and showed that some assumption of cardinal arithmetic is needed in all other cases. To be exact, the number of distinct complements of any topology on a set of cardinality \aleph_n is either 1 or \aleph_n^κ where $0 < \kappa \leq \aleph_n$ or $2^{\aleph_n + 2^{\omega_i}}$ where $0 \leq i \leq n$. In particular, on a countable set, exactly four values are possible: 1 or \aleph_0 or 2^{\aleph_0} or 2^{2^ω} .

However, this still leaves open the original 1966 question of Berri:

- ? **156. Problem 88.** (Berri, rephrased in light of new results) *Let κ be a fixed cardinal. What is the set of possible numbers of complements of topologies on a set of cardinality κ ?*

To make a specific conjecture:

- ? **157. Problem 89.** *Is the set of possible numbers of complements of topologies on a set of cardinality κ precisely:*

$$\{1\} \cup \{(\sup\{2^{2^\alpha} : \alpha < \lambda\})^\kappa : \lambda \leq \kappa\} \cup \{\kappa^\lambda : \lambda \leq \kappa\}?$$

A special case of this question is:

- ? **158. Problem 90.** *Can the number of complements of a topology on \aleph_ω be at least $(2^{\aleph_\omega})^+ + \sup\{2^{2^\kappa} : \kappa < \aleph_\omega\}$ and yet equal to neither $2^{2^{\aleph_\omega}}$ nor $(\sup\{2^{2^\kappa} : \kappa < \aleph_\omega\})^\omega$?*

Some insight into this question may be gained by pointing out that all topologies on a set of cardinality κ , except for some simple and easy to describe ones, have at least 2^κ many complements. The interested reader may find a proof of this non-trivial fact in WATSON [1989].

An additional piece of the puzzle may be provided by the fact that if κ is a regular cardinal and X is a topological space of cardinality at least κ which does not have 2^{2^κ} many complements, then, in every complement, either each point has a neighborhood of cardinality less than κ or the number of complements is exactly λ^μ for some cardinals λ and μ .

I think that an answer to this question is going to involve a mixture of set-theoretic topology and finite combinatorics. I worked on this question for quite a while and just ran out of steam when I got to the singular cardinals. Although questions 89 and 90 may seem a little technical, all they really ask is "Is the answer to question 88 a *definable* set?"

Other interesting questions on the number of complements deal with the concept of a T_1 -complement. This is the appropriate notion for T_1 spaces where the 0 in the lattice is just the cofinite topology:

Problem 91. *How many T_1 -complements can a T_1 topology on a set of cardinality κ have?* **159. ?**

In [1967], STEINER and STEINER showed that of any pair of T_1 -complements on a countable set, at least one is not Hausdorff. In [1969], ANDERSON and STEWART showed that of any pair of T_1 -complements, at least one is not first countable Hausdorff. Anderson and Stewart also asked: Can a Hausdorff topology on an uncountable set have a Hausdorff T_1 -complement? We showed in WATSON [19∞b] that there is a completely regular topology on a set of cardinality $(2^\omega)^+$ which is its own complement.

Problem 92. *It is consistent that any Hausdorff topology which is its own T_1 complement must lie on a set of cardinality at least $(2^\omega)^+$?* **160. ?**

A few months after we lectured on this result in Srní in January 1989, Aniszczyk constructed two T_1 -complementary compact Hausdorff spaces.

Problem 93. *Can two homeomorphic compact Hausdorff spaces be T_1 -complementary?* **161. ?**

Problem 94. *Does every Hausdorff topology have a T_1 -complement? What about every completely regular topology?* **162. ?**

Other information on this topic can be found in STEINER and STEINER [1968] and ANDERSON [1970].

In the notation of BIRKHOFF [1967], the maximum number of mutually T_1 -complementary topologies on a set of cardinality κ is the *complementary width* of the lattice of T_1 topological spaces on a set. In [1971] ANDERSON showed by a beautiful construction that there are at least κ mutually (T_1) complementary topologies on a set of cardinality κ . In STEPRĀNS and WATSON [19 ∞], we showed that there do not exist uncountably many mutually T_1 -complementary topologies on ω . It was also shown that it is consistent that there do not exist \aleph_2 many mutually T_1 -complementary topologies on ω_1 . On the other hand, it was shown that, under **CH**, there are 2^{\aleph_1} -many mutually T_1 -complementary topologies on ω_1 .

- ? **163. Problem 95.** *Does there exist, in **ZFC**, a cardinal κ so that there are 2^κ (or κ^+) many mutually T_1 -complementary topologies on κ ? What about if $\kappa = 2^\omega$?*

The general problem remains:

- ? **164. Problem 96.** *How many mutually T_1 -complementary topologies are there on a set of cardinality κ ?*

I think problems 95 and 96 are extremely interesting and believe that a solution will require a potent mixture of finite combinatorics and set-theoretic virtuosity.

- ? **165. Problem 97.** *Is there a triple of mutually complementary topologies which does not admit a fourth topology complementary to each of them? What are the sizes of maximal families of mutually complementary topologies?*
- ? **166. Problem 98.** *Is there a set of infinitely many but fewer than κ many mutually complementary topologies on a set of cardinality κ which does not admit another mutually complementary topology?*

In forthcoming papers with Jason Brown (BROWN and WATSON [19 ∞ , 1989b, 1989a]) we study topologies on a finite set. These are identical with preorders on a finite set (T_0 topologies are identical with partial orders on a finite set) and are thus of substantial interest to finite combinatorists. My interest in the subject originates in the fact that the somewhat difficult construction of WATSON [19 ∞ b] can be viewed as a preimage of a non-trivial topology on 18 elements. Many questions in this area have remained immune to our efforts. I mention only a few:

- ? **167. Problem 99.** *Which topology on a set of size n has the largest number of complements?*

We know which topology (other than the discrete or indiscrete) has the least number of complements but we do not know which T_0 topology (i.e., partial order) has the least number of complements.

Problem 100. *What is the maximum number of pairwise complementary T_0 topologies on a set of size n ?* **168. ?**

Specifically, we know that the answer to this question lies asymptotically between $\frac{n}{100 \log n}$ and $0.486n$ but do not know whether there is a linear lower bound. See also Anderson's beautiful paper ANDERSON [1973].

Problem 101. *If G is the graph on the set of topologies on the integers formed by putting an edge between two topologies if and only if they are complementary, then does G contain a copy of each countable graph ?* **169. ?**

Problem 102. *What is the diameter of the graph G ?* **170. ?**

We know that the answer is either 6 or 7.

We say that a topology is “self-complementary” if some complement of that topology is homeomorphic to the original topology.

Problem 103. (Jason Brown) *Characterize the self-complementary finite topologies.* **171. ?**

We have established a characterization of the self-complementary finite T_0 topologies (i.e., the self-complementary finite partial orders) and the finite equivalence relations (viewed as preorders).

Problem 104. *Can every lattice with 1 and 0 be homomorphically embedded in the lattice of topologies on some set?* **172. ?**

Problem 104 is the most important question in this section. I guess that the answer is yes but I have no idea how to prove this. Note that the image of the meet of two elements must be the topological meet of the images of the two elements, the image of the join of two elements must be the topological join of the images of the two elements and that, furthermore, the image of 0 must be the indiscrete topology and the image of 1 must be the discrete topology. It is this last requirement which is so difficult. Embedding the infinite lattice all of whose elements except 0 and 1 are incomparable means producing an infinite mutually complementary family of topologies and there are only a few ways of doing that—the intricate construction of BRUCE ANDERSON [1971] and the methods of STEPRĀNS and WATSON [19∞]. Modifying those arguments will not be easy.

13. Other Problems

? 173. **Problem 105.** *Give a topological proof of the fact that any connected metrizable manifold is the countable increasing union of compact connected manifolds.*

A few years ago, Raj Prasad asked me whether there was an elementary proof of this fact, since he had needed it and had to resort to quoting fairly deep results in low-dimensional topology to prove it. I came up with a proof which had a hole in it and then another . . . I still don't know why this fact is true. By the way, the manifold may have boundary but I can't see why this makes the problem any harder.

? 174. **Problem 106.** *Are there, under **CH** or otherwise, sets **LARGE**, **SMALL** $\subset [\omega_1]^\omega$ such that*

- (i) *If A is large and B is small, then there is a small infinite C such that $C \subset A - B$,*
- (ii) *If each A_n is small then there are finite sets F_n such that $\cup\{A_n - F_n : n \in \omega\}$ is small,*
- (iii) *If A is uncountable then there is a large $B \subset A$,*
- (iv) *If A is small and $B \subset A$ then B is small.*

I like this problem a lot. Under \diamond , the answer is yes (see OSTASZEWSKI [1976]). Deeper is the fact that under the existence of a Suslin line, the answer is yes (see RUDIN [1972], although it's hidden a little). I managed to get a few set theorists interested in this question, two of whom promptly announced that the answer is no under **PFA**. They both later withdrew this claim. The reason I was looking at this in the first place is slightly less interesting than the sheer combinatorics. Kunen's line (JUHÁSZ, KUNEN and RUDIN [1976]) had one advantage over Ostaszewski's line: it could be done under **CH**. The disadvantage is that it wasn't countably compact and worse yet it didn't have that beautiful property that closed sets are either countable or cocountable. If the answer to the above question is yes under **CH** then Ostaszewski's construction can be done under **CH**. To get a feel for the question take a \mathcal{P} -point ultrafilter on each element of a \clubsuit -sequence.

? 175. **Problem 107.** *Is there, in **ZFC**, a linear ordering in which every disjoint family of open sets is the union of countably many discrete subfamilies and yet in which there is no dense set which is the union of countably many closed discrete sets? Is there such a linear ordering if and only if there is a Suslin line?*

The Urysohn metrization theorem is to the Nagata-Smirnov-Stone metrization theorem as the Suslin problem is to this problem. It is incredible that

such a basic question about linear orderings is unsolved (and yet well-known in various disguises).

Problem 108. *Is there a topological space (or a completely regular space) in which the connected sets (with more than one point) are precisely the cofinite sets?* **176. ?**

This question was created while looking at an interesting paper of TSVID [1978]. He was asking simply whether (in a countable connected Hausdorff space) the connected sets could be a filter. That also remains unknown. I hawked this question at the 1989 Spring Topology Conference at Tennessee State University, asking for either a topological example (not necessarily even T_0) or on the other hand a proof that one couldn't find such a subset of the plane. John Kulesza at George Mason University sent a proof to me a few weeks later that there are no examples which are hereditarily normal Fréchet spaces. Later B. D. Garrett discovered a proof in ERDŐS [1944] (which Erdős attributes to Arthur Stone) of the fact that there are no metrizable examples (Kulesza rediscovered the same proof). I conjecture that there is an example (probably even completely regular) and that the existence of such an example depends on some hard finite combinatorics.

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