

A Network of Time-Varying Many-Server Fluid Queues with Customer Abandonment

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To describe the congestion in large-scale service systems, we introduce and analyze a non-Markovian open network of many-server fluid queues with customer abandonment, proportional routing and time-varying model elements. A proportion of the fluid completing service at each queue is routed immediately to each other queue, while the fluid not routed to other queues leaves the network. The fluid queue network serves as an approximation for the corresponding non-Markovian open network of many-server queues with Markovian routing, where all model elements may be time varying. We establish the existence of a unique vector of (net) arrival rate functions at each queue and the associated time-varying performance. In doing so, we provide the basis for an efficient algorithm, even for networks with many queues.

Key words: queues with time-varying arrivals; queueing networks; many-server queues; deterministic fluid model; customer abandonment; non-Markovian queues.

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1. Introduction

We introduce a new mathematical model, intended to help analyze (and thus manage) the congestion in large-scale service systems, such as in healthcare, judicial and penal systems, and both front-office and back-office operations in business systems. The model also should have other applications, because the model is both general and tractable. The model is an open network of time-varying many-server fluid queues with proportional routing. There are m queues, each with its own external fluid input. In addition a proportion $P_{i,j}(t)$ of the fluid output from queue i at time t is routed immediately to queue j , and a proportion $P_{i,0}(t) \equiv 1 - \sum_{j=1}^m P_{i,j}(t) \leq 1$ is routed out of the network (departs having successfully completed all required service). This framework permits feedback, both directly and indirectly after one or more transitions to other queues, as shown in Figure 1 for the case $m = 2$.

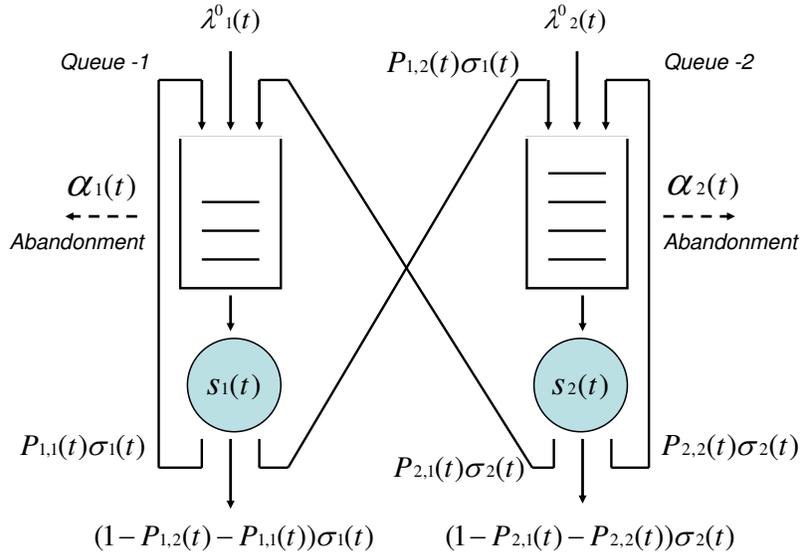


Figure 1 The open $(G_t/M_t/s_t + GI_t)^2/M_t$ fluid network.

The fluid content is intended to approximate the mean values of the corresponding many-server queueing system. For very large-scale service systems (with many servers at each queue and high arrival rates), the stochastic fluctuations about the mean values tend to be relatively small (essentially because of the law of large numbers), so that the deterministic fluid values serve as good direct approximations for the stochastic queueing quantities. The quality of approximations can be verified by simulation, as we illustrate here in §EC.6. We propose a simple heuristic stochastic refinement to estimate the full distribution at each time, beyond the mean values, in §8.

Since the model is tractable, we are providing the basis for creating a performance-analysis tool for large-scale service systems (allowing many queues and many servers at each queue) like the Queueing Network Analyzer (QNA), described in Whitt (1983). Algorithms based on performance formulas are appealing to supplement and complement computer simulation, because the models can be created and solved much more quickly. Thus they can be applied quickly in “what if” studies. They also can be efficiently embedded in optimization algorithms to systematically determine design and control parameters to meet performance objectives.

New methods are required because these large-scale service systems tend to be characterized by

many-server queues, where a large number of homogeneous servers work in parallel. For a many-server fluid queue with time-varying Markovian service rate $\mu(t)$, when the system content is $X(t)$ and the staffing is $s(t)$, the total service completion rate at time t is $\min\{X(t), s(t)\}\mu(t)$. When the system is not overloaded, the service completion rate is *not* equal to the input rate, but is instead *proportional to the system content*, cf. Chen and Mandelbaum (1991). When the service-time distribution is more general, the service completion rate depends on the *age distribution* of the fluid in service. We then focus on the two-parameter function $B(t, y)$ representing the fluid content in service at time t that has been so for at most time y , as in Whitt (2006).

When staffing is adequate in many-server systems, waiting times tend to be much shorter than service times. With few servers, congestion can be caused by only a few customers occasionally having exceptionally long service times. In contrast, congestion in many-server systems tends to be caused more by the cumulative impact of many customers and/or many servers. That cumulative impact tends to be realized through a *time-varying arrival rate* and a *time-varying staffing function*.

Hence, principal features of our model are a time-varying arrival rate and a time-varying staffing function. In fact we let all model elements be time-varying; the specific model we consider is $(G_t/M_t/s_t + GI_t)^m/M_t$, where the subscript t indicates time varying. The fluid model is intended to serve as an approximation for the corresponding many-server queueing system, having a general time-varying arrival process (the G_t), time-varying Markovian service (the first M_t), a time-varying number of servers (the s_t), a general time-varying abandonment-time distribution (the $+GI_t$) and time-varying Markovian routing (the final M_t). (We also treat GI service to some extent; see Remark 3 and §7.) We envision the model being used with only the arrival rate and the staffing function time varying; it is easy to read off results for that special case. Since proportional routing is the natural analog of Markovian routing for queueing networks, we use the model notation in Massey and Whitt (1993).

When staffing is adequate and service times are short, as in many telephone call centers, it is often possible to apply stationary models to analyze many-server queueing models with time-varying arrival rates, using some variant of the pointwise-stationary approximation, but when staffing is

occasionally inadequate or service times are longer, then other methods may be needed; see Green et al. (2007) for a review. To determine appropriate staffing levels and analyze performance in a many-server system with time-varying arrivals, we can often employ infinite-server models, as in Massey and Whitt (1993), Nelson and Taaffe (2004), Feldman et al. (2008) and references therein. However, the effectiveness of infinite-server models depends largely on the assumption that ultimately the system will be adequately staffed.

Many large-scale service systems inevitably experience periods of significant overloading, in which queues build up and customers experience significant delays. It is important to have effective mathematical methods that can predict the congestion (and provide a basis for design and control) when the system can experience periods of significant overloading. The proposed fluid models are in the spirit of early work by Newell (1982), but different in detail.

Since the model is a deterministic fluid model, the performance formulas produce the full sample path, just like simulation, but multiple replications are not required. We include the realistic feature of *customer abandonment* from each queue, which can significantly alter the overall performance, as emphasized by Garnett et al. (2002). We also treat *non-Markovian models*. The arrival processes need not be Poisson, the service-time and abandonment-time distributions need not be exponential. Unlike in familiar single-server fluid models, the service-time and abandonment-time distributions beyond their means remain essential drivers of system performance in many-server fluid models, as they do in many server systems. As discussed in §2 of Liu and Whitt (2010), theoretical support for the fluid models can be provided by many-server heavy-traffic limits, but we do not prove any here; see Pang and Whitt (2009) and references therein for related limits for queueing models.

We make two important contributions here beyond our previous paper Liu and Whitt (2010). First, we extend the model from a single fluid queue to a network of fluid queues. Second, by focusing on M_t service instead of GI service, we are able to establish the existence of a unique (computable) performance description for both one fluid queue and the network generalization without directly assuming that there are only finitely many switches between overloaded and underloaded

intervals in any finite time interval. These results are based on monotonicity and Lipschitz continuity properties of the fluid queue model in §5. Thus, we provide a stronger mathematical basis for the performance analysis methods with M_t service.

This paper is organized as follows: In §2 we introduce the $G_t/M_t/s_t + GI_t$ model of a single fluid queue. In §3 we show how the overloaded and underloaded times occur in alternating intervals of positive length, under regularity conditions, and we introduce a specific piecewise-polynomial framework for assuring that there are only finitely many switches in each finite time interval. In §4 we present the performance formulas for one queue. In §5 we extend the results to general piecewise-continuous arrival rate functions, thus providing an essential step for extending the analysis to networks. In §6 we define the network generalization and establish the existence of a unique vector of arrival rate functions at each queue and thus the performance in the network. In §7 we characterize the steady-state performance in the stationary $(G/GI/s + GI)^m/M$ fluid queue network. In §8 we propose a heuristic stochastic refinement. In §9 we present postponed proofs for §3. Finally, in §10 we draw conclusions. In the e-companion we provide (i) some proofs, (ii) some remarks, and (iii) an illustrative comparison with simulation of a large-scale queueing system.

2. The $G_t/M_t/s_t + GI_t$ Fluid Queue

The model for a single time-varying fluid queue is a variant of the model in §3 of Liu and Whitt (2010). The difference is that here service is M_t and abandonment is GI_t instead of both being GI .

Fluid is a deterministic divisible quantity, which enters the system from outside. The total fluid input over an interval $[0, t]$ is $\Lambda(t)$, where Λ is an absolutely continuous function with

$$\Lambda(t) \equiv \int_0^t \lambda(y) dy, \quad t \geq 0. \quad (1)$$

Fluid input flows directly into the service facility if the system is underloaded; otherwise it flows into the queue.

By M_t service, we mean that service is provided at the service facility at time-varying rate $\mu(t)$ per quantum of fluid in the service facility; i.e., if the total fluid content in service at time t is $B(t)$, then the total service completion rate at time t is

$$\sigma(t) \equiv B(t)\mu(t), \quad t \geq 0. \quad (2)$$

Let $S(t)$ be the total amount of fluid to complete service in the interval $[0, t]$; then

$$S(t) \equiv \int_0^t \sigma(y) dy = \int_0^t B(y)\mu(y) dy, \quad t \geq 0. \quad (3)$$

Fluid waiting in queue may abandon. Specifically, we assume that a proportion $F_t(x)$ of any fluid to enter the queue at time t will abandon by time $t+x$ if it has not yet entered service, where F_t is an absolutely continuous cumulative distribution function (cdf) for each t , $-\infty < t < +\infty$, with

$$F_t(x) = \int_0^x f_t(y) dy, \quad x \geq 0, \quad \text{and} \quad \bar{F}_t(x) \equiv 1 - F_t(x), \quad x \geq 0. \quad (4)$$

Let $h_{F_t}(y) \equiv f_t(y)/\bar{F}_t(y)$ be the hazard rate associated with the patience (abandonment) cdf F_t .

Let the staffing function (service capacity) s be an absolutely continuous function with

$$s(t) \equiv \int_0^t s'(y) dy, \quad t \geq 0. \quad (5)$$

In general, there is no guarantee that a staffing function s is feasible; i.e., having the property that no fluid that has entered service must leave without completing service, because we allow s to decrease. We directly assume that the staffing function we consider is feasible, but we also indicate how to detect the first violation and then construct the minimum feasible staffing function greater than or equal to the given staffing function; see Theorem 6.

ASSUMPTION 1. (*feasible staffing*) *The staffing function s is feasible, allowing all fluid that enters service to stay in service until service is completed; i.e., when s decreases, it never forces content out of service.*

As before, system performance will be described by a pair of two-parameter deterministic functions (\hat{B}, \hat{Q}) , where $\hat{B}(t, y)$ ($\hat{Q}(t, y)$) is the total quantity of fluid in service (in queue) at time t that has been so for time at most y , for $t \geq 0$ and $y \geq 0$. These functions will be absolutely continuous in the second parameter, so that

$$\hat{B}(t, y) \equiv \int_0^y b(t, x) dx \quad \text{and} \quad \hat{Q}(t, y) \equiv \int_0^y q(t, x) dx, \quad (6)$$

for $t \geq 0$ and $y \geq 0$. We will be characterizing performance primarily through the pair of two-parameter fluid content densities (b, q) . Let $B(t) \equiv \hat{B}(t, \infty)$ and $Q(t) \equiv \hat{Q}(t, \infty)$ be the total fluid content in service and in queue, respectively. Since service is assumed to be M_t , the performance will primarily depend on b via B . (We will not directly discuss \hat{B} .)

The system has unlimited waiting room and the FCFS service discipline. Whenever $Q(t) > 0$, we require that there be no free capacity in service, i.e., $B(t) = s(t)$. Also, whenever $B(t) < s(t)$, then the queue must be empty, i.e., $Q(t) = 0$.

ASSUMPTION 2. (*fluid dynamics constraints, FDC's*) For all $t \geq 0$,

$$(B(t) - s(t))Q(t) = 0 \quad \text{and} \quad B(t) \leq s(t). \quad (7)$$

Let $A(t)$ be the total amount of fluid to abandon in the interval $[0, t]$; then

$$A(t) \equiv \int_0^t \alpha(y) dy, \quad t \geq 0, \quad (8)$$

where $\alpha(t)$ is the abandonment rate at time t . Since $q(t, x)$ is the density of fluid in queue at time t that arrived at time $t - x$, the abandonment rate at time t is

$$\alpha(t) \equiv \int_0^\infty q(t, y) h_{F_{t-y}}(y) dy, \quad t \geq 0, \quad (9)$$

where $h_{F_t}(y)$ is the hazard rate associated with the patience cdf F_t . (Recall that F_t is defined for t extending into the past.) Hence,

$$A(t) = \int_0^t \left(\int_0^\infty q(u, y) h_{F_{u-y}}(y) dy \right) du, \quad t \geq 0. \quad (10)$$

Let $E(t)$ be the amount of fluid to enter service in $[0, t]$. We have

$$E(t) \equiv \int_0^t \gamma(u) du, \quad t \geq 0, \quad (11)$$

where $\gamma(t) \equiv b(t, 0)$ is the rate fluid enters service at time t . The rate fluid enters service depends on whether the system is underloaded or overloaded. If the system is underloaded, then the external input directly enters service; if the system is overloaded, then the fluid to enter service is determined

by the rate, $\eta(t)$, that service capacity becomes available at time t . Service capacity becomes available due to service completion and any change in the staffing function. Hence the rate service becomes available is

$$\eta(t) \equiv s'(t) + \sigma(t) = s'(t) + B(t)\mu(t), \quad t \geq 0, \quad (12)$$

so that $\eta(t) = s'(t) + s(t)\mu(t)$ if the system is overloaded at time t .

Finally, we need to specify the initial conditions. That is done via the initial fluid densities $b(0, x)$ and $q(0, x)$, $x \geq 0$; then $\hat{B}(0, y)$ and $\hat{Q}(0, y)$ are defined via (6), while $B(0) \equiv \hat{B}(0, \infty)$ and $Q(0) \equiv \hat{Q}(0, \infty)$, as defined before. Let $w(0)$ be defined in terms of $q(0, \cdot)$ as follows:

$$w(0) \equiv \inf \{x > 0 : q(0, y) = 0 \text{ for all } y > x\}. \quad (13)$$

ASSUMPTION 3. (*finite initial values*) $B(0) < \infty$, $Q(0) < \infty$ and $w(0) < \infty$.

In summary, the basic model data are in the six-tuple $(\lambda, s, \mu, F, b(0, \cdot), q(0, \cdot))$.

We will also be interested in waiting time functions. Let the *boundary waiting time* $w(t)$ be the delay experienced by the quantum of fluid at the head of the queue at time t and let the *potential waiting time* $v(t)$ be the virtual delay of a quantum of fluid arriving at time t under the assumption that the quantum has infinite patience. Informally, as in (13),

$$w(t) \equiv \inf \{x > 0 : q(t, y) = 0 \text{ for all } y > x\}. \quad (14)$$

A proper definition of q , w and v is somewhat complicated, but that has already been done in §5.2 and §5.3 in Liu and Whitt (2010), to which we refer.

Since the service discipline is FCFS, fluid leaves the queue to enter service from the right boundary of $q(t, x)$. Since the service is M_t , the proportion of fluid in service at time t that will still be in service at time $t + x$ is

$$\bar{G}_t(x) = e^{-M(t, t+x)} \quad \text{where} \quad M(t, t+x) \equiv \int_t^{t+x} \mu(y) dy, \quad t \geq 0 \quad \text{and} \quad x \geq 0. \quad (15)$$

Note that G_t coincides with the time-varying service-time cdf of a quantum of fluid that enters service at time t . The cdf G_t has density $g_t(x) = \mu(t+x)\bar{G}_t(x)$ and hazard rate $h_{G_t}(x) = \mu(t+x)$, $x \geq 0$.

Based on the way the queueing system operates, we assume that q and b satisfy the following two fundamental evolution equations.

ASSUMPTION 4. (*fundamental evolution equations*) For $t \geq 0$, $x \geq 0$ and $u \geq 0$,

$$q(t+u, x+u) = q(t, x) \frac{\bar{F}_{t-x}(x+u)}{\bar{F}_{t-x}(x)}, \quad 0 \leq x < w(t), \quad (16)$$

$$b(t+u, x+u) = b(t, x) \frac{\bar{G}_{t-x}(x+u)}{\bar{G}_{t-x}(x)} = b(t, x) e^{-M(t, t+u)}, \quad (17)$$

where M is defined in (15).

We now turn to the regularity conditions we impose on the model data. We develop a “smooth” model. For that purpose, let \mathbb{C}_p be the space of piecewise continuous real-valued functions of a real variable, by which we mean that there are only finitely many discontinuities in each finite interval, and that left and right limits exist at each discontinuity point, where the whole function is right continuous. Hence, $\mathbb{C}_p \subset \mathbb{D}$, where \mathbb{D} is the usual function space of right continuous functions with left limits; see Whitt (2002). Liu and Whitt (2010).

ASSUMPTION 5. (*smoothness*) $s', \lambda, f_t, f, \mu, b(0, \cdot), q(0, \cdot)$ in \mathbb{C}_p for each x and t .

As a consequence, $s, \Lambda, F_t, B(0, \cdot), Q(0, \cdot)$ are differentiable functions with derivatives in \mathbb{C}_p for each t ; we say that they are elements of \mathbb{C}_p^1 .

By considering M_t service instead of GI service, we no longer need the fixed point equation in Theorem 5.1 of Liu and Whitt (2010) to obtain b . That allows us to omit Assumptions 5.1-5.3 of Liu and Whitt (2010). As before, in order to treat the boundary waiting time w , we need to impose a regularity condition on the arrival rate function and the initial queue density. As in Assumption 5.5 of Liu and Whitt (2010), we make the following assumption.

ASSUMPTION 6. (*positive arrival rate and initial queue density*) For all $t \geq 0$ and $x \geq 0$,

$$\lambda_{\inf}(t) \equiv \inf_{0 \leq u \leq t} \{\lambda(u)\} > 0 \quad \text{and} \quad q_{\inf}(0, x) \equiv \inf_{0 \leq u \leq x} \{q(0, u)\} > 0.$$

In order to be sure that the PWT function v is finite, we make two more assumptions, corresponding to Assumptions 5.6 and 5.7 of Liu and Whitt (2010).

ASSUMPTION 7. (*minimum staffing level*) There exists s_L such that $s(t) \geq s_L > 0$ for all $t \geq 0$.

ASSUMPTION 8. (*minimum service rate*) There exists μ_L such that $\mu(t) \geq \mu_L > 0$ for all $t \geq 0$.

Finally to treat A with the time-varying abandonment cdf F_t , we first introduce bounds for the time-varying pdf f_t and complementary cdf \bar{F}_t . Let

$$f^\dagger \equiv \sup \{f_t(x) : x \geq 0, \quad -\infty < t \leq T\} \quad \text{and} \quad \bar{F}^\dagger(x) \equiv \inf \{\bar{F}_t(x) : -\infty \leq t \leq T\}. \quad (18)$$

ASSUMPTION 9. (*controlling the time-varying abandonment cdf*) $f^\dagger < \infty$ and $\bar{F}^\dagger(x) > 0$ for all $x > 0$, where f^\dagger and $\bar{F}^\dagger(x)$ is defined in (18).

In summary, here we have made Assumptions 3.1-3.6 and 5.4-5.7 of Liu and Whitt (2010) (with minor modifications because of M_t service and GI_t abandonment instead of both being GI). Assumption 3 above combines Assumptions 3.4 and 5.4 of Liu and Whitt (2010). We show how to relax Assumption 3.7 of Liu and Whitt (2010) in the next section. We no longer need Assumptions 5.1-5.3 because we do not need to solve the fixed point equation. Assumption 9 here is new, because of the time-varying abandonment.

3. Underloaded and Overloaded Intervals

In Assumption 3.7 of Liu and Whitt (2010), we directly assumed that the system alternates between underloaded intervals and overloaded intervals, with there being only finitely many switches in any finite interval. In this paper, we provide conditions under which that assumption can be guaranteed to hold, and then show how to treat the more general case as a limit of such systems. This extension is important to rigorously treat fluid queue networks. This extension is facilitated by having M_t service, because with M_t service we no longer need to solve the fixed point equation in Theorem 5.1 of Liu and Whitt (2010) to find b .

We initially classify the system state as overloaded or underloaded at time t as follows. Recall that the rate service capacity becomes available at time t is $\eta(t) \equiv s'(t) + \sigma(t)$, as in (12) above.

DEFINITION 1. The system is **overloaded** if either (i) $Q(t) > 0$ or

(ii) $Q(t) = 0$, $B(t) = s(t)$ and $\lambda(t) > \eta(t) = s'(t) + s(t)\mu(t)$;

the system is **underloaded** if either (i) $B(t) < s(t)$ or

$$(ii) \quad B(t) = s(t), \quad Q(t) = 0 \quad \text{and} \quad \lambda(t) \leq \eta(t) = s'(t) + s(t)\mu(t).$$

At every time t , the system is thus either overloaded or underloaded.

We now define the set of switch times. For that purpose, let $\mathcal{O}(A)$ ($\mathcal{U}(A)$) be the set of overloaded (underloaded) times t in the subset A of a designated interval $[0, T]$. From Definition 1, $\mathcal{U}(A) = A - \mathcal{O}(A)$ for each subset A (the complement relative to A).

DEFINITION 2. The subset \mathcal{S} be of **switch times** in $[0, T]$ is the subset of t for which

$$\mathcal{U}(((t - \epsilon) \vee 0, (t + \epsilon) \wedge T)) \neq \emptyset \quad \text{and} \quad \mathcal{O}(((t - \epsilon) \vee 0, (t + \epsilon) \wedge T)) \neq \emptyset \quad \text{for all} \quad \epsilon > 0. \quad (19)$$

To neatly classify the switching times, we further classify some of the underloaded times.

DEFINITION 3. An underloaded time t is **isolated** if (i) either $[0, t)$ or (a, t) is an overloaded interval and (ii) either $(t, T]$ or (t, b) is an overloaded interval.

We now reclassify all isolated underloaded points as overloaded points. When we reclassify each isolated underloaded point, we replace the two connecting overloaded intervals by the common overloaded interval; e.g., when t is an isolated underloaded time between overloaded intervals (a, t) and (t, b) , we replace the two intervals by the single interval (a, b) . In §9 we show that this procedure is well defined. In the remainder of this section we present the key results allowing us to ensure that \mathcal{S} is finite. We present the proofs in §9 and §EC.1. Our first structural result is

THEOREM 1. (partition into intervals) *After all isolated underloaded times have been reclassified as overloaded and all overloaded intervals have been increased as specified above, the interval $[0, T]$ can be partitioned into at most countably many alternating overloaded and underloaded intervals (of positive length). The resulting switch points are the boundary points between overloaded intervals and underloaded intervals.*

Our analysis above has shown how to partition the interval $[0, T]$ into alternating overloaded and underloaded intervals of positive length. Then the switch points are clearly identified as the

boundary points. It is then convenient to adopt the convention that all intervals be left closed and right open (e.g., of the form $[a, b)$), except at the interval endpoints 0 and T , so that the regime identification function $r(t) \equiv 1_{\{\mathcal{O}([0, T])\}}(t)$, where $1_{\{A\}}$ is the usual indicator function, is right continuous with left limits. This convention does not alter the switch points.

We now relate the subset \mathcal{S} to the set of discontinuity points and the zero set of the function

$$\zeta(t) \equiv \lambda(t) - s'(t) - s(t)\mu(t), \quad t \geq 0. \quad (20)$$

Note that ζ depends only on the basic model functions λ , s and μ . Also note that $\zeta = \lambda - \eta$ in the overloaded case of Definition 1. Let \mathcal{D}_ζ be the set of discontinuities of ζ in (20) and let $\mathcal{Z}_\zeta \equiv \{t \in [0, T] : \zeta(t) = 0\}$ be the zero set.

THEOREM 2. (relating switches to zeros and discontinuities of ζ) *For any interval $[0, T]$, the subsets \mathcal{S} , \mathcal{Z}_ζ and \mathcal{D}_ζ are closed subsets with $|\mathcal{S}| \leq |\mathcal{Z}_\zeta| + |\mathcal{D}_\zeta| - 1$. Moreover, the bound in is tight; i.e., there are examples for which the bound holds as an equality.*

We now introduce a convenient subset of functions in \mathbb{C}_p to represent our model data λ , μ and s' . The class is sufficiently general that it can represent any function in \mathbb{C}_p and, at the same time, it allows us to control the zeros of ζ , so that we know in advance that there are only finitely many switches between overloaded and underloaded intervals in any finite interval.

Let $\mathcal{P}_{m,n} \equiv \mathcal{P}_{T,m,n}$ be the space of *piecewise polynomials* on the interval $[0, T]$, where $[0, T]$ is partitioned into n subintervals, on each of which there is a polynomial of order at most m . We start with three elementary lemmas about $\mathcal{P}_{m,n}$. (We do not require that the overall function be continuous, but each function necessarily is in \mathbb{C}_p .) The first lemma states that any function in \mathbb{C}_p can be approximated uniformly by a function from $\mathcal{P}_{m,n}$, so that there is no practical loss of generality to restricting the model data to be in $\mathcal{P}_{m,n}$ instead of \mathbb{C}_p .

LEMMA 1. (uniform approximation) *For any function $h \in \mathbb{C}_p$ over a finite interval $[0, T]$ and any $\epsilon > 0$, there exists a function $\tilde{h} \in \mathcal{P}_{m,n}$ for some positive integers m and n such that $\|h - \tilde{h}\|_T < \epsilon$.*

The second lemma states that we can go back and forth between the functions λ, s', μ and their integrals Λ, s, M in $\mathcal{P}_{m,n}$ conveniently; i.e., the integral or derivative of a polynomial is again a polynomial. In particular, we can analytically calculate the integral for M in definition (15), as needed for the fundamental evolution equation for b in (17).

LEMMA 2. (representation of integrals) $\lambda, s', \mu \in \mathcal{P}_{m,n} \subset \mathbb{C}_p$ if and only if $\Lambda, M(t, t + \cdot), M(u - \cdot, u), s \in \mathcal{P}_{m+1,n} \cap C$.

The third lemma states that the function ζ inherits piecewise-polynomial structure assumed for the basic model functions λ, s', μ .

LEMMA 3. (preservation of piecewise-polynomial structure) If $\lambda \in \mathcal{P}_{m_1, n_1}, s' \in \mathcal{P}_{m_2, n_2},$ and $\mu \in \mathcal{P}_{m_3, n_3},$ then $\zeta \in \mathcal{P}_{m,n},$ where $n \leq n_1 + n_2 + n_3$ and $m \leq m_1 \vee m_2 \vee m_3(m_2 + 1).$

The following theorem serves as the basis for our analysis.

THEOREM 3. (finitely many switches) If $\zeta \in \mathcal{P}_{m,n}$ for ζ in (20), then $|\mathcal{S}| \leq n(m + 1) - 1.$

Hence, we can carry out the construction of the desired performance vector $(b, q, w, v, \sigma, \alpha)$ under the assumptions that the basic model functions (λ, s, μ) are such that there are only finitely many switches between overloaded intervals and underloaded intervals in any given interval $[0, T]$. It suffices to have $\lambda, s', \mu \in \mathcal{P}_{m,n}$ for some m and n . The space $\mathcal{P}_{m,n}$ is useful for the theory, but it should not be needed in applications; see Remark EC.3.

4. The Performance at One Queue

In this section we derive the performance functions under the assumption that there are only finitely many switches between overloaded and underloaded intervals. We have just seen that a sufficient condition for that is to have $\zeta \in \mathcal{P}_{m,n}$ for some m and n , for which a sufficient condition is to have $\lambda, s', \mu \in \mathcal{P}_{m,n}$ for some m and n . Here we can apply the previous results in Liu and Whitt (2010), making proper adjustments to account for the change from GI service and abandonment to M_t service and GI_t abandonment.

An underloaded interval requires modification to account for M_t service. Since the rate fluid enters service is $\gamma(t) = b(t, 0) = \lambda(t)$ when the system is underloaded, we immediately obtain an expression for $b(t, x)$ from (17). Recall that we have assumed that $b(0, \cdot) \in \mathbb{C}_p$.

PROPOSITION 1. (service content in the underloaded case) *For the fluid model with unlimited service capacity ($s(t) \equiv \infty$ for all $t \geq 0$), starting at time 0,*

$$\begin{aligned} b(t, x) &= e^{-M(t-x, t)} \lambda(t-x) \mathbf{1}_{\{x \leq t\}} + e^{-M(0, t)} b(0, x-t) \mathbf{1}_{\{x > t\}}, \\ B(t) &= \int_0^t e^{-M(t-x, t)} \lambda(t-x) dx + B(0) e^{-M(0, t)}, \quad 0 \leq t < T, \end{aligned} \quad (21)$$

where M is defined in (15). If, instead, a finite-capacity system starts underloaded, then the same formulas apply over the interval $[0, T)$, where $T \equiv \inf \{t \geq 0 : B(t) > s(t)\}$, with $T = \infty$ if the infimum is never obtained. Hence, $b(t, \cdot), b(\cdot, x), B \in \mathbb{C}_p$ for all $t \geq 0$ and $x \geq 0$, for t in the underloaded interval.

There is dramatic simplification in going from GI service to M_t service in an overloaded interval. Then we simply have $B(t) = s(t)$. The rate fluid enters service is equal to the rate service capacity becomes available: $\gamma(t) = \eta(t) = s'(t) + s(t)\mu(t)$. Paralleling equation (19) of Liu and Whitt (2010), for an overloaded interval starting at time 0, we have

PROPOSITION 2. (service content in the overloaded case) *For the fluid model in an overloaded interval, $B(t) = s(t)$ and*

$$b(t, x) = (s'(t-x) + s(t-x)\mu(t-x)) e^{-M(t-x, t)} \mathbf{1}_{\{x \leq t\}} + b(0, x-t) e^{-M(0, t)} \mathbf{1}_{\{x > t\}},$$

where M is defined in (15). Hence, $b(t, \cdot), b(\cdot, x), B \in \mathbb{C}_p$ for all $t \geq 0$ and $x \geq 0$ in an overloaded interval.

COROLLARY 1. (overall smoothness for the service content) *If there are only finitely many switches between overloaded and underloaded intervals in $[0, T]$, then $b(t, \cdot), b(\cdot, x), B \in \mathbb{C}_p$ for all $t, 0 \leq t \leq T$, and $x \geq 0$.*

We treat q , w and v just as in §5.2 and §5.3 in Liu and Whitt (2010), making adjustments for the time-varying abandonment cdf F_t . Let $\tilde{q}(t, x)$ be $q(t, x)$ during the overload interval $[0, T]$ under the assumption that no fluid enters service from queue.

PROPOSITION 3. (*queue content without transfer into service in the overloaded case*) *During an overloaded interval,*

$$\tilde{q}(t, x) = \lambda(t - x)\bar{F}_{t-x}(x)\mathbf{1}_{\{x \leq t\}} + q(0, x - t)\frac{\bar{F}_{t-x}(x)}{\bar{F}_{t-x}(x - t)}\mathbf{1}_{\{t < x\}}. \quad (22)$$

so that $\tilde{q}(t, \cdot)$ and $\tilde{q}(\cdot, x)$ belong to \mathbb{C}_p for each t and x .

We get an expression for q provided that we can find w .

COROLLARY 2. (*from \tilde{q} to q*) *Given the BWT w in an overloaded interval,*

$$\begin{aligned} q(t, x) &= \tilde{q}(t - x, 0)\bar{F}_{t-x}(x)\mathbf{1}_{\{x \leq w(t) \wedge t\}} + \tilde{q}(0, x - t)\frac{\bar{F}_{t-x}(x)}{\bar{F}_{t-x}(x - t)}\mathbf{1}_{\{t < x \leq w(t)\}} \\ &= \lambda(t - x)\bar{F}_{t-x}(x)\mathbf{1}_{\{x \leq w(t) \wedge t\}} + q(0, x - t)\frac{\bar{F}_{t-x}(x)}{\bar{F}_{t-x}(x - t)}\mathbf{1}_{\{t < x \leq w(t)\}}. \end{aligned} \quad (23)$$

Moreover, $q(t, \cdot) \in \mathbb{C}_p$ for all $t \geq 0$.

It now remains to define and characterize the BWT w . We can *define* the BWT w by postulating that two expressions for the amount of fluid to enter service over any interval $[t, t + \delta]$, namely,

$$E(t + \delta) - E(t) \equiv \int_t^{t+\delta} b(u, 0) du = I(t, w(t), \tilde{q}) - A(t, t + \delta), \quad (24)$$

where $I \equiv I(t, w(t), \tilde{q})$ is the amount of fluid removed from the right boundary of \tilde{q} during the time interval $[t, t + \delta]$ and $A(t, t + \delta)$ is the amount of the fluid content in I that abandons in the interval $[t, t + \delta]$. We then show that, if (24) holds, then w satisfies an ODE. However, our previous proof of uniqueness for the solution of that ODE does not extend directly to time-varying abandonment cdf's. Hence we assume that either (i) the abandonment cdf F_t is independent of t or (ii) extra conditions hold, allowing us to apply the classical Picard-Lindelöf theorem, Theorem 2.2 of Teschl (2000); see §EC.2. One extra requirement is that the rate fluid enters service is bounded below; we also show how to obtain that in EC.2. The proofs of Theorems 4 and 6 are in §EC.2.

THEOREM 4. (*the BWT ODE*) Consider an overloaded interval $[0, T]$. The BWT w is well defined by the relation (24), being Lipschitz continuous on $[0, T]$ with $w(t+u) \leq w(t) + u$ for all $t \geq 0$ and $u \geq 0$ with $t+u \leq T$. Moreover, w is right differentiable everywhere with right derivative

$$w'(t+) = \Psi(t, w(t)) \equiv 1 - \frac{\gamma(t+)}{\tilde{q}(t, w(t)-)}, \quad (25)$$

where $\gamma(t) = s'(t) + s(t)\mu(t)$, $t \geq 0$, and left differentiable everywhere (but not necessarily differentiable) with value

$$w'(t-) = \tilde{\Psi}(t, w(t)) \equiv 1 - \frac{\gamma(t-)}{\tilde{q}(t, w(t)+)}. \quad (26)$$

Overall, w is continuously differentiable everywhere except for finitely many t . If either (i) the abandonment cdf's F_t are independent of t or (ii) the partial derivative $\partial F_t(x)/\partial t$ is bounded over $[0, T] \times [0, c]$ for all c , and $\lambda, q(0, \cdot)$ have bounded derivatives in the intervals where they are continuous, and there exists a constant $e_L > 0$ such that $\gamma(t) \geq e_L$ for $0 \leq t \leq T$, then w is characterized as the unique solution of the initial value problem (IVP) on $[0, T]$ based on the ODE (25) and any initial value $w(0)$.

COROLLARY 3. (*end of the overloaded interval*) We can compute the end of an overloaded interval as $T \equiv \inf \{t \geq 0 : w(t) = 0 \text{ and } \lambda(t) \leq s'(t) + s(t)\mu(t)\}$.

COROLLARY 4. (*smoothness of $q(t, \cdot)$*) Under the assumptions of Theorem 4, q is given by (23) with $q(\cdot, x) \in \mathbb{C}_p$ for all x . (We have already deduced that $q(t, \cdot) \in \mathbb{C}_p$ for all t in Corollary 2.)

THEOREM 5. (*v and w*) Consider an overloaded interval. Then v is finite and v is the unique function in \mathbb{D} satisfying the equation

$$v(t - w(t)) = w(t) \quad \text{or, equivalently,} \quad v(t) = w(t + v(t)) \quad \text{for all } t \geq 0. \quad (27)$$

Moreover, v is discontinuous at t if and only if there exists $\epsilon > 0$ such that $w(t + v(t) + \epsilon) = w(t + v(t)) + \epsilon$, which in turn holds if and only if $b(u, 0) = 0$ for $t + v(t) \leq u \leq t + v(t) + \epsilon$. If $b(\cdot, 0) > 0$ a.e. with respect to Lebesgue measure, then v is continuous.

As shown in Liu and Whitt (2010), the proof of Theorem 5 provides an elementary algorithm to compute v once w has been computed. Theorem 5.6 of Liu and Whitt (2010) shows that v satisfies its own ODE under additional regularity conditions.

Overall, we can compute $b(t, x)$ and $B(t)$ directly via Propositions 1 and 2. We obtain the service completion rate directly from $\sigma(t) = B(t)\mu(t)$, $t \geq 0$, as in (2) and \tilde{q} directly from Proposition 3. We then compute the BWT w by solving the ODE in Theorem 4. We compute v by the algorithm described in the proof of Theorem 5. We compute σ from (2). Finally we compute the flow rate α by applying (9). We thus have an algorithm to compute the performance function $(b, q, w, v, \sigma, \alpha)$ in the $G_t/M_t/s_t + GI_t$ model, assuming that there are only finitely many switches in each finite interval, which holds if $\lambda, s', \mu \in \mathcal{P}_{m,n}$ for some positive integers m, n .

The construction above has been done under the assumption that the staffing function is feasible. As in §6.2 of Liu and Whitt (2010), the algorithm can detect violations of feasibility whenever they occur and can then produce the minimum feasible staffing function greater than or equal to the initial proposed staffing function. A violation is easy to detect; it necessarily occurs in an overloaded interval in $\mathcal{O}([0, T])$ at time $t^* \equiv \inf \{t \in \mathcal{O}([0, T]) : \gamma(t) < 0\}$. As in Liu and Whitt (2010), let $\mathcal{S}_{f,s}$ be the set of feasible staffing functions over the interval $[0, t]$ for $t > t^*$.

THEOREM 6. (minimum feasible staffing function) *There exist $\delta > 0$ and $s^* \in \mathcal{S}_{f,s}(t^* + \delta)$ such that $s^* = \inf \{\tilde{s} \in \mathcal{S}_{f,s}(t^* + \delta)\}$; i.e., $s^* \in \mathcal{S}_{f,s}(t^* + \delta)$ and $s^*(u) \leq \tilde{s}(u)$, $0 \leq u \leq t^* + \delta$, for all $\tilde{s} \in \mathcal{S}_{f,s}(t^* + \delta)$. In particular,*

$$s^*(t^* + u) \equiv B(t^*) \cdot e^{-M(t^*, t^* + u)}, \quad 0 \leq u \leq \delta. \quad (28)$$

Moreover, δ can be chosen so that $\delta = \inf \{u \geq 0 : s^*(t^* + u) = s(t^* + u)\}$, with $\delta \equiv \infty$ if the infimum is not attained.

COROLLARY 5. (minimum feasible staffing with M service) *For M service, i.e., with exponential service times, so that $\bar{G}(x) \equiv e^{-\mu x}$, (28) becomes simply $s^*(t^* + u) = B(t^*)e^{-\mu u}$, $0 \leq u \leq \delta$.*

Theorem 6 shows how to construct a new staffing function that (i) agrees with the proposed staffing function s over its interval of feasibility $[0, t^*)$ and (ii) itself is feasible over the longer interval $[0, t^* + \delta)$ for some $\delta > 0$. To construct the minimum feasible staffing function over $[0, T]$, this algorithm may need to be applied several times.

5. General Arrival Rate Functions

In the previous two sections we have seen that we can get a nice clean theory if we assume that $\lambda, s', \mu \in \mathcal{P}_{m,n}$. In order to treat open networks of fluid queues, we would want the service completion rate σ , which becomes the part of the input rate at other queues, to be in $\mathcal{P}_{m,n}$ for some m and n as well, but σ does not inherit this property, because $\sigma(t) = B(t)\mu(t)$ and $B(t)$ has a complicated non-polynomial form in underloaded intervals, as shown in (21). We do have $\sigma \in \mathbb{C}_p$ by virtue of Corollary 1, but we need not have $\sigma \in \mathcal{P}_{m,n}$. Hence, we show how to treat the general case in which initially we only assume that $\lambda \in \mathbb{C}_p$.

We will treat the case of general $\lambda \in \mathbb{C}_p$ as the limit of a sequence of systems with $\lambda \in \mathcal{P}_{m,n}$. In particular, for arbitrary $\lambda \in \mathbb{C}_p$, we can represent it as the limit of a sequence of functions $\{\lambda_k : k \geq 1\}$, where $\lambda_k \in \mathcal{P}_{m_k, n_k}$ and $\lambda_k \geq 0$ for each k , and $\|\lambda_k - \lambda\|_T \rightarrow 0$ as $k \rightarrow \infty$, with $\|\cdot\|_T$ denoting the uniform norm over $[0, T]$. (Positivity is no problem because of Assumption 6 and the uniform convergence.) If we also assume that $s', \mu \in \mathcal{P}_{m,n}$ for some m, n , then we will necessarily have $\zeta_k \in \mathcal{P}_{m_k, n_k}$ for all k , with $m_k < \infty$ and $n_k < \infty$ for all k . We will also have $m_k \rightarrow \infty$ and $n_k \rightarrow \infty$ as $k \rightarrow \infty$ unless $\lambda \in \mathcal{P}_{m,n}$ for some m, n .

In this section we establish results that allow us to treat the case of general arrival rate functions $\lambda \in \mathbb{C}_p$, without requiring that $\lambda \in \mathcal{P}_{m,n}$ and without directly requiring that there be only finitely many switches between overloaded and underloaded intervals in the interval $[0, T]$. To do so, we establish monotonicity and Lipschitz continuity properties, which are of independent interest. We first establish these results assuming that $\zeta \in \mathcal{P}_{m,n}$, and then we show that they extend when we allow arbitrary $\lambda \in \mathbb{C}_p$. We thus start by assuming that $\zeta \in \mathcal{P}_{m,n}$. The proofs of the three theorems in this section are relatively straightforward, but long; they appear in §EC.3.

The M_t service allows us to extend the elementary comparison results in Propositions 4.2 and 5.3 of Liu and Whitt (2010). Recall that order of functions (vectors) is defined as pointwise order for all arguments (coordinates). Let $X(t) \equiv B(t) + Q(t)$ be the total system fluid content. Let subscripts designate the model.

THEOREM 7. (fundamental comparison theorem) *Consider two $G_t/M_t/s_t + GI_t$ fluid models with common staffing function s and service rate function μ . If $\zeta_1, \zeta_2 \in \mathcal{P}_{m,n}$ with $\lambda_1 \leq \lambda_2$, $B_1(0) \leq B_2(0)$, $q_1(0, \cdot) \leq q_2(0, \cdot)$ and $h_{F_{t,1}} \geq h_{F_{t,2}}$, then*

$$(B_1(\cdot), \tilde{q}_1, q_1, Q_1(\cdot), X_1, w_1, v_1, \sigma_1) \leq (B_2(\cdot), \tilde{q}_2, q_2, Q_2(\cdot), X_2, w_2, v_2, \sigma_2). \quad (29)$$

In addition to monotonicity, the model has additional basic Lipschitz continuity properties (beyond Proposition 5).

THEOREM 8. (more Lipschitz continuity) *Consider a $G_t/M_t/s_t + GI_t$ fluid model with $\lambda, s', \mu \in \mathcal{P}_{m,n}$ for some m, n . Then the functions mapping (i) $(\lambda, B(0))$ in $\mathcal{P}_{m,n} \times \mathbb{R}$ into (B, σ) in \mathbb{C}_p^2 , (ii) $(\lambda, B(0), Q(0))$ in $\mathcal{P}_{m,n} \times \mathbb{R}^2$ into Q in \mathbb{C}_p , and (iii) $(\lambda, X(0))$ in $\mathcal{P}_{m,n} \times \mathbb{R}$ into X in \mathbb{C}_p , all over $[0, T]$, are Lipschitz continuous. In particular,*

$$\begin{aligned} \|B_1 - B_2\|_T &\leq (1 \vee T)(\|\lambda_1 - \lambda_2\|_T \vee |B_1(0) - B_2(0)|), & \|\sigma_1 - \sigma_2\|_T &\leq \mu_T^\dagger \|B_1 - B_2\|_T, \\ \|Q_1 - Q_2\|_T &\leq (1 \vee T)(\|\lambda_1 - \lambda_2\|_T \vee |B_1(0) - B_2(0)| \vee |Q_1(0) - Q_2(0)|), \\ \|X_1 - X_2\|_T &\leq 2(1 \vee T)(\|\lambda_1 - \lambda_2\|_T \vee |X_1(0) - X_2(0)|). \end{aligned} \quad (30)$$

If $B_1(0) = B_2(0)$ and $Q_1(0) = Q_2(0)$ (for Q and X), then

$$\|B_1 - B_2\|_T \leq T\|\lambda_1 - \lambda_2\|_T, \quad \|Q_1 - Q_2\|_T \leq T\|\lambda_1 - \lambda_2\|_T, \quad \|X_1 - X_2\|_T \leq 2T\|\lambda_1 - \lambda_2\|_T. \quad (31)$$

As a consequence of Theorems 3–8, we can regard the case of a general function λ as the limit of a sequence $\{\lambda_k : k \geq 1\}$, where $\zeta_k \in \mathcal{P}_{m_k, n_k}$ with $m_k \rightarrow \infty$ and $n_k \rightarrow \infty$ as $k \rightarrow \infty$. Hence, results for the k^{th} system can be “lifted” to the general case; i.e., Theorems 7–8 combine to imply the following general result.

THEOREM 9. (lifting) *For a $G_t/M_t/s_t + GI_t$ fluid model with $s', \mu \in \mathcal{P}_{m,n}$ and $\lambda \in \mathbb{C}_p$, the system performance via (B, \tilde{q}, w) , for $B \equiv \{B(t) : 0 \leq t \leq T\}$, is well defined and the conclusions of §3 and Theorems 7 and 8 remain valid.*

6. The $(G_t/M_t/s_t + GI)^m/M_t$ Fluid Queue Network.

We now introduce the open network of $G_t/M_t/s_t + GI$ fluid queues, with time-dependent proportional routing. There are m queues, where each queue has model parameters as already defined in §2, with its own external fluid input, but in addition a proportion $P_{i,j}(t)$ of the fluid output from queue i at time t is routed immediately to queue j , and a proportion $P_{i,0}(t) \equiv 1 - \sum_{j=1}^m P_{i,j}(t) \leq 1$ is routed out of the network, as shown in Figure 1 for the case $m = 2$.

ASSUMPTION 10. (proportional routing) *The routing matrix function for proportional routing, $P : [0, \infty) \rightarrow [0, 1]^{m^2}$, is in \mathbb{C}_p and $\sum_{j=1}^m P_{i,j}(t) \leq 1$ for each $t \geq 0$ and $i, 1 \leq i \leq m$.*

It is elementary to treat the basic network operations of superposition and splitting: If two input streams are combined to form a single input (superposition), then the arrival rate functions are simply added. If one stream with arrival rate function λ is split, such that a proportion $p(t)$ of that stream goes into a new split stream at time t , then the arrival-rate function of the split stream is λ_p , where $\lambda_p(t) \equiv \lambda(t)p(t)$, $t \geq 0$; just like λ , the splitting proportion can be time-dependent. Similarly, if the departure flow from one queue becomes input to another, then the resulting arrival-rate function is σ ; (We do not let the abandonment flow from one queue become input to another, but if we did, then the resulting arrival-rate function would be α .) However, converting departure rate or abandonment rate into new input rate is more complicated when feedback is allowed. We discuss that case now, for departures only.

As is usual with open queueing networks, there is an external exogenous arrival rate function to each queue (from outside the network) and there is a total arrival rate function to each queue (which we simply call the arrival rate function), taking into account the flow from other queues. Let the external arrival rate function into queue j be denoted by $\lambda_j^{(0)}$; let the arrival rate function into queue j be denoted by λ_j . The model data for the $G_t/M_t/s_t + GI_t$ fluid queues directly provides

the external arrival rate functions $\lambda_j^{(0)}$ (with the superscript 0 now added), while the arrival rate function itself satisfies a system of *traffic rate equations*. In particular,

$$\lambda_j(t) = \lambda_j^{(0)}(t) + \sum_{i=1}^m \sigma_i(t) P_{i,j}(t), \quad \text{where} \quad (32)$$

$$\sigma_i(t) = B_i(t) \mu_i(t), \quad t \geq 0. \quad (33)$$

Equations (32) and (33) produce a system of equations, with λ_j depending upon σ_i for $1 \leq i \leq m$, while σ_i in turn depends on λ_i for each i , because B_i depends on λ_i . The formulas for B_i as a function of λ_i have been given in Propositions 1 and 2, provided that we know whether the queue is overloaded or underloaded. That requirement is the major source of complexity.

Since (32) is a linear equation, it can be written in matrix notation as $\lambda = \lambda^{(0)} + \sigma P$ by omitting the argument t as below, provided that the product σP is interpreted as in (32). Moreover, we can combine (32) and (33) to express λ as the solution of a fixed point equation mapping \mathbb{C}_p^m over $[0, T]$ into itself. To see this, note that $B_i(t)$ in (33) is a function of $\lambda_i(u)$, $0 \leq u < t$, and the model data (only needed for queue i). Hence the vector $B(t) \equiv (B_1(t), \dots, B_m(t))$ is a function of λ over $[0, t)$ and the model data. Hence we can express (32) and (33) abstractly as

$$\lambda = \Psi(\lambda), \quad (34)$$

where $\Psi(x)(t)$ depends on its argument x only over $[0, t]$ for each $t \geq 0$. Here the function Ψ depends on all the model data $(\lambda_i^{(0)}, s_i, \mu_i, F_{i,\cdot}, b_i(0, \cdot), q_i(0, \cdot), P)$, $1 \leq i \leq m$.

THEOREM 10. (*contraction operator*) *If $s'_i, \mu_i \in \mathcal{P}_{m,n}$ for $1 \leq i \leq m$, then the operator Ψ in (34) is a monotone contraction operator on the m -dimensional product space \mathbb{C}_p^m over $[0, T]$ for all sufficiently small $T > 0$. Hence there exists a unique solution λ to the traffic rate equations (32) and (33) over $[0, T]$ for any fixed $T > 0$. For sufficiently short intervals, successive iterates $\Psi^{(n)}(\tilde{\lambda})$ converge uniformly, geometrically fast, to the fixed point for any initial point $\tilde{\lambda} \in \mathbb{C}_p^m$.*

Proof. We first show that Ψ actually maps \mathbb{C}_p into itself. First, if $\lambda \in \mathbb{C}_p^m$, then $B \in \mathbb{C}_p^m$ by Corollary 1 and Theorem 9. By assumption $\mu \in \mathbb{C}_p^m$, so that $\sigma \in \mathbb{C}_p^m$, so the conclusion follows

from (32) and (33). To show that Ψ is a contraction operator for sufficiently small $T > 0$, we use the norm $\|\lambda\|_T \equiv \sum_{i=1}^m \|\lambda_i\|_T$ for $\lambda \equiv (\lambda_1, \dots, \lambda_m) \in (\mathbb{C}_p)^m$. For any $\lambda_1, \lambda_2 \in (\mathbb{C}_p)^m$, the traffic rate equations in (32) and (33) imply that

$$\begin{aligned} \|\Psi(\lambda_1) - \Psi(\lambda_2)\|_T &\leq \sum_{j=1}^m \sup_{1 \leq t \leq T} \sum_{i=1}^m \mu_i(t) |B_i^1(t) - B_i^2(t)| P_{i,j}(t) \\ &\leq m \mu_T^\uparrow \sum_{i=1}^m \sup_{0 \leq t \leq T} |B_i^1(t) - B_i^2(t)| \\ &\leq m \mu_T^\uparrow T \sum_{i=1}^m \sup_{0 \leq t \leq T} |\lambda_i^1(t) - \lambda_i^2(t)| \leq m \mu_T^\uparrow T \|\lambda_1 - \lambda_2\|_T, \end{aligned}$$

where $m \mu_T^\uparrow < 1$ for all sufficiently small $T > 0$. The second inequality holds since $P_{i,j}(t) \leq 1$. The crucial third inequality follows from (31) in Theorem 8. To establish uniqueness over $[0, T]$ for any fixed $T > 0$, we consider a succession of shorter intervals, over which the contraction property holds, and apply mathematical induction. Existence, uniqueness and geometric convergence are standard consequences of the Banach contraction fixed point theorem. Finally, monotonicity follows from Theorems 7 and 9 plus the traffic rate equations (32) and (33). ■

REMARK 1. (starting at the external arrival rates) Theorem 10 implies that we can approach this system recursively. If we do so with initial vector $\tilde{\lambda} = \lambda^{(0)}$, the vector of external arrival rate functions, then the recursion has an important practical interpretation. Then the k^{th} iterate $\lambda_j^{(k)}$ is the arrival rate of fluid that has previously experienced k transitions in the fluid network. With this notation, we can write the recursive formulas

$$\lambda_j^{(n)}(t) = \Psi^{(n)}(\lambda^{(0)})_j(t) = \lambda_j^{(0)}(t) + \sum_{i=1}^m \sigma_i^{(n-1)}(t) P_{i,j}(t), \quad n \geq 1, \quad (35)$$

$$\text{where } \sigma_i^{(n)}(t) = B_i^{(n)}(t) \mu_i(t) \quad n \geq 0. \quad (36)$$

Since we necessarily have $\lambda_i^{(1)} \geq \lambda_i^{(0)}$ for each i , this recursion converges monotonically to the fixed point λ . By Theorems 7 and 9, all the performance measures increase toward their limiting values as well.

REMARK 2. (an m -dimensional ODE) Algorithmically, there is an alternative approach to $(G_t/M_t/s_t)^m/M_t$ fluid queue networks. Instead of applying nm iterations of the single-queue algorithm to achieve n iterations of the operator Ψ , we can characterize the vector of arrival rates λ

as the solution of one m -dimensional ODE. We obtain this ODE by differentiating with respect to t in the traffic rate equations in (32) and (33). We intend to discuss this approach in a subsequent paper. It yields useful explicit expressions for the special cases of one fluid queue with immediate proportional feedback and a network of two fluid queues, as depicted in Figure 1, plus an algorithm for the general case.

REMARK 3. (the $(G_t/GI/s_t + GI)^m/M_t$ Fluid Queue Network) Analogs of what we have done in this section apply to the $(G_t/GI/s_t + GI)^m/M_t$ generalization of the $G_t/GI/s_t + GI$ fluid queue considered in Liu and Whitt (2010); we only need to replace equations (33) and (36) with the more complicated expressions given for the service completion rate σ given in Theorem 6.1 of Liu and Whitt (2010). In particular, (33) should be replaced by

$$\sigma_i(t) = \int_0^\infty b_i(t, x) h_{G_i}(x) dx = \int_0^t b_i(t-x, 0) g_i(x) dx + \int_0^\infty \frac{b_i(0, y) g_i(t+y)}{\bar{G}_i(y)} dy,$$

while (36) should be replaced by

$$\sigma_i^{(n)}(t) = \int_0^\infty b_i^{(n)}(t, x) h_{G_i}(x) dx = \int_0^t b_i^{(n)}(t-x, 0) g_i(x) dx + \int_0^\infty \frac{b_i^{(n)}(0, y) g_i(t+y)}{\bar{G}_i(y)} dy, \quad n \geq 0.$$

However, the service content densities at each queue, b_i , in general are characterized only as the solution of a fixed point equation. Moreover, it remains to establish an analog of Theorem 10. The space \mathcal{P}_{mn} no longer helps immediately. So far, we must *assume* that there are finitely many switches between overloaded and underloaded intervals in any finite interval, and assume that there exists a unique solution to the new equations. However, from a practical perspective, the $(G_t/GI/s_t + GI)^m/M_t$ and even the more general $(G_t/GI_t/s_t + GI_t)^m/M_t$ model can be analyzed in the same way.

We conclude this section by establishing a network generalization of the single queue comparison in Theorem 7. The proof appears in §EC.4.

THEOREM 11. (network comparison theorem) *Consider two $(G_t/M_t/s_t + GI_t)^m + M_t$ fluid queue networks with common staffing functions s_i , service rate functions μ_i , abandonment cdf's $F_{\cdot,i}$ and*

routing matrix function P for $1 \leq i \leq m$. If $\lambda_{1,i}^{(0)} \leq \lambda_{2,i}^{(0)}$, $B_{1,i}(0) \leq B_{2,i}(0)$, $q_{1,i}(0, \cdot) \leq q_{2,i}(0, \cdot)$, $1 \leq i \leq m$, then the performance functions are ordered at each queue:

$$\begin{aligned} & (\lambda_{1,i}, B_{1,i}, \sigma_{1,i}, \tilde{q}_{1,i}, q_{1,i}, Q_{1,i}, \alpha_{1,i}, X_{1,i}, w_{1,i}, v_{1,i}) \\ & \leq (\lambda_{2,i}, B_{2,i}, \sigma_{2,i}, \tilde{q}_{2,i}, q_{2,i}, Q_{2,i}, \alpha_{2,i}, X_{2,i}, w_{2,i}, v_{2,i}) \quad \text{for } 1 \leq i \leq m. \end{aligned} \quad (37)$$

7. The Stationary $(G/GI/s + GI)^m/M$ Fluid Queue Network

This paper is primarily devoted to the time-varying fluid queue network, but the corresponding stationary fluid queue network may also be of interest. The stationary performance of a single $GI/GI/s + GI$ fluid queue was characterized in Whitt (2006). (The proof is completed by Liu and Whitt (2010) because the transient dynamics are characterized there.) The corresponding stationary $(G/GI/s + GI)^m/M$ fluid queue network is actually quite elementary given Whitt (2006). In particular, the stationary performance of this model is determined by a fixed point equation for the (now constant) arrival rates. We start by reviewing that stationary distribution of the $GI/GI/s + GI$ fluid queue.

THEOREM 12. (*steady state of the $G/GI/s + GI$ fluid queue*). *The $G/GI/s + GI$ fluid model specified with model parameter (λ, s, μ, G, F) has a unique steady state described by the vector $(b, q, B, Q, w, \sigma, \alpha)$, whose character depends on whether $\rho \equiv \lambda/s\mu \leq 1$ or $\rho > 1$.*

(a) *Underloaded and balanced cases: $\rho \leq 1$. If $\rho \leq 1$, then for $x \geq 0$*

$$B = s\rho, \quad b(x) = \lambda \bar{G}(x), \quad \sigma = B\mu = \lambda, \quad Q = \alpha = w = q(x) = 0,$$

(b) *Overloaded case: $\rho > 1$. If $\rho > 1$, then for $x \geq 0$*

$$\begin{aligned} B &= s, \quad b(x) = s\mu \bar{G}(x), \quad \sigma = s\mu, \quad \alpha = \lambda - s\mu = (\rho - 1)s\mu = \lambda \bar{F}(w), \\ w &= F^{-1}\left(1 - \frac{1}{\rho}\right), \quad Q = \lambda \int_0^w \bar{F}(x) dx \quad \text{and} \quad q(x) = \lambda \bar{F}(x) 1_{\{0 \leq x \leq w\}}. \end{aligned}$$

We now turn to the arrival rates. As can be seen from Theorem 12 above, unlike for the time-varying model, for the stationary model we can easily handle GI service, because the total service content B is independent of the service-time distribution beyond its mean. The vector of constant arrival rates λ is determined by the system of fixed point equations

$$\lambda_j = \lambda_j^{(0)} + \sum_{i=1}^m (\lambda_i \wedge s_i \mu_i) P_{i,j}, \quad 1 \leq j \leq m, \quad (38)$$

where $\lambda, \lambda^{(0)}, s, \mu \in \mathbb{R}^m$ and P is an $m \times m$ stochastic matrix. We can write (38) more compactly as

$$\lambda = \Phi(\lambda) \equiv \lambda^{(0)} + (\lambda \wedge s\mu)P. \quad (39)$$

Equation (39) was already analyzed by Goodman and Massey (1984) in the study of non-ergodic Jackson networks; also see Chen and Mandelbaum (1991) and p. 168 of Chen and Yao (2001). However, the model here is different.

THEOREM 13. (*fixed point equation for stationary arrival rates, from Goodman and Massey (1984)*) *The arrival rates in the stationary $(G/GI/s + GI)^m/M$ fluid queue network satisfy equation (38). Hence, if the stochastic matrix has spectral radius less than 1 (which holds if and only if $P^n \rightarrow 0$ as $n \rightarrow \infty$), then Φ in (39) is a monotone n -stage contraction operator on \mathbb{R}^m with an appropriate norm, so that there exists a unique solution to the fixed point equation in (38) and (39). The fixed point can be calculated by solving at most m different systems of m linear equations.*

Proof. Even for GI service, if fluid queue i is underloaded, then the stationary service content is $B_i = \lambda_i/\mu_i$ and the service completion rate is $\sigma_i = B_i\mu_i = \lambda_i$. On the other hand, if queue i is overloaded, then $B_i = s_i$ and the service completion rate is $s_i\mu_i$. In all cases, the service completion rate at queue i is $\lambda_i \wedge s_i\mu_i$. Since there is a unique solution to equation (38) or (39), that equation determines the stationary arrival rates at all queues and which queues are in fact overloaded. ■

8. Heuristic Stochastic Refinement for Many-Server Queues

As illustrated by Figure EC.2, the fluid model performance functions are remarkably effective in approximating the performance of large-scale many-server queueing systems. That is to be expected because of many-server heavy-traffic limits, as discussed in §2 of Liu and Whitt (2010). In §9 of Liu and Whitt (2010) we show that the fluid performance functions are useful more generally to describe the mean values of smaller-scale many-server queueing systems, e.g., with only 20 servers or even fewer, provided that they experience significant overloading at some times. That should be very helpful, but it is also of interest to better understand the stochastic fluctuations about those mean values in the queueing system.

We hope to prove many-server heavy-traffic limits that will provide stochastic refinements, but even if such limits are proved, it may remain very difficult to compute the resulting approximate time-varying distributions of the queueing performance measures, such as the time-varying queue length. In this section we suggest a practical heuristic approximation. The idea is very simple: We simply approximate the distribution of $X(t)$ by a *Poisson* distribution, taking the computed value from the fluid queue model as its mean. This simple Poisson approximation approach is in fact *exact* in the special case of the $M_t/GI/s_t + GI_t$ and $M_t/M_t/s_t + GI_t$ models if they are always underloaded, starting out empty at time 0 or in the past. As discussed in Liu and Whitt (2010), in that case the model reduces to the $M_t/GI/\infty$ or $M_t/M_t/\infty$ fluid model, for which the fluid values of $B(t) = X(t)$ coincide with the mean values in the stochastic model. In addition, $X(t)$ has a Poisson distribution for the stochastic infinite-server model. Finally, unless the mean is very small, we approximate the Poisson distribution by a normal distribution. For the underloaded system, this proposal coincides with §9 of Massey and Whitt (1993). It also relates to recent heavy-traffic limits for infinite-server models in Pang and Whitt (2009) and references therein.

Given the approximation for $X(t)$ in the queueing system, we approximate the random variables $Q(t)$ and $B(t)$ using $Q(t) = (X(t) - s(t))^+$ and $B(t) = X(t) \wedge s(t)$, which leads to “one-sided” normal approximations, which in regions of significant overload or underload will tend to themselves be approximately normal. This heuristic refinement should give a rough idea about the stochastic fluctuations, adequate for many engineering applications; e.g., it shows that the stochastic fluctuations in $X(t)$ should be roughly of order $\sqrt{X(t)}$, which supports the classical square-root staffing rule.

9. Proofs of Theorems 1–3

We need some basic regularity properties of Q and B , which will be valid with the assumptions in §2. For that purpose, we exploit two basic *flow-conservation equations*: (i) the queue content at time t equals the initial queue content plus input minus output to either abandonment or entering service, and (ii) the service content at time t equals the initial service content plus input minus

output. However, the input enters the queue only when the system is overloaded; otherwise it directly enters service. Thus we have the following elementary bounds and the subsequent Lipschitz continuity. We prove all propositions and lemmas in this section in §EC.1.

PROPOSITION 4. (elementary bounds) $Q(t) + A(t) + E(t) \leq Q(0) + \Lambda(t) < \infty$ and

$$B(t) + S(t) = B(0) + E(t) \leq B(0) + Q(0) + \Lambda(t) < \infty,$$

so that Q , E , A , B and S are all bounded for $0 \leq t \leq T$.

PROPOSITION 5. (Lipschitz continuity) *The functions S , E , B , A and Q are Lipschitz continuous.*

We first apply Proposition 5 to relate \mathcal{S} to the zeros of $X - s$, where $X(t) \equiv Q(t) + B(t)$.

LEMMA 4. (zeros of $X - s$) $\mathcal{S} \subseteq Z_{X-s}$.

We now characterize the overloaded times.

LEMMA 5. (overloaded intervals) *With the possible exception of 0 and T , all overloaded times appear in intervals of positive length. Hence, underloaded sets consist of either single isolated points or intervals.*

Proof of Theorem 1. We apply the results above. Since there can be at most countably many overloaded intervals of positive length in $[0, T]$, the isolated points are well defined and countably infinite. Since the isolated points are at most countably infinite, we can order them and reclassify them one by one. With that construction, we reduce the number of disjoint overloaded intervals by one at each step. Finally, all underloaded times appear in intervals too. ■

We now relate the zeros of ζ in (20) to the overloaded and underloaded intervals.

LEMMA 6. (zeros and intervals) *For each interval in the partition of $[0, T]$ into underloaded and overloaded intervals, there exists at least one zero or discontinuity point of ζ .*

Proof of Theorem 2 Since the interval $[0, T]$ can be partitioned into at most countably many intervals that alternate between overloaded and underloaded after reclassifying isolated underloaded points as overloaded, the switch points can be placed in one-to-one correspondence with the internal boundary points (excluding 0 and T). Hence the number of switch points is equal to $n - 1$, if the number of intervals in the partition is n for some $n < \infty$. Otherwise both sets are countably infinite. Next, Lemma 6 implies that there is either a discontinuity point or a zero in every overloaded and underloaded interval. Since the number of intervals is 1 greater than the number of switches, we obtain the conclusion. To see that the bound is tight, consider the common case in which ζ is differentiable on $[0, T]$ and $\zeta(t) \neq 0$ at all switch times. Then ζ has a zero where it attains its maximum in each overloaded interval, while ζ has a zero where it attains its minimum in each underloaded interval. To have the bound an equality, let ζ have no other zeros. ■

Proof of Theorem 3. First, any discontinuity points of ζ must be contained in the set of n interval boundary points. Hence, $\mathcal{D}_\zeta \leq n$. On each of the n subintervals, ζ is a polynomial of order at most m . By the fundamental theorem of algebra, on each of these intervals the zero set is either a finite set of cardinality at most m or it is the entire subinterval. If $\zeta = 0$ throughout the interval, then there can be at most a single switch in the interval, where $(Q(t), B(t))$ becomes $(0, s(t))$, after which it will remain there throughout the subinterval. In other words, the first subinterval is overloaded and the second is underloaded, so this interval produces at most a single switch. We can thus treat this interval just like any of the others; we can act as if it produces at most m zeros. Hence, $\mathcal{D}_\zeta \leq n$ and $Z_\zeta \leq mn$. Finally, Theorem 2 implies that $|\mathcal{S}| \leq mn + n - 1$, as claimed. ■

10. Conclusions

In section 2 we specified the single $G_t/M_t/s_t + GI_t$ fluid queue; it differs from Liu and Whitt (2010) by having M_t service and GI_t abandonment instead of both being GI . The M_t service eliminates the need to solve a fixed point equation to find the service content density b . In §3 and §4 we showed that a single fluid queue can be analyzed by assuming that the arrival rate function λ , the staffing function s and the service rate function μ are all piecewise polynomials. However, that did not

permit an extension to networks because the departure rate function does not inherit that property. In §5 we used asymptotic methods to show how to analyze the single fluid queue without having to assume either (i) that the arrival rate function is piecewise polynomial or (ii) that there are only finitely many switches between overloaded and underloaded intervals in each finite interval. In §6 and §7 we showed how to treat the $(G_t/M_t/s_t + GI_t)^m/M_t$ and $(G/GI/s + GI)^m/M$ networks with proportional routing. Theorem 10 established the existence of unique vector of arrival rate functions, allowing for feedback, and thus a corresponding unique performance description for the entire network. The performance functions at each queue are given in §4.

As discussed in §9 of Liu and Whitt (2010), we have conducted simulation experiments showing that the fluid model provides very accurate approximations for very large-scale many-server queueing systems; we show the results of one such experiment in §EC.6. The approximations are also excellent for the *mean values* of the corresponding queueing random variables when the scale is quite small, e.g., when there are 20 servers or fewer; e.g., see Figure 7 of Liu and Whitt (2010). We have provided a heuristic stochastic refinement in §8; it approximates the number of customers in the queueing system first by a Poisson distribution, having the fluid value as its mean, and then by a normal distribution. We remarked that this reasoning supports the square root staffing formula.

There are many directions for future research. As indicated in §2 of Liu and Whitt (2010), it remains to establish supporting many-server heavy-traffic limits, including stochastic refinements. It remains to examine the algorithms provided by Theorem 10 and Remark 2; it remains to analyze the convergence to steady state in the stationary $(G/GI/s + GI)^m/M$ fluid queue network discussed in §7; it remains to extend Theorem 10 to GI and GI_t service.

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E-Companion

This e-companion has six sections, presenting supporting material primarily in the order that it relates to the main paper. In §EC.1 we present remaining proofs for §3 and §9. In §EC.2 we present proofs for §4. In §EC.3 we present proofs for §5. In §EC.4 we present one proof for §6. In §EC.5, we make remarks about: (i) characterizing the isolated underloaded points in §3, (ii) representation of the fluid content B in an underloaded interval via an ODE, and (iii) the applied significance of the space of piecewise polynomials $\mathcal{P}_{m,n}$. In §EC.6 we compare the fluid model performance predictions to simulation results for a large-scale queueing system.

EC.1. Proofs for Sections 3 and 9.

Proof of Proposition 4. The relations follow from flow conservation. The first relation is an inequality instead of an equality because input enters the queue instead of the service facility only when the system is overloaded. ■

Proof of Proposition 5. For a nonnegative real-valued function f on $[0, \infty)$, let $f_t^\uparrow \equiv \sup_{0 \leq y \leq t} f(y)$. To treat S , recall that S is the integral of σ , where

$$\sigma(t) = B(t)\mu(t) \leq s(t)\mu(t), \quad \text{so that} \quad \sigma(t) \leq s_t^\uparrow \mu_t^\uparrow, \quad t \geq 0, \quad (\text{EC.1})$$

and

$$|S(t+u) - S(t)| = \int_t^{t+u} \sigma(y) dy \leq s_T^\uparrow \mu_T^\uparrow u, \quad 0 \leq t \leq t+u \leq T. \quad (\text{EC.2})$$

To treat E , recall that it is the integral of the rate fluid enters service, where the rate fluid enters service is either $\gamma(t) = \lambda(t)$ if the system is underloaded or $\gamma(t) = s'(t) + \sigma(t) = s'(t) + s(t)\mu(t)$ if the system is overloaded. Hence,

$$|E(t+u) - E(t)| \leq \gamma_T^\uparrow u, \quad 0 \leq t \leq t+u \leq T, \quad (\text{EC.3})$$

where $\gamma_T^\uparrow \equiv \lambda_T^\uparrow \vee (|s'|_T^\uparrow + s_T^\uparrow \mu_T^\uparrow) < \infty$. By the second equation in Proposition 4,

$$B(t+u) - B(t) = (E(t+u) - E(t)) - (S(t+u) - S(t)), \quad (\text{EC.4})$$

so that

$$|B(t+u) - B(t)| \leq |E(t+u) - E(t)| + |S(t+u) - S(t)| \leq (e_T^\dagger + s_T^\dagger \mu_T^\dagger)u \quad (\text{EC.5})$$

for $0 \leq t \leq t+u \leq T$.

Next we combine (9) with (16) to get

$$\alpha(t) = \int_0^{t \wedge w(t)} \lambda(t-x) f_{t-x}(x) dx + \int_{w(t) \wedge t}^t \frac{q(0, x-t) f_{t-x}(x)}{\bar{F}_{t-x}(x-t)} dx, \quad (\text{EC.6})$$

so that, by applying Assumption 9, we get

$$\alpha(t) \leq \alpha_t^\dagger \equiv f^\dagger \Lambda(t) + \frac{f^\dagger}{\bar{F}^\dagger(w(0))} Q(0) < \infty \quad (\text{EC.7})$$

and

$$|A(t+u) - A(t)| \leq \int_t^{t+u} \alpha(y) dy \leq \alpha_T^\dagger u, \quad 0 \leq t \leq t+u \leq T. \quad (\text{EC.8})$$

Finally, by the first relation in Proposition 4,

$$\begin{aligned} |Q(t+u) - Q(t)| &\leq |\Lambda(t+u) - \lambda(t)| + |E(t+u) - E(t)| + |A(t+u) - A(t)| \\ &\leq (\lambda_T^\dagger + \gamma_T^\dagger + \alpha_T^\dagger)u, \quad 0 \leq t \leq t+u \leq T. \quad \blacksquare \end{aligned} \quad (\text{EC.9})$$

Proof of Lemma 4. Since Q and B are continuous by Proposition 5 and s is continuous by assumption, $X - s$ is continuous. Since $X - s$ is continuous, if $X(t) - s(t) \neq 0$, then t cannot be an element of \mathcal{S} . \blacksquare

Proof of Lemma 5. If $t \in \mathcal{O}([0, T])$, then either (i) $X(t) - s(t) > 0$ or (ii) $X(t) - s(t) = 0$ and $\zeta(t) > 0$. In case (i), since $X - s$ is continuous by Proposition 5, there exists a neighborhood of t that is overloaded. In case (ii), since $\zeta(t) > 0$, we will have $X(t) - s(t) > 0$ in an interval $(t, t + \epsilon)$ for some positive ϵ . Since overloaded sets are necessarily intervals by Lemma 5, each underloaded set must fall between two overloaded intervals. \blacksquare

Proof of Lemma 6. First, consider the closure of an overloaded interval $[a, b]$. If ζ has one of its finitely many discontinuity points in $[a, b]$, then we are done. Suppose that ζ is continuous on the closed interval $[a, b]$. Necessarily, we have $X(a) - s(a) = X(b) - s(b) = 0$, $\zeta(a + \epsilon) > 0$ for all suitably

small $\epsilon > 0$ and $\zeta(b) \leq 0$. First, we could have $\zeta(b) = 0$ and we are done. If instead $\zeta(U(t)) < 0$, then there must exist t^* with $a < t^* < b$ such that $\zeta(t^*) = 0$ by the intermediate value theorem. The reasoning is essentially the same in the closure of an underloaded interval, say $[a, b]$. If ζ has one of its finitely many discontinuity points in $[a, b]$, then we are again done. Suppose that ζ is continuous on the closed interval $[a, b]$. If either $\zeta(a) = 0$ or $\zeta(b) = 0$, then we are done. Hence we must have $\zeta(a) < 0$. Since b is a switch point and ζ is continuous at b , we must have $\zeta(b) > 0$. As before, there must exist t^* with $a < t^* < b$ such that $\zeta(t^*) = 0$ by the intermediate value theorem. ■

Proof of Lemma 1. The Weierstrass approximation theorem implies that continuous functions can be approximated uniformly over bounded intervals by polynomials. That uniform approximation extends to \mathbb{C}_p provided that the boundary points of the polynomial pieces of the function in $\mathcal{P}_{m,n}$ includes the finitely many discontinuity points of the function in \mathbb{C}_p . ■

EC.2. Proofs for §4.

EC.2.1. Proof of Uniqueness in Theorem 4.

When the abandonment cdf's F_t are independent of t , the proof of uniqueness of the solution to the ODE (25) in Theorem 4 is the same as the proof of the corresponding part of Theorem 5.3 in Liu and Whitt (2010). However, that argument does not extend directly to time-varying abandonment cdf's. Hence we give a different proof under different conditions. In particular, in Theorem 4 for time-varying abandonment cdf's we imposed additional regularity conditions. With those extra regularity conditions, we can apply the classical Picard-Lindelöf theorem for the uniqueness of a solution to the ODE $w'(t) = \Psi(t, w(t))$, which requires that $\Psi(t, x)$ be locally Lipschitz in the argument x uniformly in the argument t ; e.g., Theorem 2.2 of Teschl (2000).

One regularity condition added in Theorem 4 was for the rate fluid enters service to be bounded below. We will show how to guarantee that condition in the next section. Given that the rate fluid enters service is indeed bounded below, i.e., given that $\gamma(t) \geq e_L > 0$ for all $t \in [0, T]$, from (25), there exists a constant $w_L > 0$ such that $w'(t) \leq 1 - w_L < 1$ for all $t \in [0, T]$. Since $w(0) < \infty$, by assumption, and $w(t) \leq w(0) + t$ for all t , we have $w(t) \leq w(0) + T$ for $0 \leq t \leq T$. Together with the

fact that $\lambda, q(0, \cdot) \in C_p$, that implies that the denominator in (25) is bounded above.

Since $w'(t) \leq 1 - w_L < 1$ for all t , for each x we will have $t - w(t) = x$ for at most one value of t . Since $\lambda, q(0, \cdot)$ have been assumed to have bounded derivatives where they are continuous, and since the partial derivative $\partial F_t(x)/\partial t$ of the time-varying abandonment cdf F_t as been assumed to be bounded, the mapping Ψ in (25) is Lipschitz continuous in the argument x except at only finitely many x , uniformly in t . Hence, we can deduce uniqueness of the solution of the ODE in (25) under these extra regularity conditions by applying the Picard-Lindelöf theorem.

We now elaborate on the details. Here we have

$$\Psi(t, x) \equiv 1 - \frac{\gamma(t)}{\tilde{q}(t, x)} = 1 - \frac{\mu(t)s(t) + s'(t)}{\tilde{q}(t, x)}, \quad (\text{EC.10})$$

where $\tilde{q}(t, x)$ is given in (22). Consider the region $0 \leq x_1 \leq t, 0 \leq x_2 \leq t$. In this region we have

$$\begin{aligned} |\Psi(t, x_1) - \Psi(t, x_2)| &= \frac{\mu(t)s(t) + s'(t)}{\lambda(t-x_1)\lambda(t-x_2)\bar{F}_{t-x_1}(x_1)\bar{F}_{t-x_2}(x_2)} |\lambda(t-x_1)\bar{F}_{t-x_1}(x_1) - \lambda(t-x_2)\bar{F}_{t-x_2}(x_2)| \\ &\leq \frac{\mu^\uparrow s^\uparrow + s'^\uparrow}{(\lambda^\downarrow)^2(\bar{F}^\downarrow)^2} |\lambda(t-x_1)\bar{F}_{t-x_1}(x_1) - \lambda(t-x_2)\bar{F}_{t-x_1}(x_1) \\ &\quad + \lambda(t-x_2)\bar{F}_{t-x_1}(x_1) - \lambda(t-x_2)\bar{F}_{t-x_2}(x_2)| \\ &\leq \frac{\mu^\uparrow s^\uparrow + s'^\uparrow}{(\lambda^\downarrow)^2(\bar{F}^\downarrow)^2} (|\lambda(t-x_1) - \lambda(t-x_2)| + \lambda(t-x_2)|\bar{F}_{t-x_1}(x_1) - \bar{F}_{t-x_2}(x_2)|) \\ &\leq \frac{\mu^\uparrow s^\uparrow + s'^\uparrow}{(\lambda^\downarrow)^2(\bar{F}^\downarrow)^2} (\lambda^\uparrow |x_1 - x_2| + \lambda^\uparrow |\bar{F}_{t-x_1}(x_1) - \bar{F}_{t-x_1}(x_2) + \bar{F}_{t-x_1}(x_2) - \bar{F}_{t-x_2}(x_2)|) \\ &\leq \frac{\mu^\uparrow s^\uparrow + s'^\uparrow}{(\lambda^\downarrow)^2(\bar{F}^\downarrow)^2} (\lambda^\uparrow |x_1 - x_2| + \lambda^\uparrow \frac{\partial \bar{F}^\uparrow}{\partial t} |x_1 - x_2| + \lambda^\uparrow g^\uparrow |x_1 - x_2|) \\ &\equiv C |x_1 - x_2|, \end{aligned}$$

where $C \equiv \frac{\mu^\uparrow s^\uparrow + s'^\uparrow}{(\lambda^\downarrow)^2(\bar{F}^\downarrow)^2} (\lambda^\uparrow + \lambda^\uparrow \frac{\partial \bar{F}^\uparrow}{\partial t} + \lambda^\uparrow g^\uparrow)$. The case $x_1, x_2 > t$ is similar. Hence the regularity conditions given in Theorem 4 are sufficient for Ψ to be locally Lipschitz in x uniformly in t .

EC.2.2. e_L -Feasibility of the Staffing Function s .

We have two goals in this section: first, to prove Theorem 6, showing how to construct the minimum feasible staffing function greater than or equal to any proposed staffing function s and, second, to determine the minimum feasible staffing function such that the rate fluid enters service at time

t , $\gamma(t)$, is bounded below. We use this stronger notion of feasibility to provided conditions for the ODE in (25) in Theorem 4 to have a unique solution. We treat both problems at once by introducing the notion of e_L -feasibility: A staffing function s is said to be e_L -feasible if $\gamma(t) \geq e_L \geq 0$ for all $t \in [0, T]$.

So far, we have assumed that the staffing function s is e_L -feasible (as one condition in Theorem 4) or simply feasible (e_L -feasible for $e_L \equiv 0$), yielding

$$\gamma(t) \geq s'(t) + \sigma(t) = s'(t) + \int_0^\infty b(t, x) h_G(x) dx \geq e_L \geq 0 \quad \text{when} \quad B(t) = s(t). \quad (\text{EC.11})$$

This requirement is automatically satisfied in underloaded intervals when $B(t) = s(t)$, provided that $\lambda_{inf}(T) \geq e_L$ for λ_{inf} in Assumption 6, because in that case we require that $s'(t) + \sigma(t) \geq \lambda(t)$ where necessarily $\lambda(t) \geq e_L$; see Definition 1; e_L -Feasibility is only a concern during overloaded intervals, and then only when the staffing function is decreasing, i.e., when $s'(t) < 0$.

A violation is easy to detect; it necessarily occurs in an overloaded interval in $\mathcal{O}([0, T])$ at time $t^* \equiv \inf \{t \in \mathcal{O}([0, T]) : \gamma(t) < e_L\}$. Paralleling Liu and Whitt (2010), let \mathcal{S}_{f,s,e_L} be the set of e_L -feasible staffing functions over the interval $[0, t]$ for $t > t^*$. Then

$$t^* \equiv t^*(e_L) \equiv \inf \{t \in I : \gamma(t) < e_L\}. \quad (\text{EC.12})$$

Even though we require (EC.11), so far we have done nothing to prevent having $t^* < \infty$ (violation). Thus, we compute γ and detect the first violation.

Correcting the staffing function is not difficult either (by which we mean replacing it with a higher feasible staffing function): We simply construct a new staffing function s^* consistent with reducing the input into the queue to its minimum allowed level (setting $\gamma(t) = e_L \geq 0$) starting at time t^* and lasting until the first time t after t^* at which $s^*(t) = s(t)$. (By the adjustment, we will have made $s^*(t^*+) > s(t^*+)$.) Since the system has operated differently during the time interval $[t^*, t]$, we must recalculate all the performance measures after time t , but we have now determined a feasible staffing function up to time $t > t^*$. By successive applications of this correction method

(adjusting the staffing function s and recalculating b), we can construct the minimum feasible staffing function overall.

To make this precise, let $\mathcal{S}_{f,s,e_L}(t)$ be the set of all e_L -feasible staffing functions for the system over the time interval $[0, t]$, $t > t^*$, that coincide with s over $[0, t^*]$; i.e., let

$$\mathcal{S}_{f,s,e_L}(t) \equiv \{\tilde{s} \in C_p^1(t) : \gamma_{\tilde{s}}(u) \mathbf{1}_{\{B_{\tilde{s}}(u) = \tilde{s}(u)\}} \geq e_L, \quad 0 \leq u \leq t, \quad \tilde{s}(u) = s(u), \quad 0 \leq u \leq t^*\}, \quad (\text{EC.13})$$

for t^* in (EC.12), where $\gamma_{\tilde{s}}$ and $B_{\tilde{s}}$ are the functions γ and B associated with the model with staffing function \tilde{s} .

THEOREM EC.1. (minimum e_L -feasible staffing function) *For each e_L such that $0 \leq e_L \leq \lambda_{inf}(T)$ for $\lambda_{inf}(T)$ in Assumption 6, there exist $\delta \equiv \delta(e_L)$ and $s^* \in \mathcal{S}_{f,s,e_L}(t^* + \delta)$ in (EC.13) for t^* in (EC.12) such that*

$$s^* \equiv s^*(e_L) = \inf \{\tilde{s} \in \mathcal{S}_{f,s,e_L}(t^* + \delta)\}; \quad (\text{EC.14})$$

i.e., $s^ \in \mathcal{S}_{f,s,e_L}(t^* + \delta)$ and $s^*(u) \leq \tilde{s}(u)$, $0 \leq u \leq t^* + \delta$, for all $\tilde{s} \in \mathcal{S}_{f,s,e_L}(t^* + \delta)$. In particular,*

$$s^*(t^* + u) = e_L \int_0^u e^{-M(t^*+u-x, t^*+u)} dx + B(t^*) e^{-M(t^*, t^*+u)}. \quad (\text{EC.15})$$

Moreover, δ can be chosen so that

$$\delta = \inf \{u \geq 0 : s^*(t^* + u) = s(t^* + u)\}, \quad (\text{EC.16})$$

with $\delta \equiv \infty$ if the infimum in (EC.16) is not attained.

Proof. First, since γ_s is continuous for our original s , the violation in (EC.12) must persist for a positive interval after t^* ; that ensures that a strictly positive δ can be found. We shall prove that $\tilde{s} \geq s^*$ over $[t^*, t^* + \delta]$ for s^* in (EC.15) and any feasible function \tilde{s} , and we will show that s^* itself is feasible. For $0 \leq t \leq t^* + \delta$, suppose \tilde{s} is feasible. Since the system is overloaded, system being in the overloaded regime implies that

$$\tilde{s}(t^* + u) = B_{\tilde{s}}(t^* + u) = \int_0^\infty b_{\tilde{s}}(t^* + u, x) dx$$

$$\begin{aligned}
&= \int_0^u \gamma_{\bar{s}}(t^* + u - x) \bar{G}_{t^*+u-x}(x) dx + \int_u^\infty b_{\bar{s}}(t^*, x - u) \frac{\bar{G}_{t^*+u-x}(x)}{\bar{G}_{t^*+u-x}(x - u)} dx \\
&= \int_0^u \gamma_{\bar{s}}(t^* + u - x) e^{-M(t^*+u-x, t^*+u)} dx + \int_u^\infty b_s(t^*, x - u) e^{-M(t^*, t^*+u)} dx \\
&\geq e_L \int_0^u e^{-M(t^*+u-x, t^*+u)} dx + e^{-M(t^*, t^*+u)} \int_0^\infty b_s(t^*, y) dy = s^*(t^* + u).
\end{aligned}$$

where the second equality holds because of the fundamental evolution equations in Assumption 4, the third equality holds because $b_{\bar{s}}(t^*, x) = b_s(t^*, x)$ for all x , and the inequality holds because $\gamma_{\bar{s}} \geq e_L$. On the other hand, the equality holds when $\gamma_{\bar{s}}(t^* + u) = e_L$ for all u , which yields $B(t^* + u) = s^*(t^* + u)$. Therefore, the proof is complete. ■

COROLLARY EC.1. (minimum e_L -feasible staffing with exponential service times) *For the special case of exponential service times, i.e., with $\bar{G}(x) \equiv e^{-\mu x}$, independent of t , (EC.15) becomes simply $s^*(t^* + u) = e_L(1 - e^{-\mu u})/\mu + B(t^*)e^{-\mu u}$, $0 \leq u \leq \delta$.*

EC.3. Proofs for §5.

EC.3.1. Proof of Theorem 7.

First, the assumption that $\zeta_1, \zeta_2 \in \mathcal{P}_{m,n}$ assures that there are only finitely many switches between overloaded intervals and underloaded intervals in both systems. That leads to three cases: (i) when both systems are underloaded, (ii) when the upper system is overloaded and the lower system is underloaded, and (iii) when both systems are overloaded. We apply mathematical induction over the successive alternating intervals of these three kinds. (The switch points are the union of the two separate sets of switch points.) We ensure that the initial conditions for each succeeding interval satisfy the initial ordering assumed in the theorem. If we start in an interval where both systems are underloaded, then the ordering holds while both systems are underloaded by virtue of the explicit representation in Proposition 1. Consequently, the underload termination times are ordered as well, by Proposition 1. The ordering $B_1(t) \leq B_2(t)$ necessarily remains valid when the upper system is overloaded and the lower system is underloaded, because then we have $B_1(t) \leq s(t) = B_2(t)$. For an interval where both systems are overloaded, it suffices to consider the two systems starting the first time both systems are overloaded. At that time, the initial conditions necessarily will be

ordered properly, because the system to become overloaded later has $Q_1(t) = 0$. At this initial time, $B_1(t) = B_2(t) = s(t)$.

The M_t service assumption comes to the fore in an interval where both systems are overloaded. Here we use the fact that σ and $\gamma(t) = b(t, 0)$ depend only upon s and μ during the overloaded interval, and so are the same for the two systems, because the functions s and μ have been assumed to be fixed. The rate of service completion is $\sigma(t) = s'(t) + s(t)\mu(t)$. When the two systems are both overloaded over a common interval $[t, t + u]$, the total fluid to enter service from queue, $E(t + u) - E(t)$ is therefore the same in the two systems.

When both systems are overloaded, we have the ordering $\tilde{q}_1 \leq \tilde{q}_2$ directly from Proposition 3, just as in Proposition 5.3 of Liu and Whitt (2010), exploiting the representation

$$\frac{\bar{F}_{t-x}(x)}{\bar{F}_{t-x}(x-t)} = e^{-\int_{x-t}^x h_{F_{t-x}}(y) dy}.$$

Hence, to show that $q_1 \leq q_2$, it suffices to show that $w_1 \leq w_2$, which would imply that the overload termination times are ordered as well.

Suppose we start at t_1 with $w_1(t_1) \leq w_2(t_1)$. Suppose that $w_1(t) > w_2(t)$ at some $t > t_1$. The continuity of w_1 and w_2 implies that there exists some $t_1 < t_2 < t$ such that $w_1(t_2) = w_2(t_2) \equiv \tilde{w}$. However, the ordering of \tilde{q}_1 and \tilde{q}_2 implies that $\tilde{q}_1(t_2, \tilde{w}) \leq \tilde{q}_2(t_2, \tilde{w})$. Therefore, ODE (25) implies that $w'_1(t_2) \leq w'_2(t_2)$. This contradicts with our assumption that there exists a t such that $w_1(t) > w_2(t)$.

Now we turn to v . The equation (27) in Theorem 5 implies that the ordering of w is inherited by v . That is made clear by applying the proof of Theorem 5, which shows that $v(t)$ is determined by the intersection of the function w with the linear function $L_t(u) \equiv t + u$. Clearly, if we increase the w function, then that intersection point increases as well. ■

EC.3.2. Proof of Theorem 8.

We directly prove (30); the corresponding results in (31) will be obtained along the way. To show (i), consider two models with common model data except for $\lambda, B(0)$, where $\lambda_1, \lambda_2, s', \mu \in \mathcal{P}_{m,n}$ for

some m, n . Without loss of generality, by Theorem 7, it suffices to assume that $\lambda_1 \leq \lambda_2$ and $B_1(0) \leq B_2(0)$. If that is not initially the case, consider $\tilde{\lambda}_1 \equiv \lambda_1 \wedge \lambda_2$, $\tilde{\lambda}_2 \equiv \lambda_1 \vee \lambda_2$, $\tilde{B}_1(0) \equiv B_1(0) \wedge B_2(0)$ and $\tilde{B}_2(0) \equiv B_1(0) \vee B_2(0)$ to get $\tilde{\lambda}_1 \leq \tilde{\lambda}_2$ and $\tilde{B}_1(0) \leq \tilde{B}_2(0)$ with $\|\tilde{\lambda}_1 - \tilde{\lambda}_2\|_T = \|\lambda_1 - \lambda_2\|_T$ and $|\tilde{B}_1(0) - \tilde{B}_2(0)| = |B_1(0) - B_2(0)|$.

When both systems are overloaded, we have $B_1(t) = B_2(t) = s(t)$. Hence, the overall story depends on what happens when (a) both systems are underloaded, and (b) system 1 is underloaded and system 2 is overloaded.

For simplicity, suppose that the two systems both start underloaded at time 0 with $B_1(0) \leq B_2(0)$, $\lambda_1 \leq \lambda_2$. If both systems remain underloaded over the interval $[0, t_1]$, then by Proposition 1 we have

$$\begin{aligned} |B_1(t) - B_2(t)| &\leq \|\lambda_1 - \lambda_2\|_T \int_0^t e^{-M(x)} dx + |B_1(0) - B_2(0)| \\ &\leq t \cdot \|\lambda_1 - \lambda_2\|_T + |B_1(0) - B_2(0)|, \quad 0 \leq t \leq t_1. \end{aligned} \quad (\text{EC.17})$$

Suppose system 2 becomes overloaded at $t_1 > 0$ while system 1 remains underloaded. For $t > t_1$, we have $B_1(t) \leq B_2(t) = s(t) \leq X_2(t) \equiv B_2(t) + s(t)$. Hence we have $0 \leq |B_2(t) - B_1(t)| = B_2(t) - B_1(t) \leq X_2(t) - B_1(t)$. Flow conservations of both systems implies that $B_1'(t) = \lambda_1(t) - \mu(t) B_1(t)$ and $X_2'(t) = \lambda_2(t) - \alpha_2(t) - \mu(t) s(t)$. Therefore,

$$X_2'(t) - B_1'(t) = \lambda_2(t) - \lambda_1(t) - \alpha_2(t) - \mu(t) (s(t) - B_1(t)) \leq \lambda_2(t) - \lambda_1(t),$$

which implies that

$$\begin{aligned} |B_1(t) - B_2(t)| &\leq |B_1(t_1) - B_2(t_1)| + (t - t_1) \cdot \|\lambda_1 - \lambda_2\|_T \\ &\leq t_1 \cdot \|\lambda_1 - \lambda_2\|_T + |B_1(0) - B_2(0)| + (t - t_1) \cdot \|\lambda_1 - \lambda_2\|_T \\ &\leq t \cdot \|\lambda_1 - \lambda_2\|_T + |B_1(0) - B_2(0)|, \end{aligned} \quad (\text{EC.18})$$

where the second inequality follows from (EC.17) with $t = t_1$.

If we then later start a second underloaded interval for both systems at time t_2 , where $0 < t_1 < t_2 < T$, then we will have inequality (EC.17) holding at time t_2 . Thus proceeding forward, applying (EC.17) with initial values $B_i(t_2)$, during the following underloaded interval we have for $t > t_2$

$$|B_1(t) - B_2(t)| \leq \|\lambda_1 - \lambda_2\|_T \int_{t_2}^t e^{-M(x)} dx + |B_1(t_2) - B_2(t_2)|$$

$$\begin{aligned}
&\leq (t - t_2) \cdot \|\lambda_1 - \lambda_2\|_T + t_2 \cdot \|\lambda_1 - \lambda_2\|_T + |B_1(0) - B_2(0)| \\
&\leq t \cdot \|\lambda_1 - \lambda_2\|_T + |B_1(0) - B_2(0)| \\
&\leq (1 \vee t)(\|\lambda_1 - \lambda_2\|_T \vee |B_1(0) - B_2(0)|). \tag{EC.19}
\end{aligned}$$

where the second inequality follows from (EC.18) with $t = t_2$. Applying mathematical induction over successive underloaded subintervals of $[0, T]$, using the second to last inequality, we obtain the first relation in (30), from which the desired conclusion follows.

To show (ii), when both systems are underloaded, we have $Q_1(t) = Q_2(t) = 0$. Hence, the overall story depends on what happens when (a) both systems are overloaded, and (b) system 1 is underloaded and system 2 is overloaded.

When both systems are overloaded, flow conservation implies that

$$Q'_i(t) = \lambda_i(t) - \alpha_i(t) - \gamma_i(t) = \lambda_i(t) - \alpha_i(t) - \mu(t)s(t) - s'(t).$$

Hence, we have

$$Q'_2(t) - Q'_1(t) = \lambda_2(t) - \lambda_1(t) - (\alpha_2(t) - \alpha_1(t)) \leq \lambda_2(t) - \lambda_1(t),$$

where the inequality simply follows from Theorem 7 when the two systems have common abandonment distribution. This yields

$$|Q_1(t) - Q_2(t)| = Q_2(t) - Q_1(t) \leq |Q_1(0) - Q_2(0)| + t\|\lambda_1 - \lambda_2\|_T. \tag{EC.20}$$

When system 2 is overloaded and system 1 is underloaded. For simplicity, assume at time 0 the two system have initial conditions $B_2(0) = s(0) > B_1(0)$, $Q_2(0) \geq 0 = Q_1(0)$. Let $T^* \equiv T_1 \wedge T_2$, where T_1 denotes the underload termination time of system 1 and T_2 denotes the overload termination time of system 2. Hence we know that both systems will not change regimes for $0 \leq t \leq T^*$. For $0 \leq t \leq T^*$, we have

$$\begin{aligned}
Q'_2(t) &= \lambda_2(t) - \alpha_2(t) - \gamma_2(t) \leq \lambda_2(t) - \gamma_2(t) \\
&\leq (\lambda_2(t) - \lambda_1(t)) + (\lambda_1(t) - \gamma_2(t))
\end{aligned}$$

$$\leq (\lambda_2(t) - \lambda_1(t)) + (\lambda_1(t) - \mu(t) s(t) - s'(t)),$$

which implies that

$$\begin{aligned} & |Q_2(t) - Q_1(t)| = Q_2(t) \\ & \leq Q_2(0) + t \|\lambda_2(t) - \lambda_1(t)\|_T + \int_0^t \lambda_1(u) - \mu(u) s(u) - s'(u) du \\ & \leq Q_2(0) + t \|\lambda_2(t) - \lambda_1(t)\|_T + \int_0^t \lambda_1(u) - \mu(u) B_1(u) du - (s(t) - s(0)) \\ & \leq Q_2(0) + t \|\lambda_2(t) - \lambda_1(t)\|_T + \int_0^t B_1'(u) du - s(t) + s(0) \\ & \leq Q_2(0) + t \|\lambda_2(t) - \lambda_1(t)\|_T + (s(0) - B_1(0)) - (s(t) - B_1(t)) \\ & \leq |Q_2(0) - Q_1(0)| + t \|\lambda_2(t) - \lambda_1(t)\|_T + |B_2(0) - B_1(0)|, \end{aligned} \tag{EC.21}$$

where the second inequality holds because $B_1(t) \leq s(t)$, the third inequality holds since $B_1'(t) = \lambda_1(t) - \mu(t) B_1(t)$, and the last inequality holds since $Q_1(0) = 0$, $B_2(0) = s(0)$ and $B_1(t) \leq s(t)$.

Again, the desired conclusion follows by mathematical induction.

Finally, to show (iii), (EC.18), (EC.19), (EC.20), (EC.21) imply that

$$\begin{aligned} |X_1(t) - X_2(t)| & \leq |B_1(t) - B_2(t)| + |Q_1(t) - Q_2(t)| \\ & \leq 2t \|\lambda_1 - \lambda_2\| + 2|B_1(0) - B_2(0)| + |Q_1(0) - Q_2(0)| \\ & \leq 2(1 \vee t)(\|\lambda_1 - \lambda_2\|_T \vee |X_1(0) - X_2(0)|), \end{aligned}$$

where the third inequality holds because $|X_1(0) - X_2(0)| = |B_1(0) - B_2(0)| + |Q_1(0) - Q_2(0)|$ in all regimes. ■

EC.3.3. Proof of Theorem 9.

Given $\lambda \in \mathbb{C}_p$, we choose an increasing sequence $\{\lambda_k : k \geq 1\}$ with $\lambda_k \in \mathcal{P}_{m_k, n_k}$ for each $k \geq 1$ such that $\|\lambda_k - \lambda\|_T \rightarrow 0$ as $k \rightarrow \infty$. For each $k \geq 1$, we can apply all the results above. By Theorem 8, we can define the pair (B, σ) in \mathbb{C}_p^2 as the limit of the sequence $\{(B_k, \sigma_k)$ in \mathbb{C}_p^2 with the maximum/uniform norm. There is such a limit, because the sequence is necessarily Cauchy and the

space is a complete metric space. Given the limit, the convergence holds in the space by Theorem 8.

To show that the monotonicity extends, we start with $\lambda_1 \leq \lambda_2$. We then construct sequences $\{\lambda_{i,k} : k \geq 1\}$ for $i = 1, 2$ with $\lambda_{1,k} \leq \lambda_{2,k}$ for each k and $\|\lambda_{i,k} - \lambda_i\|_T \rightarrow 0$ as $k \rightarrow \infty$. We apply Theorem 7 for each k . Since the ordering is preserved in the limit, the conclusion of Theorem 7 holds for the limiting pair by Lebesgue monotone convergence. We use a similar argument to show that the Lipschitz continuity properties in Theorem 8 extend as well: Starting with $\|\lambda_1 - \lambda_2\|_T = c$, for any $\epsilon > 0$, we construct sequences $\{\lambda_{i,k} : k \geq 1\}$ for $i = 1, 2$ with $\|\lambda_{1,k} - \lambda_{2,k}\| \leq c + \epsilon$ for each k and $\|\lambda_{i,k} - \lambda_i\|_T \rightarrow 0$ as $k \rightarrow \infty$ for $i = 1, 2$. We then can apply Theorem 8 for each $k \geq 1$, and get the conclusion there with modification by ϵ . However, since ϵ is arbitrary, we get the preservation of the Lipschitz property to the limit. ■

EC.4. one proof for §6.

Proof of Theorem 11. We recursively apply the monotone contraction operator Ψ in Theorem 10, starting with $\sigma_{j,i}^{(0)} = 0$, so that $\lambda_{1,i}^{(1)} \leq \lambda_{2,i}^{(1)}$ for all i , because $\lambda_{j,i}^{(1)} = \lambda_{j,i}^{(0)}$, $j = 1, 2$ and the external arrival rate functions have been assumed to be ordered: $\lambda_{1,i}^{(0)} \leq \lambda_{2,i}^{(0)}$. By Theorem 7 applied to each queue separately, using the assumed ordering $B_{1,i}(0) \leq B_{2,i}(0)$ for all i , we have first $B_{1,i}^{(1)} \leq B_{2,i}^{(1)}$ and then $\sigma_{1,i}^{(1)} \leq \sigma_{2,i}^{(1)}$. By (35), we then have $\lambda_{1,i}^{(2)} \leq \lambda_{2,i}^{(2)}$. We then get the order holding for all n by applying mathematical induction. However, $\lambda_{1,i}^{(n)} \rightarrow \lambda_{1,i}$ as $n \rightarrow \infty$. Since the order is preserved in the convergence, we deduce that $\lambda_{1,i} \leq \lambda_{2,i}$ for $1 \leq i \leq m$. Finally, we can apply Theorem 7 to each queue separately to get the remaining orderings. ■

EC.5. Remarks

REMARK EC.1. (characterization of isolated points)

Definition 3 implies that t is an isolated point only if $Q(t) = 0$, $B(t) = s(t)$. Moreover, if t is a discontinuity point of ζ , then $\zeta(t - \delta) < 0$ and $\zeta(t) > 0$ for some $\delta > 0$; if t is a continuity point of ζ , then $\zeta(t - \delta) < 0$, $\zeta(t) = 0$ and $\zeta(t + \delta) < 0$ for some $\delta > 0$.

REMARK EC.2. (an ODE for B in an underloaded interval)

In an underloaded interval, the total fluid content in service $B(t)$ can also be characterized via the ODE

$$B'(t) = \lambda(t) - \mu(t)B(t), \quad t \geq 0. \quad (\text{EC.22})$$

The formula in Proposition 1 provides the solution to the initial value problem determined by this ODE with initial condition $B(0)$.

REMARK EC.3. (applied significance of \mathcal{P}_{mn}) We have provided a full algorithm when $\lambda, s', \mu \in \mathcal{P}_{m,n}$. An algorithm for $\lambda \in \mathbb{C}_p$ can be developed by considering a sequence of successive approximations in $\mathcal{P}_{m,n}$, but we see no motivation for doing so. We have introduced the space $\mathcal{P}_{m,n}$ of piecewise polynomials as a device to establish mathematical results. In applications, it should suffice to use *any* convenient representations of the functions λ and s , and *assume* that there are only finitely many switches in any finite interval. While running the algorithm, that assumption can be verified, and the model can be modified if too many switches occur. However, if we start from data, then we could choose to let the functions be in $\mathcal{P}_{m,n}$ without loss of generality. Lemma 2 shows that it is convenient to work in the space $\mathcal{P}_{m,n}$, because we can obtain closed form expressions for integrals. Moreover, if we want to bound the number of switches in advance, then we can bound the parameters m and n , with the understanding that there is a tradeoff between the quality of fit and the maximum number of switches.

EC.6. Simulation Verification for the $M_t/M/s + GI$ Model

In this section we illustrate the single-queue algorithm for a relatively simple case, the $M_t/M/s + GI$ fluid queue model, in which only the arrival rate is time varying and only the abandonment cdf F is non-exponential. We let the arrival rate function λ be sinusoidal, i.e.,

$$\lambda(t) \equiv a + b \cdot \sin(c \cdot t), \quad t \geq 0, \quad (\text{EC.23})$$

where we let $b \equiv 0.6a$, $c \equiv 1$ and $a \equiv s$. By making the average input rate a coincide with the fixed staffing level s , we ensure that the system will alternate between overloaded and underloaded.

We let the service rate be $\mu \equiv 1$ and the abandonment rate $\theta \equiv 0.5$; i.e., $G(x) \equiv 1 - e^{-x}$ and $F(x) = 1 - e^{-\theta x} = 1 - e^{-0.5x}$ for $x \geq 0$. Without loss of generality, for the fluid model we let $s \equiv 1$.

For the general abandon-time cdf F , we considered two cases: Erlang-2 (E_2) and hyperexponential-2 (H_2). We fix the mean at $1/\theta$. An E_2 random variable is the sum of two i.i.d. exponential random variables, so that there are no additional parameters. An H_2 cdf is the mixture of two exponential cdf's, and so has two additional parameters beyond its mean. An H_2 pdf is of the form

$$f(x) = p \cdot \theta_1 e^{-\theta_1 x} + (1-p) \cdot \theta_2 e^{-\theta_2 x}, \quad x \geq 0,$$

We let $p = 0.5(1 - \sqrt{0.6})$, $\theta_1 = 2p\theta$, $\theta_2 = 2(1-p)\theta$, which produces “balanced means” and squared coefficient of variation (SCV, variance divided by the square of the mean) $SCV \equiv c^2 = 4$.

We only show the results for E_2 abandonment; the results for H_2 are similar. The fluid performance functions for E_2 abandonment are shown in Figure EC.1 for $t \in [0, T]$ with $T = 16$. The performance functions shown in Figure EC.1 are the boundary waiting time $w(t)$, the fluid in queue $Q(t)$, the fluid in service $B(t)$, the total fluid in the system $X(t)$, the abandonment rate $\alpha(t)$, and the rate fluid enters service (transportation rate) $\gamma(t) \equiv b(t, 0)$. We omit the departure rate $\sigma(t) = \mu B(t)$ because of the exponential service times.

In Figure EC.2 we compare the fluid approximations with results from a simulation experiment for a very large-scale queueing system. The queueing model has a nonhomogeneous Poisson arrival process with sinusoidal rate function as in (EC.23), with $a = s = 2000$, $b = 0.6a = 1200$. We compare the fluid model predictions to a single sample path of the queueing system (one simulation run). In Figure EC.2 the blue solid lines of the simulation estimations of single sample paths applied with fluid scaling, and the red dashed lines are the fluid approximations. We conclude that the fluid approximation is remarkably accurate as an approximation when the scale of the queueing model is extremely large.

As discussed in Liu and Whitt (2010), the accuracy of the fluid approximations for large-scale queueing systems can be explained by a many-server heavy-traffic limit. As discussed in §9 of Liu

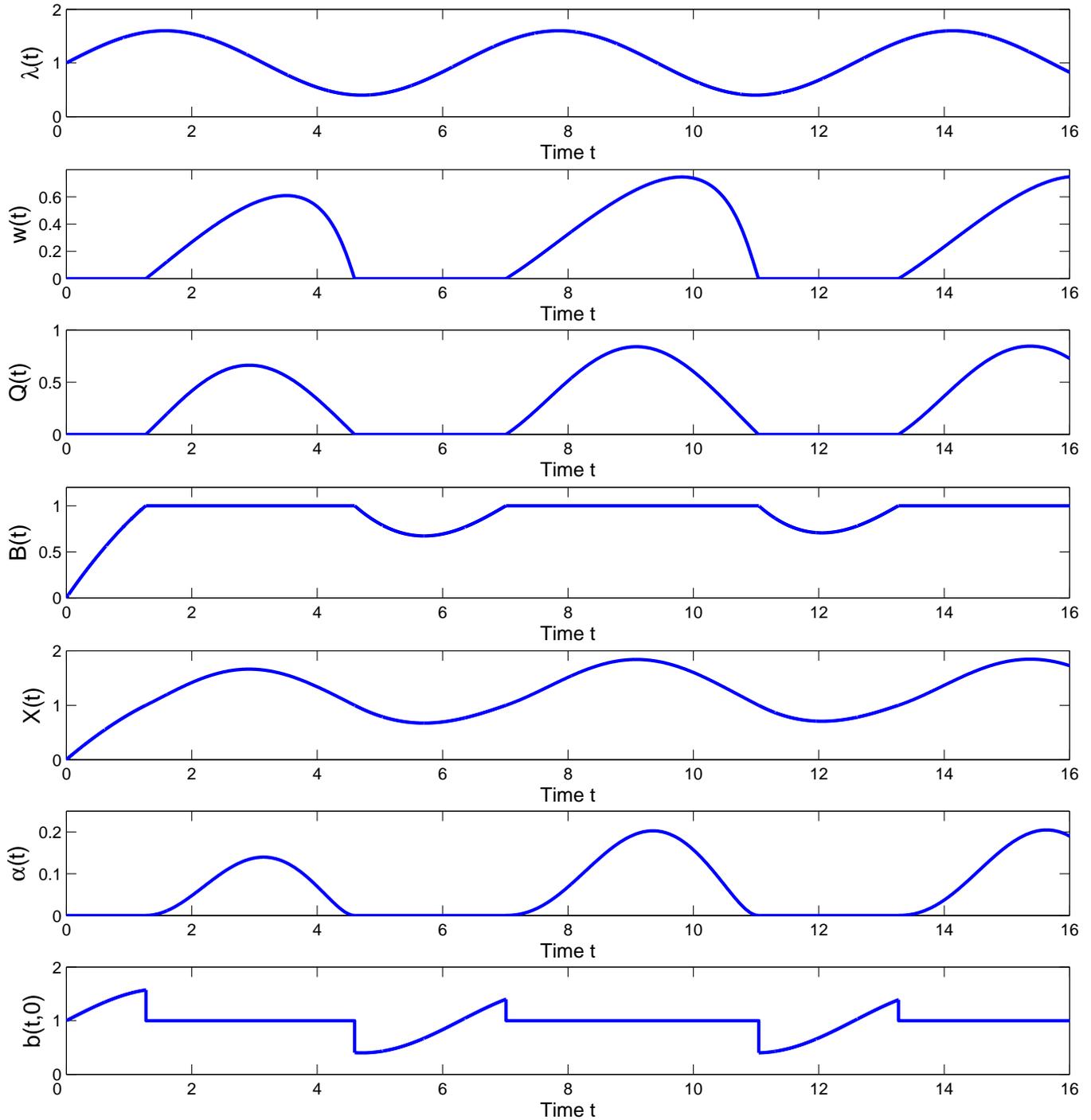


Figure EC.1 Performance for the $M_t/M/s + E_2$ fluid model with sinusoidal arrival-rate function.

and Whitt (2010), for smaller systems the queueing system has much greater stochastic fluctuations. In those cases, the fluid model performance functions quite accurately describe the mean values of the time-varying queue performance when the system experiences significant periods of overload;

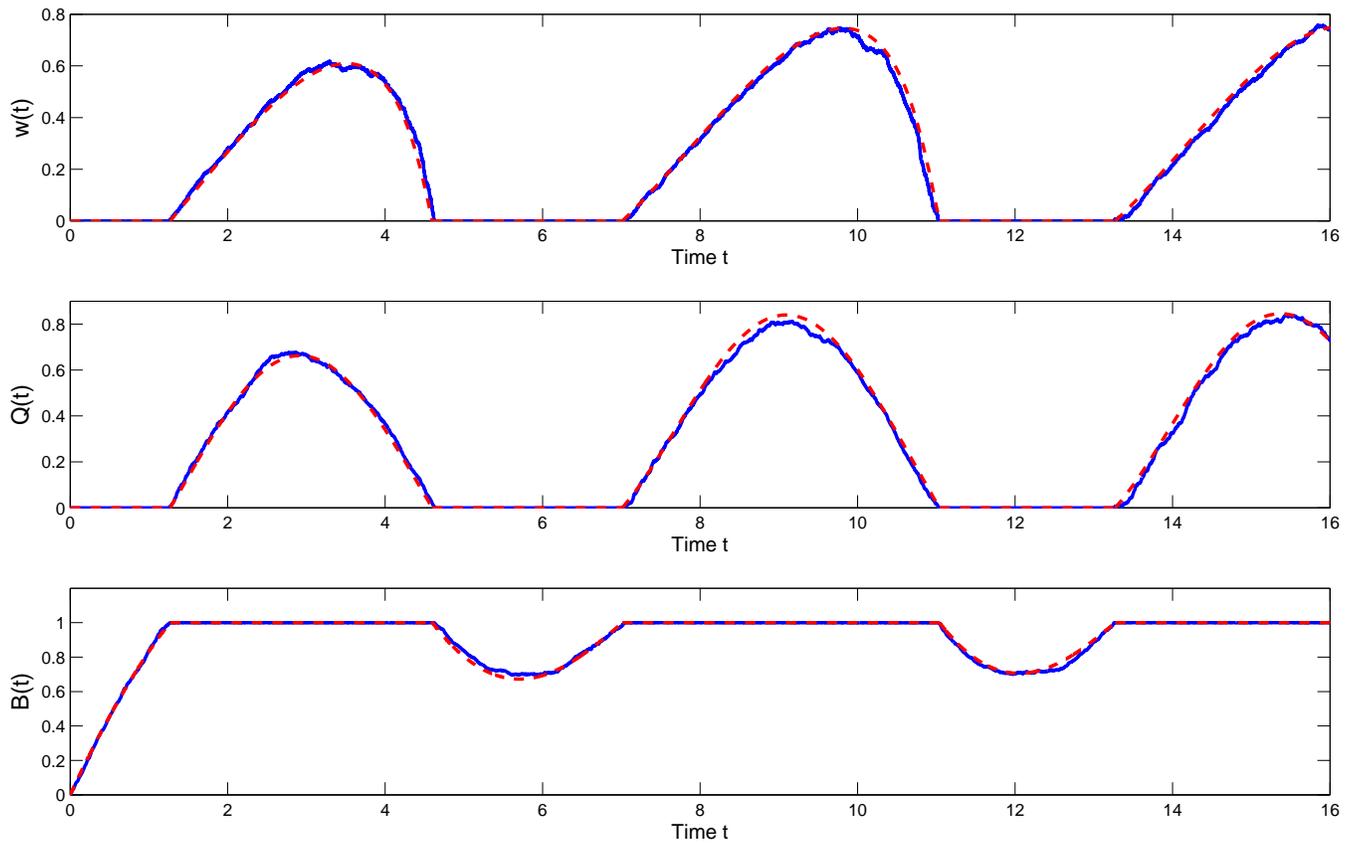


Figure EC.2 A comparison of the $M_t/M/s + E_2$ fluid model with a simulation of the large-scale queueing system.

e.g., see Figure 7 there.