DEGENERACY IN THE LENGTH SPECTRUM FOR METRIC GRAPHS

Richard Sharp

University of Manchester

Abstract. In this note we show that the length spectrum for metric graphs exhibit a very high degree of degeneracy. More precisely, we obtain an asymptotic for the number of pairs of closed geodesic (or closed cycles) with the same metric length.

0. Introduction

Let $G = (V, E)$ be a finite graph with vertices V and edges E. We write E^o for the set of oriented edges; for $e \in E^o$, $\bar{e} \in E^o$ denotes the same unoriented edge with orientation reversed. (In the physics literature, the vertices are referred to as nodes and the edges as bonds.) The degree of a vertex is the number of outgoing oriented edges. A path in G is a sequence of successive oriented edges and is called non-backtracking if an oriented edge e is not immediately followed by $\bar{e}$. A path is called closed if it returns to its starting point. A closed geodesic on G is a closed non-backtracking path, modulo the obvious cyclic permutation and we denote the (countable) set of closed geodesics by $\mathcal{C}$. For $\gamma \in \mathcal{C}$, we write $|\gamma|$ for the number of edges it traverses.

We make G into a metric graph by giving each edge $e \in E$ a length $l(e) > 0$ and thus identifying it with the interval $[0, l(e)]$. (We may also regard l as a function on E^o satisfying $l(\bar{e}) = l(e)$.) For $\gamma \in \mathcal{C}$, we define the metric length $l(\gamma)$ to be the sum of the lengths of the edges it traverses.

We impose two assumptions on G and l:

Assumption I. Each vertex has degree at least 3.

Assumption II. The set $\{l(e) : e \in E\}$ consists of $|E|$ rationally independent numbers.

Define the period of G to be $d = \text{g.c.d.}\{|\gamma| : \gamma \in \mathcal{C}\}$. Under Assumption I, d is equal to 1 or 2 [16]. If we define the growth rate λ by

$$\lambda = \lim_{n \to +\infty} \#\{\gamma \in \mathcal{C} : |\gamma| = dn\}^{1/dn}$$

then $\lambda > 1$ and

$$\#\{\gamma \in \mathcal{C} : |\gamma| = dn\} \sim \frac{\lambda^d}{\lambda^d - 1} \frac{\lambda^{dn}}{dn}, \tag{0.1}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

as $n \to +\infty$. (Later, we shall see that λ is an algebraic integer.) Alternatively, we may order closed geodesics in $\mathcal{C}$ by l. Then, under Assumptions I and II, there exists a critical exponent $\delta = \delta(G, l) > 0$ such that

$$\#\{\gamma \in \mathcal{C} : l(\gamma) \leq T\} \sim \frac{e^{\delta T}}{\delta T}, \text{ as } T \to +\infty \tag{0.2}$$

[13], [21]. (Graphs are not mentioned explicitly in [21] but the above situation is equivalent to a locally constant suspension over the subshift of finite type introduced in section 2.)

Let $\mathcal{L} = \{l \in \mathbb{R}^+ : l(\gamma) = l \text{ for some } \gamma \in \mathcal{C}\}$ and, for $l \in \mathcal{L}$, $m(l) = \#\{\gamma \in \mathcal{C} : l(\gamma) = l\}$, so that

$$\#\{\gamma \in \mathcal{C} : l(\gamma) \leq T\} = \sum_{l \in \mathcal{L},\ l \leq T} m(l).$$

A priori, the asymptotic in (0.1) may be caused by a combination of the growths of $\mathcal{L}$ and of $m(l)$. We shall show that, in this setting, the growth coming from the multiplicities $m(l)$ predominates in a very strong way. More precisely, we have the following asymptotic for the number of pairs of closed geodesics with the same metric length.

Theorem 1. *Under Assumptions I and II,*

$$\#\{(\gamma, \gamma') \in \mathcal{C} \times \mathcal{C} : |\gamma| = |\gamma'| = dn,\ l(\gamma) = l(\gamma')\} \sim C(G) \frac{\lambda^{2dn}}{n^{2+(|E|-1)/2}},$$

as $n \to +\infty$, where $C(G) > 0$ only depends on G (and not on l).

Remark. Comparing this with (0.1), we see that the proportion of pairs in $\{(\gamma, \gamma') \in \mathcal{C} \times \mathcal{C} : |\gamma| = |\gamma'| = dn\}$ satisfying $l(\gamma) = l(\gamma')$ decreases like $n^{-(|E|-1)/2}$.

If we drop the assumption of rational independence then we obtain the following weaker result. We note that, without rational independence, the multiplicities in $\mathcal{L}$ increase. More precisely, for any metric graph satisfying Assumption I, we may arbitrarily modify the edge lengths so that Assumption 2 is satisfied without increasing the number of pairs of closed geodesics $(\gamma, \gamma') = n$, for each n, with the same metric length. Thus we obtain the following corollary.

Corollary 1.1. *Under Assumption I,*

$$\liminf_{n \to +\infty} \frac{n^{2+(|E|-1)/2} \#\{(\gamma, \gamma') \in \mathcal{C} \times \mathcal{C} : |\gamma| = |\gamma'| = dn,\ l(\gamma) = l(\gamma')\}}{\lambda^{2dn}} > 0.$$

We have similar results for non-geodesic closed cycles where backtracking is allowed (see section 4). The set of cycles with a given metric length is called a *degeneracy class*. It follows from Assumption II that each cycle γ in a particular degeneracy class has the same edge length $|\gamma| = n$. The behaviour of the number of degeneracy classes as $n \to +\infty$ has been studied by Berkolaiko [1]. For binary directed graphs (i.e. directed graphs with two incoming and two outgoing edges for each vertex) with at most six vertices, Tanner analysed the asymptotics of the number of cycles in a degeneracy class [28]. Gavish and Smilansky have obtained asymptotics for the average size of a degeneracy class for a fully connected graph (i.e. a graph in which each pair of vertices is joined by a single edge) [11]; more precisely they define this average size to be the number of degeneracy classes of length n divided by the number of cycles of length n.

It is worthwhile to compare our result with the analogous problem for smooth compact surface equipped with a Riemannian metric of negative curvature. Let S be such a surface and let $\mathcal{L}(S)$ be the set of lengths of closed geodesics. For a typical variable negative curvature surface, one has $m(l) = 2$ for all $l \in \mathcal{L}(S)$, with the only multiplicity arising from reversing the orientation. In contrast, Randol has shown that for any constant negative curvature surface, $m(l)$ is unbounded [27] and Buser has the stronger result that there exists $c > 0$ and a sequence $l_n \in \mathcal{L}(S)$ such that $m(l_n) \geq c l_n^{\log 2/\log 5}$ [10]. For arithmetic surfaces, although the distribution is very irregular, the average over intervals $[l, l + \Delta l]$ is at least $ce^{l/2}/l$ [7], [8]. Numerical results also suggest such exponential growth (with a smaller exponent) for certain non-arithmetic surfaces associated to Hecke triangle groups [6], [9]. In contrast, one sees that metric graphs give rise to much greater multiplicities.

The length spectra on both metric graphs and surfaces is closely related to the spectral statistics studied in quantum chaos. It is widely conjectured that typical chaotic systems satisfy Random Matrix statistics. However, numerical results for arithmetic surfaces indicate Poissonian statistics [6] and it is believed that this may be connected to the rather high multiplicities in the length spectrum. In contrast, under certain conditions, families of progressively larger (quantum) metric graphs exhibit Random Matrix statistics in the limit. Thus it is interesting that metric graphs have much higher multiplicities than arithmetic surfaces and it would perhaps be useful to understand the phenomenon described in Theorem 1 in a uniform way over families of increasing graphs.

We shall now outline the contents of the paper. In section 1, we give a more detailed description of metric graphs and briefly mention quantum graphs. In section 2, we introduce subshifts of finite type as dynamical systems naturally associated to our graphs. We discuss their properties and, in particular, introduce a function which encodes metric lengths as vectors in $\mathbb{Z}^E$. In section 3, we proceed to the proof of Theorem 1 by introducing a product shift in order to study pairs of closed geodesics. We show that, once some conditions are checked, the theorem follows from a result in [23]. In section 4, we discuss analogous results for cycles where backtracking is allowed.

1. Metric and Quantum Graphs

In this section, we discuss some preliminary material concerning metric graphs. Let $G = (V, E)$ be a finite graph, where V is a set of vertices and E is a set of edges, assumed to be finite real interval (up to possible identification of the endpoints). The graph G is made into a metric graph by assigning a positive length $l(e)$ to each edge $e \in E$. We allow loops and multiple edges and we always assume our graphs are connected, i.e., any two vertices are joined by a path made up of successive edges. We say that G is bipartite if V is the disjoint union of two non-empty sets V_1 and V_2 such that if two vertices are joined by an edge then one of them is in V_1 and the other in V_2.

Each edge $e \in E$ has two possible orientations, which (abusing notation slightly) we denote by e and $\bar{e}$. We denote the set of oriented edges by E^o. Clearly, $|E^o| =$

$2|E|$. If $e \in E^o$ is an oriented edge going from v to v' then we write $\mathfrak{o}(e) = v$ and $\mathfrak{t}(e) = v'$. (We say that $\mathfrak{o}(e)$ is the *origin* of the edge and that $\mathfrak{t}(e)$ is the *terminus* of the edge.) We have the relations $\mathfrak{o}(\bar{e}) = \mathfrak{t}(e)$ and $\mathfrak{t}(\bar{e}) = \mathfrak{o}(e)$. (If e is a loop starting and ending at the same vertex then $\mathfrak{o}(e) = \mathfrak{t}(e)$; however, there are still two distinct orientations.) We may think of the edge lengths as defined on oriented edges but subject to the relation $l(\bar{e}) = l(e)$.

The number of oriented edges with origin a particular vertex v is called the *degree* of v, written $\deg(v)$. (Note that an edge which is loop joining a vertex v to itself contributes 2 to $\deg(v)$.)

A *path* in the graph is a sequence of edges $(e_1, e_2, \ldots, e_n)$ such that $\mathfrak{t}(e_i) = \mathfrak{o}(e_{i+1})$, $i = 1, \ldots, n-1$. We say that a path is a *non-backtracking path* if $e_{i+1} \neq \bar{e}_i$, $i = 1, \ldots, n-1$. If $\mathfrak{t}(e_n) = \mathfrak{o}(e_1)$ then we say that $(e_1, e_2, \ldots, e_n)$ is a closed path and we refer to the n cyclic permutations of the the edges $e_1, e_2, \ldots, e_n$ as the corresponding *closed cycle*, denoted by $\langle e_1, \ldots, e_n \rangle$. If $(e_1, e_2, \ldots, e_n)$ is a non-backtracking closed path and if $e_0 \neq \bar{e}_n$ then we called $\langle e_1, \ldots, e_n \rangle$ a *closed geodesic.* We shall denote the set of closed geodesics in G by $\mathcal{C} = \mathcal{C}(G)$ and the set of closed cycles in G by $\mathcal{C}^* = \mathcal{C}^*(G)$. We say that a closed geodesic is *prime* if it is not obtained by multiple repetitions of the same closed path and write $\mathcal{P}$ for the set of prime closed geodesics.

There are two lengths naturally associated to a closed geodesic γ. If γ is determined by the closed path $(e_1, e_2, \ldots, e_n)$ then the *edge length* of γ is defined to be $|\gamma| = n$ and the *metric length* of γ is defined to be

$$l(\gamma) = l(e_1) + l(e_2) + \cdots + l(e_n).$$

We define the *period* of the graph G to be the number

$$d = \text{g.c.d.}\{|\gamma| : \gamma \in \mathcal{C}\}. \tag{1.1}$$

Assumption I. We shall assume that each vertex $v \in V$ has $\deg(v) \geq 3$.

An immediate consequence of this assumption is that G does not consist of a single closed cycle and that the fundamental group of G is a free group on at least 2 generators.

Lemma 1.1 [16]**.** *If $G = (V, E)$ satisfies Assumption I then $d \in \{1, 2\}$ and $d = 2$ if and only if G is bipartite.*

Assumption II. We shall assume that $\{l(e) : e \in E\}$ consists of $|E|$ rationally independent numbers.

For an (undirected) edge $e \in E$, let $\nu_e(\gamma)$ denote the number of times that γ passes through e. Under Assumption II, $l(\gamma)$ uniquely determines the vector

$$\mathfrak{l}(\gamma) = (\nu_e(\gamma))_{e \in E} \in \mathbb{Z}^E.$$

Lemma 1.2. *For $\gamma, \gamma' \in \mathcal{C}$, we have that $l(\gamma) = l(\gamma')$ if and only if $\mathfrak{l}(\gamma) = \mathfrak{l}(\gamma')$.*

Let $\Gamma \subset \mathbb{Z}^E$ denote the abelian group generated by the set $\{\mathfrak{l}(\gamma) : \gamma \in \mathcal{C}\}$.

Lemma 1.3. *For all $e \in E$, $2\mathfrak{l}(e) \in \Gamma$. In particular, Γ has rank $|E|$.*

Proof. Let $e \in E$. If $\mathfrak{o}(e) = \mathfrak{t}(e)$ then $\mathfrak{l}(e) \in \Gamma$ and there is nothing to prove. So suppose that $v_1 = \mathfrak{o}(e) \neq v_2 = \mathfrak{t}(e)$. Since v_1 has degree at least 3, we can find a directed edge f_1 with $\mathfrak{o}(f_1) = v_1$ and a directed edge f_1' with $\mathfrak{t}(f_1') = v_1$ such that e, f_1 and f_1' represent three distinct undirected edges. By the irreducibility of G, we can find a (non-backtracking) path γ_1 in G, starting with f_1 and ending with f_1'; by construction, γ_1 is a closed geodesic and $\mathfrak{l}(\gamma_1) \in \Gamma$. Similarly, we can find a directed edge f_2 with $\mathfrak{o}(f_2) = v_2$ and a directed edge f_1' with $\mathfrak{t}(f_1') = v_2$ such that e, f_2 and f_2' represent three distinct undirected edges. By the irreducibility of G, we can find a (non-backtracking) path γ_2 in G, starting with f_2 and ending with f_2'; by construction, γ_2 is a closed geodesic and $\mathfrak{l}(\gamma_2) \in \Gamma$. However, also by construction, the concatenated path

$$\gamma^* = \gamma_1 * e * \gamma_2 * \bar{e}$$

is a closed geodesic and $\mathfrak{l}(\gamma^*) \in \Gamma$. It is clear that

$$\mathfrak{l}(\gamma^*) = \mathfrak{l}(\gamma_1) + \mathfrak{l}(\gamma_2) + 2\mathfrak{l}(e),$$

so that one obtains $2\mathfrak{l}(e) \in \Gamma$. It follows immediately that Γ has full rank E in $\mathbb{Z}^E$. $\square$

We shall now briefly describe the additional structure that makes a metric graph into a quantum graph. A quantum graph is a metric graph equipped with a self-adjoint operator. Perhaps the most popular choice is the Laplacian operator Δ, defined on the edge e, parametrized as $[0, l(e)]$, by

$$\Delta f(x_e) = -\frac{d^2 f(x_e)}{dx_e^2}, \quad x_e \in [0, l(e)],$$

with the Neumann boundary conditions

$$\sum_{e \in E^o \,:\, \mathfrak{o}(e) = v} \left.\frac{df(x_e)}{dx_e}\right|_{x_e = 0} = 0,$$

for all $v \in V$. An alternative approach, which makes the connection with shift dynamics more expicity, is to associated a unitary matrix – a quantum evolution operator – to each vertex. Usually, this construction allows backtracking but, in recent work, Harrison, Smilansky and Winn have constructed and explored quantum evolution operators which do not permit backtracking [14]. Good references for quantum graphs are [12], [17] and [19]. We mention that our assumption that the edge lengths are rationally independent is a standard one in this setting.

The key interest in the theory of quantum graphs is to understand the statistics of the eigenvalues $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{R}^+$ of Δ. A particular problem is to understand the distribution of differences $\{\lambda_n - \lambda_m\}_{n,m=0}^{\infty}$ and, via a trace formula, this is related to the differences $\{l(\gamma) - l(\gamma')\}$ for pairs of closed cycles in G [3], [4], [5], [12], [28]. Indeed, a important role is played by summations over pairs of closed cycles (γ, γ') with $l(\gamma) = l(\gamma')$. We therefore expect that our results may be useful in this theory.

2. Subshifts of Finite Type

In order to use ideas from ergodic theory to study (non-backtracking) paths in the graph G, we introduce a dynamical systems called a (one-sided) shift of finite type. Define a matrix A, indexed by $E^o \times E^o$ by

$$A(e,e') = \begin{cases} 1 & \text{if } \mathfrak{t}(e) = \mathfrak{o}(e') \text{ and } e' \neq \bar{e} \\ 0 & \text{otherwise} \end{cases}$$

and a space

$$\Sigma_A^+ = \{(e_n)_{n=0}^\infty \in (E^o)^{\mathbb{Z}^+} \, : \, \mathfrak{t}(e_n) = \mathfrak{o}(e_{n+1}), \ e_{n+1} \neq \bar{e}_n \ \forall n \in \mathbb{Z}^+\}.$$

The subshift of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$ defined by $\sigma((e_n)_{n=0}^\infty) = (e_{n+1})_{n=0}^\infty$. We define a metric on Σ_A^+ by

$$d((e_n)_{n=0}^\infty, (e_n')_{n=0}^\infty) = \sum_{n=0}^\infty \frac{1-\delta_{e_n,e_n'}}{2^n},$$

where $\delta_{i,j}$ is the Kronecker symbol: this makes Σ_A^+ into a compact space and σ into a continuous map.

Then there is a natural one-to-one correspondence between $\gamma = \langle e_1, \ldots, e_n \rangle \in \mathcal{C}$ and periodic orbits $x, \sigma x, \ldots, \sigma^{n-1}x, \sigma^n x = x$, with $|\gamma| = n$, where x is the periodic sequence $(e_1, \ldots, e_n, e_1, \ldots, e_n, \ldots)$. We write $\mathrm{Fix}_n(\sigma)$ for the number of periodic points $\sigma^n x = x$.

The fact that G is connected and Assumption I imply that A is irreducible (i.e. for each $e, e' \in E^o$ there exists $n(e,e') \geq 1$ such that $A^{n(e,e')}(e,e') = 1$) but Σ_A^+ does not consist of a single periodic orbit. Let $d(\sigma) = \text{g.c.d}\{n \geq 1 \, : \, A^n(e,e) > 0\}$, where $e \in E^o$; then $d(\sigma)$ is independent of the choice of e. In view of the correspondence between closed geodesics and periodic orbits, $d(\sigma) = d$, where $d \in \{1,2\}$ is the period of G defined by (1.1). If $d = 1$ then A is aperiodic (i.e. there exists $n \geq 1$ such that $A^n(e,e') > 0$ for all $e, e' \in E^o$) or, equivalently, in terms of the subshift of finite type, $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is topologically mixing (i.e. there exists $n_0 \geq 1$ such that, for any two non-empty open sets $U, V \subset \Sigma_i^+$, $\sigma^{-m}(U) \cap V \neq \varnothing$ for all $m \geq n$). In particular, if G is not bipartite then A is aperiodic. If $d = 2$ then the standard decomposition theorem for irreducible non-negative matrices then say that E^o may be written as a disjoint union $E^o = E_1^o \cup E_2^o$ such that if $e \in E_i^o$ and $A(e,e') = 1$ then $e' \in E_{i+1}^o$ (mod 2) and A^2 restricted to each $E_i^o \times E_i^o$ is aperiodic. Equivalently, in terms of the subshift of finite type, Σ_A^+ may be written as a disjoint union $\Sigma_A^+ = \Sigma_1^+ \cup \Sigma_2^+$ such that $\sigma : \Sigma_i^+ \to \Sigma_{i+1}^+$ (mod 2) and $\sigma^2 : \Sigma_i^+ \to \Sigma_i^+$ is topologically mixing.

If $d = 2$ then, when studying periodic orbits, we may as well consider σ^2. This is because periodic orbits for σ with period $2n$ correspond exactly to periodic orbits for σ^2 with period n. (At the level of periodic points rather than orbits, the latter undercounts by a factor of 2.) Thus we may suppose that A is aperiodic.

If A is aperiodic then it has a positive simple eigenvalue λ which is strictly maximal in modulus (i.e. every other eigenvalue has modulus strictly less that λ). Since $\#\mathrm{Fix}_n(\sigma) = \mathrm{Trace}(A^n)$, it is clear that λ agrees with the growth rate in(0.1). Furthermore, the topological entropy $h(\sigma)$ of σ is equal to $\log \lambda$.

Let $\mathcal{M}_\sigma$ denote the space of all σ-invariant Borel probability measures on Σ_A^+. For $\mu \in \mathcal{M}_\sigma$, write $h(\mu)$ for the measure theoretic entropy of μ. There is a unique measure $\mu_0 \in \mathcal{M}_\sigma$, called the measure of maximal entropy, for which

$$h(\mu_0) = \sup_{\mu \in \mathcal{M}_\sigma} h(\mu)$$

and this value coincides with the topological entropy $h(\sigma)$. For a continuous function $f : \Sigma_A^+ \to \mathbb{R}$, we define the pressure $P(f)$ of f by the formula

$$P(f) = \sup_{\mu \in \mathcal{M}_\sigma} \left(h(\mu) + \int f \, d\mu \right).$$

The supremum is uniquely attained whenever f is Hölder continuous (and the corresponding measure is called the equilibrium state of f). In particular, this holds for the functions we consider, which depend only on the first co-ordinate (i.e. $f((e_n)_{n=0}^\infty = f(e_0))$.

We say that two functions $f, g : \Sigma_A^+ \to \mathbb{R}$ are (continuously) cohomologous if there is a continuous function $u : \Sigma_A^+ \to \mathbb{R}$ such that $f = g + u \circ \sigma - u$. The cohomology class of a Hölder continuous function is determined by its values around periodic orbits. More precisely, writing $f^n = f + f \circ \sigma + \cdots + f \circ \sigma^{n-1}$, two Hölder continuous functions $f, g : \Sigma_A^+ \to \mathbb{R}$ are cohomologous if and only if $f^n(x) = g^n(x)$ whenever $x \in \mathrm{Fix}_n(\sigma)$.

We now return to the specific problem of studying $\mathfrak{l}(\gamma)$, for $\gamma \in \mathcal{C}$, in terms of $\sigma : \Sigma_A^+ \to \Sigma_A^+$. Define a function $R : \Sigma_A^+ \to \mathbb{Z}^E$ by

$$R((e_n)_{n=0}^\infty) = (\delta_{e,e_0})_{e \in E}.$$

If $x, \sigma x, \ldots, \sigma^{n-1} x$ corresponds to a closed geodesic γ then

$$R^n(x) := R(x) + R(\sigma x) + \cdots + R(\sigma^{n-1} x) = \mathfrak{l}(\gamma).$$

(If $d = 2$ then we consider the function $R + R \circ \sigma$ summed over periodic orbits of the map $\sigma^2 : \Sigma_1^+ \to \Sigma_1^+$.) Write $\Gamma_R \subset \mathbb{Z}^E$ for the additive group generated by the periodic orbit sums

$$\{R^n(x) : x \in \mathrm{Fix}_n(\sigma), \ n \in \mathbb{N}\}.$$

Clearly, $\Gamma_R = \Gamma$, where Γ is the group defined in section 1. Thus, Γ_R has rank $|E|$ in $\mathbb{Z}^E$. Furthermore, by adding a coboundary, we may suppose that R is valued in Γ. Let $\Delta_R \subset \Gamma_R$ denote the subgroup

$$\Delta_R = \{R^n(x) - R^n(y) : x, y \in \mathrm{Fix}_n(\sigma), \ n \in \mathbb{N}\}.$$

Remark. The group Γ_R was introduced in [22] and Δ_R earlier in [18], originally for products of Markov measures. The standard result that Δ_R is a group follows from the aperiodicity of A. One only needs to check that Δ_R is closed under addition. Suppose that $x, y \in \mathrm{Fix}_n(\sigma)$, $x', y' \in \mathrm{Fix}_m(\sigma)$. Fix an $e_0 \in E^o$ and modify the periodic sequences making up x, x', y and y' by adding strings of some fixed length N so that they each start and end with e_0. Concatenate the modifications of x, x' and of y, y' to obtain new periodic points $x'', y'' \in \mathrm{Fix}_{n+m+4N}(\sigma)$. It is easy to see that $(R^n(x) - R^n(y)) + (R^m(x') - R^m(y')) = R^{n+m+4N}(x'') - R^{n+m+4N}(y'') \in \Delta_R$.

The following was proved in [20], [22] for real valued functions. The proof works in arbitrary dimensions and this was carried out explicitly in [25, Lemma 3.2].

Lemma 2.1. Γ_R / Δ_R *is a cyclic group.*

3. The Product Shift

In order to study *pairs* of closed geodesics, we now introduce a product dynamical system $\widetilde{\sigma} : \widetilde{\Sigma}_A^+ \to \widetilde{\Sigma}_A^+$, where $\widetilde{\Sigma}_A^+ = \Sigma_A^+ \times \Sigma_A^+$ and $\widetilde{\sigma}(x,y) = (\sigma x, \sigma y)$. (This approach was used to study pairs of closed geodesics on surfaces in [24].) Clearly, $\widetilde{\sigma} : \widetilde{\Sigma}_A^+ \to \widetilde{\Sigma}_A^+$ is the subshift of finite type, with symbols $E^o \times E^o$, defined by the matrix

$$\widetilde{A}((e,e'),(f,f')) = A(e,e')A(f,f').$$

Furthermore, the topological entropy of $\widetilde{\sigma}$ is given by $h(\widetilde{\sigma}) = 2h(\sigma)$.

Define a function $F : \widetilde{\Sigma}_A^+ \to \mathbb{Z}^E$ by

$$F(x,y) = R(x) - R(y).$$

If $x, \sigma x, \dots, \sigma^{n-1}x$ and $y, \sigma y, \dots, \sigma^{n-1} y$ correspond to closed geodesics γ and γ', respectively, then

$$F^n(x,y) = \mathfrak{l}(\gamma) - \mathfrak{l}(\gamma').$$

Lemma 3.1. *If periodic orbits $x, \sigma x, \dots, \sigma^{n-1}x$ and $y, \sigma y, \dots, \sigma^{n-1}y$ correspond to closed geodesics γ and γ', respectively, then $l(\gamma) = l(\gamma')$ if and only if $F^n(x,y) = 0$. In particular,*

$$\#\{(\gamma,\gamma') \,:\, |\gamma| = |\gamma'| = dn,\ l(\gamma) = l(\gamma')\} = \frac{\#\{(x,y) \in \mathrm{Fix}_{dn}(\widetilde{\sigma}) \,:\, F^{dn}(x,y) = 0\}}{(dn)^2} + O(e^{3h(\sigma)dn/2}).$$

Proof. We have that $F^n(x,y) = 0$ if and only if $\mathfrak{l}(\gamma) = \mathfrak{l}(\gamma')$. Thus the first part of the result follows by Lemma 1.2. For the second part, note that

$$\#\{\gamma : |\gamma| = dn\} = \sum_{m|dn} \frac{\#\mathrm{Fix}_m^*(\sigma)}{m} = \frac{\#\mathrm{Fix}_{dn}^*(\sigma)}{dn} + O(e^{h(\sigma)dn/2})$$
$$= \frac{\#\mathrm{Fix}_{dn}(\sigma)}{dn} + O(e^{h(\sigma)dn/2}),$$

where $\mathrm{Fix}_m^*(\sigma)$ is the set of periodic points for σ with *least* period m. The estimate on pairs then follows. □

Let $\widetilde{\mu}_0$ denote the measure of maximal entropy for $\widetilde{\sigma}$. Since $h(\widetilde{\sigma}) = 2h(\sigma)$ and $h_{\widetilde{\sigma}}(\mu_0 \times \mu_0) = 2h_\sigma(\mu_0) = 2h(\sigma)$, it is clear that $\widetilde{\mu}_0 = \mu_0 \times \mu_0$. The next result is that F averages to zero with respect to $\widetilde{\mu}_0$; this will be important when we later apply results from [23].

Lemma 3.2.

$$\int F(x,y)\ d\widetilde{\mu}_0(x,y) = 0.$$

Proof. The lemma follows from an easy calculation. We have

$$\int F(x,y) d\widetilde{\mu}_0(x,y) = \int R(x) - R(y) d\widetilde{\mu}_0(x,y)$$
$$= \int R(x) d\mu_0(x) - \int R(y) d\mu_0(y) = 0,$$

as required. □

In order to apply [23], we need to understand the analogues of the groups Γ_R and Δ_R when R is replaced by F. Notice that

$$\{F^n(x,y) : (x,y) \in \mathrm{Fix}_n(\widetilde{\sigma}),\ n \in \mathbb{N}\} = \{R^n(x) - R^n(y) : x,y \in \mathrm{Fix}_n(\sigma),\ n \in \mathbb{N}\}.$$

Thus, if Γ_F denotes the additive group generated by

$$\{F^n(x,y) : (x,y) \in \mathrm{Fix}_n(\widetilde{\sigma}),\ n \in \mathbb{N}\}$$

then we have $\Gamma_F = \Delta_R$.

Lemma 3.3. *Γ_F has rank $|E|-1$ and*

$$\Gamma_F \otimes \mathbb{R} = \left\{ t = (t_e)_{e\in E} \in \mathbb{R}^E : \sum_{e\in E} t_e = 0 \right\}.$$

Proof. Recall that Γ_R has rank $|E|$. Now, by Lemma 2.1, $\Gamma_R/\Gamma_F = \Gamma_R/\Delta_R$ is either a finite cyclic group or isomorphic to $\mathbb{Z}$. Thus Γ_F has rank at least $|E|-1$.

On the other hand, for any $F^n(x,y) = \mathfrak{l}(\gamma) - \mathfrak{l}(\gamma') \in \Gamma_F$, we have $F^n(x,y) = (\nu_e(\gamma) - \nu_e(\gamma'))_{e\in E}$, so

$$\sum_{e\in E} (\nu_e(\gamma) - \nu_e(\gamma')) = \sum_{e\in E} \nu_e(\gamma) - \sum_{e\in E} \nu_e(\gamma') = n - n = 0.$$

Thus $\Gamma_F \subset \{t = (t_e)_{e\in E} \in \mathbb{R}^E : \sum_{e\in E} t_e = 0\}$ and, in particular, the rank of Γ_F is at most $|E|-1$. Combining these facts gives the required expression for the tensor product. □

Let

$$\Delta_F = \{F^n(x,y) - F^n(x',y') : (x,y),(x',y') \in \mathrm{Fix}_n(\widetilde{\sigma}),\ n \in \mathbb{N}\}.$$

Lemma 3.4. *$\Delta_F = \Gamma_F$.*

Proof. An arbitrary element of Γ_F is of the the form $F^n(x,y)$, where $(x,y) \in \mathrm{Fix}_n(\widetilde{\sigma})$. Choose $x' \in \mathrm{Fix}_n(\sigma)$; then $(x',x') \in \mathrm{Fix}_n(\widetilde{\sigma})$ and $F^n(x',x') = 0$. Thus

$$F^n(x,y) = F^n(x,y) - F^n(x',x') \in \Delta_F,$$

as required. □

The constant C in Theorem 1 is determined by a particular pressure function $\mathfrak{p} : \Gamma_F \otimes \mathbb{R} \to \mathbb{R}$. This is defined by $\mathfrak{p}(t) = P(\langle t, F\rangle)$, where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^E$. Define $\sigma(G) \geq 0$ by

$$\sigma(G)^{2(|E|-1)} = \det \nabla^2 \mathfrak{p}(0).$$

Lemma 3.5. $\sigma(G) > 0$

Proof. It is a standard result that $\det \nabla^2 \mathfrak{p}(0) > 0$ unless there exists $t \in V\backslash\{0\}$ such that $\langle t, F\rangle$ is cohomologous to a constant $c \in \mathbb{R}$. In particular, $\langle t, \mathfrak{l}(\gamma) - \mathfrak{l}(\gamma')\rangle = c|\gamma|$, for every $\gamma, \gamma' \in \mathcal{CG}$ with $|\gamma| = |\gamma'|$. Switching the order of γ and γ' gives $c = 0$, so that $\Gamma_F \subset \{m \in \mathbb{R}^E : \langle t, m\rangle = 0\}$. This cannot hold for $t \neq 0$, since it would force Γ_F to have rank at most $|E| - 2$, contradicting Lemma 3.3. □

By adding a coboundary if necessary, we may assume that F is valued in Γ_F. The next result then follows by applying the analysis of [23] to the function $F : \widetilde{\Sigma}_A^+ \to \Gamma_F \cong \mathbb{Z}^{|E|-1}$. Then the obvious modification of condition (A1) of that paper is automatically satisfied and condition (A2) follows directly from Lemma 3.2.

Proposition 3.1. *We have that*

$$\#\{(x,y) \in \mathrm{Fix}_{dn}(\widetilde{\sigma}) \,:\, F^{dn}(x,y) = 0\} \sim \frac{1}{(2\pi)^{(|E|-1)/2}\sigma(G)^{|E|-1}} \frac{e^{dnh(\widetilde{\sigma})}}{(dn)^{(|E|-1)/2}},$$

as $n \to +\infty$.

Theorem 1 follows from Lemma 3.1 and Proposition 3.1, with the value of C given by

$$C = \frac{1}{(2\pi)^{(|E|-1)/2}\sigma(G)^{|E|-1}}.$$

It is clear from its definition that C depends only on G and not on the edge lengths $l(e)$.

Remarks.

(i) In [26], Pollicott and Weiss considered multiplicities for locally constant functions $f : \Sigma_A^+ \to \mathbb{R}$ on an arbitrary mixing subshift of finite type. They showed that one may always find periodic points $\sigma^n x = x$, $\sigma^n y = y$ such that $f^n(x) = f^n(y)$ and that the number of such periodic points is unbounded [26, Proposition 4.2]. Using the methods described above, one may show that if the values of f are rationally independent then

$$\#\{x, y \in \mathrm{Fix}_n(\sigma) \,:\, f^n(x) = f^n(y)\} \sim C(f)\frac{e^{2h(\sigma)}}{n^{p/2}},$$

as $n \to +\infty$, where $C(f) > 0$ and $p \geq 1$ is the rank of the group

$$\{f^n(x) - f^n(y) \,:\, x, y \in \mathrm{Fix}_n(\sigma),\ n \in \mathbb{N}\}.$$

(ii) One may deduce from standard results on periodic obits for subshifts of finite type that $l(\gamma)$ is typically close to $|\gamma| \left(\int r d\mu_0\right)$, where $r : \Sigma_A^+ \to \mathbb{R}$ is the function defined by $r((e_n)_{n=0}^\infty) = l(e_0)$. More precisely, for $\epsilon > 0$, one has the large deviation result

$$\frac{\#\{\gamma \in \mathcal{C} \,:\, |\gamma| = n,\ \left|\frac{l(\gamma)}{n} - \int r d\mu_0\right| \geq \epsilon\}}{\#\{\gamma \in \mathcal{C} \,:\, |\gamma| = n\}} \to 0$$

exponentially fast, as $n \to +\infty$. More refined local limit type results also hold. Note that, in general, $\int r d\mu_0$, is *not* equal to $\sum_{e \in E} l(e)/|E|$.

4. Non-Geodesic Cycles

Our analysis applies, essentially unchanged, to the closed cycles $\mathcal{C}^*$ in G. In this case we use the shift of finite type defined by the matrix B, indexed by $E^o \times E^o$, given by

$$B(e,e') = \begin{cases} 1 & \text{if } \mathfrak{t}(e) = \mathfrak{o}(e') \\ 0 & \text{otherwise.} \end{cases}$$

(In contrast to A, the matrix B allows backtracking.) We now use the shift of finite type $\sigma : \Sigma_B^+ \to \Sigma_B^+$

It is clear that if $\gamma \in \mathcal{C}^*$ then, by successively removing each backtracking step, $|\gamma| = |\gamma_0| + 2m$, where $\gamma_0 \in \mathcal{C}$ and $m \in \mathbb{Z}^+$. In particular,

$$d = \text{g.c.d.}\{|\gamma| : \gamma \in \mathcal{C}\} = \text{g.c.d.}\{|\gamma| : \gamma \in \mathcal{C}^*\}.$$

Let $\Gamma_* \subset \mathbb{Z}^E$ denote the abelian group generated by the set $\{\mathfrak{l}(\gamma) : \gamma \in \mathcal{C}\}$. Since $\Gamma \subset \Gamma_*$, it is clear the Γ_* has rank $|E|$.

In we write λ_* for the largest positive eigenvalue of B then

$$\{\gamma \in \mathcal{C} : |\gamma| = dn\} \sim \frac{\lambda_*^d}{\lambda_*^d - 1} \frac{\lambda_*^{dn}}{dn},$$

as $n \to +\infty$, and we have the following.

Theorem 3. *Under Assumptions I and II,*

$$\#\{(\gamma,\gamma') \in \mathcal{C}^* \times \mathcal{C}^* : |\gamma| = |\gamma'| = dn,\ l(\gamma) = l(\gamma')\} \sim C_* \frac{\lambda_*^{2dn}}{n^{2+(|E|-1)/2}},$$

as $n \to +\infty$, where $C_* > 0$ only depends on G (and not on l).

Theorem 3 also applies to a class of graphs popular in quantum chaos but which fail to satisfy Assumption I, namely *star graphs* [3], [15]. These consist of $k+1$ vertices ($k \geq 3$ and k edges arranged so that there is a central vertex of degree k with the other vertices each having degree one. Clearly such a graph does not have closed geodesics but if backtracking is allowed then one may study the closed cycles. The corresponding matrix B is irreducible with period 2 and $\lambda_* = \sqrt{k}$. If the k edge lengths are rational independent, one obtains

$$\#\{(\gamma,\gamma') \in \mathcal{C}^* \times \mathcal{C}^* : |\gamma| = |\gamma'| = 2n,\ l(\gamma) = l(\gamma')\} \sim C_* \frac{k^{2n}}{n^{2+(k-1)/2}},$$

as $n \to +\infty$.

References

1. G. Berkolaiko, *Quantum star graphs and related systems*, PhD Thesis, University of Bristol (2000).
2. G. Berkolaiko, *Form factor expansions for large graphs: a diagrammatic approach*, Quantum Graphs and Their Applications (G. Berkolaiko, R. Carlson, S. Fulling and P. Kuchment, eds.), Contemp. Math. 415, AMS, Providence, RI, 2006.
3. G. Berkolaiko and J. Keating, *Two-point spectral correlations for star graphs*, J. Phys. A **32** (1999), 7827-7841.

4. G. Berkolaiko, H. Schanz and R. Whitney, *Leading off-diagonal correction to the form factor of large graphs*, Phys. Rev. Lett. **82** (2002), 104101.
5. G. Berkolaiko, H. Schanz and R. Whitney, *Form factor for a family of quantum graphs: an expansion to third order*, J. Phys. A: Math. Gen. **36** (2003), 8373-8392.
6. E. Bogomolny, *Quantum and arithmetical chaos*, Frontier in Number Theory, Physics and Geometry I, Les Houches 2003, Springer-Verlag, Berlin, 2006, pp. 3-106.
7. E. Bogomolny, B. Georgeot, M.-J. Giannoni and C. Schmit, *Quantum chaos on constant negative curvature surfaces*, Chaos Solitons Fractals **5** (1995), 1311-1323.
8. E. Bogomolny, B. Georgeot, M.-J. Giannoni and C. Schmit, *Arithmetical chaos*, Phys. Rep. **291** (1997), 219-324.
9. E. Bogomolny and C. Schmit, *Multiplicities of periodic orbit length for non-arithmetic models*, J. Phys. A: Math. Gen. **37** (2004), 4501-4536.
10. P. Buser, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics, 106, Birkhauser, Boston, 1992.
11. U. Gavish and U. Smilansky, *Degeneracies in the length spectra of metric graphs*, J. Phys. A (2007), 10009-10020.
12. S. Gnutzmann and U. Smilansky, *Quantum Graphs: Applications to Quantum Chaos and Universal Spectral Statistics*, Advances in Physics **55** (2006), 527-625.
13. L. Guillopé, *Entropies et spectres*, Osaka J. Math. **31** (1994), 247-289.
14. J. Harrison, U. Smilansky and B. Winn, *Quantum graphs where back-scattering is prohibited*, J. Phys. A **40** (2007), 14181-14193.
15. J. Keating, *Fluctuation statistics for quantum star graphs*, Quantum Graphs and Their Applications (G. Berkolaiko, R. Carlson, S. Fulling and P. Kuchment, eds.), Contemp. Math. 415, AMS, Providence, RI, 2006.
16. M. Kotani and T. Sunada, *Zeta functions of finite graphs*, J. Math. Sci. Univ. Tokyo **7** (2000), 7-25.
17. T. Kottos and U. Smilansky, *Periodic orbit theory and spectral statistics for quantum graphs*, Ann. Phys. **274** (1999), 76-124.
18. W. Krieger, *On the finitary isomorphisms of Markov shifts that have finite expected coding time*, Z. Wahrsch. Verw. Gebiete **65** (1983), 323-328.
19. P. Kuchment, *Quantum graphs I. Some basic structures*, Waves in Random Media **14** (2004), S107.
20. B. Marcus and S. Tuncel, *The weight-per-symbol polytope and scaffolds of invariants associated with Markov chains*, Ergodic Theory Dynam. Syst. **11** (1991), 129-180.
21. W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque **187-188** (1990), 1-268.
22. W. Parry and K. Schmidt, *Natural coefficients and invariants for Markov-shifts*, Invent. Math. **76** (1984), 15-32.
23. M. Pollicott and R. Sharp, *Rates of recurrence for $\mathbb{Z}^q$ and $\mathbb{R}^q$ extensions of subshifts of finite type*, J. London Math. Soc. **49** (1994), 401-416.
24. M. Pollicott and R. Sharp, *Correlations for pairs of closed geodesics*, Invent. Math. **163** (2006), 1-24.
25. M. Pollicott and R. Sharp, *Pseudo-Anosov foliations on periodic surfaces*, Topology Appl. **154** (2007), 2365-2375,.
26. M. Pollicott and H. Weiss, *Free energy as a dynamical invariant (or Can you hear the shape of a potential?)*, Comm. Math. Phys. **240** (2003), 457-482.
27. B. Randol, *The length spectrum of a surface is always of unbounded multiplicity*, Proc. Amer. Math. Soc. **78** (1980), 455-456.
28. G. Tanner, *Spectral statistics for unitary transfer matrices of binary graphs*, J. Phys. A **33** (2000), 3567-3586.

Richard Sharp, School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL