Contractions in the 2-Wasserstein length space and thermalization of granular media

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March 8, 2004

Abstract

An algebraic decay rate is derived which bounds the time required for velocities to equilibrate in a spatially homogeneous flow-through model representing the continuum limit of a gas of particles interacting through slightly inelastic collisions. This rate is obtained by reformulating the dynamical problem as the gradient flow of a convex energy on an infinite-dimensional manifold. An abstract theory is developed for gradient flows in length spaces, which shows how degenerate convexity (or even non-convexity) — if uniformly controlled — will quantify contractivity (limit expansivity) of the flow.

1 Introduction

It has been known since the work of Otto [38] that various familiar diffusion equations can be considered, at least heuristically, to be gradient flows on the space of probability measures, endowed with a manifold structure and local metric whose arc length distance coincides with the quadratic Wasserstein distance,

$$\mathrm{dist}_2(\rho_0,\rho_1) = \inf\left\{ \int |v-w|^2\, d\gamma(v,w); \qquad \gamma \in \Gamma(\rho_0,\rho_1)\right\}^{1/2}; \tag{1.1}$$

here $\Gamma(\rho_0,\rho_1)$ is the set of probability measures on $\mathbf{R^d}\times\mathbf{R^d}$ having marginals ρ_0 and ρ_1. Otto showed how to use these heuristics to study the long-time behavior of nonlinear porous-medium type equations. His work has inspired numerous developments, many of which are reviewed in [43].

The present paper deals with applications of this point of view to diffusion equations whose nonlinearities may also present a nonlocal structure, as found in the kinetic models

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of Benedetto et al for equilibration of velocities in granular media [8, 9, 10]. It is the sequel to our previous work [19], in which we studied these equations by means of entropy methods, or more precisely, convergence of the energy functional towards its infimum as time becomes large. In the present paper we shall pursue two goals. The first of these is to complement our previous study by estimating rates of convergence in quadratic Wasserstein distance. Although convergence in Wasserstein distance may be weaker than convergence in entropy sense, as explained and illustrated in [19], this approach offers several advantages: 1) Wasserstein distance is the natural distance associated with the gradient flow structure under examination; 2) the assumption of finite Wasserstein distance is much more general than the assumption of finite entropy; 3) most importantly, this approach enables one to directly compare two different solutions, instead of just comparing each solution to the stationary one. Thus much information is gained about the short-time behaviour of the flow as well as its long-time asymptotics. For instance, when we can show that the distance between any two solutions does not grow too quickly as a function of time, uniqueness of solutions and extension of the flow to singular initial data follow immediately. If these distances actually decrease — which is often the case — then existence and uniqueness of a fixed point may also be inferred from contractivity.

This investigation will lead us to examine in fine detail the structure of the space of probability measures equipped with the Wasserstein distance. Thus the second goal of our paper is to develop a formal mathematical framework for Otto's ideas. To do this, we shall study of the space of probability measures $\mathcal{P}_2(\mathbf{R}^d)$ with finite second order moments, viewed as a *length space*. This provides a conceptual setting in which many known results, and some new ones, fall into place naturally. We introduce an additional structure, which we call a *Riemannian* length space, to serve as a framework for converting the heuristical arguments of Otto directly into rigorous theorems — at least on bounded domains $\Omega \subset\subset \mathbf{R}^d$ or in smooth situations. Until now, they have typically functioned as a collection of powerful motivational tools for guessing inequalities that require tedious verification a posteriori.

We mention that a related study of this length space structure was recently performed by Ambrosio, Gigli and Savarè [4, 5], as we learned when much of this work had been completed. Their construction has a lot to do with ours, even if the goal is quite different: the authors in [5] do not wish to study rates of convergence, but rather to establish general results of existence of gradient flows, for convex energy functionals on this length space. For this reason, the authors develop a somewhat more general theory, since it is important for them to be able to handle singular measures, while our analysis is largely restricted to probability measures which are absolutely continuous with respect to Lebesgue measure. (With additional work, our Riemannian length space structure can be adapted to singular measures also, though various geometrical pathologies seem to arise). We refer to [5] for more explanations. Our main theorem gives an explicit estimate on the growth or decay of the distance between any two solutions of a subgradient flow in the Riemannian length setting we have introduced. To apply it in the particular case of the 2-Wasserstein for probability measures and to the PDE models we are dealing with, we need to perform a series of approximations to the equations which does not close all the possible cases (see last section for precise open problems). A possibility to explore would be to use the general existence result of gradient flows in [5] as an alternative tool for proving this kind of contractivity estimates for the distance.

Other authors exploring similar themes include Carlen and Gangbo, who in their investigation of the kinetic Fokker-Planck equation show the length space $(\mathcal{P}_2(\mathbf{R}^d), \mathrm{dist}_2)$ posseses a conical structure [15] analogous to a warped product in Riemannian geometry, and Cordero-Erausquin, Gangbo, and Houdre [23], who establish various expressions of *uniform* displacement convexity for entropies $E : \mathcal{P}_2(\mathbf{R}^d) \longrightarrow \mathbf{R}$ with respect to more general costs on $\mathcal{P}_2(\mathbf{R}^d)$. When the cost is given by dist_2, these relate to our rate of convergence results. The *displacement convexity* of such entropies — which amounts to convexity along geodesics in $(\mathcal{P}_2(\mathbf{R}^d), \mathrm{dist}_2)$ — originated in work of McCann, where it was established using a particular geodesic structure without reference to an underlying metric [35]. The application of displacement convexity to rates of convergence in nonlinear evolution equations begun by Otto [38], was recently explored for more general costs associated with different nonlinearities by Agueh [1, 3] and Agueh, Ghoussoub, Kang [2].

Typical equations to which the present considerations apply take the form

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\rho \nabla \left(A'\left(\rho\right) + B + C * \rho\right)\right], \tag{1.2}$$

where $\rho : [0,T] \times \Omega \longrightarrow [0, +\infty]$ is an integrable density, $\Omega \subset \mathbf{R}^d$, and $A : \mathbf{R} \longrightarrow \mathbf{R}$ and $B, C : \mathbf{R}^d \longrightarrow \mathbf{R}$ are convex potentials. In the one-dimensional models for granular media which motivated our original interest, $\rho(t, v)$ represents a distribution of velocities $v \in \Omega$ at each time, and the three potentials model the effects of: A) random interactions of the granules with their environment (a fluid or heat bath), B) friction, and C) inelastic collisions between granules with different velocities — the nonlocal source of nonlinearity. Notice equation (1.2) is appropriate to spatially homogeneous initial conditions — meaning $\rho(t, x, v) = \rho(t, v)$ depends only on the velocity coordinate v in phase space and not the position x — so it would be natural to study the evolution on the entire space tangent space of velocities $v \in \Omega = \mathbf{R}^d$. However, for technical reasons, as in [38, 18], it is more convenient to begin by formulating the problem on a bounded convex domain of velocities $\Omega \subset\subset \mathbf{R}^d$ with no-flux boundary conditions,

$$\rho \nu_\Omega \cdot \nabla \left(A'\left(\rho\right) + B + C * \rho\right) = 0, \qquad \text{on } (t, v) \in [0, T] \times \partial\Omega, \tag{1.3}$$

and study the large domain limit $\Omega \longrightarrow \mathbf{R}^d$ subsequently. Here ν_Ω denotes the outer unit normal at v to Ω. Later on in the text, we may use the variables x or y in place of v; regardless of its name, our independent variable always represents a velocity in kinetic models.

The spirit of our results is captured by the following examples. We assume power law potentials here for simplicity; more general potentials are addressed in later sections.

Example 1.1. *Take* $B(v) = \beta|v|^{b+2}/(b+2)$ *and* $C(v) = \gamma|v|^{c+2}/(c+2)$*, and*

$$A(\varrho) = \begin{cases} \alpha\varrho^a/(a-1) & 1 \neq a \geq \max\{\frac{d-1}{d}, \frac{d}{d+2}\} \\ \alpha\varrho\log\varrho & a = 1 \end{cases} \tag{1.4}$$

with $\alpha, \beta, \gamma \geq 0$ *and* $b, c \geq -1$*. The Wasserstein* L^2 *distance* $d_t := \mathrm{dist}_2(\rho_1(t), \rho_2(t)$ *between any two solutions of (1.2–1.3) on* $\Omega \subset\subset \mathbf{R}^d$ *decays like*

$$d_t \leq \begin{cases} e^{-\beta t} d_0 & b = 0 \\ d_0(1 + \beta t b (d_0/2)^b)^{-1/b} \quad \sim 2(\beta t b)^{-1/b} & b > 0 \end{cases} \tag{1.5}$$

in the presence of friction $\beta > 0$*. In the absence of friction* $\beta = 0$*, the inelastic collisions* $\gamma > 0$ *alone yield a decay rate*

$$d_t \leq \begin{cases} e^{-\gamma t} d_0 & c = 0 \\ d_0(1 + \gamma t c (d_0/\sqrt{2})^c)^{-1/c} \quad \sim \sqrt{2}(\gamma t c)^{-1/c} & c > 0, \end{cases} \tag{1.6}$$

provided the center of masses of the two solutions coincide at each point in time; this will be true if, for example, we assume reflection symmetry $\Omega = -\Omega$ *and* $\rho(0, v) = \rho(0, -v)$ *initially (and hence for all time).*

In the most relevant case, the diffusive term is linear ($a = 1$), and the conclusions apply equally well to the entire space $\Omega = \mathbf{R}^d$. In the most interesting cases of interaction potentials $C(v) = \gamma|v|^{c+2}/(c+2)$, we are able to overcome the restriction of reflection symmetry by approximating the solution using very smooth fixed center of mass solutions of the same equation which decay quickly at infinity on all of $\mathbf{R}^d$ [19]. Precise statements are given in the last section.

Choosing $d = 1$, $a = 1$, $b = 0$, $c = 1$ produces the one-dimensional granular models of Benedetto, Caglioti, Carrillo, and Pulvirenti [8, 9]. There the presence or absence of friction can mean the difference between exponentially fast and algebraically slow thermalization: indeed, Benedetto, Caglioti and Pulvirenti's original calculation shows that neither the constants nor the exponent of the algebraic bound (1.6) can be improved when $\alpha = \beta = 0$. In this special case, all velocities converge to a single equilibrium value, and the slow convergence results from the rate of collisions dwindling to zero along with the dissipated energy per collision. The mathematical reason for this algebraic rate is collapse of the relative sand grain velocities $v - \overline{v}$ onto the unique point where the second derivative of the collision potential $C(v - \overline{v}) = |v - \overline{v}|^3/3$ vanishes. We eventually showed in our companion paper how exceptional this example is: the algebraic bound (1.6) can be improved to an exponential bound provided $\alpha > 0$; the presence of a heat bath speeds up thermalization by ensuring that neither the rate of collisions nor the dissipated energy becomes too small. The resulting bound differs from (1.5) however, in that the exponential rate of contraction we derive in this case is not global, but depends on the initial entropy of $\rho_1(0)$ and $\rho_2(0)$ [19].

For $a \neq 1$, it may also be possible to extract the same results in the large domain limit $\Omega \to \mathbf{R}^d$, but a complete discussion of most general conditions which permit this would form the subject of separate treatise; the presence of friction $\beta > 0$ above is sufficient if $b \geq 0$. When $\beta = 0$ the center of mass condition is necessary for convergence in Wasserstein distance on $\Omega = \mathbf{R}^d$: translation invariance implies the average velocities $\langle v \rangle_{\rho_1(t)}$ and $\langle v \rangle_{\rho_2(t)}$ from (5.5) do not change; if they differ initially then $d_t \geq |\langle v \rangle_{\rho_1(0)} - \langle v \rangle_{\rho_2(0)}|$ cannot converge to zero. Compare how barycenter enters explicitly in the inequalities formulated by Agueh, Ghoussoub and Kang [2].

Let us also mention that Wasserstein contraction estimates have been obtained recently by H. Li and G. Toscani [33] for the family of one dimensional granular media models introduced in [42]. Their main idea is to use the particular explicit formula of the Wasserstein distance in one dimension. In fact, the optimal transport map in one dimension is always the same for all convex costs and is defined in terms of the inverse distribution functions of the measures involved. A short review of these ideas applied to one dimensional nonlinear diffusion-dominated equations can be found in [21]. Wasserstein contraction estimates play a role in controlling the expansion of the support of solutions for one dimensional nonlinear diffusions as recently point out in [17] for the porous medium equation and in [16] for diffusion-dominated equations.

Finally, let us also point out that a related equation that it is also included in this theory (at least formally) is the one dimensional nonlinear Fokker-Planck equation arising in free probability [11], also called the free Fokker-Planck equation. The linear diffusion term is replaced by the Hilbert transform in this equation. The free Fokker-Planck equation has also a formal gradient flow structure with respect to a logarithmic interaction energy functional. In one dimension, this energy happens to be displacement convex in the sense of [35], as observed and exploited by Blower in the context of random matrix theory [12].

2 A schematic cartoon of the rate arguments

Before attempting to construct an abstract argument in a context fraught with perils of nonsmoothness, infinite dimensions, and degenerate convexity, it is instructive to recall the ideas behind the convergence arguments in their simplest form. The setting will be so simple that not only are the results well-known, they could all be deduced by a good sophomore calculus student. Nevertheless, they serve to contrast the contraction strategy developed hereafter with the Bakry-Emery [7] type entropy production analysis employed in Otto [38] and in our previous work [19].

Fix $E \in C^2(\mathbf{R}^d)$ and consider solutions of the ordinary differential equation

$$\frac{dx_t}{dt} = -\nabla E(x_t) \tag{2.1}$$

corresponding to *steepest descent* or *gradient flow* on the energy (entropy) landscape determined by E. Here I will denote the $d \times d$ identity matrix.

Proposition 2.1 (Bounding contraction / expansion rates). *Fix $k \in \mathbf{R}$. If $E \in C^2(\mathbf{R}^d)$ satisfies $D^2E(x) \geq kI$ throughout $\mathbf{R}^d$, and the curves x_t and $t \in [0,\infty) \longrightarrow y_t \in \mathbf{R}^d$ both solve the differential equation* (2.1)*, then $|x_t - y_t| \leq e^{-kt}|x_0 - y_0|$.*

Proof. Set $f(t) = |x_t - y_t|^2/2$. Then

$$\begin{aligned} f'(t) &= -\langle x_t - y_t, \nabla E(x_t) - \nabla E(y_t)\rangle \\ &= -\langle x_t - y_t, \int_0^1 D^2E[(1-s)x_t + sy_t]\,(y_t - x_t)ds\rangle \\ &\leq -2kf(t)\int_0^1 ds. \end{aligned}$$

Gronwall's inequality (integration) implies the desired result: $f(t) \leq e^{-2kt}f(0)$. □

Corollary 2.2 (Contraction in a convex valley). *Taking $k = 0$ in the preceding proposition implies $|x_t - y_t|$ is monotone nonincreasing as a function of $t \in [0,\infty)$.*

Proof. Obviously $|x_t - y_t| \leq |x_0 - y_0|$. Since the equation is autonomous, time translation invariance implies $|x_{T+t} - y_{T+t}| \leq |x_T - y_T|$ for all $t, T \geq 0$. □

If $k > 0$, more can be achieved. The convexity of E is said to be 2-*uniform*, and we have shown that the solution map $x_0 \in \mathbf{R}^d \longrightarrow X_t(x_0) = x_t$ of the initial value problem (2.1) defines a uniform contraction on $\mathbf{R}^d$ for each $t > 0$. The C^2 smoothness of E ensures that the solution map is well-defined locally in space and time; the map is globally defined for all future times since x_t is constrained to lie in the level set $\{x \mid E(x) \leq E(x_0)\}$, whose compactness follows from the coercivity of $E(x) \geq E(x_0) + \langle\nabla E(x_0), x - x_0\rangle + k|x-x_0|^2/2$. Since $\mathbf{R}^d$ is complete, the contraction mapping principle dictates that this map has a unique fixed point $X_t(x_\infty) = x_\infty \in \mathbf{R}^d$, and each solution curve $x_t = X_t(x_0)$ must converge to x_∞ in the long time limit $t \to \infty$. If we are only interested in the rate of convergence to x_∞, an alternative to Proposition 2.1 can be based on the Bakry and Emery entropy production approach. We give that argument here for comparison's sake. The quantity estimated is the decay rate of the slope $|\nabla E(x_t)| \to 0$; by the analogy discussed at the end of this section, the square of this slope is called the *information*.

Proposition 2.3 (Entropy production and information decay rate). *Let $E \in C^2(\mathbf{R}^d)$ satisfy $D^2E(x) \geq kI > 0$ throughout $\mathbf{R}^d$. Then any solution $t \in [0,\infty) \longrightarrow x_t \in \mathbf{R}^d$ of* (2.1) *satisfies $|\nabla E(x_t)| \leq e^{-kt}|\nabla E(x_0)|$.*

Proof. Let $f(t) := |\nabla E(x_t)|^2/2$. Then

$$\begin{aligned} f'(t) &= \langle \nabla E(x_t), D^2E(x_t)\,\dot{x}_t\rangle \\ &= -\langle \nabla E(x_t), D^2E(x_t)\nabla E(x_t)\rangle \\ &\leq -2kf(t), \end{aligned}$$

and Gronwall's inequality proves the desired estimate: $f(t) \leq e^{-2kt}f(0)$. □

While the conclusions of these two propositions are not immediately comparable, the following consequence (2.2) of 2-uniform convexity relates them. It shows that information dominates the altitude or *relative entropy* $E(x) - E(x_\infty)$, which in turn dominates horizontal distance squared. Thus in its limited range of validity — $k > 0$ and $y_t := x_\infty$ — and apart from constants, Proposition 2.3 trumps Proposition 2.1. On the other hand, (2.3) also shows that if information remains bounded, then convergence in the weakest sense, namely of distance (unsquared), also implies convergence in the stronger sense of relative entropy.

Lemma 2.4 (Manifestations of 2-uniform convexity). *Let $0 \le f \in C^2(\mathbf{R})$ satisfy $f(0) = 0$ and $f''(s) > k > 0$ for all $s \in \mathbf{R}$. Then $ks^2 \le 2f(s) \le k^{-1}|f'(s)|^2$ and*

$$f(s) \le sf'(s) - ks^2/2.$$

Proof. Let $g(s) := f(s) - ks^2/2$. Taking two derivatives shows $g(s)$ is convex, so its critical point at the origin must be a minimum: $g(s) \ge g(0) = 0$. This proves the first inequality.

Since $f(s) \ge 0$ is strictly convex, its minimum $f(0) = 0$ is its only critical point. Defining $h(s) := |f'(s)|^2/2 - kf(s)$, we see $h'(s) = f'(s)(f''(s) - k)$ can vanish only where $f'(s)$ does — namely, at zero. Since $h''(0) = f''(0)(f''(0) - k) + 0f'''(0) > 0$, the unique critical point of $h(s)$ is a strict local minimum; it must be a global minimum since the absence of other critical points ensures that monotonicity of $h(s)$ changes only at zero. Thus $h(s) \ge h(0) = 0$, which establishes the second inequality.

Finally, let $e(s) = sf'(s) - ks^2/2 - f(s)$. Then $e'(s) = s(f''(s) - k)$ vanishes only when $s = 0$. A second derivative $e''(0) = f''(0) - k > 0$ shows this unique critical point of $e(s)$ to be a strict local minimum, hence a global minimum as above: $e(s) \ge e(0) = 0$ to complete the proof of the lemma. □

Corollary 2.5 (Log Sobolev, transportation, and HWI inequalities). *Suppose $E(x_\infty) \le E(x) \in C^2(\mathbf{R}^d)$ and $D^2E(x) \ge kI > 0$ for all $x \in \mathbf{R}^d$. Then*

$$\frac{k}{2}|x - x_\infty|^2 \;\le\; E(x) - E(x_\infty) \;\le\; \frac{1}{2k}|\nabla E(x)|^2 \tag{2.2}$$

$$\text{and} \qquad E(x) - E(x_\infty) \;\le\; |x - x_\infty||\nabla E(x)| - k|x - x_\infty|^2/2. \tag{2.3}$$

Proof. The conclusions of the lemma continue to hold under the relaxed hypothesis $f(s) \ge k$, as is easily seen by replacing k with $k - 1/n$ and taking a limit $n \to \infty$. Given $x \in \mathbf{R}^d$, the function $f(s) := E(x_\infty + s\frac{x - x_\infty}{|x - x_\infty|}) - E(x_\infty)$ satisfies the hypothesis $f''(s) \ge k$. Setting $s = |x - x_\infty|$ in the conclusion of the lemma, Cauchy-Schwarz yields the desired inequalities (2.2–2.3). □

For the reader familiar with Riemannian geometry, it is not hard to extend the results of this section to a C^2 function $E : M \longrightarrow \mathbf{R}$ on a complete Riemannian manifold $(M, \langle\ ,\ \rangle)$ satisfying the Hessian bound $D^2E \ge k\langle\ ,\ \rangle$. For example, (2.3) takes the form

$$E(x) - E(x_\infty) \le \operatorname{dist}(x, x_\infty)|\nabla E(x)| - k \operatorname{dist}(x, x_\infty)^2/2. \tag{2.4}$$

where $\mathrm{dist}(x, x_\infty)$ denotes arclength (geodesic) distance between x and $x_\infty \in M$ induced by the Riemannian metric $\langle\ ,\ \rangle$. Our primary task will be to extend the argument of Proposition 2.1 to the length space $M = \mathcal{P}_2(\mathbf{R}^d)$ of probability measures metrized by quadratic Wasserstein distance, to obtain optimal contraction rates under a range of degenerate convexity assumptions. Following Otto's work [38], analogs of Propositions 2.3–2.5 have been explored in this setting by Agueh [1], Agueh, Ghoussoub, and Kang [2], Carrillo, Jüngel, Markowich, Toscani and Unterreiter [18], Cordero-Erausquin, Gangbo and Houdre [23], Otto and Villani [39], and our parallel work [19]. In the classical case of linear diffusion with quadratic confinement (Example 1.1 with $\alpha = \beta = a = 1$ and $\gamma = b = 0$), the relative energy $E(\rho) - E(\rho_\infty)$ reduces to the Boltzmann entropy (5.1) of $\rho = f^2\rho_\infty$, and $|\nabla E(\rho)|^2$ to its Fisher information. As explained in these references, the first inequality in (2.2) becomes Talagrand's transportation inequality [41], the second the log-Sobolev inequality of Gross and others [31], while (2.4) becomes the HWI inequality of Otto and Villani [39]. In particular, (2.4) indicates how 2-uniform displacement convexity on a neighbourhood of x_∞ converts convergence in the weak metric dist_2 to convergence in relative entropy. From there it can often be converted to strong convergence in $L^1(\mathbf{R}^d)$ via a Csiszar-Kullback inequality, as in [18, 38]. This helps to explain part of the interest in Wasserstein contraction rates.

3 Gradient flows on Riemannian length spaces

In this section we develop an abstract theory governing gradient flows on Riemannian manifolds. By gradient flow we refer to a family of maps $X_t : M \longrightarrow M$ parameterized by $t \in (a, b) \subset \mathbf{R}$ solving the differential equation

$$\frac{dX_t}{dt} \;=\; -\mathrm{grad}\, E(X_t) \tag{3.1}$$

associated to some energy $E : M \longrightarrow \mathbf{R} \cup \{+\infty\}$ and satisfying the initial condition $X_0(x) = x$. Our immediate goal is to show how convexity of E along geodesics governs contractivity of the flow X_t. In particular, we recover the result mentioned above that $D^2E \geq k > 0$ implies

$$\mathrm{dist}(X_t(x), X_t(y)) \leq e^{-kt}\, \mathrm{dist}(x, y). \tag{3.2}$$

More importantly, we show the degenerate convexity present in our model for granular media implies a corresponding expression with algebraic (instead of exponential) decay.

Since our ultimate plan is to apply these ideas in an infinite-dimensional setting which corresponds only loosely to a Riemannian manifold, it is necessary to develop our theory in a more general setting. The basic structure we need is given by the concept of length spaces [29, 30]. However, this is not enough to make rigorous our approach to equilibration in granular media. We therefore introduce some additional structure to define subgradient flows and relate the geodesic distance to a distance induced by the nominally "Riemannian" metric.

Given a continuous curve $u : [a,b] \longrightarrow M$ in a metric space (M, dist), its *length* $\mathcal{L}(u)$ is defined as a supremum over finite partitions $\Pi = \{s_i \mid a = s_0 < s_1 < \cdots < s_k = b\}$ by

$$\mathcal{L}(u) := \sup_{\Pi \subset [a,b]} \sum_{i=1}^{k} \mathrm{dist}(u_{s_i}, u_{s_{i-1}}).$$

Obviously, this length depends only on the curve and is invariant under reparameterizations. Moreover, $\mathcal{L}(u) \geq \mathrm{dist}(u_a, u_b)$ by the triangle inequality.

Definition 3.1 (Length Space). *A metric space* (M, dist) *is called a* length space [29] *(c.f* path metric space [30]*) if each* $x, y \in M$ *satisfy*

$$\mathrm{dist}(x,y) = \inf_{\substack{u_0 = x \\ u_1 = y}} \mathcal{L}(u), \tag{3.3}$$

where the infimum is over all continuous curves $u_s \in M$ *joining* $u_0 = x$ *to* $u_1 = y$*.*

Example 3.2 (Minimal Geodesics). *Fix* (M, dist)*, and suppose a continuous curve* $u_s \in M$ *exists satisfying* $\mathrm{dist}(u_s, u_{s+t}) = t\,\mathrm{dist}(u_0, u_1)$ *for* $0 \leq s \leq s+t \leq 1$ *and linking any given pair of endpoints* $u_0, u_1 \in M$*. Then* $\mathcal{L}(u) = \mathrm{dist}(u_0, u_1)$ *achieves the infimum* (3.3) *so* (M, dist) *is a length space. Such curves (and their affine reparameterizations) are called* minimal geodesics*.*

The convexity properties to be required along minimal geodesics are laid out in the following definitions. The term *modulus of convexity* refers to any function ϕ taking a single sign on the positive reals and satisfying three conditions (ϕ_0–ϕ_2):

$$(\phi_0) \qquad \phi : [0,\infty) \longrightarrow \mathbf{R} \text{ is continuous and vanishes only at } \phi(0) = 0; \tag{3.4}$$

$$(\phi_1) \qquad \phi(x) \geq -kx \text{ for some } k < \infty; \tag{3.5}$$

$$(\phi_2) \qquad \phi(x) + \phi(y) \leq \phi(x+y) \quad \text{(superadditivity)}; \tag{3.6}$$

$$(\phi_3) \qquad \chi_s(x) := \frac{1}{2}\int_{|1-2s|\sqrt{x}}^{\sqrt{x}} \phi(t)dt \text{ is convex on } x \geq 0 \text{ for each fixed } s \in [0,1]. \tag{3.7}$$

For our main application discussed in §6, we shall also require the additional hypothesis (ϕ_3). It is therefore convenient to remark that if ϕ is convex then (ϕ_0) and (ϕ_1) together imply all four conditions (ϕ_0–ϕ_3) have been satisfied.

Definition 3.3 (ϕ-Uniform Convexity). *A lower-semicontinuous energy* $E : M \to \mathbf{R} \cup \{+\infty\}$ *on the length space* M *is said to be* ϕ-uniformly convex *if*

$$E(u_0) - E(u_s) - E(u_{1-s}) + E(u_1) \geq \frac{1}{2}\int_{|1-2s|L}^{L} \phi(t)dt \quad , 0 \leq s \leq 1 \ , \tag{3.8}$$

along each minimal geodesic $u_s \in M$ *of length* $L = \mathrm{dist}(u_0, u_1)$ *linking endpoints of finite energy.*

Example 3.4 (Geodesic convexity; 2-uniform convexity; semiconvexity).

1. *If $\phi := 0$, then* (3.8) *with $s = 1/2$ asserts midpoint convexity of E. Lower semicontinuity then implies convexity of E as a function of arclength along all minimal geodesics in M. Thus* (3.8) *with $\phi = 0$ will be called* geodesic convexity *of E, or* displacement convexity *in the context of the Wasserstein length space* (1.1).

2. *Condition* (3.8) *with $\phi(s) = ks \geq 0$ is called* 2-uniform convexity with constant k.

3. *Condition* (3.8) *with $\phi(s) = -ks \leq 0$ is called* semiconvexity with constant k.

Conditions equivalent to ϕ-uniform convexity can also be given on derivatives of E:

Lemma 3.5 (Differential characterization of ϕ-uniform convexity). *The following condition on a lower-semicontinuous $E : M \longrightarrow \mathbf{R} \cup \{+\infty\}$ is equivalent to ϕ-uniform convexity, provided it holds on all minimal geodesics $s \in [0, 1] \longrightarrow u_s \in M$ whose endpoints have finite energy: (i) $E(u_s)$ is continuous on $[0, 1]$, its distributional derivative belongs to $BV_{loc}(0, 1)$, and the left and right derivatives, when they exist, satisfy*

$$\frac{d}{ds}E(u_s)\Big|_{1^-} - \frac{d}{ds}E(u_s)\Big|_{0^+} \geq \phi(\operatorname{dist}(u_0, u_1)) \operatorname{dist}(u_0, u_1). \tag{3.9}$$

Proof. Let $s \in [0, 1] \longrightarrow u_s \in M$ be a minimal geodesic whose endpoints have finite energy, and set $L := \operatorname{dist}(u_0, u_1)$. To begin, assume E is ϕ-uniformly convex. From hypothesis (ϕ_1) in (3.5) and (3.8), we see that $E(u_s) + kL^2 s^2/2$ is a convex function on $s \in [0, 1]$ as in Example 3.4.1. Any real-valued lower-semicontinous convex funtion on the unit interval is actually continuous and has a non-decreasing derivative: more precisely, the left and right derivatives are given everywhere by two nondecreasing functions which differ only on a countable set. It follows immediately that $E(u_s)$ has left and right derivatives everywhere which agree a.e., and $\frac{d}{ds}E(u_s)$ is $BV_{loc}(0, 1)$.

To deduce (3.9), rewrite (3.8) as

$$\begin{aligned}\frac{E(u_1) - E(u_{1-s})}{s} - \frac{E(u_s) - E(u_0)}{s} \quad &\geq \quad \frac{1}{2s}\int_{|1-2s|L}^{L} \phi(t)dt \\ &\to \quad L\phi(L) \qquad \text{as } s \to 0,\end{aligned}$$

and take the limit $s \to 0$.

Conversely, assume $E(u_s)$ is a continuous function of $s \in [0, 1]$ with $\frac{d}{ds}E(u_s)$ in $BV_{loc}(0, 1)$ and (3.9) holds. Noting that $s \in [0, 1] \longrightarrow v_s := u_{\tau+s(1-2\tau)}$ gives the minimal geodesic linking u_τ to $u_{(1-\tau)}$, we have

$$\begin{aligned}\frac{d}{dt}E(u_t)\Big|_{(1-\tau)^-} - \frac{d}{dt}E(u_t)\Big|_{\tau^+} \quad &= \quad \frac{1}{1-2\tau}\left[\frac{dE(v_s)}{ds}\right]_{0^+}^{1^-} \\ &\geq \quad L\,\phi((1-2\tau)L)\end{aligned}$$

for each $\tau \in (0, 1/2)$. Integrating this inequality over $(\delta, s) \subset (0, 1/2)$ yields

$$E(u_\delta) - E(u_s) - E(u_{1-s}) + E(u_{1-\delta}) \geq \frac{1}{2} \int_\delta^s \phi((1-2\tau)L) 2L d\tau.$$

Letting $\delta \to 0$ and changing variables to $t = (1-2\tau)L$ we recover (3.8). This shows that (3.9) implies ϕ-uniform convexity and completes the proof. □

Example 3.6 (ϕ-Uniform convexity on the line). *For a smooth enough function $E : \mathbf{R} \longrightarrow \mathbf{R}$, a simple arclength rescaling shows ϕ-uniform convexity to be equivalent to the following condition: for each $x_0, x_1 \in \mathbf{R}$ with $x_0 < x_1$,*

$$\int_{x_0}^{x_1} E''(x) dx \;=\; \left.\frac{dE}{dx}\right|_{x_1} - \left.\frac{dE}{dx}\right|_{x_0} \;\geq\; \phi(x_1 - x_0). \tag{3.10}$$

This characterization of ϕ-uniform convexity via second derivatives shows why superadditivity is a natural restriction on $\phi(s)$: (3.6) merely implies that the mass of $E''(ds)$ on each interval of length $x+y$ is no less than the sum of the masses required on disjoint intervals of length x and y.

Example 3.7 (Powers). *The second derivative condition* (3.10) *also makes clear that:*
(a) For $\phi(s) = ks$, a smooth energy on a Riemannian manifold is ϕ-uniformly convex if and only if $D^2 E \geq k$.
(b) For $\phi(s) = ks^{q-1} \geq 0$ with $q \geq 2$, definition (3.8) *coincides with the q-uniform convexity discussed in Ball, Carlen and Lieb [6]. In particular, $C(x) = |x|^q/q$ is ϕ-uniformly convex on $\mathbf{R}^d$ with constant $k = 2^{2-q}$. This notion also coincides with the c-uniform convexity of potentials in $\mathbf{R}^d$ used in [23, 1].*

At this point, let us introduce the additional structures on M required for the sequel.

Definition 3.8 (Riemannian length spaces). *Let $\langle \cdot\,, \cdot \rangle$ and $|\cdot|$ denote the inner product and norm on some Hilbert space $\mathcal{H}$. A subset M of a length space (N, dist) is called* Riemannian *if each $x \in M$ is associated with a map $\exp_x : \mathcal{H} \longrightarrow N$ which gives a surjection from a star-shaped subset $\mathcal{K}_x \subset \mathcal{H}$ onto M such that the curve $x_s = \exp_x(sp)$ defines an (affinely parameterized) minimizing geodesic $s \in [0,1] \longrightarrow x_s$ linking $x = x_0$ to $y = x_1$ for each $p \in \mathcal{K}_x$. We moreover assume there exists $q \in \mathcal{K}_y$ such that $x_s = \exp_y(1-s)q$ and that*

$$\mathrm{dist}^2(\exp_x u, \exp_y v) \leq \mathrm{dist}^2(x,y) - 2\langle v, q\rangle - 2\langle u, p\rangle + o(\sqrt{|u|^2 + |v|^2}), \tag{3.11}$$

for all $u \in \mathcal{H}$ and $v \in \mathcal{H}$ as $|u| + |v| \to 0$. It is often convenient to allow the Hilbert space and scalar product to depend on the base point $x \in M$, with the understanding that each such tangent space $T_x M = (\mathcal{H}_x, \langle \cdot, \cdot \rangle_x)$ is isomorphic to a standard Hilbert space $\mathcal{H}$ through a tacit but fixed isomorphism.

Remark 3.9 (Riemannian structure inherited by geodesically convex subsets). *As a corollary to the preceding definition, a Riemannian length space M contains a minimal geodesic $s \in [0,1] \longrightarrow x_s \in M$ linking each pair of points x and $y \in M$. If $M' \subset M$ is a* geodesically convex *subset, meaning any such geodesic lies in M' whenever its endpoints do, then it is easy to check that M' is itself a Riemannian length space with the same tangent space and exponential map as M, but*

$$K'_x := \{p \in \mathcal{K}_x \mid \exp_x p \in M'\}.$$

Remark 3.10 (Convex sets and complete manifolds). *Thus Definition 3.8 simultaneously encompasses convex sets $M \subset N = \mathbf{R}^\mathbf{d}$ in Euclidean space and complete manifolds $M = N$. Clearly the surjections $\exp_x : \mathcal{K}_x \longrightarrow M$ are intended to occupy the role played by Riemannian normal coordinates on an ordinary manifold. We remark furthermore that the only connection between the scalar product $\langle \cdot\,, \cdot \rangle$ and the metrical distance we shall need is encoded in* (3.11). *In fact,* (3.11) *is nothing but superdifferentiability of the distance* dist*, which holds on Riemannian manifolds (see* [37]*).*

Now, we introduce the more general notions of super and subdifferentiability of functions on a Riemannian length space M that we need to set up our model problem.

Fix $x \in M$. A function $E : M \longrightarrow \mathbf{R} \cup \{-\infty\}$ is said to be *superdifferentiable* at x with *supergradient* $p \in T_xM$ if

$$E(\exp_x tv) \leq E(x) + t\langle p, v\rangle_x + o(t) \tag{3.12}$$

holds for all $v \in \mathcal{K}_x$, $t \geq 0$ as $t \to 0$. Such (supergradient, point) pairs (p, x) form a subset $\overline{\partial} E \subset TM$ of the tangent bundle; we also express their relationship (3.12) by writing $p \in \overline{\partial} E_x$. If the opposite inequality

$$E(\exp_x tv) \geq E(x) + t\langle q, v\rangle_x + o(t)$$

holds, E is said to be *subdifferentiable* with *subgradient* $q \in \underline{\partial} E_x \subset T_xM$. When both inequalities hold and the closed convex hull of $\mathcal{K}_x$ contains a neighbourhood of $0 \in \mathcal{H}$, then the super and subgradients of E coincide, $p = q = \operatorname{grad} E(x)$; in this case we can think of them as giving the gradient of E at $x \in M$.

Definition 3.11 (Tangent vector). *A continuous curve $t \in [0,T] \longrightarrow x_t \in M$ is* right differentiable *at $t = 0$ with tangent vector $\frac{dx_t}{dt}\big|_{t=0^+} := v$ if there exists $v \in \mathcal{H}$ with* $\operatorname{dist}(x_t, \exp_x tv) = o(t)$ *as $t \to 0^+$. Note that we do not insist on uniqueness of such a tangent vector for the curve to be differentiable. We none the less use the notation $v \in T_xM$ and $|v|_x^2 = \langle v, v\rangle$. The left derivative $\frac{dx_t}{dt}\big|_{t=T^-}$ is analogously defined.*

Finally, we come to the main result of this section, linking convexity of E to the contraction properties of its subgradient flow. Notice that E must be subdifferentiable along the paths u_t and v_t, but not necessarily elsewhere in M.

Theorem 3.12 (Rate of contraction for gradient flows). *Fix a Riemannian length space* (M, dist) *and a ϕ-uniformly convex energy functional* $E : M \longrightarrow \mathbf{R} \cup \{+\infty\}$. *Given two continuous and right differentiable paths u_t and $v_t \in M$, if the differential inclusions $-\dot{u}_{t^+} \cap \underline{\partial} E_{u_t} \neq \emptyset$ and $-\dot{v}_{t^+} \cap \underline{\partial} E_{v_t} \neq \emptyset$ hold for all $t \in [0, T)$ then*

$$\mathrm{dist}(u_t, v_t) \leq \begin{cases} \Phi^{-1}(\Phi(\mathrm{dist}(u_0, v_0)) - t) & \text{if } \mathrm{dist}(u_0, v_0) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.13}$$

where

$$\Phi(x) = \int^x \frac{dy}{\phi(y)}. \tag{3.14}$$

Proof. Choose tangent vectors $\dot{u}_0 \in -\underline{\partial} E_{u_0}$ and $\dot{v}_0 \in -\underline{\partial} E_{v_0}$ to the curves u_t and v_t at $t = 0^+$. The definition of right differentiability together with the triangle inequality imply

$$\mathrm{dist}(u_t, v_t) = \mathrm{dist}(\exp_{u_0} t\dot{u}_0, \exp_{v_0} t\dot{v}_0) + o(t).$$

As the length space M is Riemannian, there must be vectors $p, q \in \mathcal{H}$ which generate a minimal geodesic $\sigma_s = \exp_{u_0} sp = \exp_{v_0}(1-s)q$ linking $\sigma_0 = u_0$ to $\sigma_1 = v_0$. This curve is differentiable and has tangents $p \in \dot{\sigma}_0 \cap \mathcal{K}_{u_0}$ and $-q \in \dot{\sigma}_1 \cap \mathcal{K}_{v_0}$ at its endpoints. Furthermore, superdifferentiability of the square distance (3.11) yields

$$\mathrm{dist}^2(\exp_{u_0} t\dot{u}_0, \exp_{v_0} t\dot{v}_0) \leq \mathrm{dist}^2(u_0, v_0) - 2t\langle \dot{v}_0, q\rangle_{v_0} - 2t\langle \dot{u}_0, p\rangle_{u_0} + o(t).$$

The first inequality squared combines with the second to give

$$\begin{aligned} \left.\frac{d^+}{dt}\right|_0 \mathrm{dist}^2(u_t, v_t)/2 &:= \limsup_{t \to 0^+} \frac{\mathrm{dist}^2(u_t, v_t) - \mathrm{dist}^2(u_0, v_0)}{2t} \\ &\leq -\langle \dot{v}_0, q\rangle_{v_0} - \langle \dot{u}_0, p\rangle_{u_0}. \end{aligned} \tag{3.15}$$

The differential inclusion $-\dot{u}_0 \in \underline{\partial} E_{u_0}$ asserts

$$E(\sigma_s) = E(\exp_{u_0} sp) \geq E(u_0) - s\langle \dot{u}_0, p\rangle + o(s),$$

since $sp \in \mathcal{K}_x$, so the convex function $E(\sigma_s)$ has right derivative

$$\left.\frac{dE(\sigma_s)}{ds}\right|_{s=0^+} \geq -\langle \dot{u}_0, p\rangle. \tag{3.16}$$

Similarly, $-\dot{v}_0 \in \underline{\partial} E_{v_0}$ and $\sigma_s = \exp_{v_0}(1-s)q$ imply

$$\left.\frac{dE(\sigma_s)}{ds}\right|_{s=1^-} \leq \langle \dot{v}_0, q\rangle. \tag{3.17}$$

Using (3.16–3.17) to estimate (3.15) yields

$$\left.\frac{d^+}{dt}\right|_0 \mathrm{dist}^2(u_t, v_t)/2 = \mathrm{dist}(u_0, v_0) \left.\frac{d^+}{dt}\right|_0 \mathrm{dist}(u_t, v_t)$$

$$
\begin{aligned}
&\leq -\left.\frac{dE(\sigma_s)}{ds}\right|_{s=0+}^{s=1^-} \\
&\leq -\phi(\operatorname{dist}(u_0, v_0))\operatorname{dist}(u_0, v_0),
\end{aligned}
$$

by ϕ-uniform convexity (3.9) of E along the geodesic σ_s of length $\operatorname{dist}(u_0, v_0)$. Time-translation invariance shows the same estimate must hold at any other time $t = t_0$ that we derived at $t = 0$. Thus when $\phi \geq 0$ (resp. $\phi \leq 0$),

$$
\left.\frac{d^+}{dt}\right|_{t_0} \Phi(\operatorname{dist}(u_t, v_t)) = \frac{1}{\phi(\operatorname{dist}(u_{t_0}, v_{t_0}))} \left.\frac{d^+}{dt}\right|_{t_0} \operatorname{dist}(u_t, v_t) \leq -1 \quad (\text{resp. } \geq) \tag{3.18}
$$

holds at each instant $t_0 \in I := \{t \in [0, T) \mid u_{t_0} \neq v_{t_0}\}$.

In case $\phi \geq 0$, the primitive equation (3.14) defines a continuously increasing function $\Phi : (0, \infty) \longrightarrow \mathbf{R}$ in view of hypothesis (ϕ_0) (3.4), but its limit $\Phi(0) = -\infty$ is unbounded due to the Lipschitz continuity of ϕ near $\phi(0) = 0$ implied by (ϕ_2). Thus the inverse $\Phi^{-1} : (-\infty, \Phi(\infty)) \longrightarrow \mathbf{R}$ is also a continuously increasing function.

If $\phi \leq 0$, (3.18) is reversed but Φ decreases monotonically from $\Phi(0) = +\infty$, and we may need to extend $\Phi^{-1}(s)$ to $s \leq \Phi(\infty)$ by setting $\Phi^{-1}(s) = +\infty$. In this case the growing bound (3.13) may only remain finite for a short time. Using hypothesis (ϕ_1) (3.5), we obtain that this growth is no larger than exponential and thus, it remains finite for all times.

Either way, Gronwall's inequality completes the proof as long as $I = [0, b) \subset [0, T)$.

The only remaining possibility is that the relatively open subset $I \subset [0, T)$ contains a non-empty connected component $(a, b) \subset [0, T)$. We claim this cannot happen. To see why, observe that if $\phi \geq 0$ then Gronwall's inequality yields $t + \Phi(\operatorname{dist}(u_t, v_t))$ non-increasing so

$$
s + \Phi(\operatorname{dist}(u_s, v_s)) \geq t + \Phi(\operatorname{dist}(u_t, v_t)) \tag{3.19}
$$

for $a < s < t < b$. Letting $s \to a$ shows $\Phi(0) \geq t - a + \Phi(\operatorname{dist}(u_t, v_t))$, contradicting $\Phi(0) = -\infty$. On the other hand, if $\phi \leq 0$, then (3.18–3.19) are reversed. Taking the limit $s \to a$ contradicts $\Phi(0) = +\infty$, to conclude the proof of the theorem. □

Example 3.13 (Exponential versus algebraic convergence).

a) $\phi(x) = kx$ with $k \in \mathbf{R}$ implies $\Phi(x) = \frac{1}{k}\log x$ and $\Phi^{-1}(y) = e^{ky}$ so (3.13) *becomes*

$$
\operatorname{dist}(u_t, v_t) \leq e^{-kt} \operatorname{dist}(u_0, v_0). \tag{3.20}
$$

b) $\phi(x) = (k/r)x^{r+1}$ with $k, r > 0$ implies $\Phi(x) = -\frac{1}{k}x^{-r}$ and $\Phi^{-1}(y) = (-ky)^{-1/r}$ so (3.13) *becomes*

$$
\operatorname{dist}(u_t, v_t) \leq \frac{\operatorname{dist}(u_0, v_0)}{(1 + tk \operatorname{dist}^r(u_0, v_0))^{1/r}}.
$$

Remark 3.14 (Rates of expansion). *Theorem* 3.12 *covers semiconvex functionals as well as convex ones. Thus* (3.20) *with $k < 0$ provides exponential control on the growth of separation between two initial conditions under the subgradient flow. In particular, taking $u_0 = v_0$ shows the time evolution defined by the flow is unique, when it exists.*

4 Probability measures form a Riemannian length space

As discussed in the introduction, we are interested in the evolution of probability measures verifying certain partial differential equations. Our objective is to formulate this evolution as a subgradient flow on a Riemannian length space. In this section we introduce the relevant Riemannian length space structure on subsets of the space of all Borel probability measures on $\mathbf{R}^d$, i.e., $\mathcal{P}(\mathbf{R}^d)$.

To begin we recall the Kantorovich-Rubinstein-Wasserstein L^2 distance $\operatorname{dist}_2(\rho, \rho')$ [32, 44] between two measures $\rho, \rho' \in \mathcal{P}(\mathbf{R}^d)$: its square is defined as an infimum

$$\operatorname{dist}_2^2(\rho, \rho') := \inf_{\gamma \in \Gamma(\rho,\rho')} \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^2 \, d\gamma(x, y) \tag{4.1}$$

over the set $\Gamma(\rho, \rho')$ of joint measures $\gamma \geq 0$ on $\mathbf{R}^d \times \mathbf{R}^d$ with left and right marginals ρ and ρ', respectively. It is not hard to see that dist_2 satisfies the triangle inequality and makes $\mathcal{P}(\mathbf{R}^d)$ a complete metric space [25, 28]. However $\operatorname{dist}_2(\rho, \rho') = +\infty$ whenever one measure has finite second moment and the other does not, so henceforth we restrict our attention to the connected component

$$\mathcal{P}_2(\mathbf{R}^d) := \left\{ \rho \in \mathcal{P}(\mathbf{R}^d) \mid \int_{\mathbf{R}^d} |x|^2 d\rho(x) < +\infty \right\}, \tag{4.2}$$

itself a complete metric space on which dist_2 is finite. Let $\mathcal{P}^{ac}(\mathbf{R}^d)$ denote the set of Borel probability measures on $\mathbf{R}^d$ which are absolutely continuous with respect to Lebesgue. The intersection $\mathcal{P}_2(\mathbf{R}^d) \cap \mathcal{P}^{ac}(\mathbf{R}^d)$ is denoted $\mathcal{P}_2^{ac}(\mathbf{R}^d)$.

It is also easy to see that $N := \mathcal{P}_2(\mathbf{R}^d)$ is a length space: the infimum (4.1) is attained, and the projection ρ_s of the optimal joint measure γ onto the subspace

$$\{((1 - s)x, sx) | x \in \mathbf{R}^d\} \subset \mathbf{R}^d \times \mathbf{R}^d$$

for each $s \in [0, 1]$ yields a minimal geodesic in $\mathcal{P}_2(\mathbf{R}^d)$. What is more subtle are the following facts established in McCann's thesis [35], where these paths were first introduced and described from a different point of view under the name *displacement interpolation*: (i) $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ is geodesically convex; (ii) a minimal geodesic is uniquely determined by its endpoints if either (or both) of them lie in $\mathcal{P}_2^{ac}(\mathbf{R}^d)$; (iii) in this case, the entire geodesic lies in $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ except perhaps for its second endpoint. For the present point of view, the most relevant articulation and proof of (ii) is the one given by Carlen and Gangbo [15].

Taking $N = \mathcal{P}_2(\mathbf{R}^d)$ as our complete length space, the subset $M = \mathcal{P}_2^{ac}(\mathbf{R}^d)$ of absolutely continuous probability measures will carry our Riemannian length space structure. Here we recall the formal Riemannian structure introduced on $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ by Otto [38], who first realized the connection between this structure and nonlinear diffusions as gradient flows. Although Otto used this connection in a purely formal manner to motivate detailed rate calculations in [38], for the rigorous theory developed hereafter it is necessary to state somewhat more precisely the nature of the tangent space, exponential mapping, and topological structure of M.

According to Definition 3.8 we only need to define the exponential mapping over the subset M. At any $\rho \in \mathcal{P}_2^{ac}(\mathbf{R}^d)$, the tangent space $T_\rho M$ to M is identified with the Hilbert space $\mathcal{H} = W^{1,2}_\rho := W^{1,2}(\mathbf{R}^d, d\rho)$ closure of the smooth compactly supported functions $C_c^\infty(\mathbf{R}^d)$ with respect to the inner product

$$\langle \psi, \psi \rangle_\rho = \int_{\mathbf{R}^d} |\nabla \psi|^2 d\rho(x). \tag{4.3}$$

The exponential map generates a curve $s \in \mathbf{R} \longrightarrow \rho_s \in N$ passing through $\rho_0 = \rho$ in direction $\psi \in W^{1,2}_\rho$ defined by imagining a collection of infinitesimally small particles comprising ρ, which evolve freely in time (both future and past) and having velocity profile $\nabla\psi$ at time $s = 0$. More precisely, the Borel map $F(x) := x + s\nabla\psi(x)$ is used to push forward the measure ρ on $\mathbf{R}^d$ to yield

$$\exp_\rho s\psi := [Id + s\nabla\psi]_\# \rho, \tag{4.4}$$

where by definition, the pushed-forward measure $F_\#\rho \in \mathcal{P}(\mathbf{R}^d)$ assigns mass

$$F_\#\rho[K] := \rho[F^{-1}(K)] \tag{4.5}$$

to each Borel set $K \subset \mathbf{R}^d$.

Observe $\rho_s = \exp_\rho s\psi$ belongs to $\mathcal{P}_2(\mathbf{R}^d)$ by finiteness of the kinetic energy (4.3). Thus $\exp_\rho : W^{1,2}_\rho \longrightarrow N$ is well-defined, and surjective as a consequence of Brenier's theorem [13], which associates to each $\rho \in \mathcal{P}_2(\mathbf{R}^d)$ and $\rho' \in \mathcal{P}_2(\mathbf{R}^d)$ a convex function $\psi(x) + |x|^2/2$ on $\mathbf{R}^d$ whose gradient pushes ρ forward to ρ' [34]. This motivates the identification of the star-shaped set

$$\mathcal{K}_\rho = \left\{ \psi(x) \in W^{1,2}_\rho \mid \Psi(x) = \frac{1}{2}|x|^2 + \psi(x) \text{ convex on } \mathbf{R}^d,\ \nabla\Psi_\#\rho \in \mathcal{P}_2^{ac}(\mathbf{R}^d) \right\} \tag{4.6}$$

which allows us to verify the conditions over the exponential map necessary for $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ to be a Riemannian length space :

Proposition 4.1 (Wasserstein distance metrizes a Riemannian length space). *The absolutely continuous measures $M = \mathcal{P}_2^{ac}(\mathbf{R}^d)$ form a Riemannian length space metrized by $\operatorname{dist}_2(\rho, \rho')$. In particular, the squared Wasserstein distance is superdifferentiable on the product manifold $M \times M$: letting ρ_s denote the minimal geodesic joining $\rho_0 = \rho$ to $\rho_1 = \rho'$ yields*

$$\operatorname{dist}_2^2(\exp_\rho t\psi, \exp_{\rho'} t\psi') \leq \operatorname{dist}_2^2(\rho, \rho') - 2t\langle \psi', \frac{d\rho_s}{ds}\Big|_{1^-} \rangle_{\rho'} - 2t\langle \psi, \frac{d\rho_s}{ds}\Big|_{0^+} \rangle_\rho + 4t^2, \tag{4.7}$$

or equivalently

$$\operatorname{dist}_2^2(\exp_\rho t\psi, \exp_{\rho'} t\psi') \leq \operatorname{dist}_2^2(\rho, \rho') + 2t\langle \psi', \varphi' \rangle_{\rho'} - 2t\langle \psi, \varphi \rangle_\rho + 4t^2,$$

for each pair of unit tangent vectors $\psi \in T_\rho M$ and $\psi' \in T_{\rho'} M$, where φ, φ' are such that $\rho_s = \exp_\rho s\varphi = \exp_{\rho'}(1-s)\varphi'$.

Proof. Given $\rho, \rho' \in \mathcal{P}_2^{ac}(\mathbf{R}^d)$, let $\gamma_0 \in \Gamma(\rho, \rho')$ denote the joint measure which achieves the infimum (4.1) defining the Wasserstein distance. This measure can also be expressed in the form $\gamma_0 = (id \times (\nabla\varphi + id))_\# \rho = ((\nabla\varphi' + id) \times id)_\# \rho'$, where the functions $\varphi(x) + x^2/2$ and $\varphi'(y) + y^2/2$ are convex and Legendre transforms, according to Brenier's theorem [14]; c.f. [34] or Rachev and Rüschendorf [40]. (The same theorem has a converse that we also require: every $\tilde{\varphi} \in \mathcal{K}_\rho$ gives rise to a $\tilde{\gamma}_0$ achieving the Wasserstein distance $\operatorname{dist}_2^2(\rho, (\nabla\tilde{\varphi} + id)_\# \rho) = \langle \tilde{\varphi}, \tilde{\varphi} \rangle_\rho$.)

Our prescription for constructing minimal geodesics yields

$$\rho_s := [id + s\nabla\varphi]_\# \rho = [id + (1-s)\nabla\varphi']_\# \rho'; \tag{4.8}$$

indeed $\operatorname{dist}_2^2(\rho_s, \rho_{s+t}) = |s-t|^2 \langle \varphi, \varphi \rangle_\rho = |s-t|^2 \operatorname{dist}_2^2(\rho, \rho') < +\infty$ as in Example 3.2, and $\rho_s \in \mathcal{P}_2^{ac}(\mathbf{R}^d)$ is absolutely continuous according to [35, Prop 1.3]. This shows $\mathcal{K}_\rho$ is star-shaped, $\psi \in \mathcal{K}_\rho$, and the exponential (4.4) maps $\mathcal{K}_\rho$ onto M, taking rays onto minimal geodesics as desired. Also $\psi' \in \mathcal{K}_{\rho'}$, and (4.8) shows the geodesic $\rho_s = \exp_{\rho'}(1-s)\psi'$ can be parameterized from the other end equally well, as required in the Riemannian length space definition 3.8. It remains only to establish (4.7), which will imply (3.11) to complete the proof.

Given $\psi \in W_\rho^{1,2}$ and $\psi' \in W_{\rho'}^{1,2}$ normalized, the map $F(x,y) = (x + t\nabla\psi(x), y + t\nabla\psi'(y))$ on $\mathbf{R}^d \times \mathbf{R}^d$ can be used to define a pushed-forward measure $\gamma_t := F_\# \gamma_0$ via (4.5). Then $\gamma_t \in \Gamma(\exp_\rho t\psi, \exp_{\rho'} t\psi')$, so (4.1) implies

$$\begin{aligned}
&\operatorname{dist}_2^2(\exp_\rho t\psi, \exp_{\rho'} t\psi') \le \int_{\mathbf{R}^d \times \mathbf{R}^d} |x' - y'|^2 \, d\gamma_t(x', y') \\
&= \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y + t(\nabla\psi(x) - \nabla\psi'(y))|^2 \, d\gamma_0(x, y) \\
&= \operatorname{dist}_2^2(\rho, \rho') + \int_{\mathbf{R}^{2n}} 2t\langle x - y, \nabla\psi(x) - \nabla\psi'(y) \rangle + t^2 |\nabla\psi(x) - \nabla\psi'(y)|^2 \, d\gamma_0(x, y) \\
&\le \operatorname{dist}_2^2(\rho, \rho') + 2t \int_{\mathbf{R}^d} \Big[\langle -\nabla\varphi(x), \nabla\psi(x) \rangle d\rho(x) + \langle \nabla\varphi'(y), \nabla\psi'(y) \rangle d\rho'(y) \Big] + 4t^2 \\
&= \operatorname{dist}_2^2(\rho, \rho') + 2t\langle \varphi', \psi' \rangle_{\rho'} - 2t\langle \varphi, \psi \rangle_\rho + 4t^2,
\end{aligned}$$

yielding the proof of (4.7). □

Remark 4.2. *We consider* $\exp_\rho$ *as a coordinate chart covering* $M = \mathcal{P}_2^{ac}(\mathbf{R}^d)$*. The reason why this atlas cannot be used to define a Hilbert manifold is that the convex set* $\mathcal{K}_\rho \subset \mathcal{H}$ *on which the exponential map gives a bijection, although dense, is too thin to contain an* open *neighbourhood of the origin in* $\mathcal{H} = W_\rho^{1,2}(\mathbf{R}^d)$*.*

For technical reasons, it is convenient in certain applications to be able to restrict our attention to compactly supported measures. The following corollary to Remark 3.9 shows that probability measures on any convex set $\Omega \subset \mathbf{R}^d$ also form a Riemannian length space

$$\mathcal{P}^{ac}(\Omega) := \{\rho \in \mathcal{P}_2^{ac}(\mathbf{R}^d) \mid \rho[\mathbf{R}^d \setminus \Omega] = 0\}.$$

Corollary 4.3 (Geodesic convexity of measures on a convex domain). *Let $\Omega \subset \mathbf{R}^d$ be convex, and $(M, \mathrm{dist}) = (\mathcal{P}_2^{ac}(\mathbf{R}^d), \mathrm{dist}_2)$. Then $\mathcal{P}^{ac}(\Omega)$ forms a geodesically convex subset of M, and hence a Riemannian length space.*

Proof. Let $\rho, \rho' \in \mathcal{P}^{ac}(\Omega)$. Carlen and Gangbo [15, Theorem 2.2] assert the existence of a unique minimal geodesic $s \in [0,1] \longrightarrow N = \mathcal{P}_2(\mathbf{R}^d)$ joining $\rho_0 = \rho$ to $\rho_1 = \rho'$. We claim that $\rho_s \in \mathcal{P}^{ac}(\Omega) \subset M$. The previous proof (4.8) asserts that $\rho_s = [id + s\nabla\varphi]_{\#}\rho \in M$ is absolutely continuous and given in terms of a function $\varphi \in \mathcal{K}_\rho$. To see

$$0 = \rho_s[\mathbf{R}^d \setminus \Omega] = \rho[(id + s\nabla\varphi)^{-1}(\mathbf{R}^d \setminus \Omega)], \tag{4.9}$$

observe that it holds for $s = 0$ and $s = 1$ by hypothesis. This means for ρ-a.e. $x \in \mathbf{R}^d$ that $x \in \Omega$ and $x + \nabla\varphi(x) \in \Omega$. Convexity of Ω implies $(1-s)x + s(x + \nabla\varphi(x)) \in \Omega$ on the same set where ρ has full measure, thus extending (4.9) to all $s \in [0,1]$. This proves $\mathcal{P}^{ac}(\Omega)$ is geodesically convex. That $M' := \mathcal{P}^{ac}(\Omega)$ inherits the Riemannian length space structure from $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ now follows from Proposition 4.1 by Remark 3.9. □

4.1 Differentiable curves on M.

Finally, we make contact with Otto's formalism [38] by pointing out that the charts described above correspond to normal coordinates around the point $\rho \in \mathcal{P}_2^{ac}(\mathbf{R}^d)$, in the sense that the metric assumes the canonical form (4.3). If one chooses to parameterize $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ by some other set of coordinates near ρ, a corresponding linear transformation is induced on the components $\psi(x)$ of each tangent vector. In particular, the linear transformation

$$\psi \in W^{1,2}(\mathbf{R}^d, d\rho) \longrightarrow \mathrm{div}\,[\rho\nabla\psi] \in \mathcal{Z}_2(\mathbf{R}^d, dx) \tag{4.10}$$

of the tangent space has a distinguished role, since the geodesic path ρ_t defined by (4.4) satisfies the instantaneous transport equation

$$\left.\frac{\partial\rho_t}{\partial t}\right|_{t=0} + \mathrm{div}\,[\rho\nabla\psi] = 0 \tag{4.11}$$

weakly on $\mathbf{R}^d$. Here $\mathcal{Z}_2(\mathbf{R}^d, dx)$ denotes the space of neutral (i.e., signed, but with total mass zero) Borel measures with finite total variation and second moments. Given any non-geodesic path $\rho_t \in M$ through $\rho_0 = \rho$ smooth enough that $(\partial\rho/\partial t)_{t=0}$ belongs to $\mathcal{Z}_2(\mathbf{R}^d, dx)$, Otto's prescription is to solve equation (4.11) for $\psi(x) \in W^{1,2}(\mathbf{R}^d, d\rho)$ to express the tangent vector to the curve in normal coordinates. Since $\rho \in \mathcal{P}_2^{ac}(\mathbf{R}^d)$ is assumed fixed, ψ depends linearly on $(\partial\rho/\partial t)_{t=0}$ as desired.

On a smooth bounded domain $\Omega \subset\subset \mathbf{R}^d$, the following lemma gives sufficient conditions for differentiability of such a curve, and identifies its tangent vector. The outward unit normal to the domain boundary is denoted by $\nu_\Omega(x)$ at $x \in \partial\Omega$.

Lemma 4.4 (Tangent to a smooth curve; c.f. [38]). *Fix $\Omega \subset\subset \mathbf{R}^d$ bounded smooth domain. Suppose a C^2 smooth function $\psi_t(x) := \psi(t,x) \in \mathbf{R}$ and a smooth C^2 curve of* *probability densities* $\rho_t(x) := \rho(t,x) \geq 0$ *on* $[0,T]\times\overline{\Omega}$ *are related by the transport equation and no-flux (Neumann) condition*

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}[\rho \nabla \psi] = 0 \qquad & \text{on } [0,T]\times\Omega, \\ \nabla\psi(t,x)\cdot \nu_\Omega(x) = 0 \qquad & \text{on } [0,T]\times\partial\Omega. \end{aligned} \tag{4.12}$$

Then $t \to \rho_t$ *is a differentiable curve in* $\mathcal{P}_2^{ac}(\mathbf{R}^d)$*. Its tangent vector* $\dot\rho_t$ *at* $t\in[0,T]$ *is given by* $\psi_t \in W^{1,2}_{\rho_t} =: \mathcal{H}$*.*

Proof. Without loss of generality, we'll establish right differentiability of the curve at $t=0$, and show its tangent vector is $\psi_0 \in W^{1,2}_{\rho_0}$. To compare ρ_t with the geodesic $\tilde\rho_t := \exp_{\rho_0} t\nabla\psi_0$, integrate

$$\begin{aligned} \frac{dX_t(x)}{dt} &= \nabla\psi_t(X_t(x)) \\ X_0(x) &= x \end{aligned} \tag{4.13}$$

to find the one-parameter family of diffeomorphisms $X_t : \Omega \longrightarrow \Omega$ generated by $\nabla\psi_t$. The Wasserstein distance between ρ_t and $\tilde\rho_t = (Y_t)_\#\rho_0$ is estimated using the joint measure $\gamma_t := (X_t\times Y_t)_\#\rho_0$ constructed from X_t and $Y_t(x) := x + t\nabla\psi_0(x)$. Note that Taylor's theorem and (4.13) imply $X_t(x) = Y_t(x) + O(t^2)$; the C^2 smoothness of $\psi(t,x)$ allows the error term to be estimated uniformly in $x\in\overline{\Omega}$ as $t\to 0$. By definition (4.1),

$$\begin{aligned} \operatorname{dist}_2^2(\rho_t,\tilde\rho_t) &\leq \int_{\mathbf{R}^d\times\mathbf{R}^d} |x-y|^2 d\gamma_t(x,y) \\ &= \int_\Omega |X_t(z)-Y_t(z)|^2 d\rho_0(z) \\ &= O(t^4), \end{aligned}$$

with is more than the definition (3.11) of differentiability requires. [Mere continuity of dX/dt on $[0,T]\times\overline{\Omega}$ is enough to yield $\operatorname{dist}_2(\rho_t,\tilde\rho_t) = o(t)$]. □

For certain applications, we will also be interested in proving differentiability of paths of measures defined on the whole space $\mathbf{R}^d$.

Lemma 4.5 (Tangent to a smooth curve in $\mathbf{R}^d$). *Suppose a* C^2 *smooth function* $\psi_t(x) := \psi(t,x)\in\mathbf{R}$ *and a smooth* C^2 *curve of probability densities* $\rho_t(x) := \rho(t,x)\geq 0$ *on* $[0,T]\times\mathbf{R}^d$ *are related by the transport equation*

$$\frac{\partial\rho}{\partial t} + \operatorname{div}[\rho\nabla\psi] = 0 \text{ on } [0,T]\times\mathbf{R}^d.$$

Assume that

$$\nabla\psi_t(x) \leq C_0(1+|x|) \tag{4.14}$$

and

$$\int_{\mathbf{R}^d} |x|^2 \ \rho(t,x)\, dx < C_0 \tag{4.15}$$

for any $t \in [0,T]$. *Then* $t \to \rho_t$ *is a differentiable curve in* $\mathcal{P}_2^{ac}(\mathbf{R}^d)$. *Its tangent vector* $\dot\rho_t$ *at* $t \in [0,T]$ *is given by* $\psi_t \in W^{1,2}_{\rho_t} =: \mathcal{H}$.

Proof. Let us follow the notation of the previous lemma. We shall denote by C various positive constants only depending on C_0 and T.

Since $\nabla\psi_t$ is of class C^1 and linearly growing at ∞, standard classical results of ODE's ensure the global existence in $[0,T]$, uniqueness and regularity of the solutions of the initial value problem (4.13). Therefore, the family X_t of C^1 diffeomorphisms is well defined for any $t \in [0,T]$ and there is no difficulty in deducing from the transport equation that $X_t \# \rho_0 = \rho_t$ for any $t \in [0,T]$.

Again, the Wasserstein distance between ρ_t and $\tilde\rho_t = Y_t \# \rho_0$ is estimated using the joint measure $\gamma_t := (X_t \times Y_t)\#\rho_0$ constructed from X_t and $Y_t(x) := x + t\nabla\psi_0(x)$. Note that

$$X_t - Y_t = \int_0^t [\nabla\psi_s(X_s) - \nabla\psi_0(X_0)]\, ds,$$

and together with our bounds on $\nabla\psi$ this implies in particular

1. $|X_t - Y_t| \le Ct(1+|x|)$,
2. $(X_t - Y_t)/t$ converges towards 0 as $t \to 0$, for all x.

By Lebesgue dominated convergence theorem, it follows that

$$\lim_{t\to 0} \frac{1}{t^2} \int |X_t(z) - Y_t(z)|^2 \, d\rho_0(z) = 0,$$

which is what the definition (3.11) of differentiability requires. □

Remark 4.6 (Differentiable curves defined by gradient flows). *The previous lemma remains valid under less stringent conditions on the growth of* $\nabla\psi$ *in* x. *For instance,* (4.14) *can be replaced by any hypothesis implying a well-defined flow map for the ODE system* (4.13) *in the whole interval* $[0,T]$ *for any* $T > 0$. *We refer to this as* global existence *for* (4.13).

Linear growth of the function defining an ODE system is the simplest assumption implying global existence of (4.13). *The use of a Liapunov functional* $L(x)$ *is one of the standard tools for proving global existence for* (4.13). *In particular, any autonomous gradient-flow, i.e.,*

$$\frac{dX_t(x)}{dt} = \nabla\psi(X_t(x))$$

has a Liapunov functional given by $L(x) = -\psi(x)$. *Coercivity of* $L(x)$*, i.e. boundedness of its sublevel sets, is enough to ensure a well-defined family of diffeomorphisms* X_t *for* *any $t \in [0,T]$. Therefore, in this case the growth of $L(x) = -\psi(x)$ when $|x| \to \infty$ need not be restrictive. Nonetheless, hypothesis (4.15) needs to be strengthened by suitably bounded moments related to the growth of $\nabla\psi(x)$ at infinity. In particular, a set of hypotheses for autonomous gradient-flows ensuring the conclusion of the previous lemma is $\psi(x) \simeq -A|x|^k$, when $|x| \to \infty$, $|\nabla\psi(x)| \le C(1+|x|^{k-1})$ with $k \ge 2$ and uniform time estimates on the $2(k-1)$th-moment of the densities ρ_t.*

In the case of non-autonomous gradient-flow ODE's systems, i.e.,

$$\frac{dX_t(x)}{dt} = \nabla\psi_t(X_t(x)),$$

the conditions on $L(t,x) = -\psi_t(x)$ which imply global existence are

$$-\frac{d\psi_t(x)}{dt} - |\nabla\psi_t(x)|^2 \le 0 \tag{4.16}$$

and $-\psi_t(x) \ge -\tilde\psi(x)$ for any $t \in [0,T]$ with $-\tilde\psi(x)$ coercive. Therefore, a set of hypotheses for non-autonomous gradient-flows ensuring the thesis of previous lemma can also be written in the same spirit as for the autonomous case by adding to (4.16) uniform bounds in time for the gradient of $\psi_t(x)$ and suitable uniform time estimates of moments of ρ_t. However, these assumptions are difficult to meet in applications.

5 Energy functionals on M

In this section we turn to the model for granular media which motivates the foregoing theory. The energy functional $\mathcal{E}(\rho)$ that we consider is a sum of three terms:

$$\begin{aligned} \mathcal{E}(\rho) &= \quad \mathcal{A}(\rho) \quad + \quad \mathcal{B}(\rho) \quad + \quad \mathcal{C}(\rho) \\ &= \int_{\mathbf{R}^d} A(\rho_{ac}(x))\,dx + \int_{\mathbf{R}^d} B(x)d\rho(x) + \frac{1}{2}\int_{\mathbf{R}^d\times\mathbf{R}^d} C(x-y)d\rho(x)d\rho(y), \end{aligned} \tag{5.1}$$

which can be defined on $\mathcal{P}^{ac}(\mathbf{R}^d)$, though we only need it on $M = \mathcal{P}_2^{ac}(\mathbf{R}^d)$. Here ρ_{ac} denotes the Radon-Nikodym derivative of ρ with respect to Lebesgue measure.

Let us first clarify the assumptions over each of these three terms. Assume the internal energy $A(\varrho)$ is lower semicontinuous, satisfies $A(0) = 0$, and

$$\lambda \longmapsto \lambda^d A(\lambda^{-d}) \quad \text{is convex nonincreasing on } \lambda \in (0,\infty). \tag{5.2}$$

It follows that $A(\varrho)$ is proper and convex throughout $[0,\infty)$. Also, in terms of the pressure function $P(\varrho) := A'(\varrho)\varrho - A(\varrho)$, (5.2) becomes equivalent to

$$P(\varrho) \ge 0 \quad \text{and} \quad \frac{P(\varrho)}{\varrho^{1-1/d}} \qquad \text{is nondecreasing on } \varrho \in (0,\infty).$$

Convexity properties of the internal energy functional $\mathcal{A}(\rho)$ in $\mathcal{P}^{ac}(\mathbf{R}^d)$ were studied in [35] and we refer to it for the proof of:

Theorem 5.1 (Convexity of entropy [35, Theorem 2.2]). *If $A(\varrho)$ satisfies* (5.2), *then $\mathcal{A}(\rho)$ is displacement convex on $\mathcal{P}^{ac}(\mathbf{R}^d)$.*

The external and interaction potentials B and C are assumed to satisfy

$$\begin{array}{ll} \textbf{(B1)} & B : \mathbf{R}^d \longrightarrow \mathbf{R} \cup \{+\infty\} \text{ is semiconvex on } \mathbf{R}^d; \\ \textbf{(C1)} & C : \mathbf{R}^d \longrightarrow \mathbf{R} \cup \{+\infty\} \text{ is semiconvex on } \mathbf{R}^d. \end{array} \tag{5.3}$$

Due to the symmetry of the functional $\mathcal{C}(\rho)$, we will consider included in hypothesis **(C1)** that $C(x) = C(-x)$ for all $x \in \mathbf{R}^d$ and $C(0) = 0$ without any loss of generality.

To apply Theorem 3.12 to this energy functional over the Riemannian length space $M = \mathcal{P}_2^{ac}(\mathbf{R}^d)$, or its subspaces $\mathcal{P}^{ac}(\Omega)$, we still need to verify two important hypotheses: convexity and subdifferentiability of $\mathcal{E}$. This is accomplished in the next subsections. Under suitable hypothesis, a main conclusion will be that the variational derivative (5.11) $\delta\mathcal{E}/\delta\rho \in W^{1,2}_\rho =: \mathcal{H}$ gives a subgradient to $\mathcal{E}$ at $\rho \in \mathcal{P}^{ac}(\Omega)$.

5.1 Displacement convexity of interaction energies.

Assumption (ϕ_3) on our modulus of convexity will play a key role in deriving uniform displacement convexity of the functionals $\mathcal{B}(\rho)$ and $\mathcal{C}(\rho)$ from uniform convexity of the interaction potentials $B(x)$ and $C(x)$. Notice that $\mathcal{C}(\rho)$ is translation invariant, so its convexity degenerates along the geodesic joining two translates of the same measure. To derive *uniform* convexity we need to fix a center of mass. Therefore, let $\mathcal{P}_0(\mathbf{R}^d) \subset \mathcal{P}(\mathbf{R}^d)$ denote the measures with center of mass at the origin; similarly $\mathcal{P}_{2,0}(\mathbf{R}^d) := \mathcal{P}_2(\mathbf{R}^d) \cap \mathcal{P}_0(\mathbf{R}^d)$ and $\mathcal{P}_0^{ac}(\Omega) := \mathcal{P}^{ac}(\Omega) \cap \mathcal{P}_0(\Omega)$ for each $\Omega \subset \mathbf{R}^d$. Although we need only convexity properties of $\mathcal{B}(\rho)$ and $\mathcal{C}(\rho)$ on $M = \mathcal{P}_2^{ac}(\mathbf{R}^d)$, we can also prove them without absolute continuity (i.e. on $N = \mathcal{P}_2(\mathbf{R}^d)$).

Lemma 5.2 (Uniform convexity of potential energies). *Let ϕ be a modulus of convexity satisfying (ϕ_0–ϕ_3). Then*

a) ϕ-uniform convexity of B on $\mathbf{R}^d$ implies ϕ-uniform convexity of

$$\mathcal{B}(\rho) = \int_{\mathbf{R}^d} B(x) d\rho(x)$$

on $(\mathcal{P}_2(\mathbf{R}^d), \mathrm{dist}_2)$.

b) $\mathcal{P}_{2,0}(\mathbf{R}^d)$ is a geodesically convex subset of $\mathcal{P}_2(\mathbf{R}^d)$;

c) $\sqrt{2}\phi(\cdot/\sqrt{2})$-uniform convexity of C on $\mathbf{R}^d$ implies ϕ-uniform convexity of

$$\mathcal{C}(\rho) = \frac{1}{2} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} C(x-y) d\rho(x) d\rho(y)$$

on $\mathcal{P}_{2,0}(\mathbf{R}^d)$.

Proof. Given two $\rho, \rho' \in \mathcal{P}_2(\mathbf{R}^d)$, the minimal geodesic joining ρ to ρ' is given by the projection onto the subspace

$$\{((1-s)x, sx) \mid x \in \mathbf{R}^d\} \subset \mathbf{R}^d \times \mathbf{R}^d$$

of the optimal mass transference plan γ achieving the infimum (4.1). Any joint measure can be written in the form $\gamma = (\eta_1 \times \eta_2)_{\#}\mu$ with $\mu \in \mathcal{P}(\mathcal{S})$, $\mathcal{S}$ a probability space and $\eta_1, \eta_2 : \mathcal{S} \longrightarrow \mathbf{R}^d$ measurable functions. For example, take $(\mathcal{S}, \mu) = (\mathbf{R}^d \times \mathbf{R}^d, \gamma)$ and $(\eta_1(x, y), \eta_2(x, y)) = (x, y)$. In order to prove a), we express $\mathcal{B}(\rho_s)$ as

$$\mathcal{B}(\rho_s) = \int_{\mathcal{S}} B[(1-s)\eta_1(\omega) + s\eta_2(\omega)]\, d\mu(\omega)$$

for any $s \in [0, 1]$. Let us denote by $R(\omega)$ the function

$$R(\omega) \;=\; B[\eta_1(\omega)] - B[(1-s)\eta_1(\omega) + s\eta_2(\omega)] - B[s\eta_1(\omega) + (1-s)\eta_2(\omega)] + B[\eta_2(\omega)].$$

Using the ϕ-uniform convexity of B on $\mathbf{R}^d$, we deduce

$$\begin{aligned}
\mathcal{B}(\rho_0) - \mathcal{B}(\rho_s) - \mathcal{B}(\rho_{1-s}) + \mathcal{B}(\rho_1) &= \int_{\mathcal{S}} R(\omega)\, d\mu(\omega) \\
&\geq \int_{\mathcal{S}} \frac{1}{2} \int_{|1-2s|S(\omega)}^{S(\omega)} \phi(t)\, dt\, d\mu(\omega) \\
&= \int_{\mathcal{S}} \chi_s\left(S(\omega)^2\right)\, d\mu(\omega),
\end{aligned}$$

with $S(\omega) = |\eta_1(\omega) - \eta_2(\omega)|$. Hypothesis (ϕ_3) over the modulus of convexity ϕ allows us to use Jensen's inequality for $\chi_s(x)$ giving

$$\begin{aligned}
\int_{\mathcal{S}} \chi_s\left(|\eta_1(\omega) - \eta_2(\omega)|^2\right)\, d\mu(\omega) &\geq \chi_s\left(\int_{\mathcal{S}} |\eta_1(\omega) - \eta_2(\omega)|^2\, d\mu(\omega)\right) \\
&= \chi_s\left(\operatorname{dist}_2^2(\rho_0, \rho_1)\right),
\end{aligned}$$

and thus,

$$\mathcal{B}(\rho_0) - \mathcal{B}(\rho_s) - \mathcal{B}(\rho_{1-s}) + \mathcal{B}(\rho_1) \;\geq\; \frac{1}{2} \int_{|1-2s|L}^{L} \phi(t)\, dt,$$

with $L = \operatorname{dist}_2(\rho_0, \rho_1)$. This proves ϕ-uniform convexity of $\mathcal{B}(\rho)$.

Part b) can be deduced from part a) as follows. set $B(x) = x_i$ for $i \in \{1, 2, \ldots, d\}$. Note $B \in L^1(\mathbf{R}^d, d\rho_s)$ since ρ_s has second moments. Furthermore, $B(x)$ is continuous and simultaneously convex and concave, so part a) shows that the same must be true for $\mathcal{B}(\rho_s)$: it can only be an affine function of $s \in [0, 1]$. If ρ_0 and $\rho_1 \in \mathcal{P}_{2,0}(\mathbf{R}^d)$, then the affine function $\mathcal{B}(\rho_s)$ vanishes at both endpoints and hence everywhere in between. This shows $\mathcal{P}_{2,0}(\mathbf{R}^d)$ is geodesically convex.

Part c) is proved similarly to part a): Given the function

$$\begin{aligned} 2R(\omega,\omega') &= C[\eta_1(\omega)-\eta_1(\omega')] - C[(1-s)(\eta_1(\omega)-\eta_1(\omega')) + s(\eta_2(\omega)-\eta_2(\omega'))] \\ &+ C[\eta_2(\omega)-\eta_2(\omega')] - C[s(\eta_1(\omega)-\eta_1(\omega')) + (1-s)(\eta_2(\omega)-\eta_2(\omega'))], \end{aligned}$$

for any $\omega,\omega' \in \mathcal{S}$, we have

$$\mathcal{C}(\rho_0) - \mathcal{C}(\rho_s) - \mathcal{C}(\rho_{1-s}) + \mathcal{C}(\rho_1) = \int_{\mathcal{S}\times\mathcal{S}} R(\omega,\omega')\, d(\mu(\omega)\times\mu(\omega')).$$

Thus, by using the $(1/\sqrt{2})\phi(\cdot/\sqrt{2})$-uniform convexity of $C/2$ on $\mathbf{R}^d$, we deduce

$$\begin{aligned} \mathcal{C}(\rho_0) - \mathcal{C}(\rho_s) - \mathcal{C}(\rho_{1-s}) + \mathcal{C}(\rho_1) &= \int_{\mathcal{S}\times\mathcal{S}} R(\omega,\omega')\, d(\mu(\omega)\times\mu(\omega')) \\ &\geq \int_{\mathcal{S}\times\mathcal{S}} \int_{|1-2s|S(\omega,\omega')}^{S(\omega,\omega')} \frac{\phi\left(\frac{t}{\sqrt{2}}\right)}{2\sqrt{2}}\, dt\, d(\mu(\omega)\times\mu(\omega')) \\ &= \int_{\mathcal{S}\times\mathcal{S}} \chi_s\left(\frac{1}{2}S(\omega,\omega')^2\right) d(\mu(\omega)\times\mu(\omega')), \end{aligned}$$

with $S(\omega,\omega') = |\eta_1(\omega)-\eta_1(\omega')-\eta_2(\omega)+\eta_2(\omega')|$. Taking into account the convexity of $\chi_s(x)$ in (ϕ_3), Jensen's inequality gives us

$$\begin{aligned} \int_{\mathcal{S}\times\mathcal{S}} \chi_s\left(\frac{S(\omega,\omega')^2}{2}\right) d(\mu\times\mu)(\omega,\omega') &\geq \chi_s\left(\int_{\mathcal{S}\times\mathcal{S}} \frac{S(\omega,\omega')^2}{2}\, d(\mu\times\mu)(\omega,\omega')\right) \\ &= \chi_s\left(\mathrm{dist}_2^2(\rho_0,\rho_1) - |\langle x\rangle_{\rho_0} - \langle x\rangle_{\rho_1}|^2\right), \end{aligned} \tag{5.4}$$

where $\langle x\rangle_\rho$ is the center of mass of the density ρ, i.e.,

$$\langle x\rangle_\rho = \int_{\mathbf{R}^d} x\, d\rho(x). \tag{5.5}$$

Since we have assumed that our densities $\rho,\rho' \in \mathcal{P}_0(\mathbf{R}^d)$, then $\langle x\rangle_{\rho_0} = \langle x\rangle_{\rho_1} = 0$ and

$$\mathcal{C}(\rho_0) - \mathcal{C}(\rho_s) - \mathcal{C}(\rho_{1-s}) + \mathcal{C}(\rho_1) \geq \frac{1}{2}\int_{|1-2s|L}^{L} \phi(t)\, dt$$

with $L = \mathrm{dist}_2(\rho_0,\rho_1)$, which proves the ϕ-uniform convexity of $\mathcal{C}(\rho)$ on $\mathcal{P}_{2,0}(\mathbf{R}^d)$. □

Remark 5.3 (Displacement convexity without moments). *In the previous lemma, existence of second moments was used only to ensure* $\mathrm{dist}_2(\rho,\rho') < \infty$ *so the Wasserstein geodesics were uniquely defined. The displacement interpolation [35] can be used to extend this notion of* geodesic *to all of* $\mathcal{P}(\mathbf{R}^d)$*. The lemma continues to hold by the same proof in this greater generality, assuming first moments only for parts (b–c) so the center of mass is well-defined. The fact that mere convexity of* B *or of* C *implies the displacement convexity of* $\mathcal{B}(\rho)$ *or* $\mathcal{C}(\rho)$ *throughout* $\mathcal{P}(\mathbf{R}^d)$ *was already in [35, 36].*

Remark 5.4 (Semiconvexity). *Taking $\phi(s) = -ks$ in the previous lemma shows that semiconvexity of $B(x)$ and $C(x)$ on $\mathbf{R}^d$ implies displacement semiconvexity with the same constant k of the functionals $\mathcal{B}(\rho)$ and $\mathcal{C}(\rho)$ on $\mathcal{P}_2(\mathbf{R}^d)$, and not merely on $\mathcal{P}_{2,0}(\mathbf{R}^d)$. The last observation follows directly from* (5.4) *since $\chi_s(t) = -ks(1-s)t$ varies inversely with t when $k > 0$.*

5.2 Lower semicontinuity of energies

The following standard lemma is a required preparation for arguments of the next section. We will denote by $C_o(\mathbf{R}^d)$ the set of continuous with limit zero at $+\infty$ functions on $\mathbf{R}^d$ and by $C_c(\mathbf{R}^d)$ the subset of compactly supported functions in $C_o(\mathbf{R}^d)$.

Lemma 5.5 (Semiconvex integrands yield lower semicontinuous functionals). *Assumptions* **(B1-C1)** *and* (5.2) *imply lower semicontinuity of the energies* (5.1) *with respect to the metric* $\mathrm{dist}_2(\rho, \rho')$ *on* $\mathcal{P}_2^{ac}(\mathbf{R}^d)$.

Proof. Convergence in Wasserstein metric $\mathrm{dist}_2(\rho_n, \rho) \to 0$ is equivalent to weak-$*$ convergence of ρ_n in $C_o(\mathbf{R}^d)^*$ plus convergence of second moments [43, Theorem 73]:

$$\langle x^2 \rangle_\rho := \int_{\mathbf{R}^d} |x|^2 d\rho(x) = \lim_{n\to\infty} \langle x^2 \rangle_{\rho_n}. \tag{5.6}$$

Lower semicontinuity of $\mathcal{A}(\rho)$ therefore follows directly from [35, Lemma 3.4].

Turning to $\mathcal{B}(\rho)$, suppose first that $B(x)$ is convex and bounded below on $\mathbf{R}^d$, adding a constant if necessary so that $B(x) > 0$. Then $B(x)$ agrees with a lower semicontinuous function except on a set of measure zero. Although $B(x)$ does not tend to zero at infinity, it can be approximated pointwise a.e. by an increasing sequence of positive functions $B_r(x) \in C_c(\mathbf{R}^d)$ which do. Define $\mathcal{B}_r(\rho)$ analogously to $\mathcal{B}(\rho)$ but with B_r replacing B. For fixed r, $\mathcal{B}_r(\rho) = \lim_n \mathcal{B}_r(\rho_n) \leq \liminf_n \mathcal{B}(\rho_n)$ if $\rho_n \to \rho$ weak-$*$ in $\mathcal{P}_2^{ac}(\mathbf{R}^d)$. By Lebesgue's monotone convergence theorem, $\mathcal{B}_r(\rho)$ increases to $\mathcal{B}(\rho)$ as $r \to \infty$, proving the lemma for $B(x)$ convex.

If $B(x)$ is semiconvex or unbounded below, then $\tilde{B}(x) := B(x) + k|x|^2$ will be convex if k is large enough, and bounded below for k larger. The preceding argument shows lower semicontinuity of $\tilde{\mathcal{B}}(\rho) := \mathcal{B}(\rho) + k\langle x^2 \rangle_\rho$. But the difference $\tilde{\mathcal{B}}(\rho) - \mathcal{B}(\rho)$ is continuous on $(\mathcal{P}_2^{ac}(\mathbf{R}^d), \mathrm{dist}_2)$ according to (5.6), so the lower semicontinuity of $\mathcal{B}(\rho)$ is established.

The lower semicontinuity of $\mathcal{C}(\rho)$ is established in a similar way. For $C(x)$ convex this was done in [35, Lemma 3.6]. Otherwise $\tilde{C}(x) := C(x) + k|x|^2$ is convex, whence

$$\tilde{\mathcal{C}}(\rho) \;=\; \mathcal{C}(\rho) + k \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} |x - y|^2 d\rho(x) d\rho(y)$$

is lower semicontinuous on $(\mathcal{P}_2^{ac}(\mathbf{R}^d), \mathrm{dist}_2)$, and differs from $\mathcal{C}(\rho)$ by the continuous function $2k[\langle x^2 \rangle_\rho - {\langle x \rangle_\rho}^2]$. □

5.3 Subdifferentiability of energies.

In this section we first prove subdifferentiability of the energy functional $\mathcal{E}(\rho)$ in two different geodesically convex subsets of $\mathcal{P}_2^{ac}(\mathbf{R}^d)$. On one hand, we analyze a dense subset of smooth positive functions in the Riemannian length space $M = \mathcal{P}^{ac}(\Omega)$ with $\Omega \subset\subset \mathbf{R}^d$ a bounded, smooth, convex domain with outward unit normal $\nu_\Omega(x)$ at $x \in \partial\Omega$. On the other hand, we consider smooth densities on the Riemannian length space $M = \mathcal{P}_2^{ac}(\mathbf{R}^d)$ with suitable decay assumptions at $+\infty$.

The following technical lemma simplifies the subdifferentiability analysis by lifting the problem on a Riemannian length space into its tangent space.

Lemma 5.6 (Subdifferentiability test). *Let* (M, dist) *be a Riemannian length space and* $\mathcal{E} : M \longrightarrow \mathbf{R} \cup \{+\infty\}$ *lower semicontinuous and geodesically semiconvex. Fix* $x \in M$ *of finite energy* $\mathcal{E}(x) < \infty$*, the star-shaped set* $\mathcal{K}_x \subset T_x M$ *mapped by* $\exp_x$ *onto* M*, and let* $E^x : \mathrm{cone}(\mathcal{K}_x) \longrightarrow \mathbf{R} \cup \{\pm\infty\}$ *denote the positively homogeneous function of degree 1 defined by*

$$E^x(v) := \lim_{t \to 0^+} t^{-1}(\mathcal{E}(\exp_x tv) - \mathcal{E}(x)) \tag{5.7}$$

on $\mathrm{cone}(\mathcal{K}_x) := \{tv \mid t > 0, v \in \mathcal{K}_x\}$*. Then the subdifferentials* $(\underline{\partial}\mathcal{E})_x = (\underline{\partial}E^x)_0$ *coincide.*

Proof. Lower semicontinuity and semiconvexity imply $\mathcal{E}(\exp_x tv) + kt^2|v|^2$ is convex on $t \in [0, 1]$ for some $k \geq 0$ and all $v \in \mathcal{K}_x$. Thus

$$\frac{\mathcal{E}(\exp_x tv) + kt^2|v|^2 - \mathcal{E}(x)}{t} \geq \frac{\mathcal{E}(\exp_x sv) + ks^2|v|^2 - \mathcal{E}(x)}{s} \geq E^x(v) \tag{5.8}$$

for each $0 \leq s \leq t \leq 1$. Indeed, this monotonicity ensures that the limit (5.7) converges so $E^x(v)$ is well-defined. Now suppose $p \in (\underline{\partial}E^x)_0$, meaning

$$E^x(\tau w) \geq \tau\langle p, w\rangle + o(\tau) \tag{5.9}$$

for $w \in \mathcal{K}_x$ and $\tau \geq 0$ small enough. Taking $t = 1$, $v = \tau w$ in (5.8) shows

$$\mathcal{E}(\exp_x \tau w) + k\tau^2|w|^2 - \mathcal{E}(x) \geq \tau\langle p, w\rangle_x + o(\tau), \tag{5.10}$$

so $p \in (\underline{\partial}\mathcal{E})_x$. Conversely, if we begin by assuming $p \in (\underline{\partial}\mathcal{E})_x$, then (5.10) holds with $k = 0$ for all $w \in \mathcal{K}_x$ and τ small enough. The limit $\tau \to 0$ yields

$$E^x(w) \geq \langle p, w\rangle,$$

completing the proof that $p \in (\underline{\partial}E^x)_0$, while also showing the error terms to be unnecessary in (5.9) and hence, in (5.10). □

We use the previous lemma to study the subdifferentiability of each of the three terms in our energy functional.

5.3.1 Subdifferentiability of energies in a bounded domain.

Let $W^{1,2}_\rho(\Omega)$ be the space $W^{1,2}(\Omega, d\rho)$. Lemmas 5.7-5.9 show in more suggestive notation that the variational derivative $\delta\mathcal{E}/\delta\rho \in W^{1,2}_\rho(\Omega)$ given by

$$\frac{\delta\mathcal{E}}{\delta\rho}(\rho(x)) = A'(\rho(x)) + B(x) + (\rho * C)(x) \tag{5.11}$$

is a subgradient $\delta\mathcal{E}/\delta\rho \in \underline{\partial}\mathcal{E}_\rho$ at any $\rho \in \mathcal{P}^{ac}(\Omega)$ under the corresponding smoothness assumptions.

Lemma 5.7 (Entropy subgradient). *Let* $(M, \mathrm{dist}) = (\mathcal{P}^{ac}(\Omega), \mathrm{dist}_2)$*, with* $\Omega \subset\subset \mathbf{R}^d$ *smooth and convex. Fix* $0 < \rho(x) \in C^1(\overline{\Omega})$ *and* $A \in C^2(0,\infty)$ *satisfying* (5.2)*. Then* $\varphi(x) := A'(\rho(x)) \in W^{1,2}_\rho(\Omega)$ *is a subgradient of the entropy (5.1):* $\varphi \in \underline{\partial}\mathcal{A}_\rho \subset T_\rho M$*.*

Proof. We always assume A is lower semicontinuous and satisfies $A(0) = 0$. Convexity of $A : [0,\infty) \longrightarrow \mathbf{R}$ then follows from (5.2). The functional $\mathcal{E}(\rho) := \mathcal{A}(\rho)$ is displacement convex and lower semicontinuous on $\mathcal{P}^{ac}_2(\mathbf{R}^d)$ by Theorem 5.1 and Lemma 5.5. Thus it suffices to show $\varphi \in (\underline{\partial}E^\rho)_0$, according to Lemma 5.6. Let us therefore compute the directional derivative $E^\rho(\psi)$ of the entropy (5.7) in some arbitrary direction $\psi \in \mathcal{K}_\rho \subset W^{1,2}(\mathbf{R}^d, d\rho)$. Recall from section §4 that $\exp_\rho s\psi := [(1-s)Id + s\nabla\Psi]_\# \rho$, where $\Psi(x) = \psi(x) + |x|^2/2$ is a convex function on $\mathbf{R}^d$ and $\exp_\rho \psi \in \mathcal{P}^{ac}(\Omega)$. By [27, Theorem 1.1], it costs no generality to assume $\nabla\Psi(x) \in \mathrm{spt}\,[\exp_\rho \psi] \subset \overline{\Omega}$ a.e. on $\mathbf{R}^d$. Since Ω is convex, this implies

$$\frac{\partial\psi}{\partial\nu}(x) \leq 0 \qquad \forall x \in \partial\Omega, \tag{5.12}$$

where $\partial\psi/\partial\nu$ denotes the outward normal derivative of ψ as computed from *inside* the domain Ω. Convexity of Ψ implies $\partial\psi/\partial\nu$ exists and gives appropriate boundary terms when integrating by parts, even if $\nabla\psi(x)$ is not defined; obviously $\partial\psi(x)/\partial\nu = \nabla\psi(x) \cdot \nu_\Omega(x)$ if the latter exists.

Let $X \subset \Omega$ denote the set where $\Psi(x)$ can be approximated to second order by a quadratic polynomial; this set has full measure according to Aleksandrov's theorem. Define $v_s(x) = \det[(1-s)I + sD^2\Psi(x)]$ at $x \in X$. For each $s < 1$ the monotone change of variables theorem [35, Theorem 4.4] yields

$$\mathcal{E}(\exp_\rho s\psi) = \int_{X\subset\Omega} A\left(\frac{\rho(x)}{v_s(x)}\right) v_s(x) dx. \tag{5.13}$$

We shall shortly justify interchange of the integral

$$E^\rho(\psi) = \lim_{s\to 0} \int_X \frac{A\left(\rho(x)/v_s(x)\right) v_s(x) - A(\rho(x))}{s} dx \tag{5.14}$$

with the limit

$$\lim_{s\to 0} \frac{A\left(\rho/v_s\right) v_s - A(\rho)}{s} \quad = \quad [A(\rho) - \rho A'(\rho)] \frac{\partial v_s}{\partial s}\bigg|_{s=0}$$

$$= \quad -P(\rho)\frac{\partial v_s}{\partial s}\bigg|_{s=0}.$$

Let us first assume the legitimacy of this interchange, to complete the proof. Note that $\partial v_s(x)/\partial s|_{s=0} = \operatorname{tr} D^2\psi(x)$ for each $x \in X$. Now the convexity of $A(\varrho)$ with $A(0) = 0$ yield $A(\varrho) \le \varrho A'(\varrho)$, and convexity of Ψ implies the distributional Laplacian $\Delta\Psi$ is a non-negative Radon measure on Ω with $\operatorname{tr} D^2\Psi|_X$ as its absolutely continuous part. Thus

$$\begin{aligned}
E^\rho(\psi) &= -\int_X P(\rho)\operatorname{tr} D^2\psi\, dx \\
&\ge -\int_\Omega P(\rho)\Delta\psi\, dx \\
&= \int_\Omega \langle \nabla P(\rho), \nabla\psi\rangle\, dx - \int_{\partial\Omega} P(\rho)\frac{\partial\psi}{\partial\nu}\, d\mathcal{H}^{d-1}(x) \\
&\ge \int_\Omega \langle \nabla A'(\rho(x)), \nabla\psi(x)\rangle\, \rho(x) dx \\
&=: \langle\varphi, \psi\rangle_\rho.
\end{aligned}$$

Here the last inequality follows from $P(\rho) \ge 0$, (5.12), and the identity $P'(\rho) = \rho A''(\rho)$.

Finally, let us justify the exchange of the integral with the limit in (5.14). As in the proof of [35, Theorem 2.2], hypothesis (5.2) implies the integrand of (5.13) is convex as a function of $s \in [0,1]$. It follows that the integrand in (5.14) is non-decreasing on $s \in [0,1]$. In particular, $0 < s < 1/2$ yields

$$\begin{aligned}
-P(\rho)\operatorname{tr} D^2\psi &= [A(\rho) - \rho A'(\rho)][(\operatorname{tr} D^2\Psi) - d] \\
&\le \frac{A(\rho/v_s)\, v_s - A(\rho)}{s} \\
&\le 2A\left(\rho/v_{1/2}\right) v_{1/2} - 2A(\rho) \\
&\le 2A\left(2^d\rho\right)/2^d - 2A(\rho)
\end{aligned}$$

for all $x \in X$, and thus,

$$-P(\rho)\operatorname{tr} D^2\psi \le \frac{A(\rho/v_s)\, v_s - A(\rho)}{s} \le 2A\left(2^d\rho\right)/2^d - 2A(\rho), \tag{5.15}$$

for all $x \in X$. Since $A(\rho(x))$ and $P(\rho(x))$ are $C^1(\overline{\Omega})$ on a compact domain, $\operatorname{tr} D^2\psi \ge -d$, and

$$\begin{aligned}
\int_X \operatorname{tr} D^2\psi\, dx &\le \int_\Omega \Delta\psi\, dx \\
&= \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\, d\mathcal{H}^{d-1}(x) \\
&< \mathcal{H}^{d-1}(\partial\Omega) \sup_{x,y\in\Omega} |x-y|,
\end{aligned}$$

we have $L^1(\Omega)$ bounds above and below throughout $X \subset \Omega$. Lebesgue's dominated convergence theorem completes the justification. □

Let us verify the same result for the other two terms of the energy functional.

Lemma 5.8 (Friction subgradient). *Given $\rho \in \mathcal{P}^{ac}(\Omega)$ on $\Omega \subset\subset \mathbf{R}^d$ and $B : \mathbf{R}^d \longrightarrow \mathbf{R}$ semiconvex, $\varphi := B \in W^{1,2}_\rho(\Omega)$ is a subgradient of the potential energy: $\varphi \in \underline{\partial}\mathcal{B}_\rho \subset T_\rho M$.*

Proof. Semiconvexity of the integrand B implies lower semicontinuity and displacement semiconvexity of the functional $\mathcal{E}(\rho) := \mathcal{B}(\rho)$ on $\mathcal{P}^{ac}_2(\mathbf{R}^d)$, by Remark 5.4 and Lemma 5.5. Thus it suffices to show $\varphi \in (\underline{\partial}E^\rho)_0$, according to Lemma 5.6. Semiconvexity also implies B is Lipschitz since is locally finite, so both the function and its derivative are uniformly bounded on the bounded domain Ω. A straightforward application of Lebesgue's dominated convergence theorem proves that

$$\begin{aligned}
E^\rho(\psi) &= \lim_{s\to 0}\int_\Omega \frac{B(x+s\nabla\psi(x)) - B(x)}{s}\, d\rho(x) \\
&= \int_\Omega \lim_{s\to 0}\left\{\frac{B(x+s\nabla\psi(x)) - B(x)}{s}\right\} d\rho(x) \\
&= \int_\Omega \langle \nabla B(x), \nabla\psi\rangle\, d\rho(x) \\
&=: \langle \varphi, \psi\rangle_\rho .
\end{aligned}$$

Thus $\varphi := B \in (\underline{\partial}E^\rho)_0$. □

Lemma 5.9 (Collision subgradient). *Given $\rho \in \mathcal{P}^{ac}(\Omega)$ on $\Omega \subset\subset \mathbf{R}^d$ and $C : \mathbf{R}^d \longrightarrow \mathbf{R}$ semiconvex, $\varphi := \rho * C \in W^{1,2}_\rho(\Omega)$ is a subgradient of the interaction energy: $\varphi \in \underline{\partial}\mathcal{C}_\rho \subset T_\rho M$.*

Proof. Semiconvexity of the integrand C implies lower semicontinuity and displacement semiconvexity of the functional $\mathcal{E}(\rho) := \mathcal{C}(\rho)$ on $\mathcal{P}^{ac}_2(\mathbf{R}^d)$, by Remark 5.4 and Lemma 5.5. Thus it suffices to show $\varphi \in (\underline{\partial}E^\rho)_0$, according to Lemma 5.6. Again C is locally Lipschitz, so its derivative is uniformly bounded in the domain $\Omega \subset\subset \mathbf{R}^d$, and it is straightforward to prove using Lebesgue's dominated convergence theorem that $\varphi := \rho * C \in W^{1,\infty}(\Omega)$ with $\nabla\varphi = \rho * \nabla C$ and

$$\begin{aligned}
E^\rho(\psi) &= \lim_{s\to 0}\frac{1}{2}\int_{\Omega\times\Omega} \frac{C\left[x-y+s(\nabla\psi(x)-\nabla\psi(y))\right] - C(x-y)}{s}\, d\rho(x)\, d\rho(y) \\
&= \frac{1}{2}\int_{\Omega\times\Omega} \lim_{s\to 0}\left\{\frac{C\left[x-y+s(\nabla\psi(x)-\nabla\psi(y))\right] - C(x-y)}{s}\right\} d\rho(x)\, d\rho(y) \\
&= \int_\Omega \langle \nabla C(x-y), \nabla\psi(x)\rangle\, d\rho(x)\, d\rho(y) \\
&=: \langle \varphi, \psi\rangle_\rho .
\end{aligned}$$

□

5.3.2 Subdifferentiability of energies in $\mathbf{R}^d$.

The treatment of subdifferentiability for the whole space problem poses new challenges due to the need of control for the densities and tangent vectors behavior at $+\infty$.

Lemmas 5.8 and 5.9 are easily generalized to $\mathbf{R}^d$ provided we restrict to a smaller suitable geodesically convex subset of $\mathcal{P}_2^{ac}(\mathbf{R}^d)$.

In fact, let us change the hypothesis on the confinement and interaction potentials:

$$\begin{array}{ll}(\mathbf{B2}) & B : \mathbf{R}^d \longrightarrow \mathbf{R} \text{ is semiconvex on } \mathbf{R}^d \text{ and } |\nabla B|^2 \text{ convex on } \mathbf{R}^d; \\ (\mathbf{C2}) & C : \mathbf{R}^d \longrightarrow \mathbf{R} \text{ is semiconvex on } \mathbf{R}^d \text{ and } |\nabla C|^2 \text{ convex on } \mathbf{R}^d.\end{array} \tag{5.16}$$

Let us first note that mere convexity of $|\nabla B|^2$ and $|\nabla C|^2$ implies that the functionals

$$\tilde{\mathcal{B}}(\rho) := \int_{\mathbf{R}^d} |\nabla B(x)|^2 d\rho(x)$$

and

$$\tilde{\mathcal{C}}(\rho) := \int_{\mathbf{R}^d \times \mathbf{R}^d} |\nabla C(x-y)|^2 d\rho(x) d\rho(y),$$

are displacement convex defined on $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ (Lemma 5.2). Therefore, the set

$$M' := \{\rho \in \mathcal{P}_2^{ac}(\mathbf{R}^d) \mid \tilde{\mathcal{B}}(\rho) < +\infty \;\text{ and }\; \tilde{\mathcal{C}}(\rho) < +\infty\}$$

is a geodesically convex subset of $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ and hence, it inherits the structure of length space and the star-shaped set $\mathcal{K}_\rho$ of the tangent space is restricted to those tangent vectors joining measures in M' and therefore, lying inside the subset M'.

Lemma 5.10 (Friction subgradient in $\mathbf{R}^d$). *Given $\rho \in M'$, $\varphi := B$ is a subgradient of the potential energy: $\varphi \in \underline{\partial}\mathcal{B}_\rho \subset T_\rho M'$.*

Proof. Following the same proof as in Lemma 5.8, we need just to justify the application of Lebesgue's dominated convergence theorem to interchange the limit $s \to 0$ and the integral over $\mathbf{R}^d$. Since $B \in C^1$, we can estimate the integrand as follows

$$\begin{aligned}\left|\frac{B(x+s\nabla\psi(x)) - B(x)}{s}\right| &= |\langle \nabla B(x+s(x)\nabla\psi(x)), \nabla\psi(x)\rangle| \leq \\ |\nabla B(x+s(x)\nabla\psi(x))||\nabla\psi(x)| &\leq \max\{|\nabla B(x)|, |\nabla B(x+\nabla\psi(x))|\}|\nabla\psi(x)|\end{aligned}$$

by convexity of $|\nabla B(x)|^2$ where $s(x) \in (0,1)$ possibly depending on x. Since both ends of the geodesic lie on M' then the right-hand side of the inequality is integrable with respect to ρ and thus, we have $L^1(\mathbf{R}^d)$ control uniformly in s. □

The generalization of the subdifferentiability of the collision functional follows the same lines of the previous lemma, so we omit the proof.

Lemma 5.11 (Collision subgradient in $\mathbf{R}^d$). *Given $\rho \in M'$, $\varphi := \rho * C$ is a subgradient of the interaction energy: $\varphi \in \underline{\partial}\mathcal{C}_\rho \subset T_\rho M'$.*

Regarding the subdifferentiability of the entropy functional we will assume additional smoothness hypotheses on the density $\rho \in M'$. Assume

(A2) $A \in C^2(0,\infty) \cap C[0,\infty)$ is strictly convex, satisfies $A(0) = 0$ and

$$P(\varrho) \geq 0 \quad \text{and} \quad \frac{P(\varrho)}{\varrho^{1/2}} \qquad \text{is integrable in } (0,1). \tag{5.17}$$

It follows that $A''(\varrho) > 0$ on $(0,\infty)$. Thus the primitives $P(\varrho)$ and $Q(\varrho)$ of the differential equations $P'(\varrho) = \varrho A''(\varrho)$ and $Q'(\varrho) = \varrho^{1/2}A''(\varrho)$ define diffeomorphisms on $(0,\infty)$. Moreover, assumption (5.17) allows us to normalize, so that $P(0) = Q(0) = 0$. Indeed $P(\varrho) := \varrho A'(\varrho) - A(\varrho)$ and

$$Q(\varrho) := \int_0^\varrho s^{1/2}A''(s)ds = \int_0^\varrho \frac{P'(s)}{s^{1/2}}ds, \tag{5.18}$$

where the last integral converges due to (5.17). For example if $m > 1/2$ then

$$A(\varrho) = (\varrho^m - \varrho)/(m-1), \qquad P(\varrho) = \varrho^m, \qquad Q(\varrho) := \frac{2m}{2m-1}\rho^{(2m-1)/2}.$$

Lemma 5.12 (Integration by parts in the whole space). *Fix $\rho \in \mathcal{P}_2^{ac}(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$ locally Lipschitz on $\{\rho > 0\}$, with the boundary of $\{\rho > 0\}$ having zero volume. Assume $\varphi, \psi \in W^{1,2}_\rho(\mathbf{R}^d)$ where $\Psi(x) = \psi(x) + |x|^2/2$ is a convex function finite on $\{\rho > 0\}$, and $\varphi(x) := A'(\rho(x))$. Moreover, if $d = 1$ we assume that $\rho \in L^\infty(\mathbf{R}^d)$. If A satisfies* **(A2)** *and $D^2\psi(x)$ denotes the Hessian of ψ in the a.e. sense of Aleksandrov, then*

$$\int_{\mathbf{R}^d} P(\rho(x)) \mathrm{tr}\, D^2\psi(x)\, dx \leq -\int_{\mathbf{R}^d} \langle \nabla P(\rho(x)), \nabla\psi(x)\rangle\, dx.$$

Proof. The argument (cf. [24]) hinges on being able to express the second integrand $\langle \nabla P(\rho), \nabla\psi\rangle = \langle \nabla Q(\rho), \rho^{1/2}\nabla\psi\rangle$ as the inner product of the two $L^2(\mathbf{R}^d; \mathbf{R}^d)$ vector fields $\nabla Q(\rho) = \rho^{1/2}\nabla A'(\rho)$ and $\rho^{1/2}\nabla\psi$ where Q is defined by (5.18). This is made possible by our assumptions on ρ, ψ and the differentiability of P and Q on $(0,\infty)$.

From the Sobolev inequality $W^{1,2}(\mathbf{R}^d)$ embeds continuously in $L^{2^*}(\mathbf{R}^d)$, where $2^* = 2d/(d-2)$ for $d \geq 3$. The Moser-Trudinger inequality embeds $W^{1,2}(\mathbf{R}^d)$ continuously onto $L^p(\mathbf{R}^d)$ for any $p \in [2,\infty)$ with $d = 2$. Finally, since Q is continuous and non decreasing with $Q(0) = 0$ and $\rho \in L^\infty(\mathbf{R}^d)$ for $d = 1$ we get $Q(\rho) \in L^\infty(\mathbf{R}^d)$. Thus $f := Q(\rho) \in L^{2^*}(\mathbf{R}^d)$, $d \geq 3$, $f := Q(\rho) \in L^p(\mathbf{R}^d)$, $d = 2$ for any $p \in [2,\infty)$ and $f := Q(\rho) \in L^\infty(\mathbf{R}^d)$, $d = 1$. We shall truncate and mollify $f \geq 0$ as follows.

Note that f vanishes outside the convex domain $\Omega := \mathrm{int}\{x \mid \Psi(x) < \infty\}$, and translate the whole problem without loss of generality so that $\mathbf{0} \in \Omega$. Truncate f to have compact support inside Ω by setting

$$f_\epsilon(x) := \min\{f(\frac{x}{1-\epsilon}), \chi(\epsilon x) f(x)\}$$

where $\chi \in C_c^\infty(\mathbf{R}^d)$ is a bump function supported in the unit ball being unity on $B_{1/2}(0)$.

Clearly, the sequence f_ϵ converges pointwise to f. Now, we prove that the sequence f_ϵ is bounded uniformly in ϵ in $\dot W^{1,2}(\mathbf{R}^d)$.

Take first $d \geq 3$, then $f_\epsilon \leq f \in L^{2^*}(\mathbf{R}^d)$ converges pointwise hence strongly to f in $L^{2^*}(\mathbf{R}^d)$ by the dominated convergence theorem. We also claim $\chi(\epsilon x) f(x) \to f$ strongly in $\dot W^{1,2}(\mathbf{R}^d)$ as $\epsilon \to 0$. Indeed,

$$\begin{aligned}
\|\nabla(f[1-\chi(\epsilon x)])\|_{L^2}^2 &\leq 2\int_{\frac{1}{2\epsilon}<|x|} |\nabla f|^2\,dx + 2\epsilon^2 \int_{\frac{1}{2\epsilon}<|x|<\frac{1}{\epsilon}} f^2 |(\nabla\chi)(\epsilon x)|^2\,dx \\
&\leq 2\int_{\frac{1}{2\epsilon}<|x|} |\nabla f|^2\,dx + 2\|\nabla\chi\|_{L^d(\mathbf{R}^d)}^2 \left(\int_{\frac{1}{2\epsilon}<|x|<\frac{1}{\epsilon}} f^{2^*}\right)^{2/2^*} \to 0
\end{aligned}$$

with ϵ. For $d=2$, one can repeat the above argument using now, any $p \in (2,\infty)$ to get

$$\|\nabla(f[1-\chi(\epsilon x)])\|_{L^2}^2 \leq 2\int_{\frac{1}{2\epsilon}<|x|} |\nabla f|^2\,dx + 2\epsilon^{4/p}\|\nabla\chi\|_{L^{2p/p-2}(\mathbf{R}^d)}^2 \left(\int_{\frac{1}{2\epsilon}<|x|<\frac{1}{\epsilon}} f^{p}\right)^{2/p} \to 0$$

with ϵ. Finally, for $d=1$ we have

$$\|\nabla(f[1-\chi(\epsilon x)])\|_{L^2}^2 \leq 2\int_{\frac{1}{2\epsilon}<|x|} |\nabla f|^2\,dx + 2\epsilon\|\nabla\chi\|_{L^2(\mathbf{R}^d)}^2 \|f\|_{L^\infty(\mathbf{R}^d)}^2 \to 0$$

with ϵ.

Since $f(x/(1-\epsilon))$ also belongs to $\dot W^{1,2}(\mathbf{R}^d)$, the formula $\min\{f,g\} = (f+g)/2 - |f-g|/2$ shows f_ϵ remains bounded in $\dot W^{1,2}(\mathbf{R}^d)$ as $\epsilon \to 0$. We conclude $\nabla f_\epsilon \uparrow \nabla f$ weakly in $L^2(\mathbf{R}^d;\mathbf{R}^d)$ for a subsequence, that we denote with the same index, as we shall need later on in the proof.

Smooth $f_\epsilon^\delta = \eta_\delta * f_\epsilon$ by convolving with a standard mollifier $\eta_\delta = c\chi(x/\delta)/\delta^d$ normalized to have unit mass. For $\delta > 0$ small enough, $f_\epsilon^\delta \in C^\infty(\mathbf{R}^d)$ continues to be supported inside a neighbourhood Ω_ϵ of $\overline{\{f_\epsilon > 0\}}$ whose closure is compactly contained in Ω.

Once we have smoothed positive densities, we compose f_ϵ^δ with the diffeomorphism $P \circ Q^{-1}$ yielding a $C^1(\mathbf{R}^d)$ function, because — and this is crucial — $d(P\circ Q^{-1})/ds = \sqrt{Q^{-1}(s)} \to 0$ remains bounded as $s \to 0$. Since $0 \leq \mathrm{tr}\, D^2\Psi \leq \Delta\Psi$ where Δ denotes the distributional Laplacian, we have

$$\begin{aligned}
\int_{\Omega_\epsilon} (P\circ Q^{-1}\circ f_\epsilon^\delta)\,\mathrm{tr}\, D^2\Psi\,dx &\leq \int_{\Omega_\epsilon} (P\circ Q^{-1}\circ f_\epsilon^\delta)\,\Delta\Psi\,dx \\
&= -\int_{\Omega_\epsilon} \langle \nabla(P\circ Q^{-1}\circ f_\epsilon^\delta), \nabla\Psi\rangle\,dx \\
&= -\int_{\Omega_\epsilon} \langle \nabla f_\epsilon^\delta, \sqrt{Q^{-1}\circ f_\epsilon^\delta}\,\nabla\Psi\rangle\,dx. \qquad (5.19)
\end{aligned}$$

Now $\nabla\psi$ and $Q^{-1}\circ f_\epsilon^\delta \leq \|\rho\|_{L^\infty(\Omega_\epsilon)}$ are both bounded on the domain $\Omega_\epsilon \subset\subset \Omega$, since Ψ is locally Lipschitz on Ω. Moreover, $f_\epsilon^\delta \to f_\epsilon$ pointwise a.e. and strongly in

$\dot{W}^{1,2}(\mathbf{R}^d)$ as $\delta \to 0$, e.g. by adapting [26, §4.2, Theorem 1] to the homogeneous Sobolev space $\dot{W}^{1,2}(\mathbf{R}^d)$. Thus $\sqrt{Q^{-1} \circ f_\epsilon^\delta}\ \nabla\Psi \to \sqrt{Q^{-1} \circ f_\epsilon}\ \nabla\Psi$ in $L^2(\mathbf{R}^d; \mathbf{R}^d)$ by Lebesgue's dominated convergence theorem, and we have no trouble taking the limit $\delta \to 0$ in the final expression of (5.19). In the initial expression, pointwise a.e. convergence of $P \circ Q^{-1} \circ f_\epsilon^\delta \to P \circ Q^{-1} \circ f_\epsilon$ and Fatou's lemma yield

$$\int_{\mathbf{R}^d} P \circ Q^{-1} \circ f_\epsilon \operatorname{tr} D^2\Psi \, dx \le -\int_{\mathbf{R}^d} \langle \nabla f_\epsilon, \sqrt{Q^{-1} \circ f_\epsilon}\ \nabla\Psi\rangle \, dx. \tag{5.20}$$

Recall that $f_\epsilon \le f$ converges to $f = Q(\rho)$ pointwise a.e. and weakly in $\dot{W}^{1,2}(\mathbf{R}^d)$. Applied to $\sqrt{Q^{-1} \circ f_\epsilon}\ |\nabla\Psi| \le \sqrt{\rho}\ |\nabla\psi + x| \in L^2(\mathbf{R}^d)$, the dominated convergence theorem yields $L^2(\mathbf{R}^d)$ convergence of $\sqrt{Q^{-1} \circ f_\epsilon}\ \nabla\Psi \to \sqrt{\rho}\ \nabla\Psi$. Thus the right expression in (5.20) converges as $\epsilon \to 0$, and applying Fatou's lemma to the left side gives

$$\int_{\mathbf{R}^d} P(\rho) \operatorname{tr} D^2\Psi \, dx \ \ \le \ \ -\int_{\mathbf{R}^d} \langle \nabla f, \sqrt{\rho}\ \nabla\Psi\rangle \, dx \ < \ +\infty \tag{5.21}$$

Recalling $\nabla P(\rho) = \sqrt{\rho}\ \nabla f$ a.e. completes the proof of the lemma with the convex function Ψ in place of the semiconvex function ψ.

To produce the same inequality for $\psi = \Psi - |x|^2/2$, it is enough to check that (5.21) becomes an equality when $\Psi(x) = |x|^2/2$. In this case there is no difference between the Aleksandrov and distributional Laplacians $\operatorname{tr} D^2\Psi(x) = \Delta\Psi(x) = d$. The only possible inequalities arise from Fatou's lemma in the regularization $\delta \to 0$ and truncation $\epsilon \to 0$ limits. We must therefore show the extra information $\Delta\Psi(x) = d$ allows these limits to be taken using the dominated convergence theorem instead. Since the limit $\delta \to 0$ happens on a bounded set $\Omega_\epsilon \subset\subset \Omega$, and the integrands $P \circ Q^{-1} \circ f_\epsilon^\delta$ are bounded uniformly by $P(\|\rho\|_{L^\infty(\Omega_\epsilon)})$, we conclude that equality holds in (5.20) with $\Psi(x) = |x|^2/2$. Furthermore, since $0 \le P \circ Q^{-1} \circ f_\epsilon \le P(\rho) \in L^1(\mathbf{R}^d)$ from the special case $\Psi(x) = |x|^2/2$ of (5.21), the dominated convergence theorem yields equality in (5.21) as the $\epsilon \to 0$ limit of equality (5.20). This concludes the proof of the lemma. □

Let us point out that the main technical difficulty in previous lemma comes from the fact that the density might vanish at certain points. If the density is positive everywhere we have the following easier answer.

Lemma 5.13 (Integration by parts in the whole space for positive densities). *Fix $\rho \in \mathcal{P}_2^{ac}(\mathbf{R}^d) \cap C^\infty(\mathbf{R}^d)$ with $\rho > 0$ on $\mathbf{R}^d$. Assume $\varphi, \psi \in W_\rho^{1,2}(\mathbf{R}^d)$ where $\Psi(x) = \psi(x) + |x|^2/2$ is a finite convex function and $\varphi(x) := A'(\rho(x))$. Moreover, assume that $P(\rho)^2\rho^{-1} \in L^1(\mathbf{R}^d)$. If A satisfies* **(A2)** *and $D^2\psi(x)$ denotes the Hessian of ψ in the a.e. sense of Aleksandrov, then*

$$\int_{\mathbf{R}^d} P(\rho(x)) \operatorname{tr} D^2\psi(x) \, dx \le -\int_{\mathbf{R}^d} \langle \nabla P(\rho(x)), \nabla\psi(x)\rangle \, dx.$$

Proof. Take the sequence of bump functions $\chi_n(x) = \chi(x/n)$, $n \in N$, with $\chi(x)$ defined as in previous lemma. Thus, we have $\nabla\chi_n(x)$ bounded uniformly in n in $L^\infty(\mathbf{R}^d)$ and being unity on $B_{n/2}(0)$ with support inside $B_n(0)$. Since $\rho > 0$ and smooth then $P(\rho)$ does too, and then we can use that $0 \le \mathrm{tr}\, D^2\Psi \le \Delta\Psi$ where Δ denotes the distributional Laplacian, and thus we have

$$\begin{aligned}\int_{\mathbf{R}^d} P(\rho)\chi_n \,\mathrm{tr}\, D^2\Psi \,dx &\le \int_{\mathbf{R}^d} P(\rho)\chi_n \,\Delta\Psi \,dx \\ &= -\int_{\mathbf{R}^d} \chi_n \,\langle \nabla P(\rho), \nabla\Psi\rangle \,dx \\ &\quad -\int_{\mathbf{R}^d} P(\rho)\,\langle \nabla\chi_n, \nabla\Psi\rangle \,dx.\end{aligned}$$

Since by hypotheses we have that $\varphi, \Psi \in W^{1,2}_\rho(\mathbf{R}^d)$ and we can rewrite the first term of the right-hand side as $\langle \nabla P(\rho), \nabla\Psi\rangle = \langle \nabla Q(\rho), \rho^{1/2}\nabla\Psi\rangle = \langle \rho^{1/2}\nabla A'(\rho), \rho^{1/2}\nabla\Psi\rangle$, then the dominated convergence theorem proves that

$$\int_{\mathbf{R}^d} \chi_n \,\langle \nabla P(\rho), \nabla\Psi\rangle \,dx \to \int_{\mathbf{R}^d} \langle \nabla P(\rho), \nabla\Psi\rangle \,dx$$

when $n \to \infty$. To pass to the limit in the second term, we first notice that $\nabla\chi_n(x)$ is bounded uniformly in n in $L^\infty(\mathbf{R}^d)$ and converges pointwise to zero when $n \to \infty$. Moreover, by the assumptions and Hölder's inequality $|\nabla\Psi|P(\rho) \in L^1(\mathbf{R}^d)$, and thus, the second term vanishes when $n \to \infty$ by Lebesgue dominated convergence theorem. Application of Fatou's lemma finally results in the desired inequality for Ψ. Then, we get the same inequality for ψ by analogous arguments to the previous lemma. □

Finally, the proof of Lemma 5.7 can be applied without any change to obtain the same conclusion based on the integration by parts inequality proved in the last two lemmas 5.12-5.13.

Lemma 5.14 (Entropy subgradient in $\mathbf{R}^d$). *Assume A satisfies* (A2) *and $\rho \in M'$ satisfies the additional hipotheses of either Lemma 5.12 or Lemma 5.13. Moreover we assume that $A(2^d\rho) \in L^1(\mathbf{R}^d)$. Then $\varphi(x) := A'(\rho(x)) \in W^{1,2}_\rho(\Omega)$ is a subgradient of the entropy (5.1): $\varphi \in \underline{\partial}\mathcal{A}_\rho \subset T_\rho M'$.*

Let us remark that the additional assumption $A(2^d\rho) \in L^1(\mathbf{R}^d)$ is needed for the L^1 bound from above of the integral in Lemma 5.7 and trivially satisfied in homogeneous cases.

Therefore, in a more suggestive notation, we have shown that the variational derivative $\delta\mathcal{E}/\delta\rho \in W^{1,2}_\rho$ given by $\frac{\delta\mathcal{E}}{\delta\rho}(\rho(x)) = A'(\rho(x)) + B(x) + (\rho * C)(x)$ is a subgradient $\delta\mathcal{E}/\delta\rho \in \underline{\partial}\mathcal{E}_\rho$ at any $\rho \in M'$ with the additional smoothness assumptions of Lemmas 5.10-5.14.

6 Application to granular media

Having identified conditions which guarantee the desired convexity and differentiability properties of the functional $E(\rho)$ on our Riemannian length space $\mathcal{P}^{ac}(\Omega)$ or $\mathcal{P}_2^{ac}(\mathbf{R}^d)$, we can finally apply them to the nonlinear nonlocal evolution equations described in the introduction.

Let us consider the Cauchy problem for (1.2), i.e.,

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\rho \nabla \left(A' \left(\frac{d\rho}{dx} \right) + B + C * \rho \right) \right],$$

with an initial data $\rho(t=0,x) = \rho_o(x) \geq 0$ satisfying $(1+|x|^2)\rho_o \in L^1(\mathbf{R^d})$. We refer to Appendix A of [19] for details on the existence theory for this Cauchy problem in the different cases: linear diffusion, nonlinear diffusion, no diffusion.

In order to apply Theorem 3.12 to this Cauchy problem we will proceed in two different ways: bounded domain approximation (subsections 6.1-6.3) or approximation by smoother solutions in the whole $\mathbf{R}^d$ (subsection 6.4).

On one hand, we can approximate this problem by smoother problems in bounded domains. Then we will demonstrate that solution curves $\rho(t)$ in this smooth setting are differentiable in $\mathcal{P}^{ac}(\Omega)$ under the structure introduced in sections §3–§4, and moreover they are subgradient flows for the energy functional $\mathcal{E}(\rho)$ of §5. Comparing (4.12) with (5.11), we see that for this subgradient flow it is crucial to impose no-flux boundary conditions (1.3) on a bounded domain $\Omega \subset\subset \mathbf{R}^d$. Then we can apply Theorem 3.12 in the smooth setting. Finally, the global contraction and decay estimate for the Wasserstein L^2-distance will be obtained for (1.2) by a limiting procedure.

However, this procedure encounters a problem when one has to fix the center of mass of the solution (see discussion below), therefore we show how to perform a different approximation by smooth positive fast decaying solutions in the whole $\mathbf{R}^d$ in the most interesting case from the applications point of view, i.e., the linear diffusion case. We prove rigourously our decay estimate of the difference between two solutions in Wasserstein sense in $\mathcal{P}_2^{ac}(\mathbf{R}^d)$ for a particular case: no confining potential, general ϕ-convex interaction potential and linear diffusion.

6.1 Smooth approximations on bounded velocity domains

As shown in Appendix A [19] (under suitable additional assumptions) one can approximate weak solutions of the Cauchy problem (1.2) by sequences of regularized problems in bounded domains that have smooth solutions bounded away from zero and infinity yet for which the equation retains the relevant aspects of its structure. We refer also to [38, 18] for similar approximation procedures in particular situations of the Cauchy problem (1.2). In fact, one can construct a sequence of smooth positive functions ρ^n, which are solutions of regularized equations of the form

$$\frac{\partial \rho^n}{\partial t} = \nabla \cdot \left[\rho^n \nabla \left(A_n'(\rho^n) + B_n + C_n * \rho^n \right) \right], \tag{6.1}$$

in bounded smooth convex domains $\Omega_n \subset \mathbf{R}^d$ with no flux boundary conditions

$$\rho^n \nabla \left(A_n'(\rho^n) + B_n + C_n * \rho^n \right) \cdot \nu_{\Omega_n} = 0 \quad \text{on } \partial\Omega.$$

The solutions to this regularized sequence of problems $\hat{\rho}^n$ converges towards a solution ρ of equation (1.2) at least verifying

$$\hat{\rho}^n(t) \to \rho(t) \quad \text{weakly in } L^1(\mathbf{R}^d) \text{ a.e. } t > 0. \tag{6.2}$$

Moreover the sequence of regularized initial data satisfies

$$(1 + |x|^2)\hat{\rho}_o^n \to (1 + |x|^2)\rho_o \quad \text{strongly in } L^1(\mathbf{R}^d). \tag{6.3}$$

Here, $\hat{\rho}$ refers to the extension of the function ρ to $\mathbf{R}^d$ by 0. Moreover, one can keep the convexity properties of the potentials B and C in the following sense: B_n and C_n are respectively ϕ_n^B- and ϕ_n^C-convex potentials such that ϕ_n^B and ϕ_n^C are converging uniformly on compact subsets of of $\mathbf{R}_\mathbf{o}^+$ to the respective moduli ϕ^B and ϕ^C of convexity of the potentials B and C.

In principle, the regularized solutions are not probability densities since their mass will be equal to the mass of the regularized sequence of initial data. Therefore, both regularized solutions are not directly comparable. However, we can normalize both regularized initial data by their masses and consider the new corresponding regularized solutions for these normalized initial data. It is easy to check in [19, 38, 18] that the approximation procedure works for these new solutions keeping the same convergence properties as stated above. Therefore, we assume without loss of generality that the approximated sequences also satisfy

$$\|\hat{\rho}_o^n\|_{L^1(\mathbf{R}^d)} = \|\hat{\rho}^n(t)\|_{L^1(\mathbf{R}^d)} = 1. \tag{6.4}$$

for any $t > 0$.

In the following, we will assume that we are given a smooth approximation sequence of problems $\{P_n\}$ in an expanding sequence of bounded convex domains with no flux boundary conditions and convergence properties (6.2–6.4). We refer to appendix A of [19], [38] and [18] for sufficient conditions on the potentials and initial data to perform this approximation procedure.

6.2 Application of the main theorem

Let us consider two approximating sequences of solutions $\rho_1^n(t,x)$ and $\rho_2^n(t,x)$ of (6.1) converging to two solutions $\rho_1(t,x)$ and $\rho_2(t,x)$ of our goal problem (1.2) in the sense explained above.

Let us consider the smooth velocity fields

$$u_i = -\nabla\psi_i \quad \text{with} \quad \psi_i = A_n'(\rho_i^n) + B_n + C_n * \rho_i^n$$

that verify $u_i \cdot \nu_{\Omega_n} = 0$ on the boundary of Ω_n for $i = 1, 2$. Moreover, these positive solutions and their vector fields are smooth on $\overline{\Omega_n}$ and verify

$$\frac{\partial \rho_i^n}{\partial t} + \nabla \cdot [\rho_i^n u_i] = 0,$$

with $u_i \cdot \nu_{\Omega_n} = 0$ on the boundary for $i = 1, 2$. Therefore, Lemma 4.4 ensures that both solutions are differentiable curves in $M = \mathcal{P}^{ac}(\Omega)$ with tangent vectors given by $\dot{\rho}_1^n = -\psi_1$ and $\dot{\rho}_2^n = -\psi_2$ respectively. From Lemmas 5.7–5.9 of subsection §5.3, we deduce that both curves are subgradient flows for the mollified energy functional $\mathcal{E}^n(\rho)$ since

$$\psi_i = A_n'(\rho_i^n) + B_n + C_n * \rho_i^n = \frac{\delta \mathcal{E}^n}{\delta \rho}(\rho_i^n(x))$$

for $i = 1, 2$ and thus, $-\dot{\rho}_i^n \in \underline{\partial} \mathcal{E}^n_{\rho_i^n}$.

We have checked that our two curves on M are differentiable and subgradient flows with respect to the regularized energy functional $\mathcal{E}^n(\rho)$. Since B_n and C_n satisfy convexity properties converging uniformly in n to those of B and C, we conclude that all the hypotheses of Theorem 3.12 are verified in this smooth setting.

Therefore, the sequences $\hat{\rho}_1^n(t)$ and $\hat{\rho}_2^n(t)$ verify the decay estimate

$$\operatorname{dist}_2(\rho_1^n(t), \rho_2^n(t)) \le \begin{cases} \Phi_n^{-1}(\Phi_n(\operatorname{dist}_2(\rho_1^n(0), \rho_2^n(0))) - t) & \text{if } \operatorname{dist}_2(\rho_1^n(0), \rho_1^n(0)) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{6.5}$$

for Φ_n given by (3.14).

Careful comparison of Examples 3.7 and 3.13 yield the conclusions of Example 1.1 when the evolutions are smooth — as for $\alpha \neq 0$ and initial data bounded away from zero and infinity. For instance, if $c > 0$ then $C(x) = \gamma |x|^{c+2}/(c+2)$ is $\sqrt{2}\phi(s/\sqrt{2})$-uniformly convex with $\phi(s) = \gamma 2^{-c/2} s^{c+1}$. In the absence of friction $\beta = 0$ we need net momentum to vanish $\langle x \rangle_{\rho(t)} = \langle x \rangle_{\tilde{\rho}(t)} = 0$ since Lemma 5.2 shows $\mathcal{C}(\rho)$ to be ϕ-uniformly convex on $\mathcal{P}_0^{ac}(\Omega) \subset \mathcal{P}_{2,0}^{ac}(\mathbf{R}^d)$ but not generally on $\mathcal{P}^{ac}(\Omega) \subset \mathcal{P}_2^{ac}(\mathbf{R}^d)$. As we point out in Corollary 6.2 below, it is not difficult to remove the smoothness assumption or boundedness requirement from the initial data.

6.3 Limiting procedure for degenerate parabolicity on all of $\mathbf{R}^d$

Let us now take the limit $n \to \infty$, which simultaneously relaxes the assumptions of uniform parabolicity and bounded velocity domain satisfied by the approximate problems.

Since the modulus of convexity ϕ_n converges to ϕ uniformly in compacts of $\mathbf{R}^+$, then Φ_n does so to Φ. This fact together with the convergence of the solutions and initial data, i.e., $\hat{\rho}_i^n(t) \to \rho_i(t)$ weakly in $L^1(\mathbf{R}^d)$ a.e. in $t > 0$ and $\hat{\rho}_i^n(0) \to \rho_i(0)$ and $|x|^2 \hat{\rho}_i^n(0) \to |x|^2 \rho_i(0)$ strongly in $L^1(\mathbf{R}^d)$, we conclude that

$$\operatorname{dist}_2(\rho_1(t), \rho_2(t)) \le \begin{cases} \Phi^{-1}(\Phi(\operatorname{dist}_2(\rho_1(0), \rho_2(0))) - t) & \text{if } \operatorname{dist}_2(\rho_1(0), \rho_1(0)) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{6.6}$$

a.e. $t > 0$. Here, we have used well-known properties of the Wasserstein distance with respect to weak-$*$ limits one can see in e.g. Givens and Shortt [28]: namely, weak-$*$ lower semicontinuity in both arguments and continuity when weak-$*$ convergence is augmented by convergence of second order moments (5.6).

Therefore, we have proved that the decay rate (6.6) holds for solutions of the Cauchy problem (1.2) assuming that the approximation procedure explained above can be carried over. Applying Theorem 3.12 with different degrees of convexity yields our main results concerning applications to granular media models. The following theorem is the analog of Proposition 2.1.

Theorem 6.1 (Exponential contraction / expansion rates for gradient flows). *Assume $\rho_1(t)$ and $\rho_2(t)$ are solutions of the Cauchy problem for* (1.2) *obtained by smoothing approximations $\{P_n\}$ with the properties described in subsection §6.1. If $B : \mathbf{R}^d \longrightarrow \mathbf{R}$ and $C : \mathbf{R}^d \longrightarrow \mathbf{R}$ are (semi)convex, say $D^2B(x) \geq \beta I$ and $D^2C(x) \geq \gamma I$ for a.e. $x \in \mathbf{R}^d$, some $\beta \in \mathbf{R}$ and $\gamma \leq 0$, then*

$$\mathrm{dist}_2(\rho_1(t), \rho_2(t)) \leq e^{-(\beta+\gamma)t}\, \mathrm{dist}_2(\rho_1(0), \rho_2(0)) \tag{6.7}$$

holds for all $t \geq 0$. If the hypotheses are strengthened by insisting $\gamma > 0$, the stronger conclusion (6.7) *will be true provided the centers of mass $\langle x\rangle_{\rho_1^n(t)} = \langle x\rangle_{\rho_2^n(t)} = 0$ remain fixed for all $t \geq 0$ in the approximating problems (hence in the limit $n \to \infty$ a fortiori).*

Corollary 6.2 (Uniqueness and extension to general initial data). *The preceding theorem, applied with with $\min\{\gamma, 0\}$ in place of γ, asserts $\rho_1(t) = \rho_2(t)$ for all $t > 0$ if it holds at initial time $t = 0$. Thus the theorem implies that any solution $\rho(t)$ obtained as a limit of smooth approximations as per §6.1 is uniquely determined by its initial condition $\rho(0)$. It also shows the time t solution map $X_t(\rho(0)) := \rho(t)$, if defined on a dense subset of $\mathcal{P}_2^{ac}(\mathbf{R}^d)$, has a unique continuous extension to the metric space completion $\mathcal{P}_2(\mathbf{R}^d)$.*

Remark 6.3 (Preservation of symmetry). *When the confining potential $B(x) = B(-x)$ is even, the equation shares this symmetry. If the initial condition $\rho(0, x) = \rho(0, -x)$ is also even, uniqueness forces this parity to be preserved for all time: $\rho(t, x) = \rho(t, -x)$. Choosing smooth approximations which respect this invariance, $\langle x\rangle_{\rho^n(t)} = 0$ is enforced and the stronger form of the theorem applies.*

If no confining potential is present $B(x) = 0$, the center of mass of any solution will be preserved due to translation invariance of the limiting flow regardless of parity. However, constructing a sequence of approximate problems which conserve center of mass without even parity remains an open problem.

Remark 6.4 (Compensating convexities and existence of equilibria). *The previous theorem shows that 2-uniform convexity of one of the potentials can compensate for lack of convexity in the other one to produce a uniform contraction if $\beta + \gamma > 0$. Then the solution map $X_t : \mathcal{P}_2(\mathbf{R}^d) \longrightarrow \mathcal{P}_2(\mathbf{R}^d)$ of Corollary 6.2 — restricted to even distributions if $B(x) = B(-x)$ — has a (unique) fixed point $X_t(\rho_\infty) = \rho_\infty \in \mathcal{P}_2(\mathbf{R}^d)$, according to the contraction mapping principle.*

Theorem 6.5 (Algebraic contraction by gradient flow).
Assume $\rho_1(t)$ and $\rho_2(t)$ are solutions of the Cauchy problem for (1.2) *obtained by smoothing approximations $\{P_n\}$ with the properties described in subsection §6.1. In addition let $\phi(s) = (k/r)s^{r+1}$, $k, r > 0$, and assume that two convex functions $B : \mathbf{R}^d \longrightarrow \mathbf{R}$ and $C : \mathbf{R}^d \longrightarrow \mathbf{R}$ satisfy one of the following conditions:*

(i) $B(x)$ is ϕ-uniformly convex on $\mathbf{R}^d$, or

(ii) $C(x)$ is ϕ-uniformly convex on $\mathbf{R}^d$, and the approximating solutions $\rho_1^n(t)$ and $\rho_2^n(t)$ verify $\langle x\rangle_{\rho_1^n(t)} = \langle x\rangle_{\rho_2^n(t)} = 0$ for all $t \geq 0$.

Then for all $t \geq 0$ the solutions $\rho_1(t)$ and $\rho_2(t)$ verify

$$\operatorname{dist}_2^2(\rho_1(t), \rho_2(t)) \leq \frac{\operatorname{dist}_2^2(\rho_1(0), \rho_2(0))}{(1 + tk \operatorname{dist}_2^r(\rho_1(t), \rho_2(0)))^{2/r}}.$$

Remark 6.6 (Convergence to equilibrium). *Corollary 6.2 and Remark 6.3 apply equally well under the hypotheses of Theorem 6.5. Uniqueness of a fixed point $\rho_\infty \in \mathcal{P}_2(\mathbf{R}^d)$ follows as before, but its existence requires some compactness, since the contraction is not uniform. When ρ_∞ exists, the rate of convergence to equilibrium can be estimated by choosing $\rho_2(t) = \rho_\infty$ to be the stationary solution in either theorem.*

6.4 Approximation procedure on $\mathbf{R}^d$

In this last subsection, we describe how to perform an approximation with smooth solutions in the whole space $\mathbf{R}^d$. This is done in the the special case $A(\rho) = \rho \log \rho$, $B(\rho) = 0$ and $C(x) = |x|^{r+2}$ with $r \geq 0$ which is the most relevant to applications.

We first consider the case: C uniformly convex, D^2C bounded from above and $|D^3C| \leq R/(1 + |x|)$ for a given constant R. Let us denote by ρ_∞ the unique minimizer of $\mathcal{E}(\rho)$ having zero center of mass in $\mathcal{P}_2^{ac}(\mathbf{R}^d)$. At the end of this subsection the other potentials $C(x) = |x|^{r+2}$ with $r > 0$ will be approximated by potentials of this form.

From the analysis in Appendix A.1 of our companion paper [19], we can construct a smooth solution decaying fast enough, say with all finite moments, for the Cauchy problem:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\rho \nabla \left(\log \rho + C * \rho\right)\right],$$

with a smooth positive initial data $\rho(t = 0, x) = \rho_o(x)$ satisfying additionally the assumptions ρ_0/ρ_∞ and $|\nabla(\rho_0/\rho_\infty)|$ bounded with zero center of mass.

Let us remark that previous assumptions imply that all moments of the initial data are bounded. Boundedness of moments was proved [19] to propagate in time and thus, moments of the solution are bounded in any time interval $[0, T]$.

For the rest of the paper R will denote several constants possibly depending on the initial data and the time interval $[0, T]$ to be considered through moments of the solution.

Lemma 6.7 (The solution defines a smooth curve in $\mathcal{P}_2^{ac}(\mathbf{R}^d)$). *The velocity field $\nabla\psi_t$ with*

$$\psi(t,x) = -(\log\rho(t,x) + C*\rho(t,x))$$

is of class C^1 and satisfies the bound $|\nabla\psi(t,x)| \le C(1+|x|)$ for $0 \le t \le T$.

Proof. Since C is locally Lipschitz and grows not faster than quadratically, it is easy to prove that $C*\rho$ is Lipschitz with respect to the x variable. It is also Lipschitz with respect to the t variable: indeed,

$$\begin{aligned} C*\rho_t(x) - C*\rho_s(x) &= \int_s^t d\tau \int_{\mathbf{R}^d} \partial_\tau\rho(\tau,y)\, C(x-y)\, dy \\ &= \int_s^t d\tau \int_{\mathbf{R}^d} \rho(\tau,y)[\Delta C(x-y) - \nabla(C*\rho)(y)\cdot\nabla C(x-y)]\, dy. \end{aligned}$$

From our bounds we deduce that

$$\Delta C(x-y) - \nabla(C*\rho)(y)\cdot\nabla C(x-y) \le R(1+|x|^2+|y|^2).$$

Combining this with the moment bound on ρ, we obtain

$$|(C*\rho)(t,x) - (C*\rho)(s,x)| \le R(t-s)(1+|x|^2).$$

From parabolic regularity theory we deduce that ρ is locally of class $C^{1+\alpha,2+\alpha}$ (i.e. $C^{1+\alpha}$ with respect to time, $C^{2+\alpha}$ with respect to x) for all $\alpha \in (0,1)$. By strong maximum principle, it is positive everywhere, and it follows that $\nabla_x \log\rho$ is a C^1 function. There is no problem in checking that $\nabla C*\rho$ is also a C^1 function.

Let us now address the linear growth of the gradient of ψ. To prove this estimate, we use a classical scheme based on Bernstein's method, after a change of unknown. Let us remark first that since

$$\nabla\rho_\infty + \nabla C*\rho_\infty = 0,$$

and since C is Lipschitz, then

$$|\nabla\log\rho_\infty(x)| = |\nabla C*\rho_\infty| \le R(1+|x|).$$

We also note that ρ_∞ has all its moments finite; this can be seen for instance by writing down the equation

$$\Delta\rho_\infty + \nabla\cdot(\rho_\infty\nabla C*\rho_\infty) = 0$$

and integrating it against $(1+|x|^2)^\alpha$. Easy computations, using the uniform convexity of C, as in [19], lead to

$$\int \rho_\infty(1+|x|^\alpha) \le R\int \rho(1+|x|^{\alpha-2}).$$

From Jensen's inequality it results that $M_\alpha := \int \rho_\infty(1+|x|^\alpha)$ satisfies

$$M_\alpha \le C M_\alpha^{1-2/\alpha},$$

in particular $M_\alpha \le C^{\alpha/2}$.

Let $h = \rho/\rho_\infty$. Since $\nabla(\log\rho) = \nabla(\log\rho_\infty) + \nabla(\log h)$, and since $\nabla(\log\rho_\infty) = -\nabla(C * \rho_\infty)$ satisfies the desired bound, it is sufficient for us to prove that

$$|\nabla(\log h)| \le R \tag{6.8}$$

and

$$|\nabla C * (\rho - \rho_\infty)| \le R. \tag{6.9}$$

Let $\partial C = \partial_i C$ for some index i. Since ρ and ρ_∞ have the same mass and the same center of mass, we can write

$$\begin{aligned}
\partial C * (\rho - \rho_\infty) &= \int_{\mathbf{R^d}} \partial C(x-y)\,(\rho-\rho_\infty)(y)\,dy \\
&= \int_{\mathbf{R^d}} [\partial C(x-y) - \partial C(x) - \nabla\partial C(x)\cdot y]\,(\rho-\rho_\infty)(y)\,dy.
\end{aligned}$$

By Taylor's formula and the uniform bound on $D^2\partial C$, we can bound this expression by

$$R\int_{\mathbf{R^d}} |y|^2|\rho-\rho_\infty|(y)\,dy,$$

which is bounded by a uniform constant and (6.8) is proved.

Let us proceed to estimate h. We will use the notation $\overline{C} = C*\rho$ and $\overline{C}_\infty = C*\rho_\infty$. Some tedious but easy computations lead to the equations

$$\begin{aligned}
\partial_t h &= \Delta h + (2\nabla\log\rho_\infty + \nabla\overline{C})\cdot\nabla h + (\Delta\log\rho_\infty + |\nabla\log\rho_\infty|^2 + \nabla\log\rho_\infty\cdot\nabla\overline{C} + \Delta\overline{C})h \\
&= \Delta h + (\nabla\overline{C} - 2\nabla\overline{C}_\infty)\cdot\nabla h + (|\nabla\overline{C}_\infty|^2 - \Delta\overline{C}_\infty - \nabla\overline{C}_\infty\cdot\nabla\overline{C} + \Delta\overline{C})h,
\end{aligned}$$

then, with $u = \log h$,

$$\partial_t u = \Delta u + |\nabla u|^2 + (\nabla\overline{C} - 2\nabla\overline{C}_\infty)\cdot\nabla u + (|\nabla\overline{C}_\infty|^2 - \Delta\overline{C}_\infty - \nabla\overline{C}_\infty\cdot\nabla\overline{C} + \Delta\overline{C}).$$

Let $b := \nabla\overline{C} - 2\nabla\overline{C}_\infty$ and $c := |\nabla\overline{C}_\infty|^2 - \Delta\overline{C}_\infty - \nabla\overline{C}_\infty\cdot\nabla\overline{C} + \Delta\overline{C}$. Another calculation yields

$$\partial_t\frac{|\nabla u|^2}{2} = \Delta\frac{|\nabla u|^2}{2} - \|D^2u\|^2 + \nabla u\cdot\nabla|\nabla u|^2 + (2\nabla u + b)\cdot\nabla\frac{|\nabla u|^2}{2} + \langle\nabla b\cdot\nabla u, \nabla u\rangle + \nabla c\cdot\nabla u,$$

where $\|\cdot\|$ stands for the Hilbert-Schmidt norm. Let $a := 2\nabla u + b$, we find

$$(\partial_t - \Delta - a\cdot\nabla)\cdot\frac{|\nabla u|^2}{2} \le (\|\nabla b\| + 1)|\nabla u|^2 + |\nabla c|^2.$$

Our goal is to prove that $|\nabla u|$ remains bounded on each interval $[0,T]$, knowing that it is bounded at time $t=0$. If we manage to prove that both $\|\nabla b\|$ and $|\nabla c|$ are bounded, then the conclusion will follow by maximum principle.

From our assumptions, ∇b is bounded. Let us estimate ∇c: the terms $\nabla\Delta\overline{C}$ and $\nabla\Delta\overline{C}_\infty$ are bounded, so we only have to estimate

$$\nabla\Big[|\nabla\overline{C}_\infty|^2 - \nabla\overline{C}_\infty\cdot\nabla\overline{C}\Big] = \nabla\Big[\nabla\overline{C}_\infty\cdot\nabla C*(\rho-\rho_\infty)\Big].$$

And in view of the bounds $|\nabla\overline{C}_\infty|\le C(1+|x|)$, $|D^2\overline{C}_\infty|\le C$ and (6.8) we only have to prove

$$|D^2C*(\rho-\rho_\infty)|\le\frac{R}{1+|x|}.$$

Similarly to (6.8), if $\partial^2C=\partial^2_{ij}C$ for some indices i and j, we can write

$$|\partial^2C(x)|\le R\int_{\mathbf{R^d}}\frac{|y|^3}{1+\min(|x|,|x-y|)}\,|\rho(y)-\rho_\infty(y)|\,dy.$$

Assume $|x|\ge1$. The contribution of those y's such that $|y|\le|x|/2$ to the integral above is bounded by

$$\frac{R}{1+|x|}\int_{\mathbf{R^d}}|y|^3|\rho(y)-\rho_\infty(y)|\,dy.$$

On the other hand, by Chebyshev's inequality, the contribution of those y's such that $|y|\ge|x|/2$ is bounded by

$$\frac{R}{|x|}\int_{\mathbf{R^d}}\frac{|y|^4}{1+\min(|x|,|x-y|)}\,|\rho(y)-\rho_\infty(y)|\,dy\le\frac{R}{|x|}\int_{\mathbf{R^d}}|y|^4\,|\rho(y)-\rho_\infty(y)|\,dy.$$

We conclude that indeed $|D^2\overline{C}(x)-D^2\overline{C}_\infty(x)|\le R/(1+|x|)$, as was announced. □

The previous lemma allows us to apply Lemma 4.5 showing that the solution $\rho_t(x):=\rho(t,x)$ is a differentiable curve on $\mathcal{P}_2^{ac}(\mathbf{R}^d)$. Subdifferentiability of the energy functional results directly from Lemmas 5.10-5.14 due to the smoothness of the solution $\rho(t)$ and the hypotheses on the potentials. Therefore, given two curves for two different initial data, with assumptions above, the main Theorem 3.12 can be applied and the conclusion of Theorem 6.1 follows. Finally, simple density arguments can be used to approximate less regular initial data. The convergence properties discussed in subsection 6.1 extend this result to positive initial data in $\mathcal{P}_2^{ac}(\mathbf{R}^d)$. In summary, we obtain the following improvement on Theorem 6.1 in the linear diffusion case without confining potentials:

Theorem 6.8 (Exponential contraction in $\mathbf{R}^d$; linear diffusion, no confinement). *Assume $\rho_1(t)$ and $\rho_2(t)$ are solutions of the Cauchy problem for* (1.2) *with $A(\rho)=\rho\log\rho$ and $B(\rho)=0$ and zero center of mass. If $C:\mathbf{R}^d\longrightarrow\mathbf{R}$ is uniformly convex, i.e.,*

$D^2C(x) \geq \gamma I$ for a.e. $x \in \mathbf{R}^d$, $\gamma \leq 0$, and $|D^3C| \leq R/(1+|x|)$ for a given constant R, then

$$\mathrm{dist}_2(\rho_1(t), \rho_2(t)) \leq e^{-\gamma t}\,\mathrm{dist}_2(\rho_1(0), \rho_2(0))$$

holds for all $t \geq 0$.

Let us finally remark that this approximation procedure in $\mathbf{R}^d$ overcomes the difficulty of fixing the center of mass in the sequence of approximations on bounded domains. However, as a trade-off we need to face a new challenging problem, that is, to show the existence of a well-defined global-in-time flow map for the velocity field $\nabla\psi_t$. In order to do so, we needed to impose hypotheses on the interaction potential C for which we are able to prove linear growth in x of the velocity field.

We finally generalize the kind of interaction potentials by a further approximation. Given a general interaction potential of the form $C(x) = |x|^{r+2}$ with $r > 0$, we approximate it by a sequence of smooth interaction potentials C_n with quadratic behavior at ∞, in such a way that the modulus of convexity ϕ_n of C_n converges uniformly in compact subsets of $\mathbf{R}_+$ to the modulus of convexity ϕ of C.

This can be accomplished in this radial case by radial approximating functions obtained by smoothly truncating the second radial derivative near zero and outside a large interval $[0, n]$.

In this way, we obtain potentials C_n satisfiying the quadratic growth at ∞ for which the theory developed in this subsection applies. Therefore by Lemma 6.7, we ensure that our evolution defines smooth enough curves for the differentiability structure we need. One now proceeds analogously to subsections 6.1 to 6.3, obtaining the conclusions of Theorem 6.5. We skip all the details since most of the work has already been done either in subsections 6.1-6.3 or in our companion paper [19] for the properties of the solutions and approximation. The final result is summarized as:

Theorem 6.9 (Algebraic contraction in $\mathbf{R}^d$, No confinement, Linear diffusion). *Assume $\rho_1(t)$ and $\rho_2(t)$ are solutions of the Cauchy problem for (1.2) with $A(\rho) = \rho\log\rho$ and $B(\rho) = 0$ and zero center of mass. Let $\phi(s) = (k/r)s^{r+1}$, $k, r > 0$, and assume that $C : \mathbf{R}^d \longrightarrow \mathbf{R}$ is ϕ-uniformly convex on $\mathbf{R}^d$, then for all $t \geq 0$ the solutions $\rho_1(t)$ and $\rho_2(t)$ verify*

$$\mathrm{dist}_2^2(\rho_1(t), \rho_2(t)) \leq \frac{\mathrm{dist}_2^2(\rho_1(0), \rho_2(0))}{(1 + tk\,\mathrm{dist}_2^r(\rho_1(t), \rho_2(0)))^{2/r}}.$$

Let us finally remark that even in the presence of linear diffusion we have not been able to show exponential convergence towards equilibrium with degenerately convex interaction potential. This was done by the entropy method in our companion paper [19] and it remains an open problem to derive this result by means of measuring the convexity of the involved functionals in the approach just presented. Feasibility of the latter approach was explored in collaboration with NSERC summer undergraduate research assistant Tim Capes at the University of Toronto, who showed that an apriori bound on $\|\rho(t)\|_{L^\infty}$ allows 2-uniform convexity of the entropy to be quantified, since the bound keeps us far away from the Dirac measures δ_{v_0} where the convexity degenerates.

Acknowledgements: JAC and CV acknowledge support from the European IHP network "Hyperbolic and Kinetic Equations: Asymptotics, Numerics, Applications" HPRN-CT-2002-00282. JAC acknowledges the support from the Spanish DGI-MCYT/FEDER project BFM2002-01710. RJM gratefully acknowledges the support of Natural Sciences and Engineering Research Council of Canada grant 217006-03 RGPIN and NSF grant DMS 0074037. JAC and RJM thank the Pacific Institute for the Mathematical Sciences where part of this work was achieved.

References

[1] Agueh, M. Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory. PhD Thesis, Georgia Institute of Technology, 2002.

[2] Agueh, M., Ghoussoub, N., and Kang, X. Geometric inequalities via a general comparison principle for interacting gases. To appear in GAFA (2003).

[3] Agueh, M. Asymptotic behavior for doubly degenerate parabolic equations. *C. R. Math. Acad. Sci. Paris, Ser. I 337*, (2003), 331-336.

[4] Ambrosio, L.A., Gigli, N., and Savaré, G., Gradient flows with metric and differentiable structures, and applications to the Wasserstein space. To appear in the proceedings of the meeting "Nonlinear Evolution Equations" held in the Academy of Lincei in Rome.

[5] Ambrosio, L.A., Gigli, N., and Savaré, G., Gradient flows in metric spaces and in the Wasserstein spaces of probability measures. To appear in Lecture Notes in Mathematics.

[6] Ball, K., Carlen, E.A., and Lieb, E.H. Sharp uniform convexity and smoothness inequalities for trace norms. *Invent. Math. 115*, (1994), 463–482.

[7] Bakry, E. and Emery, M. Diffusions hypercontractives in *Sem. Probab. XIX* LNM 1123 Springer, New York, 1985, pp 177–206.

[8] Benedetto, D., Caglioti, E., and Pulvirenti, M. A kinetic equation for granular media. *RAIRO Modél. Math. Anal. Numér. 31*, 5 (1997), 615–641.

[9] Benedetto, D., Caglioti, E., Carrillo, J. A., and Pulvirenti, M. A non-maxwellian steady distribution for one-dimensional granular media. *J. Stat. Phys. 91*, (1998), 979-990.

[10] Benedetto, D., Caglioti, E., Golse, F., and Pulvirenti, M. A hydrodynamic model arising in the context of granular media *Comput. Math. Appl. 38*, (1999), 121–131.

[11] Biane, P., Speicher, R. Free diffusions, free entropy and free Fisher information. *Ann. Inst. H. Poincar Probab. Statist. 37*, (2001), 581-606.

[12] Blower, G. Displacement convexity for the generalized orthogonal ensemble. Preprint (2002).

[13] Brenier, Y. The least action principle and the related concept of generalized flows for incompressible perfect fluids. *J. Amer. Math. Soc. 2*, (1989), 225–255.

[14] Brenier, Y. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math. 44*, (1991), 375–417.

[15] Carlen, E. and Gangbo, W. Constrained steepest descent in the 2-Wasserstein metric. *Annals Math. 157*, (2003), 807-846.

[16] Carrillo, J. A., Fellner, K. Long time asymptotics via entropy methods for diffusion dominated equations. HYKE preprint (2003) (www.hyke.org).

[17] Carrillo, J. A., Gualdani, M. P., and Toscani, G. Finite speed of propagation for the porous medium equation by mass transportation methods. HYKE preprint (2003) (www.hyke.org).

[18] Carrillo, J. A., Jüngel, A., Markowich, P. A., Toscani, G., and Unterreiter, A. Entropy dissipation methods for degenerate parabolic systems and generalized Sobolev inequalities. *Monatsh. Math.* 133 (2001) 1–82.

[19] Carrillo, J. A., McCann, R. J., and Villani, C. Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Rev. Matemática Iberoamericana* 19 (2003), 1-48.

[20] Carrillo, J. A., and Toscani, G. Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity. *Indiana Univ. Math. J. 49*, (2000), 113-141.

[21] Carrillo, J. A., and Toscani, G. Wasserstein metric and large–time asymptotics of nonlinear diffusion equations. HYKE preprint (2003) (www.hyke.org).

[22] Cordero-Erausquin, D. Some applications of mass transport. To appear in *Arch. Rational Mech. Anal.*

[23] Cordero-Erausquin, D., Gangbo, W., and Houdre, C. Inequalities for generalized entropy and optimal transportation. To appear in Proceedings of the Workshop: Mass Transportation Methods in Kinetic Theory and Hydrodynamics.

[24] Cordero-Erausquin, D., Nazaret, B., and Villani, C. A mass transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. To appear in Adv. Math.

[25] Dudley, R.M. *Probabilities and metrics - Convergence of laws on metric spaces, with a view to statistical testing.* Universitet Matematisk Institut, Aarhus, Denmark, 1976.

[26] Evans, L.C., and Gariepy, R.F. *Measure Theory and Fine Properties of Functions.* CRC Press, Boca Raton, 1992.

[27] Gangbo, W., and McCann, R. J. Shape recognition via Wasserstein distance. *Quart. J. Appl. Math. 4*, (2000), 705-737.

[28] Givens, C.R., and Shortt, R.M. A class of Wasserstein metrics for probability distributions. *Michigan Math. J. 31*, (1984), 231–240.

[29] Gromov, M. *Structures métriques pour les variétés riemanniennes.* J. Lafontaine and P. Pansu, eds. Cedic/Fernand Nathan, Paris, 1981.

[30] Gromov, M. *Metric Structures for Riemannian and non-Riemannian Spaces.* J. Lafontaine and P. Pansu, eds. With appendices by S. Semmes. Birkhauser, Boston, 1999.

[31] Gross, L. Logarithmic Sobolev inequalities. *Amer. J. of Math. 97*, (1975), 1061-1083.

[32] Kantorovich, L.V., and Rubinstein, G.S. On a space of completely additive functions. *Vestnik Leningrad. Univ. 13*, (1958), 52–59.

[33] Li, H., and Toscani, G. Long–time asymptotics of kinetic models of granular flows. *Arch. Rat. Mech. Anal.*, to appear.

[34] McCann, R. J. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J. 80*, (1995), 309–323.

[35] McCann, R. J. A convexity principle for interacting gases. *Adv. Math. 128*, 1 (1997), 153–179.

[36] McCann, R. J. Equilibrium shapes for planar crystals in an external field. *Comm. Math. Phys. 195* (1998), 699–723.

[37] McCann, R. J. Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.* 11 (2001) 589–608.

[38] Otto, F. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations* 26 (2001) 101–174.

[39] Otto, F., and Villani, C. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal* 173 (2001) 361–400.

[40] Rachev, S. T., and Rüschendorf, L. *Mass transportation problems.* Probability and its Applications. Springer-Verlag, New York, 1998.

[41] Talagrand, M. Transportation cost for Gaussian and other transport measures. *Geom. Func. Anal. 6*, (1996), 587-600.

[42] Toscani, G. One-dimensional kinetic models of granular flows. *RAIRO Modél. Math. Anal. Numér. 34*, 6 (2000), 1277–1291.

[43] Villani, C. *Topics in optimal transportation.* Graduate Studies in Mathematics Vol. 58, Amer. Math. Soc, Providence, 2003.

[44] Wasserstein, L.N. Markov processes over denumerable products of spaces describing large systems of automata. *Problems of Information Transmission 5*, (1969), 47–52.