

# On the implementation and usage of SDPT3 – a MATLAB software package for semidefinite-quadratic-linear programming, version 4.0

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## Abstract

This software is designed to solve conic programming problems whose constraint cone is a product of semidefinite cones, second-order cones, nonnegative orthants and Euclidean spaces; and whose objective function is the sum of linear functions and log-barrier terms associated with the constraint cones. This includes the special case of determinant maximization problems with linear matrix inequalities. It employs an infeasible primal-dual predictor-corrector path-following method, with either the HKM or the NT search direction. The basic code is written in MATLAB, but key subroutines in C are incorporated via Mex files. Routines are provided to read in problems in either SDPA or SeDuMi format. Sparsity and block diagonal structure are exploited. We also exploit low-rank structures in the constraint matrices associated the semidefinite blocks if such structures are explicitly given. To help the users in using our software, we also include some examples to illustrate the coding of problem data for our SQLP solver. Various techniques to improve the efficiency and stability of the algorithm are incorporated. For example, step-lengths associated with semidefinite cones are calculated via the Lanczos method. Numerical experiments show that this general purpose code can solve more than 80% of a total of about 300 test problems to an accuracy of at least  $10^{-6}$  in relative duality gap and infeasibilities.

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# 1 Introduction

The current version of SDPT3, version 4.0, is designed to solve conic programming problems whose constraint cone is a product of semidefinite cones, second-order cones, nonnegative orthants and Euclidean spaces; and whose objective function is the sum of linear functions and log-barrier terms associated with the constraint cones. It solves the following standard form of such problems, henceforth called SQLP problems:

$$\begin{aligned}
 (P) \quad \min \quad & \sum_{j=1}^{n_s} [\langle c_j^s, x_j^s \rangle - \nu_j^s \log \det(x_j^s)] + \sum_{i=1}^{n_q} [\langle c_i^q, x_i^q \rangle - \nu_i^q \log \gamma(x_i^q)] \\
 & + \langle c^l, x^l \rangle - \sum_{k=1}^{n_l} \nu_k^l \log x_k^l + \langle c^u, x^u \rangle \\
 \text{s.t.} \quad & \sum_{j=1}^{n_s} \mathcal{A}_j^s(x_j^s) + \sum_{i=1}^{n_q} A_i^q x_i^q + A^l x^l + A^u x^u = b, \\
 & x_j^s \in K_s^{s_j} \quad \forall j, \quad x_i^q \in K_q^{q_i} \quad \forall i, \quad x^l \in K_l^{n_l}, \quad x^u \in \mathbb{R}^{n_u}.
 \end{aligned}$$

Here,  $c_j^s, x_j^s$  are symmetric matrices of dimension  $s_j$  and  $K_s^{s_j}$  is the cone of positive semidefinite symmetric matrices of the same dimension. Similarly,  $c_i^q, x_i^q$  are vectors in  $\mathbb{R}^{q_i}$  and  $K_q^{q_i}$  is the quadratic or second-order cone defined by  $K_q^{q_i} := \{x = [x_0; \bar{x}] \in \mathbb{R}^{q_i} : x_0 \geq \sqrt{\bar{x}^T \bar{x}}\}$ . Finally,  $c^l, x^l$  are vectors of dimension  $n_l$ ,  $K_l^{n_l}$  is the nonnegative orthant  $\mathbb{R}_+^{n_l}$ , and  $c^u, x^u$  are vectors of dimension  $n_u$ . In the notation above,  $\mathcal{A}_j^s$  is the linear map from  $K_s^{s_j}$  to  $\mathbb{R}^m$  defined by

$$\mathcal{A}_j^s(x_j^s) = \left[ \langle a_{j,1}^s, x_j^s \rangle; \dots; \langle a_{j,m}^s, x_j^s \rangle \right],$$

where  $a_{j,1}^s, \dots, a_{j,m}^s \in \mathcal{S}^{s_j}$  are constraint matrices associated with the  $j$ th semidefinite block variable  $x_j^s$ . The matrix  $A_i^q$  is an  $m \times q_i$  dimensional constraint matrix corresponding to the  $i$ th quadratic block variable  $x_i^q$ , and  $A^l$  and  $A^u$  are the  $m \times n_l$  and  $m \times n_u$  dimensional constraint matrices corresponding to the linear block variable  $x^l$  and the unrestricted block variable  $x^u$ . The notation  $\langle p, q \rangle$  denotes the standard inner product in the appropriate space. For a given vector  $u = [u_0; \bar{u}]$  in a second order cone, we define  $\gamma(u) := \sqrt{u_0^2 - \bar{u}^T \bar{u}}$ . In the problem (P),  $\nu_j^s, \nu_i^q$ , and  $\nu_k^l$  are given nonnegative parameters.

In this paper, the vector 2-norm and Frobenius norm are denoted by  $\|\cdot\|$  and  $\|\cdot\|_F$ , respectively. We use the MATLAB notation  $[U; V]$  to denote the matrix obtained by appending  $V$  below the last row of  $U$ . For a given matrix  $U$ , we use the notation  $U(k, :)$  and  $U(:, k)$  to denote the  $k$ th row and column of  $U$ , respectively.

Let  $\text{svec} : \mathcal{S}^n \rightarrow \mathbb{R}^{n(n+1)/2}$  be the vectorization operator on symmetric matrices defined by  $\text{svec}(X) = [X_{11}, \sqrt{2}X_{12}, X_{22}, \dots, \sqrt{2}X_{1n}, \dots, \sqrt{2}X_{n-1,n}, X_{nn}]^T$ . For computational purpose, it is convenient to identify  $\mathcal{A}_j^s$  with the following  $m \times \bar{s}_j$  matrix (where  $\bar{s}_j = s_j(s_j + 1)/2$ ):

$$A_j^s = \left[ \text{svec}(a_{j,1}^s); \dots; \text{svec}(a_{j,m}^s) \right].$$

With the matrix representation of  $\mathcal{A}_j^s$ , we have that  $\mathcal{A}_j^s(x_j^s) = A_j^s \mathbf{svec}(x_j^s)$ .

The software also solves the dual problem associated with the problem above:

$$\begin{aligned}
(D) \quad & \max b^T y + \sum_{j=1}^{n_s} [\nu_j^s \log \det(z_j^s) + s_j \nu_j^s (1 - \log \nu_j^s)] \\
& + \sum_{i=1}^{n_q} [\nu_i^q \log \gamma(z_i^q) + \nu_i^q (1 - \log \nu_i^q)] + \sum_{k=1}^{n_l} [\nu_k^l \log z_k^l + \nu_k^l (1 - \log \nu_k^l)] \\
\text{s.t.} \quad & (\mathcal{A}_j^s)^T y + z_j^s = c_j^s, \quad z_j^s \in K_s^{s_j}, \quad j = 1 \dots, n_s \\
& (A_i^q)^T y + z_i^q = c_i^q, \quad z_i^q \in K_q^{q_i}, \quad i = 1 \dots, n_q \\
& (A^l)^T y + z^l = c^l, \quad z^l \in K_l^{n_l}, \\
& (A^u)^T y = c^l, \quad y \in \mathbb{R}^m.
\end{aligned}$$

In the notation above,  $(\mathcal{A}_j^s)^T$  is the adjoint of  $\mathcal{A}_j^s$  defined by  $(\mathcal{A}_j^s)^T y = \sum_{k=1}^m y_k a_{j,k}^s$ .

For later convenience, we introduce the following notation:

$$x^s = (x_1^s; \dots; x_{n_s}^s), \quad x^q = [x_1^q; \dots; x_{n_q}^q], \quad A^q = [A_1^q, \dots, A_{n_q}^q],$$

where the notation  $(x_1^s; \dots; x_{n_s}^s)$  means that the objects  $x_j^s$  are placed in a column format. We define  $c^s, z^s, c^q$ , and  $z^q$  analogously. Let  $\mathcal{A}^s(x^s) = \sum_{j=1}^{n_s} \mathcal{A}_j^s(x_j^s)$ ,  $(\mathcal{A}^s)^T y = ((\mathcal{A}_1^s)^T y; \dots; (\mathcal{A}_{n_s}^s)^T y)$ , and  $c = (c^s; c^q; c^l; c^u)$ ,  $x = (x^s; x^q; x^l; x^u)$ ,  $z = (z^s; z^q; z^l; 0)$ . Finally, we define

$$\mathcal{A}(x) = \mathcal{A}^s(x^s) + A^q x^q + A^l x^l + A^u x^u, \quad \mathcal{A}^T(y) = ((\mathcal{A}^s)^T y; (A^q)^T y; (A^l)^T y; (A^u)^T y),$$

$$K = K_s^{s_1} \times \dots \times K_s^{s_{n_s}} \times K_q^{q_1} \times \dots \times K_q^{q_{n_q}} \times K_l^{n_l} \times \mathbb{R}^{n_u},$$

$$K^* = K_s^{s_1} \times \dots \times K_s^{s_{n_s}} \times K_q^{q_1} \times \dots \times K_q^{q_{n_q}} \times K_l^{n_l} \times \{0\},$$

so that the equality constraints of (P) and (D) can be written more compactly as follows:

$$\mathcal{A}(x) = b, \quad x \in K, \quad \mathcal{A}^T(y) + z = c, \quad z \in K^*. \quad (1)$$

The matrix representation of  $\mathcal{A}$  is given by

$$A = [A_1^s, \dots, A_{n_s}^s, A^q, A^l, A^u]. \quad (2)$$

The software package was originally developed to provide researchers in semidefinite programming with a collection of reasonably efficient and robust algorithms that could solve general SDPs (semidefinite programming problems) with matrices of dimensions of the order of a hundred. The current release expands the family of problems solvable by the software in several dimensions.

1. This version is faster than the previous releases, e.g. [27], [30], especially on large sparse problems, and consequently can solve larger problems.

2. The current release can also solve problems that have explicit log-barrier terms in the objective functions. Hence determinant maximization problems can be solved.
3. The solver is more robust in handling unrestricted variables.
4. Low-rank structures in the constraint matrices associated with the semidefinite blocks can be exploited to improve computational efficiency and memory requirement.

All the numerical experiments in this paper are performed on a Pentium IV 3.0GHz personal computer with 4GB of physical memory using MATLAB version 7 on a Linux operating system.

**Organization of the paper.** In Section 2, we describe the representation of SQLP data by cell arrays. The SQLP solver `sqlp.m` in our software is described in Section 3. In Section 4, we present a few SQLP examples to illustrate the usage of our software. Implementation details such as the computation of search directions are given in Section 5. Finally, the last section reports computational results obtained by SDPT3 on about 300 SQLP problems.

**Installation.** The current version is written in MATLAB version 6.5 and is compatible with MATLAB version 7.0. It is available from the internet sites:

<http://www.math.nus.edu.sg/~mattohkc/sdpt3.html>  
<http://www.math.cmu.edu/~reha/sdpt3.html>

Our software uses a number of Mex routines generated from C programs written to carry out certain operations that MATLAB is not efficient at. In particular, operations such as extracting selected elements of a matrix, and performing arithmetic operations on these selected elements are all done in C. As an example, the vectorization operation `svec` is coded in the C program `mexsvec.c`. To install SDPT3 and generate these Mex routines, the user can simply follow the steps below:

- (a) unzip SDPT3-4.0.zip;
- (b) run MATLAB in the directory SDPT3-4.0;
- (c) run the m-file `Installmex.m`.

After that, to see whether you have installed SDPT3 correctly, run the m-files:

```
>> startup  
>> sqlpdemo
```

## 2 Cell array representation of problem data

Our implementation exploits the block structure of the given SQLP problem. In the internal representation of the problem data, we classify each semidefinite block into one of the following two types:

1. a dense or sparse matrix of dimension greater than or equal to 100;
2. a sparse block-diagonal matrix consisting of numerous sub-blocks each of dimension less than 100.

The reason for using the sparse matrix representation to handle the case when we have numerous small diagonal blocks is that it is less efficient for MATLAB to work with a large number of cell array elements compared to working with a single cell array element consisting of a large sparse block-diagonal matrix. Technically, no problem will arise if one chooses to store the small blocks individually instead of grouping them together as a sparse block-diagonal matrix.

For the quadratic part, we typically group all quadratic blocks (small or large) into a single block, though it is not mandatory to do so. If there are a large number of small blocks, it is advisable to group them all together as a single large block consisting of numerous small sub-blocks for the same reason we just mentioned.

Let  $L$  be the total number of blocks in the SQLP problem. If all the various types of blocks are present in  $(P)$ , then  $L = n_s + n_q + 2$ . For each SQLP problem, the block structure of the problem data is described by an  $L \times 2$  cell array named `blk`. The content of each of the elements of the cell arrays is given as follows. If the  $j$ th block is a semidefinite block consisting of a single block of size  $\mathbf{s}_j$ , then

$$\begin{aligned} \text{blk}\{j,1\} &= \text{'s'}, & \text{blk}\{j,2\} &= [\mathbf{s}_j], \\ \text{At}\{j\} &= [\bar{\mathbf{s}}_j \times m \text{ sparse}], \\ \text{C}\{j\}, \text{X}\{j\}, \text{Z}\{j\} &= [\mathbf{s}_j \times \mathbf{s}_j \text{ double or sparse}], \end{aligned}$$

where  $\bar{\mathbf{s}}_j = \mathbf{s}_j(\mathbf{s}_j + 1)/2$ .

If the  $j$ th block is a semidefinite block consisting of numerous small sub-blocks, say  $p$  of them, of dimensions  $\mathbf{s}_{j1}, \mathbf{s}_{j2}, \dots, \mathbf{s}_{jp}$  such that  $\sum_{k=1}^p \mathbf{s}_{jk} = \mathbf{s}_j$ , then

$$\begin{aligned} \text{blk}\{j,1\} &= \text{'s'}, & \text{blk}\{j,2\} &= [\mathbf{s}_{j1}, \mathbf{s}_{j2}, \dots, \mathbf{s}_{jp}], \\ \text{At}\{j\} &= [\bar{\mathbf{s}}_j \times m \text{ sparse}], \\ \text{C}\{j\}, \text{X}\{j\}, \text{Z}\{j\} &= [\mathbf{s}_j \times \mathbf{s}_j \text{ sparse}], \end{aligned}$$

where  $\bar{\mathbf{s}}_j = \sum_{k=1}^p \mathbf{s}_{jk}(\mathbf{s}_{jk} + 1)/2$ .

Notice that we store all the constraint matrices associated with the  $j$ th semidefinite block in vectorized form as a single  $\bar{\mathbf{s}}_j \times m$  matrix where the  $k$ th column of this matrix corresponds to the  $k$ th constraint matrix. That is,  $\text{At}\{j\}(:,k) = \text{svec}(a_{j,k}^s)$ .

The above storage scheme for the data matrix  $A_j^s$  associated with the semidefinite blocks of the SQLP problem represents a departure from earlier versions (version 2.3 or earlier) of our implementation, such as the one described in [27]. Previously, the constraint matrices were stored in an  $n_s \times m$  cell array `AA`, where  $\text{AA}\{j,k\} = a_{j,k}^s$ , and it was stored as an individual matrix in either dense or sparse format. The data format we used in earlier versions of SDPT3 was more natural, but our current data representation was adopted for the sake of computational efficiency. The reason for such a change is again due to the fact that it is less efficient for MATLAB to work with a single cell array with many cells.

But note that it is easy to use the function `svec.m` provided in SDPT3 to convert `AA` into the new storage scheme as follows: `At(j) = svec(blk(j,:),AA(j,:))`.

While we now store the constraint matrix in vectorized form, the parts of the iterates `X` and `Z` corresponding to semidefinite blocks are still stored as matrices, since that is how the user wants to access them.

The data storage scheme corresponding to quadratic, linear, and unrestricted blocks is rather straightforward. If the  $i$ th block is a quadratic block consisting of numerous sub-blocks, say  $p$  of them, of dimensions  $q_{i1}, q_{i2}, \dots, q_{ip}$  such that  $\sum_{k=1}^p q_{ik} = q_i$ , then

$$\begin{aligned} \text{blk}\{i,1\} &= \text{'q'}, & \text{blk}\{i,2\} &= [q_{i1}, q_{i2}, \dots, q_{ip}], \\ \text{At}\{i\} &= [q_i \times m \text{ sparse}], \\ \text{C}\{i\}, \text{X}\{i\}, \text{Z}\{i\} &= [q_i \times 1 \text{ double or sparse}]. \end{aligned}$$

If the  $k$ th block is the linear block, then

$$\begin{aligned} \text{blk}\{k,1\} &= \text{'l'}, & \text{blk}\{k,2\} &= n_1, \\ \text{At}\{k\} &= [n_1 \times m \text{ sparse}], \\ \text{C}\{k\}, \text{X}\{k\}, \text{Z}\{k\} &= [n_1 \times 1 \text{ double or sparse}]. \end{aligned}$$

Similarly, if the  $k$ th block is the unrestricted block, then

$$\begin{aligned} \text{blk}\{k,1\} &= \text{'u'}, & \text{blk}\{k,2\} &= n_u, \\ \text{At}\{k\} &= [n_u \times m \text{ sparse}], \\ \text{C}\{k\}, \text{X}\{k\}, \text{Z}\{k\} &= [n_u \times 1 \text{ double or sparse}]. \end{aligned}$$

(It is possible to have several linear or unrestricted blocks, but it is more efficient to reformulate such a problem by combining all linear blocks and similarly all unrestricted blocks.)

## 2.1 Specifying the block structure of problems

Our software requires the user to specify the block structure of the SQLP problem. Although no technical difficulty will arise if the user choose to lump a few blocks together and consider it as a single large block, the computational time can be dramatically different. For example, the problem `qpG11` in the `SDPLIB` library [2] actually has block structure `blk{1,1} = 's'`, `blk{1,2} = 800`, `blk{2,1} = 'l'`, `blk{2,2} = 800`, but the structure specified in the library is `blk{1,1} = 's'`, `blk{1,2} = 1600`. That is, in the former, the linear variables are explicitly identified, rather than being part of a large sparse semidefinite block. The difference in the running time for specifying the block structure differently is dramatic: the former representation is at least six times faster when the HKM direction is used, besides using much less memory space.

It is thus crucial to present problems to the algorithms correctly. We could add our own preprocessor to detect this structure, but believe users are aware of linear

variables present in their problems. Unfortunately the versions of `qpG11` (and also `qpG51`) in SDPLIB do not show this structure explicitly. In our software, we provide an m-file, `detect_diag.m`, to detect problems with linear variables. The user can call this m-file after loading the problem data into MATLAB as follows:

```
>> [blk,At,C,b] = read_sdpa('./sdplib/qpG11.dat-s');
>> [blk,At,C,b] = detect_diag(blk,At,C,b);
```

## 2.2 Storing constraint matrices with low-rank structures

A new feature of the current version of SDPT3 is that it can exploit low-rank structures present in the constraint matrices associated with the semidefinite blocks. To do so, the user needs to specify the low-rank structures in the constraint matrices explicitly when coding the problem data. The purpose here is to explain how this is done.

Suppose the  $j$ th row of `blk` corresponds to a semidefinite block. To simplify implementation, we exploit possible low-rank structures only when this semidefinite block is a single block. That is, `blk{j,2} = [sj]`. Suppose that the first  $p$  matrices,  $a_{j,1}^s, \dots, a_{j,p}^s$ , have no low-rank structures, and the remaining matrices  $a_{j,p+1}^s, \dots, a_{j,m}^s$  have such structures with

$$a_{j,k}^s = V_k D_k V_k^T, \quad k = p + 1, \dots, m,$$

where  $V_k \in \mathbb{R}^{s_j \times r_{j,k}}$  is a low-rank matrix with  $r_{j,k} \ll s_j$ , and  $D_k \in \mathbb{R}^{r_{j,k} \times r_{j,k}}$  is a symmetric matrix. The low-rank structures of these matrices should be recorded as follows:

$$\begin{aligned} \text{blk}\{j,1\} &= \text{'s'}, & \text{blk}\{j,2\} &= [s_j], & \text{blk}\{j,3\} &= [r_{j,p+1}, \dots, r_{j,m}], \\ \text{At}\{j,1\} &= [\bar{s}_j \times p \text{ sparse}], & \text{At}\{j,2\} &= [V_{j,p+1}, \dots, V_{j,m}], & \text{At}\{j,3\} &= \text{dd}, \end{aligned}$$

where `dd` is a 4-column matrix that records the non-zero elements of  $D_k$ ,  $k = p + 1, \dots, m$ , and a row (say,  $i$ th row) of `dd` has the following form:

$$d(i,:) = [\text{constraint number} - p, \text{row index}, \text{column index}, \text{non-zero value}].$$

If all the matrices  $D_k$  are diagonal, then the user can simply set `dd` to be the following column vector:

$$\text{dd} = [\text{diag}(D_{p+1}); \dots; \text{diag}(D_m)].$$

In the subdirectory `Examples`, we give an m-file `randlowranksdp.m` to generate random SDP problems with low-rank constraint matrices, whose calling syntax is:

$$[\text{blk}, \text{At}, \text{C}, \text{b}, \text{bblk}, \text{AAAt}] = \text{randlowranksdp}(n, p, m2, r)$$

It will generate an SDP where the first  $p$  constraint matrices have no low-rank structures, and the remaining  $m2$  matrices have low-rank structures and each matrix has rank  $r$ . The output `[blk,At,C,b]` explicitly describes the low-rank structure as above, while `[bblk,AAAt,C,b]` encodes the same SDP, but without including the low-rank structure information.

### 3 The main function: `sqlp.m`

The algorithm implemented in SDPT3 is an infeasible primal-dual path-following algorithm, described in detail in Appendix A. At each iteration, we first compute a *predictor* search direction aimed at decreasing the duality gap as much as possible. After that, the algorithm generates a Mehrotra-type corrector step [17] with the intention of keeping the iterates close to the central path. However, we do not impose any neighborhood restrictions on our iterates.<sup>1</sup> Initial iterates need not be feasible — the algorithm tries to achieve feasibility and optimality of its iterates simultaneously. It should be noted that in our implementation, the user has the option to use a primal-dual path-following algorithm that does not use corrector steps.

The main routine that corresponds to Algorithm IPC described in Appendix A is `sqlp.m`, whose calling syntax is as follows:

```
[obj,X,y,Z,info,runhist] = sqlp(blk,At,C,b,OPTIONS,X0,y0,Z0).
```

#### Input arguments.

`blk`: a cell array describing the block structure of the SQLP problem.

`At`, `C`, `b`: SQLP data.

`OPTIONS`: a structure array of parameters (optional).

`X0`, `y0`, `Z0`: an initial iterate (optional).

If the input argument `OPTIONS` is omitted, default values specified in the function `sqlparameters.m` are used. More detail on `OPTIONS` is given in Section 3.1.

#### Output arguments.

The names chosen for the output arguments explain their contents. The argument `info` is a structure array containing performance information such as `info.termcode`, `info.obj`, `info.gap`, `info.pinfeas`, `info.dinfeas`, `info.cputime` whose meanings are explained in `sqlp.m`. The argument `runhist` is a structure array which records the history of various performance measures during the run; for example, `runhist.gap` records the complementarity gap at each interior-point iteration.

Note that, while  $(\mathbf{X}, \mathbf{y}, \mathbf{Z})$  normally gives approximately optimal solutions, if `info.termcode` is 1 the problem is suspected to be primal infeasible and  $(\mathbf{y}, \mathbf{Z})$  is an approximate certificate of infeasibility, with  $\mathbf{b}^T \mathbf{y} = 1$ ,  $\mathbf{Z}$  in the appropriate cone, and  $\mathbf{A}^T \mathbf{y} + \mathbf{Z}$  small, while if `info.termcode` is 2 the problem is suspected to be dual infeasible and  $\mathbf{X}$  is an approximate certificate of infeasibility, with  $\langle \mathbf{C}, \mathbf{X} \rangle = -1$ ,  $\mathbf{X}$  in the appropriate cone, and  $\mathbf{A}\mathbf{X}$  small. Note that  $\mathbf{A}$  is defined in (2).

#### Caveats.

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<sup>1</sup>This strategy works well on most of the problems we tested. However, it should be noted that the occasional failure of the software on problems with poorly chosen initial iterates is likely due to the lack of a neighborhood enforcement in the algorithm.



(a) The user should be aware that SQLP is more complicated than linear programming. For example, it is possible that both primal and dual problems are feasible, but their optimal values are not equal. Also, either problem may be infeasible without there being a certificate of that fact (so-called weak infeasibility). In such cases, our software package is likely to terminate after some iterations with an indication of short step-length or lack of progress. Also, even if there is a certificate of infeasibility, our infeasible-interior-point methods may not find it. In our very limited testing on strongly infeasible problems, our algorithms have been quite successful in detecting infeasibility.

(b) Since our algorithm is a primal-dual method storing the primal iterate  $\mathbf{X}$ , it cannot exploit common sparsity in  $\mathbf{C}$  and the constraint matrices as well as dual methods or nonlinear-programming based methods. Thus our software may not be able to handle dense or sparse semidefinite blocks (with a single block) with dimension more than 2000 on an average PC available today.

(c) Our interior-point algorithms are designed based on the existence of a central path in the interior of the primal-dual feasible region of  $(P)$  and  $(D)$ . For problems where the primal-dual feasible region is non-empty but has an empty interior, our SQLP solver can generally still deliver a reasonably good approximate optimal solution, but the solver tends to encounter numerical difficulties before a high accuracy solution can be obtained.

### 3.1 The structure array `OPTIONS` for parameters

`sqlp.m` uses a number of parameters which are specified in a MATLAB structure array called `OPTIONS` in the m-file `sqlparameters.m`. If desired, the user can change the values of these parameters. The meaning of the specified fields in `OPTIONS` are given in the m-file itself. As an example, if the user does not wish to use corrector steps in Algorithm IPC, then he can do so by setting `OPTIONS.predcorr = 0`. If the user wants to use a fixed value, say 0.98, for the step-length parameter  $\gamma$  in Algorithm IPC instead of the adaptive strategy used in the default, he can achieve that by setting `OPTIONS.gam = 0.98`. Similarly, if the user wants to solve the SQLP problem to an accuracy tolerance of  $1e-4$  instead of the default value of  $1e-8$  while using the default values for all other parameters, he only needs to set `OPTIONS.gaptol = 1e-4`.

The defaults in `sqlparameters.m` assume that the parameters  $\nu_j^s, \nu_i^q, \nu_k^l$  in  $(P)$  are all 0. For an SQLP problem where some of the parameters  $\nu_j^s, \nu_i^q, \nu_k^l$  are positive, the user needs to specify an  $L \times 1$  cell array `OPTIONS.parbarrier` to store the values of these parameters (including zeros) as follows. If the  $j$ th block is a semidefinite block consisting of one or more sub-blocks, say  $p$  of them, of dimensions  $\mathbf{s}_{j1}, \mathbf{s}_{j2}, \dots, \mathbf{s}_{jp}$ , then

$$\text{OPTIONS.parbarrier}\{j\} = [\nu_{j1}^s, \nu_{j2}^s, \dots, \nu_{jp}^s].$$

If the  $i$ th block is a quadratic block consisting of one or more sub-blocks, say  $p$  of them, of dimensions  $\mathbf{q}_{i1}, \mathbf{q}_{i2}, \dots, \mathbf{q}_{ip}$ , then

$$\text{OPTIONS.parbarrier}\{i\} = [\nu_{i1}^q, \nu_{i2}^q, \dots, \nu_{ip}^q].$$

If the  $k$ th block is the linear block, then

$$\text{OPTIONS.parbarrier}\{k\} = [\nu_1^1, \nu_2^1, \dots, \nu_{n_1}^1],$$

while if the  $k$ th block is the unrestricted block, then

$$\text{OPTIONS.parbarrier}\{k\} = \text{zeros}(1, n_u).$$

The reader is referred to Section 4.2 for an example where the objective function in ( $D$ ) is given by  $\log \det(z^s)$ .

### 3.2 Running problems in SDPA and SeDuMi format

We provide two m-files, `read_sdpa.m` and `read_sedumi.m`, to respectively convert problem data stored in SDPA [6] and SeDuMi [22] format into MATLAB cell arrays described above. The subdirectory `sdplib` in SDPT3 contains a few problems in SDPA format that are extracted from the SDPLIB library [2], while the subdirectory `dimacs` contains problems in SeDuMi format that are extracted from the DIMACS library [20].

Assuming that the current directory is SDPT3-4.0, we can read in and run the test problem `mcp250-1.dat-s` in the subdirectory `sdplib` as follows:

```
>> startup      % set up Matlab paths
>> [blk,At,C,b] = read_sdpa('./sdplib/mcp250-1.dat-s');
>> [obj,X,y,Z,info] = sqlp(blk,At,C,b);

num. of constraints = 250
dim. of sdp      var = 250,   num. of sdp blk = 1
*****
SDPT3: Infeasible path-following algorithms
*****
version predcorr gam expon scale_data
HKM      1      0.000  1      0
it  pstep dstep p_infeas d_infeas gap      mean(obj)      cputime
-----
 0  0.000 0.000 1.4e+03 9.5e+01 7.0e+05 -1.462827e+04 0:0:0 spchol 1 1
 1  0.981 1.000 2.6e+01 9.8e-15 1.7e+04 -2.429708e+03 0:0:0 spchol 1 1
 2  1.000 1.000 5.0e-14 0.0e+00 2.4e+03 -1.352811e+03 0:0:1 spchol 1 1
 :      :      :      :      :      :      :      :
13  1.000 0.996 9.4e-13 5.1e-16 2.1e-05 -3.172643e+02 0:0:4 spchol 1 1
14  1.000 1.000 2.5e-12 4.3e-16 6.5e-07 -3.172643e+02 0:0:5
Stop: max(relative gap, infeasibilities) < 1.00e-08
-----
number of iterations = 14
primal objective value = -3.17264340e+02
dual objective value = -3.17264340e+02
gap := trace(XZ) = 6.45e-07
relative gap = 1.02e-09
actual relative gap = 1.02e-09
rel. primal infeas = 2.52e-12
```

```

rel. dual   infeas   = 4.29e-16
norm(X), norm(y), norm(Z) = 1.3e+02, 2.3e+01, 1.3e+01
norm(A), norm(b), norm(C) = 1.6e+01, 1.6e+01, 1.4e+01
Total CPU time (secs) = 4.7
CPU time per iteration = 0.3
termination code = 0
DIMACS: 2.5e-12  0.0e+00  4.3e-16  0.0e+00  1.0e-09  1.0e-09

```

---

We can solve a DIMACS test problem in a similar manner.

```

>> OPTIONS.vers = 2; % use NT direction
>> [blk,At,C,b] = read_sedumi('./dimacs/nb.mat');
>> [obj,X,y,Z,info] = sglp(blk,At,C,b,OPTIONS);

```

### 3.3 Stopping criteria

We define

$$n = \sum_{\{j:\nu_j^s=0\}} s_j + \sum_{\{i:\nu_i^q=0\}} q_i + |\{k:\nu_k^l=0\}| \quad (3)$$

$$\mu(x, z) = \frac{1}{n} \sum_{\alpha \in \{s, q, l\}} \sum_{j=1}^{n_\alpha} \begin{cases} \langle x_j^\alpha, z_j^\alpha \rangle & \text{if } \nu_j^\alpha = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

$$\begin{aligned} \text{gap} &= \langle x, z \rangle - \sum_{\{j:\nu_j^s>0\}} \nu_j^s \left( s_j + \log \det(x_j^s z_j^s / \nu_j^s) \right) \\ &\quad - \sum_{\{i:\nu_i^q>0\}} \nu_i^q \left( 1 + \log(\gamma(x_i^q) \gamma(z_i^q) / \nu_i^q) \right) - \sum_{\{k:\nu_k^l>0\}} \nu_k^l \left( 1 + \log(x_k^l z_k^l / \nu_k^l) \right). \end{aligned} \quad (5)$$

Note that if  $n = 0$ , we define  $\mu(x, z) = 0$ .

The algorithm is stopped when any of the following cases occur.

1. solutions with the desired accuracy have been obtained, i.e.,

$$\phi := \max \{ \text{relgap}, \text{pinfeas}, \text{dinffeas} \} \leq \text{OPTIONS.gaptol}, \quad (6)$$

where

$$\text{relgap} = \frac{\text{gap}}{1 + |\langle c, x \rangle| + |b^T y|}, \quad \text{pinfeas} = \frac{\|\mathcal{A}(x) - b\|}{1 + \|b\|}, \quad \text{dinffeas} = \frac{\|\mathcal{A}^T(y) + z - c\|}{1 + \|c\|}.$$

2. primal infeasibility is suggested because

$$b^T y / \|\mathcal{A}^T y + z\| > 10^8;$$

3. dual infeasibility is suggested because

$$-c^T x / \|\mathcal{A}x\| > 10^8;$$

4. slow progress is detected, measured by a rather complicated set of tests including

$$\text{relgap} < \max\{\text{pinfeas}, \text{dinfeas}\};$$

5. numerical problems are encountered, such as the iterates not being positive definite or the Schur complement matrix not being positive definite; or

6. the step sizes fall below  $10^{-6}$ .

### 3.4 Initial iterates

Our algorithms can start with an infeasible starting point. However, the performance of these algorithms is quite sensitive to the choice of the initial iterate. As observed in [7], it is desirable to choose an initial iterate that at least has the same order of magnitude as an optimal solution of the SQLP. If a feasible starting point is not known, we recommend that the following initial iterate be used:

$$\begin{aligned} y^0 &= 0, \\ (x_j^s)^0 &= \xi_j^s I_{s_j}, \quad (z_j^s)^0 = \eta_j^s I_{s_j}, \quad j = 1, \dots, n_s, \\ (x_i^q)^0 &= \xi_i^q e_i^q, \quad (z_i^q)^0 = \eta_i^q e_i^q, \quad i = 1, \dots, n_q, \\ (x^l)^0 &= \xi^l e^l, \quad (z^l)^0 = \eta^l e^l, \quad (x^u)^0 = 0, \end{aligned}$$

where  $I_{s_j}$  is the identity matrix of order  $s_j$ ,  $e_i^q$  is the first  $q_i$ -dimensional unit vector,  $e^l$  is the vector of all ones, and

$$\begin{aligned} \xi_j^s &= \max \left\{ 10, \sqrt{s_j}, s_j \max_{1 \leq k \leq m} \frac{1 + |b_k|}{1 + \|a_{j,k}^s\|_F} \right\}, \\ \eta_j^s &= \max \left\{ 10, \sqrt{s_j}, \max\{\|c_j^s\|_F, \|a_{j,1}^s\|_F, \dots, \|a_{j,m}^s\|_F\} \right\}, \\ \xi_i^q &= \max \left\{ 10, \sqrt{q_i}, \sqrt{q_i} \max_{1 \leq k \leq m} \frac{1 + |b_k|}{1 + \|A_i^q(k, :)\|} \right\}, \\ \eta_i^q &= \max \{ 10, \sqrt{q_i}, \max\{\|c_i^q\|, \|A_i^q(1, :)\|, \dots, \|A_i^q(m, :)\|\} \}, \\ \xi^l &= \max \left\{ 10, \sqrt{n_l}, \sqrt{n_l} \max_{1 \leq k \leq m} \frac{1 + |b_k|}{1 + \|A^l(k, :)\|} \right\}, \\ \eta^l &= \max \left\{ 10, \sqrt{n_l}, \max\{\|c^l\|, \|A^l(1, :)\|, \dots, \|A^l(m, :)\|\} \right\}. \end{aligned}$$

By multiplying the identity matrix  $I_{s_j}$  by the factors  $\xi_j^s$  and  $\eta_j^s$  for the semidefinite blocks, and similarly for the quadratic and linear blocks, the initial iterate has a better chance of having the appropriate order of magnitude.

The initial iterate above is set by calling `infeaspt.m`, with syntax

```
[X0,y0,Z0] = infeaspt(blk,At,C,b,options,scalefac),
```

where `options = 1` (default) corresponds to the initial iterate just described, and `options = 2` corresponds to the choice where the blocks of `X0`, `Z0` are `scalefac` times identity matrices or unit vectors, and `y0` is a zero vector.

### 3.5 Preprocessing

#### Nearly dependent constraints.

The primal-dual path-following algorithm we implemented assumes that the matrix  $A$  in (2) has full column rank. But in our software, the presence of (nearly) dependent constraints is detected automatically, and warning messages are displayed if such constraints exist. When this happens, the user has the option of removing these (nearly) dependent constraints by calling a preprocessing routine to remove them by setting `OPTIONS.rmdepconstr = 1`. The routine, `checkdepconstr.m`, we have coded to detect nearly dependent constraints is based on computing the sparse  $LDL^T$  factorization of  $AA^T$ . Such a method is fast but is not as reliable as the method that is based on sparse LU factorization of  $A$ .

#### Detecting diagonal blocks.

We provide the m-file, `detect_diag.m`, to look for diagonal blocks in semidefinite blocks in the data: see Subsection 2.1 for the use of this subroutine.

#### Detecting unrestricted blocks.

We have provided a routine, `detect_ublk.m`, to detect unrestricted variables that have been modeled as the difference of two nonnegative variables. The calling syntax is:

```
[bblk,AAAt,CC] = detect_ublk(blk,At,C);
```

#### Complex data.

In earlier versions, 2.3 or earlier, SDPT3 can directly handle complex data in SDP, i.e., the constraint matrices are hermitian matrices. However, as problems with complex data rarely occur in practice, and in an effort to simplify the code, we removed this flexibility from subsequent versions. But we intend to keep version 2.3 of the code available for users who wish to solve SDP problem (with no quadratic blocks) with complex data directly.

Users can also solve an SDP with complex data using SDPT3-4.0. This is done by calling the m-file `convertcmpsd.m` to convert the SDP into one with real data. But unlike the earlier versions, here we convert the problem into one with real data by doubling the size of the constraint matrices. Let  $B$  be an  $n \times n$  hermitian matrix. The conversion is based on the following equivalence:

$$B \text{ is positive semidefinite} \Leftrightarrow \begin{bmatrix} B^R & -B^I \\ B^I & B^R \end{bmatrix} \text{ is positive semidefinite,}$$

where  $B^R$  and  $B^I$  denote the real and imaginary parts of  $B$ , respectively. Note that since  $B$  is hermitian,  $B^R$  is symmetric and  $B^I$  is skew-symmetric.

Now suppose  $C, A_1, \dots, A_m$  are given  $n \times n$  hermitian matrices. Then  $C - \sum_{k=1}^m y_k A_k \succeq 0$  if and only

$$\begin{bmatrix} C^R & -C^I \\ C^I & C^R \end{bmatrix} - \sum_{k=1}^m y_k^R \begin{bmatrix} A_k^R & -A_k^I \\ A_k^I & A_k^R \end{bmatrix} - \sum_{k=1}^m y_k^I \begin{bmatrix} -A_k^I & -A_k^R \\ A_k^R & -A_k^I \end{bmatrix} \succeq 0. \quad (7)$$

Notice that the matrices  $[-A_k^I, -A_k^R; A_k^R, -A_k^I]$  are skew-symmetric. For a complex SDP, the vector  $b$  must necessarily be real, and the linear term in the objective function in (D) is replaced by  $\langle b, y^R \rangle$ . Since the skew symmetric matrices in (7) do not affect the positiveness condition and  $y^I$  does not appear in the objective function in (D), we can take  $y_j^k = 0, k = 1, \dots, m$ .

Note that the conversion of a complex SDP into a real SDP based on (7) would double the storage and if the data is dense, the cost of each interior-point iteration for solving the resulting real SDP is about twice as expensive as that for solving the complex SDP directly.

Suppose **AA** is an  $1 \times m$  cell array such that  $\mathbf{AA}\{\mathbf{k}\} = A_k$ . Then `convertcmpsd.m` has the calling syntax:

$$[\mathbf{bblk}, \mathbf{AAt}, \mathbf{CC}, \mathbf{bb}] = \text{convertcmpsd}(\mathbf{blk}, \mathbf{AA}, \mathbf{C}, \mathbf{b});$$

where **AAt** corresponds to the first  $m$  real symmetric constraint matrices in (7), **CC** corresponds to the real constant matrix in (7), and **bb** =  $b^R$ .

### Rotated cones.

Let  $K_r^n$  ( $n \geq 3$ ) be the rotated cone defined by

$$K_r^n = \{x^r = [u; v; w] \in \mathbb{R}^n : \|w\|^2 \leq 2uv, u, v \geq 0\}.$$

Note the constant "2" above. Define the symmetric orthogonal matrix  $T_n \in \mathbb{R}^{n \times n}$  as follows:

$$T_n = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & & \\ 1/\sqrt{2} & -1/\sqrt{2} & & \\ & & & \\ & & & I_{n-2} \end{bmatrix}.$$

It is easy to show that  $x^r \in K_r^n$  if and only if  $x^q := T_n x^r \in K_q^n$ , i.e.,  $T_n K_r^n = K_q^n$ . Thus we can always convert a rotated cone variable into one belonging to a second-order cone.

In SDPT3-4.0, the user can code a rotated cone block consisting of several sub-blocks, say  $p$  of them of dimension  $\mathbf{r}_{i1}, \dots, \mathbf{r}_{ip}$ , as follows:

$$\mathbf{blk}\{\mathbf{i}, 1\} = \mathbf{r}'; \quad \mathbf{blk}\{\mathbf{i}, 2\} = [\mathbf{r}_{i1}, \dots, \mathbf{r}_{ip}];$$

Let  $D$  be the block diagonal matrix defined by  $D = \text{diag}(T_{r_{i1}}, \dots, T_{r_{ip}})$ . Internally, SDPT3 would convert such a rotated cone block and its associated data into a second-order cone block as follows:

```

blk{i,1} = 'q'; blk{i,2} = [ri1, ..., rip];
At{i,1} = D*At{i,1};
C{i,1} = D*C{i,1};

```

## 4 Examples

For an user to solve his SQLP problem using SDPT3, the first task he needs to do is to code the problem data corresponding to the standard form in (P). The simplest way to learn how to generate the data of an SQLP problem in SDPT3 format is through examples. The subdirectory `Examples` in SDPT3 contains many such example files. Here we will just mention a few.

Note that the user can also store the problem data in either the SDPA or SeDuMi format, and then use the m-files `read_sdpa.m` or `read_sedumi.m` to read the data into SDPT3.

### 4.1 MAXCUT problem

Let  $\mathcal{S}_+^n$  be the space of  $n \times n$  symmetric positive semidefinite matrices. Let  $B$  be the weighted adjacency matrix of a graph. The SDP relaxation of the MAXCUT problem associated with  $B$  has the following form:

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \text{diag}(X) = e, \quad X \in \mathcal{S}_+^n, \end{aligned}$$

where  $e$  is the vector of ones, and  $C = -(\text{Diag}(Be) - B)/4$ . It is clear that we need the cell array, `blk{1,1}='s'`, `blk{1,2}=n`, to record the block structure of the problem. The constraint matrices can be constructed conveniently via an  $1 \times n$  cell array as follows:

```

AA = cell(1,n);
for k=1:n; AA{k}=spconvert([k,k,1;n,n,0]); end
At = svec(blk,AA);

```

For more details, see the m-file `maxcut.m` in the subdirectory `Examples`. (We could also create a version of the problem explicitly showing the low-rank structure; however, as the constraint matrices are so sparse, this would not be more efficient.)

## 4.2 D-optimal experiment design - an example with an explicit barrier term

Given a set of points  $\{v_1, \dots, v_p\}$  in  $\mathbb{R}^n$  with  $n \leq p$ , the D-optimal experiment design problem [31] needs to solve the following dual SQLP:

$$\begin{aligned} \max \quad & \log \det(Z) \\ \text{s.t.} \quad & \sum_{k=1}^p y_k (-v_k v_k^T) + Z = 0, \quad Z \in \mathcal{S}_{++}^n \\ & -y + z^l = 0, \quad z^l \in \mathbb{R}_+^p \\ & e^T y = 1, \quad y \in \mathbb{R}^p. \end{aligned}$$

The associated problem data can be coded in SDPT3 format as follows:

```
b = zeros(p,1);
blk{1,1} = 's'; blk{1,2} = n;
AA = cell(1,p); for k=1:p; AA{k} = -vk*vk'; end
At(1) = svec(blk(1,:),AA); C{1,1} = sparse(n,n);
blk{2,1} = '1'; blk{2,2} = p;
At{2,1} = -speye(p); C{2,1} = zeros(p,1);
blk{3,1} = 'u'; blk{3,2} = 1;
At{3,1} = ones(1,p); C{3,1} = 1;
```

Because the problem contains an explicit log-barrier term in the objective function, we also need to set up `OPTIONS.parbarrier` as follows:

```
OPTIONS.parbarrier{1} = 1;
OPTIONS.parbarrier{2} = zeros(1,p);
OPTIONS.parbarrier{3} = 0;
```

For more details, see the m-file `Doptdesign.m` in the subdirectory `Examples`.

The constraint matrices corresponding to the semidefinite block in this example are all rank-one matrices. The user can explicitly code such structures for SDPT3 as follows:

```
blk{1,1} = 's', blk{1,2} = n; blk{1,3} = ones(1,p);
At{1,1} = []; At{1,2} = [v1, ..., vp]; At{1,3} = -ones(p,1);
```

## 4.3 An LMI example

Consider the following LMI problem [3]:

$$\begin{aligned} \max \quad & -\eta \\ \text{s.t.} \quad & GY + YG^T \preceq 0 \\ & -Y \preceq -I \\ & Y - \eta I \preceq 0 \\ & Y_{11} = 1, \quad Y \in \mathcal{S}^n, \end{aligned} \tag{8}$$



where  $G \in \mathbb{R}^{n \times n}$ . This problem can be viewed as a dual SDP with  $Y$  identified as a vector  $y$  in  $\mathbb{R}^{n(n+1)/2}$ . In this case, we have  $(A_1^s)^T y = \mathbf{svec}(G\mathbf{smat}(y) + \mathbf{smat}(y)G^T)$ , where  $\mathbf{smat}$  is the inverse of  $\mathbf{svec}$ . The SDP data can be generated for SDPT3 as follows:

```
blk{1,1} = 's'; blk{1,2} = n;    blk{2,1} = 's'; blk{2,2} = n;
blk{3,1} = 's'; blk{3,2} = n;    blk{4,1} = 'u'; blk{4,2} = 1;
n2 = n*(n+1)/2; zz = sparse(n2,1); I = speye(n);
At{1,1} = [lmifun(G,I), zz];
At{2,1} = [-lmifun(I/2,I), zz];
At{3,1} = [lmifun(I/2,I), svec(blk(1,:),-I)];
At{4,1} = [1, zz'];
C{1,1} = sparse(n,n); C{2,1} = -I; C{3,1} = sparse(n,n); C{4,1} = 1;
b = [zz; -1];
```

In the above,  $\text{lmifun}(G,H)$  is a function (available in `Examples`) that generates the matrix representation of the linear map  $y \in \mathbb{R}^{n(n+1)/2} \mapsto \mathbf{svec}(G\mathbf{smat}(y)H^T + H\mathbf{smat}(y)G^T)$ .

For more details, see the m-file `lmiexamp1.m` in the subdirectory `Examples`.

## 4.4 Nearest correlation matrix problem

Given an  $n \times n$  symmetric matrix  $H$ , the nearest correlation matrix problem is the following [12]:

$$\min_X \{\|H - X\|_F : \text{diag}(X) = e, X \in \mathcal{S}_+^n\}.$$

The above problem can be converted to the following SQLP:

$$\min\{e_1^T y : \text{diag}(X) = e, \mathbf{svec}(X) + [0, I_{n_2}]y = \mathbf{svec}(H), X \in \mathcal{S}_+^n, y \in Q^{n_2+1}\},$$

where  $n_2 = n(n+1)/2$ ,  $I_{n_2}$  denotes the identity matrix of dimension  $n_2$ , and  $Q^{n_2+1}$  denotes the second-order cone of dimension  $n_2 + 1$ . The corresponding SQLP data can be coded for SDPT3 as follows:

```
blk{1,1} = 's'; blk{1,2} = n; n2 = n*(n+1)/2;
for k=1:n; AA{1,k} = spconvert([k,k,1; n,n,0]); end;
matrepdiag = svec(blk(1,:),AA);
At{1,1} = [matrepdiag{1}, speye(n2)];
blk{2,1} = 'q'; blk{2,2} = n2+1;
At{2,1} = [sparse(n,n2+1); sparse(n2,1), speye(n2)];
b = [ones(n,1); svec(blk(1,:),H)];
C{1,1} = sparse(n,n); C{2,1} = [1; zeros(n2,1)];
```

For more details, see the m-file `corrmat.m` in the subdirectory `Examples`.

## 4.5 An example from distance geometry

Consider a graph  $G = (V, \mathcal{E}, D)$  where  $V = \{1, 2, \dots, n\}$ ,  $\mathcal{E}$ , and  $D = (d_{ij})$  denote the nodes, edges, and associated weight matrix on the edges, respectively. The problem is to find points  $x_1, \dots, x_n$  in  $\mathbb{R}^p$  (for some  $p$ ) such that the pairwise Euclidean distance between  $x_i$  and  $x_j$  is as close as possible to  $d_{ij}$  if  $(i, j) \in \mathcal{E}$ . The associated minimization problem is the following:

$$\min \left\{ \sum_{(i,j) \in \mathcal{E}} \left| \|x_i - x_j\|^2 - d_{ij}^2 \right| : X := [x_1, \dots, x_n] \in \mathbb{R}^{p \times n} \right\}.$$

By noting that  $\|x_i - x_j\|^2 = e_{ij}^T X^T X e_{ij}$ , where  $e_{ij} = e_i - e_j$ , the above problem can equivalently be written as  $\min \{ \sum_{(i,j) \in \mathcal{E}} |\langle e_{ij} e_{ij}^T, Y \rangle - d_{ij}^2| : Y = X^T X, X \in \mathbb{R}^{p \times n} \}$ . One of the SDP relaxations of the above problem is  $\min \{ \sum_{(i,j) \in \mathcal{E}} |\langle e_{ij} e_{ij}^T, Y \rangle - d_{ij}^2| : Y \in \mathcal{S}_+^n \}$ , where the corresponding problem in standard form is given by

$$\min \left\{ \sum_{(i,j) \in \mathcal{E}} \alpha_{ij}^+ + \alpha_{ij}^- : \langle e_{ij} e_{ij}^T, Y \rangle - \alpha_{ij}^+ + \alpha_{ij}^- = d_{ij}^2, \alpha_{ij}^+, \alpha_{ij}^- \geq 0 \forall (i, j) \in \mathcal{E}, Y \in \mathcal{S}_+^n \right\}.$$

Let  $m = |\mathcal{E}|$ . The SQLP data can be coded as follows:

```
blk{1,1} = 's'; blk{1,2} = n;
AA = cell(1,m); b = zeros(m,1); cnt = 0;
for i = 1:n-1
    for j = i+1:n
        if (D(i,j) ~= 0)
            cnt = cnt + 1;
            AA{cnt} = spconvert([i,i,1; i,j,-1; j,i,-1; j,j,1; n,n,0]);
            b(cnt) = D(i,j)^2;
        end
    end
end
end
At(1) = svec(blk(1,:),AA); C{1,1} = sparse(n,n);
blk{2,1} = '1'; blk{2,2} = 2*m;
At{2,1} = [-speye(m), speye(m)]; C{2,1} = ones(2*m,1);
```

## 4.6 Norm minimization problem with complex data

Let  $B_0, \dots, B_m$  be  $p \times q$  matrices that are possibly complex. Consider the norm minimization problem:

$$\min \{ t : \left\| \sum_{k=1}^m x_k B_k + B_0 \right\|_2 \leq t, x \in \mathbb{C}^m, t \in \mathbb{R} \},$$

where  $\|\cdot\|_2$  denotes the matrix 2-norm. The problem above can equivalently be written as follows:

$$\begin{aligned} \max \quad & -t \\ \text{s.t.} \quad & \sum_{k=1}^m x_k^R \begin{bmatrix} 0 & B_k \\ B_k^* & 0 \end{bmatrix} + \sum_{k=1}^m x_k^I \begin{bmatrix} 0 & iB_k \\ (iB_k)^* & 0 \end{bmatrix} - tI \preceq - \begin{bmatrix} 0 & B_0 \\ B_0^* & 0 \end{bmatrix}. \end{aligned}$$

This is a complex SDP written in the format as in (D). Such a problem can be solve by SDPT3 as follows:

```
blk{1,1} = 's'; blk{1,2} = p+q;
AA = cell(1,2*m+1);
for k = 1:m; AA{1,k} = [zeros(p), Bk; B'k, zeros(q)]; end
for k = 1:m; AA{1,m+k} = [zeros(p), i*Bk; -i*B'k, zeros(q)]; end
AA{1,2*m+1} = -speye(p+q);
C{1} = -[zeros(p), B0; B'0, zeros(q)]; b = [zeros(2*m,1); -1];
[bblk,AAAt,CC,bb] = convertcmpsdp(blk,AA,C,b);
[obj,X,y,Z] = sglp(bblk,AAAt,CC,bb);
x = y(1:m) + i*y(m+[1:m]); t = y(2*m+1);
```

For more details, see the m-file `norm_min.m` in the subdirectory `Examples`.

## 4.7 Logarithmic Chebyshev approximation problem

This is an example that contains variables in a rotated cone. The original problem [16] is given by  $\min_{x \in \mathbb{R}^m} \max\{\log(f_i^T x) - \log(d_i) : i = 1, \dots, p\}$ , which can equivalently be formulated as:  $\min\{s : t \leq f_i^T x / d_i \leq s, i = 1, \dots, p, 1 \leq st\}$ . Let  $F = [f_1^T; \dots; f_p^T]$  and  $d = [d_1; \dots; d_p]$ . The latter problem can be formulated in the following standard dual form:

$$\begin{aligned} \max \quad & -s \\ \text{s.t.} \quad & \begin{bmatrix} -F & d & 0 & 0 \end{bmatrix} [x; s; t; u] \geq 0 \\ & \begin{bmatrix} F & 0 & -d & 0 \end{bmatrix} [x; s; t; u] \geq 0 \\ & \begin{bmatrix} 0 & I_3 \end{bmatrix} [x; s; t; u] \in K_r^3 \\ & \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} [x; s; t; u] = \sqrt{2}. \end{aligned}$$

The corresponding data for SDPT3 can be coded as follows:

```
blk{1,1} = '1'; blk{1,2} = p;
blk{2,1} = '1'; blk{2,2} = p;
blk{3,1} = 'r'; blk{3,2} = 3;
blk{4,1} = 'u'; blk{4,2} = 1;
zz = sparse(p,1);
At{1,1} = [-F, d, zz, zz]; C{1,1} = zeros(p,1);
At{2,1} = [F, zz, -d, zz]; C{2,1} = zeros(p,1);
At{3,1} = [sparse(3,m), speye(3)]; C{3,1} = zeros(3,1);
At{4,1} = [sparse(1,m+2), 1]; C{4,1} = sqrt(2);
b = [zeros(m,1); -1; 0; 0];
```

For more details, see the m-file `logchebyRcone.m` in the subdirectory `Examples`.

## 4.8 Maximizing the geometric mean of affine functions

Another example with rotated cone variables comes from maximizing the geometric mean of nonnegative affine functions [16]:

$$\begin{aligned} \max_{x \in \mathbb{R}^m} \quad & \prod_{i=1}^4 (a_i^T x + b_i)^{1/4} \\ \text{s.t.} \quad & a_i^T x + b_i \geq 0, \quad i = 1, \dots, 4. \end{aligned}$$

The above can be reformulated as the following SQLP problem with rotated cone constraints:

$$\max \left\{ t_3/\sqrt{2} : \sqrt{2}y_i = a_i^T x + b_i, y_i \geq 0, i = 1, \dots, 4, t_1^2 \leq 2y_1y_2, t_2^2 \leq 2y_3y_4, t_3^2 \leq 2t_1t_2 \right\}.$$

The corresponding standard (dual) form is as follows:

$$\begin{aligned} \max \quad & t_3/\sqrt{2} \\ \text{s.t.} \quad & \begin{bmatrix} a_1^T/\sqrt{2} & 0 & 0 & 0 \\ a_2^T/\sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_3^T/\sqrt{2} & 0 & 0 & 0 \\ a_4^T/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t_1 \\ t_2 \\ t_3 \end{bmatrix} + \begin{bmatrix} b_1/\sqrt{2} \\ b_2/\sqrt{2} \\ 0 \\ b_3/\sqrt{2} \\ b_4/\sqrt{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in K_r^3 \times K_r^3 \times K_r^3. \end{aligned}$$

## 4.9 Conversion of problems into the standard form

SDP problems are usually not formulated in the standard form (P) or (D). It is often quite tedious to convert such problems into the standard form. As such, there is an increasing demand for interfaces to automate the conversion process. Currently, one of the few interfaces that is publicly available for SQLP solvers is YALMIP [15].

Here we shall just give an example on how an SDP with linear inequality constraints can be converted into the standard form given in the Introduction. Suppose we have an SDP of the following form:

$$\begin{aligned} (P_1) \quad \min \quad & \langle c^s, x^s \rangle \\ \text{s.t.} \quad & \mathcal{A}^s(x^s) \leq b, \\ & x^s \in K_s^n. \end{aligned}$$

That is, it has inequality instead of equality constraints. But by introducing a slack

variable  $x^l$ , we can easily convert  $(P_1)$  into standard form, namely,

$$\begin{aligned} (P_1^*) \quad & \min \quad \langle c^s, x^s \rangle + \langle c^l, x^l \rangle \\ & \text{s.t.} \quad \mathcal{A}^s(x^s) + A^l x^l = b, \\ & \quad x^s \in K_s^n. \quad x^l \in K_l^m, \end{aligned}$$

where  $c^l = 0$ , and  $A^l = I_{m \times m}$ . With our use of cell arrays to represent SQLP data, it is easy to take the problem data of  $(P_1)$  and use them for the standard form  $(P_1^*)$  as follows:

$$\begin{aligned} \text{blk}\{1,1\} &= 's'; & \text{blk}\{1,2\} &= \mathbf{n}; \\ \text{At}\{1\} &= (A^s)^T; & \mathbf{C}\{1\} &= c^s; \\ \text{blk}\{2,1\} &= 'l'; & \text{blk}\{2,2\} &= \mathbf{m}; \\ \text{At}\{2\} &= \text{speye}(\mathbf{m}); & \mathbf{C}\{2\} &= c^l; \end{aligned}$$

## 5 Implementation details

The main step at each iteration of our algorithms is the computation of the search direction  $(\Delta x, \Delta y, \Delta z)$  from the *symmetrized Newton equation* with respect to some invertible block diagonal scaling matrix that is usually chosen as a function of the current iterate  $x, z$ .

### 5.1 The HKM search direction

Let

$$J_i^q = \begin{bmatrix} 1 & 0 \\ 0 & -I_{q_i-1} \end{bmatrix}. \quad (9)$$

For the choice of the HKM scaling matrix [10, 14, 18, 28, 29], the search direction  $(\Delta x, \Delta y, \Delta z)$  is obtained from the following system of equations:

$$\begin{aligned} \mathcal{A}^s(\Delta x^s) + A^q(\Delta x^q) + A^l \Delta x^l + A^u \Delta x^u &= R_p := b - \mathcal{A}^s(x^s) - A^q(x^q) - A^l x^l - A^u x^u \\ (A^s)^T \Delta y + \Delta z^s &= R_d^s := c^s - z^s - (A^s)^T y \\ (A^q)^T \Delta y + \Delta z^q &= R_d^q := c^q - z^q - (A^q)^T y \\ (A^l)^T \Delta y + \Delta z^l &= R_d^l := c^l - z^l - (A^l)^T y \\ (A^u)^T \Delta y &= R_d^u := c^u - (A^u)^T y \\ \Delta x^s + H^s(\Delta z^s) &= R_c^s := \left( \max\{\sigma\mu(x, z), \nu_j^s\} (z_j^s)^{-1} - x_j^s \right)_{j=1}^{n_s} \\ \Delta x^q + H^q(\Delta z^q) &= R_c^q := \left( \max\{\sigma\mu(x, z), \nu_i^q\} (z_i^q)^{-1} - x_i^q \right)_{i=1}^{n_q} \\ \Delta x^l + H^l(\Delta z^l) &= R_c^l := \left( \max\{\sigma\mu(x, z), \nu_k^l\} (z_k^l)^{-1} - x_k^l \right)_{k=1}^{n_l}, \end{aligned} \quad (10)$$

where  $\sigma \in (0, 1)$  is the centering parameter;  $(z_j^s)^{-1}$  and  $(z_k^l)^{-1}$  have the usual meaning, and  $(z_i^q)^{-1} := J_i^q z_i^q / \gamma(z_i^q)^2$ . In the above,

$$\begin{aligned} H^s(\Delta z^s) &= \left( H_j^s(\Delta z_j^s) := \frac{1}{2}((z_j^s)^{-1} \Delta z_j^s x_j^s + x_j^s \Delta z_j^s (z_j^s)^{-1}) \right)_{j=1}^{n_s}, \\ H^q(\Delta z^q) &= \left( H_i^q(\Delta z_i^q) := -\frac{\langle x_i^q, z_i^q \rangle}{\gamma(z_i^q)^2} J_i^q \Delta z_i^q + x_i^q ((z_i^q)^{-1})^T \Delta z_i^q + (z_i^q)^{-1} (x_i^q)^T \Delta z_i^q \right)_{i=1}^{n_q}, \\ H^l(\Delta z^l) &= \text{Diag}(x^l) \text{Diag}(z^l)^{-1} \Delta z^l. \end{aligned} \quad (11)$$

We compute the search direction via a Schur complement equation as follows (the reader is referred to [1] and [23] for details). First compute  $\Delta y$  from the Schur complement equation

$$\underbrace{\begin{bmatrix} M & A^u \\ (A^u)^T & 0 \end{bmatrix}}_{\mathcal{M}} \begin{bmatrix} \Delta y \\ \Delta x^u \end{bmatrix} = \begin{bmatrix} h \\ R_d^u \end{bmatrix} \quad (12)$$

where

$$M = \sum_{j=1}^{n_s} \underbrace{\mathcal{A}_j^s H_j^s (\mathcal{A}_j^s)^T}_{M_j^s} + \sum_{i=1}^{n_q} \underbrace{A_i^q H_i^q (A_i^q)^T}_{M_i^q} + \underbrace{A^l H^l (A^l)^T}_{M^l} \quad (13)$$

$$h = R_p + \mathcal{A}^s (H^s(R_d^s) - R_c^s) + A^q (H^q(R_d^q) - R_c^q) + A^l (H^l(R_d^l) - R_c^l).$$

(The notation in (13) should be interpreted as follows: the  $k$ th columns of  $M_j^s$  and  $M_i^q$  are  $\mathcal{A}_j^s H_j^s(a_{j,k}^s)$  and  $A_i^q H_i^q(a_{i,k}^q)$ , with  $a_{i,k}^q$  the  $k$ th column of  $(A_{i,k}^q)^T$ . Note that the matrices  $M_j^s$ ,  $M_i^q$ ,  $M^l$  are all symmetric positive definite.) Then compute  $\Delta x$  and  $\Delta z$  from the equations

$$\begin{aligned} \Delta z^s &= R_d^s - (\mathcal{A}^s)^T \Delta y, & \Delta z^q &= R_d^q - (A^q)^T \Delta y, & \Delta z^l &= R_d^l - (A^l)^T \Delta y \\ \Delta x^s &= R_c^s - H^s(\Delta z^s), & \Delta x^q &= R_c^q - H^q(\Delta z^q), & \Delta x^l &= R_c^l - H^l(\Delta z^l). \end{aligned}$$

## 5.2 The NT search direction

The user also has the choice of using the NT direction [21, 28, 29]. Let  $w_j^s$  be the unique positive definite matrix such that  $w_j^s z_j^s w_j^s = x_j^s$ . Let

$$\omega_i^q = \sqrt{\frac{\gamma(z_i^q)}{\gamma(x_i^q)}}, \quad \xi_i^q = \frac{1}{\omega_i^q} z_i^q + \omega_i^q J_i^q x_i^q, \quad t_i^q = \frac{\sqrt{2}}{\omega_i^q \gamma(\xi_i^q)} J_i^q \xi_i^q. \quad (14)$$

In this case, the search direction  $(\Delta x, \Delta y, \Delta z)$  satisfies the same system as in (10) except that  $H^s$  and  $H^q$  are replaced by

$$\begin{aligned} H^s(\Delta z^s) &= \left( H_j^s(\Delta z_j^s) := w_j^s \Delta z_j^s w_j^s \right)_{j=1}^{n_s}, \\ H^q(\Delta z^q) &= \left( H_i^q(\Delta z_i^q) := -\frac{1}{(\omega_i^q)^2} J_i^q \Delta z_i^q + t_i^q (t_i^q)^T \Delta z_i^q \right)_{i=1}^{n_q}. \end{aligned} \quad (15)$$

### 5.3 Choice of search direction

The current version of the code allows only two search directions, HKM and NT. In older versions, version 2.3 or earlier, also allowed the AHO direction of Alizadeh, Haeberly, and Overton [1] and the GT direction [25], but these are uncompetitive when the problems are of large scale. We intend to keep version 2.3 of the code available for those who wish to experiment with these other search directions, which tend to give more accurate results on smaller problems.

For the HKM and NT search directions, our computational experience on problems tested in Section 7 is that the HKM direction is almost universally faster than NT on problems with semidefinite blocks, especially for sparse problems with large semidefinite blocks. The reason that the latter is slower is because computing the NT scaling matrix  $w_j^s$  requires a full eigenvalue decomposition. This computation can dominate the work at each interior-point iteration when the problem is sparse.

The NT direction, however, was faster on sparse SOCP problems. The reason for this behavior is not hard to understand. By comparing the formulae for  $H_i^q$  for the HKM and NT directions in (11) and (15), it is clear that more computation is required to assemble the Schur complement matrix and more low-rank updating is necessary for the former direction.

### 5.4 Computation of the Schur complement matrix

Generally, the most expensive part in each iteration of Algorithm IPC lies in the computation and factorization of the Schur complement matrix  $M$  defined in (12). And this depends critically on the size and density of  $M$ . Note that the density of this matrix depends on two factors: (i) The density of  $\mathcal{A}^s$ ,  $A^q$ , and  $A^l$ , and (ii) any additional fill-in introduced because of the terms  $H^s$ ,  $H^q$ , and  $H^l$  in (13).

#### 5.4.1 Semidefinite blocks

For problems with semidefinite blocks, the contribution by the  $j$ th semidefinite block to  $M$  is given by  $M_j^s := \mathcal{A}_j^s H_j^s (\mathcal{A}_j^s)^T$ . The matrix  $M_j^s$  is generally dense even if  $\mathcal{A}_j^s$  is sparse. The computation of each entry of  $M_j^s$  involves matrix products, which has the form

$$(M_j^s)_{\alpha\beta} = \begin{cases} \langle a_{j,\alpha}^s, x_j^s a_{j,\beta}^s (z_j^s)^{-1} \rangle & \text{for the HKM direction.} \\ \langle a_{j,\alpha}^s, w_j^s a_{j,\beta}^s w_j^s \rangle & \text{for the NT direction.} \end{cases}$$

This computation can be very expensive if it is done naively without properly exploiting the sparsity that is generally present in the constraint matrices in  $\mathcal{A}_j^s$ . In our earlier papers [23, 27], we discussed briefly how sparsity of  $\mathcal{A}_j^s$  is exploited in our implementation by following the ideas presented by Fujisawa, Kojima, and Nakata in [7]. Further details on the efficient implementation of these ideas are given in [30].

When the constraint matrices have low-rank structures as described in Section 2.2, we can also compute the element  $(M_j^s)_{\alpha,\beta}$  as follows:

$$(M_j^s)_{\alpha\beta} = \text{Tr}(\tilde{V}_\beta^T V_\alpha D_\alpha V_\alpha^T \tilde{V}_\beta D_\beta),$$

where  $\tilde{V}_\beta = x_j^s V_\beta$ , and  $\hat{V}_\beta = (z_j^s)^{-1} V_\beta$  if the HKM direction is used; and  $\tilde{V}_\beta = \hat{V}_\beta = w_j^s V_\beta$  if the NT direction is used. Assume for simplicity that all the constraint matrices associated with the  $j$ th semidefinite block are dense and low-rank, i.e.,  $p = 0$  in Section 2.2. Suppose that the matrices  $\tilde{V}_k, \hat{V}_k, k = 1, \dots, m$ , are pre-computed (at the cost of  $\Theta(s_j^2 \sum_{k=1}^m r_{j,k})$  flops). Then it would take an additional  $\Theta(s_j (\sum_{k=1}^m r_{j,k})^2)$  flops to compute  $M_j^s$  since each element  $(M_j^s)_{\alpha\beta}$  can be computed at  $\Theta(s_j r_{j,\alpha} r_{j,\beta})$  flops. In contrast, without exploiting the low-rank structures, it would take  $\Theta(s_j^3 m) + \Theta(s_j^2 m^2)$  flops to compute  $M_j^s$ . If the average rank of the constraint matrices is  $r$ , then the latter complexity is  $\Theta(s_j/r^2)$  times larger than the former of  $\Theta(s_j^2 mr) + \Theta(s_j m^2 r^2)$ . Thus it is definitely advantageous to exploit low-rank structures.

As example, we generated a random SDP with low rank structure using the m-file `randlowranksdp.m` described in Section 2.2 with  $n = 200$  and  $m = 1000$ , the solver `sqlp.m` runs about 6 times faster when the low-rank structure is exploited.

#### 5.4.2 Quadratic and linear blocks

For the linear block,  $H^l$  is a diagonal matrix and it does not introduce any additional fill-in. This matrix does, however, affect the conditioning of the Schur complement matrix.

From equation (13), it is easily shown that the contribution of the quadratic blocks to the matrix  $M$  is given by

$$M_i^q = \begin{cases} -\frac{\langle x_i^q, z_i^q \rangle}{\gamma^2 (z_i^q)} A_i^q J_i^q (A_i^q)^T + u_i^q (v_i^q)^T + v_i^q (u_i^q)^T, & \text{for HKM direction} \\ -\frac{1}{(\omega_i^q)^2} A_i^q J_i^q (A_i^q)^T + w_i^q (w_i^q)^T & \text{for NT direction} \end{cases} \quad (16)$$

where  $u_i^q = A_i^q x_i^q$ ,  $v_i^q = A_i^q (z_i^q)^{-1}$ ,  $w_i^q = A_i^q t_i^q$ .

In the rest of this subsection, we focus our discussion on the HKM direction, but the same holds true for the NT direction.

The appearance of the outer-product terms in the equation above is potentially alarming. If the vectors  $u_i^q, v_i^q$  are dense, then even if  $A_i^q$  is sparse, the corresponding matrix  $M_i^q$ , and hence the Schur complement matrix  $M$ , will be dense. A direct factorization of the resulting dense matrix will be very expensive for even moderately large  $m$ .

The density of the matrix  $M_i^q$  depends largely on the particular problem structure. When the problem has many small quadratic blocks, it is often the case that each block appears in only a small fraction of the constraints. In this case, all  $A_i^q$  matrices are sparse and the vectors  $u_i^q$  and  $v_i^q$  turn out to be sparse vectors for each  $i$ . Consequently, the matrices  $M_i^q$  remain relatively sparse for these problems. As a result,  $M$  is also sparse and it can be factorized directly with reasonable cost. This behavior is typical for all `nql` and `qssp` problems from the DIMACS library.

The situation is drastically different for problems where one of the quadratic blocks, say the  $i$ th block, is large. For such problems the vectors  $u_i^q, v_i^q$  are typically dense, and therefore,  $M_i^q$  is likely be a dense matrix even if the data  $A_i^q$  is sparse. However, observe that  $M_i^q$  is a rank-two perturbation of a sparse matrix when  $A_i^q (A_i^q)^T$  is sparse. In such a situation, it is advantageous to use the dense-column handling



technique described in Section 5.7 to reduce the computational cost in solving (12). This approach helps tremendously on the scheduling problems from the DIMACS library.

To apply the dense-column handling technique, we need to modify the sparse portion of the matrix  $M_i^q$  slightly. Since the diagonal matrix  $-J_i$  has a negative component, the matrix  $-A_i^q J_i^q (A_i^q)^T$  need not be a positive definite matrix, and therefore the Cholesky factorization of the sparse portion of  $M_i^q$  need not exist. To overcome this difficulty, we use the following identity:

$$M_i^q = \frac{\langle x_i^q, z_i^q \rangle}{\gamma^2(z_i^q)} A_i^q (A_i^q)^T + u_i^q (v_i^q)^T + v_i^q (u_i^q)^T - k_i k_i^T, \quad (17)$$

where  $k_i = \sqrt{2} \langle x_i^q, z_i^q \rangle / \gamma^2(z_i^q) A_i^q(:, 1)$ . Note that if  $A_i^q$  is a large sparse matrix with a few dense columns, we can also explicitly separate the outer-product terms contributed by these dense columns from the sparse part of  $A_i^q (A_i^q)^T$  in (17).

## 5.5 Solving the Schur complement equation

The linear system (12) typically becomes more and more ill-conditioned as  $\mu$  decreases to 0. Thus iterative refinement is generally recommended to improve the accuracy of the computed solution. An even better approach to solve (12) is via a preconditioned symmetric quasi-minimal residual method (PSQMR) [5] with the preconditioner computed based on the following analytical expression of  $\mathcal{M}^{-1}$ :

$$\mathcal{M}^{-1} = \begin{bmatrix} M^{-1} - M^{-1} A^u S^{-1} (A^u)^T M^{-1} & M^{-1} A^u S^{-1} \\ S^{-1} (A^u)^T M^{-1} & -S^{-1} \end{bmatrix}, \quad (18)$$

where  $S = (A^u)^T M^{-1} A^u$ . Note that for a given vector  $[u; v]$ ,  $\mathcal{M}^{-1}[u; v]$  can be evaluated efficiently as follows:

$$\begin{aligned} \hat{u} &= M^{-1} u \\ t &= S^{-1} \left( (A^u)^T \hat{u} - v \right) \\ \mathcal{M}^{-1}[u; v] &= [\hat{u} - M^{-1} A^u t; t]. \end{aligned}$$

Thus if the Cholesky factorization of  $M$  and that of  $S$  are computed, then each evaluation involves solving four triangular linear systems for  $M$  and two triangular linear systems for  $S$ .

We should mention that the state-of-the-art Cholesky factorization softwares are highly developed and optimized. Thus our preference is to solve a linear system via Cholesky factorizations whenever possible. For most SDP problems, the matrix  $M$  is typically dense even when the constraint matrices are sparse. In this case, we use the routine `chol` (based on the LAPACK routine `dpotrf`) in MATLAB to compute the Cholesky factorization of a dense matrix.

For most sparse SOCP problems, the matrix  $M$  is usually sparse after dense-column handling. Let  $M_{sp}$  be the sparse part of  $M$  after dense-column handling.

In this case, the Cholesky factorization routine `chol` for a dense matrix is not efficient enough since it does not exploit sparsity. To factorize the sparse matrix  $M_{sp}$  more efficiently, we imported (with slight modifications) the sparse Cholesky solver in SeDuMi [22], which is a C translation of the Fortran programs developed by Esmond Ng, Barry Peyton, and Joseph Liu for sparse Cholesky factorization [19]. When SDPT3 uses the sparse Cholesky solver, it first performs a symmetric re-ordering on  $M_{sp}$  and generates a symbolic factorization of the re-ordered  $M_{sp}$  to determine the pivot order by examining the sparsity structure of this matrix carefully. Then, this pivot order is re-used in later iterations to compute the Cholesky factors. Contrary to the case of linear programming, however, the sparsity structure of  $M_{sp}$  can change during the course of interior-point iterations for SOCP problems. If this happens, the symbolic factorization has to be recomputed. We detect changes in the sparsity structure of  $M_{sp}$  by monitoring the nonzero elements of  $M_{sp}$ . Since the default initial iterates we use for an SOCP problem are unit vectors but subsequent iterates are not, there is always a change in the sparsity pattern of  $M_{sp}$  after the first iteration. After the second iteration, the sparsity pattern remains unchanged for most problems, and only one more change occurs in a small fraction of the test problems in the DIMACS library.

The effect of using a sparse Cholesky solver for sparse SOCP problems was dramatic. We observed speed-ups of up to two orders of magnitude. In our implementation, SDPT3 automatically makes a choice between MATLAB's built-in `chol` routine and the sparse Cholesky solver based on the density of  $M$ . The cutoff density is specified in the parameter `OPTIONS.spdensity`.

The approach of solving (12) by the SQMR method with preconditioner (18) works reasonably well if the following conditions are satisfied: (i) the number of columns of  $A^u$  is small and  $A^u$  is well-conditioned; (ii) the matrix  $M$  is not extremely ill-conditioned. (iii) the matrix  $S$  is not extremely ill-conditioned. However, when these conditions are not satisfied, preconditioning (12) based on (18) may not be advisable because either (a) computing  $S$  becomes very expensive due to large number of columns in  $A^u$ , or (b) the computed preconditioner based on (18) is no longer an accurate approximation of  $\mathcal{M}^{-1}$ . Note that  $S$  is typically much more ill-conditioned than  $A^u$ , especially when  $A^u$  is ill-conditioned. When conditions (i)–(iii) are not satisfied, it is advisable to use an  $LDL^T$  factorization of the symmetric indefinite matrix  $\mathcal{M}$  to compute an approximation of  $\mathcal{M}^{-1}$ . But unfortunately the state-of-the-art  $LDL^T$  factorization software available in the public domain is not as highly developed as its counterpart for Cholesky factorization, especially for sparse matrices. When the use of  $LDL^T$  factorization of  $\mathcal{M}$  is unavoidable, we use the publicly available Fortran subroutine `MA47` from the Harwell Subroutine Library [9] to compute such a factorization if  $\mathcal{M}$  is sparse. When  $\mathcal{M}$  is dense, the implementation we have at the moment is a little bit complicated. By rights, we should use the  $LDL^T$  factorization routine `dsysvx` in LAPACK to compute the required factorization for  $\mathcal{M}$ , but we have yet to modify and incorporate that routine into SDPT3. At the moment, we simply use the MATLAB routine `lu` to compute an LU factorization of  $\mathcal{M}$ , and use the computed LU factors to precondition the BiCGSTAB iterative method used to solve (12). Note that because the preconditioner in the latter case is no longer symmetric,

we have to replace the SQMR method by a nonsymmetric iterative method such as the BiGSTAB method.

## 5.6 Internal handling of unrestricted blocks

As mentioned in the last sub-section, solving the symmetric indefinite system (12) can potentially be very expensive when  $A^u$  is ill-conditioned or has a large number of columns because computing an  $LDL^T$  factorization of a sparse matrix can be much more costly than that for a symmetric positive definite matrix of the same dimension. It is possible to avoid the need to solve a symmetric indefinite system if we reformulate the equality constraint in (D) as

$$\begin{aligned} (A^u)^T y + z_+^u &= c^u, & z_+^u &\geq 0 \\ -(A^u)^T y + z_-^u &= -c^u, & z_-^u &\geq 0, \end{aligned}$$

with the corresponding primal variable  $x^u$  expressed as

$$x^u = x_+^u - x_-^u, \quad x_+^u, x_-^u \geq 0.$$

In this case, the system (12) is replaced by

$$\left( M + A^u \text{Diag}(x_+^u) \text{Diag}(z_+^u)^{-1} (A^u)^T + A^u \text{Diag}(x_-^u) \text{Diag}(z_-^u)^{-1} (A^u)^T \right) \Delta y = \text{rhs}$$

where rhs denotes the right-hand-side vector. Notice that in contrast to (12), the coefficient matrix is now symmetric positive definite.

But such a reformulation is not without difficulties. In fact, the variables  $x_+^u, x_-^u$  tend to become very large and  $z_+^u, z_-^u$  tend to become extremely small as the interior-point iteration progresses, and this generally makes the component matrices,  $A^u \text{Diag}(x_+^u) \text{Diag}(z_+^u)^{-1} (A^u)^T$  and  $A^u \text{Diag}(x_-^u) \text{Diag}(z_-^u)^{-1} (A^u)^T$ , extremely ill-conditioned. Fortunately, the following heuristic to modify the vectors  $x_+^u, x_-^u$  can typically ameliorate such an ill-conditioning problem:

$$x_+^u := x_+^u - 0.8 \min(x_+^u, x_-^u), \quad x_-^u := x_-^u - 0.8 \min(x_+^u, x_-^u).$$

This modification does not change the original variable  $x^u$  but does slow down the growth of  $x_+^u, x_-^u$ . After these modified vectors have been obtained, we also add positive perturbations to the vectors  $z_+^u, z_-^u$ . Such a modification in  $z_+^u, z_-^u$  ensures that they approach 0 at the same rate as  $\mu$ , and thus prevents the dual problem (D) from approaching the equality constraint too closely prematurely.

In the current implementation of SDPT3, we always reformulate an unrestricted vector by the difference of 2 non-negative vectors.

## 5.7 Dense-column handling

Here we describe our technique to handle dense columns when  $M$  is a low-rank perturbation of a sparse matrix. In such a case, the Schur complement matrix  $M$  can be written in the form

$$M = M_{sp} + UDU^T \tag{19}$$

where  $M_{sp}$  is a sparse symmetric positive semidefinite matrix,  $U$  has only few columns, and  $D$  is a non-singular matrix. If  $M_{sp}$  is positive definite, then we can solve (12) by solving a slightly larger but sparse linear system as follows. Let  $\lambda = DU^T\Delta y$ . It is easy to show that (12) is equivalent to the following linear system:

$$\begin{bmatrix} M_{sp} & A^u & U \\ (A^u)^T & 0 & 0 \\ U^T & 0 & -D^{-1} \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta x^u \\ \lambda \end{bmatrix} = \begin{bmatrix} h \\ R_d^u \\ 0 \end{bmatrix}. \quad (20)$$

We can use the same method described in Section 5.5 to solve (20).

## 5.8 User supplied routine to compute Schur complement matrix

The current version of SDPT3 allows the user to supply specialized routines to compute the Schur complement matrices  $M_j^s$  corresponding to the semidefinite blocks.

The specialized routine to compute  $M_j^s$  should have first line that look like:

```
function schurmat = schurfun_jth(U,V,schurfun_jth_par);
```

where the input arguments  $U$  and  $V$  should correspond to  $x_j^s$  and  $(z_j^s)^{-1}$  if the HKM direction is used; and they should correspond to the NT scaling matrix  $w_j^s$  if the NT direction is used. The third optional argument `schurfun_jth_par` can be a structure array that stores the parameters needed inside the function `schurfun_jth.m`.

The user can tell SDPT3 to use the specialized routine by setting the  $L \times 1$  cell array `OPTIONS.schurfun` as follows: set `OPTIONS.schurfun{j} = schurfun_jth` if  $M_j^s$  is to be computed by the specialized routine coded in the function `schurfun_jth.m`; otherwise set `OPTIONS.schurfun{j} = []`. If the function `schurfun_jth.m` requires some parameters, then the  $L \times 1$  cell array `OPTIONS.schurfun_par` must also be set correspondingly as follows: set `OPTIONS.schurfun_par{j} = schurfun_jth_par`; otherwise set `OPTIONS.schurfun_par{j} = []`.

Below is an example on we how use the specialized routine `mcp schur.m` in the subdirectory `Examples` to compute the Schur complement matrix when solving the SDP problem `mcp250-1.dat-s`.

```
>> [blk,At,C,b] = read_sdpa('./sdplib/mcp250-1.dat-s');
>> OPTIONS.schurfun{1} = 'mcp schur';
>> [obj,X,y,Z]=sqlp(blk,At,C,b,OPTIONS);
```

In the above example, there is no need to set the cell array `OPTIONS.schurfun_par` since the function `mcp schur.m` does not need any additional parameters.

## 5.9 Step-length computation

Once a direction  $\Delta x$  is computed, a full step will not be allowed if  $x + \Delta x$  violates the conic constraints. Thus, the next iterate must take the form  $x + \alpha\Delta x$  for an

appropriate choice of the step-length  $\alpha$ . In this subsection, we discuss an efficient strategy to compute the step-length  $\alpha$ .

For semidefinite blocks, it is straightforward to verify that, for the  $j$ th block, the maximum allowed step-length that can be taken without violating the positive semidefiniteness of the matrix  $x_j^s + \alpha_j^s \Delta x_j^s$  is given as follows:

$$\alpha_j^s = \begin{cases} \frac{-1}{\lambda_{\min}((x_j^s)^{-1} \Delta x_j^s)}, & \text{if the minimum eigenvalue } \lambda_{\min} \text{ is negative} \\ \infty & \text{otherwise.} \end{cases} \quad (21)$$

If the computation of eigenvalues necessary in  $\alpha_j^s$  above becomes expensive, then we resort to finding an approximation of  $\alpha_j^s$  by estimating extreme eigenvalues using Lanczos iterations [24]. This approach is quite accurate in general and represents a good trade-off between the computational effort versus quality of the resulting stepsizes.

For quadratic blocks, the largest step-length  $\alpha_i^q$  that keeps the next iterate feasible with respect to the  $i$ th quadratic cone can be computed as follows. Let

$$a_i = \gamma(\Delta x_i^q)^2, \quad b_i = \langle \Delta x_i^q, J_i^q x_i^q \rangle, \quad c_i = \gamma(x_i^q)^2, \quad d_i = b_i^2 - a_i c_i,$$

where  $J_i^q$  is the matrix defined in (9). We want the largest positive  $\bar{\alpha}$  for which  $a_i \alpha^2 + 2b_i \alpha + c_i > 0$  for all smaller positive  $\alpha$ 's, which is given by

$$\alpha_i^q = \begin{cases} \frac{-b_i - \sqrt{d_i}}{a_i} & \text{if } a_i < 0 \text{ or } b_i < 0, a_i \leq b_i^2/c_i \\ \frac{-c_i}{2b_i} & \text{if } a_i = 0, b_i < 0 \\ \infty & \text{otherwise.} \end{cases}$$

For the linear block, the maximum allowed step-length  $\alpha_k^l$  for the  $k$ th component is given by

$$\alpha_k^l = \begin{cases} \frac{-x_k^l}{\Delta x_k^l}, & \text{if } \Delta x_k^l < 0 \\ \infty & \text{otherwise.} \end{cases}$$

Finally, an appropriate step-length  $\alpha$  that can be taken in order for  $x + \alpha \Delta x$  to satisfy all the conic constraints takes the form

$$\alpha = \min \left( 1, \gamma \min_{1 \leq j \leq n_s} \alpha_j^s, \gamma \min_{1 \leq i \leq n_q} \alpha_i^q, \gamma \min_{1 \leq k \leq n_l} \alpha_k^l \right), \quad (22)$$

where  $\gamma$  (known as the step-length parameter) is typically chosen to be a number slightly less than 1, say 0.98, to ensure that the next iterate  $x + \alpha \Delta x$  stays strictly in the interior of all the cones.

For the dual direction  $\Delta z$ , we let the analog of  $\alpha_j^s$ ,  $\alpha_i^q$  and  $\alpha_k^l$  be  $\beta_j^s$ ,  $\beta_i^q$  and  $\beta_k^l$ , respectively. Similar to the primal direction, the step-length that can be taken by the dual direction  $\Delta z$  is given by

$$\beta = \min \left( 1, \gamma \min_{1 \leq j \leq n_s} \beta_j^s, \gamma \min_{1 \leq i \leq n_q} \beta_i^q, \gamma \min_{1 \leq k \leq n_l} \beta_k^l \right). \quad (23)$$

## 6 Distances to infeasibilities

Let  $d = (A, c, b)$  be the SQLP data associated with  $(P)$  and  $(D)$ . Let  $F_P(d)$  and  $F_D(d)$  be the feasible regions of  $(P)$  and  $(D)$  respectively. It is often of interest to know whether the interiors,  $F_P^\circ(d)$  and  $F_D^\circ(d)$ , are empty, and how “thick” these regions are. A natural quantitative measure of “thickness” of  $F_P(d)$  and  $F_D(d)$  is the concept of primal distance to infeasibility defined by Renegar [?]:

$$\rho_P(d) = \inf\{\|\Delta d\| : F_P(d + \Delta d) = \emptyset\}, \quad \rho_D(d) = \inf\{\|\Delta d\| : F_D(d + \Delta d) = \emptyset\}.$$

With appropriately chosen norm in the above definitions, the computation of  $\rho_D(d)$  amounts to solving an SQLP problem with roughly the same dimension and structure as the original primal instance. Unfortunately, the computation of  $\rho_P(d)$  is extremely costly, which requires solving  $2m$  SQLP problems each with roughly the same dimension and structure as the original dual instance; see [4].

However, one is typically interested in the magnitudes of  $\rho_P(d)$  and  $\rho_D(d)$  rather than the exact values. It turns out that the following cheaper upper estimates proposed by Freund [?] usually give enough information about the size of  $F_P(d)$  and  $F_D(d)$ :

$$g_P(d) = \inf \left\{ \max \left\{ \|x\|, \frac{\|x\|}{\text{dist}(x, \partial K)}, \frac{1}{\text{dist}(x, \partial K)} \right\} : \mathcal{A}(x) = b, x \in K \right\}$$

$$g_D(d) = \inf \left\{ \max \left\{ \|z\|, \frac{\|z\|}{\text{dist}(x, \partial K)}, \frac{1}{\text{dist}(x, \partial K)} \right\} : \mathcal{A}^T(y) + z = c, z \in K^* \right\}.$$

It is readily shown that  $1/g_P(d) \geq \rho_P(d)$  and  $1/g_D(d) \geq \rho_D(d)$ , and  $g_P(d) = \infty \Leftrightarrow \rho_P(d) = 0$ ,  $g_D(d) = \infty \Leftrightarrow \rho_D(d) = 0$ .

Freund showed that  $g_P(d)$  ( $g_D(d)$ ) can be computed at the cost of solving an SQLP problem with roughly the same dimension and structure as the original primal (dual) instance.

In the current release of SDPT3, we include the following m-files to compute  $g_P(d)$  and  $g_D(d)$ :

```
function gp = gpcomp(blk,At,C,b);
function gd = gdcomp(blk,At,C,b);
```

## 7 Computational results

Here we describe the results of our computational testing of SDPT3-4.0 using the default parameters, on problems from the following sources:

1. SDPLIB collection of Borchers, available at <http://www.nmt.edu/~borchers/sdplib.html>
2. DIMACS Challenge test problems, available at <http://dimacs.rutgers.edu/Challenges/Seventh/Instances/>

3. Sparse SDPs from structural optimization, available at <http://www2.am.uni-erlangen.de/~kocvara/pennon/problems.html>
4. Sparse SDP collection of Hans Mittelmann, available at <ftp://plato.asu.edu/pub/sdp/>
5. SDPs from electronic structure calculations, available at <http://www.cims.nyu.edu/~mituhiro/software.html>
6. SDPs from polynomial optimizations [11].
7. SOCP problems generated by the MATLAB FIR filter toolbox, available at <http://www.csee.umbc.edu/~dscho12/opt.html>

Our results were obtained on a Pentium IV PC (3.0GHz) with 4G of memory running Linux, using MATLAB 7.0. Figure 1 shows the performance of SDPT3-4.0 on a total of about 300 SQLP problems. It shows that SDPT3 was able to solve 80% of the problems to an accuracy of at least  $10^{-6}$  in the measure  $\phi$  defined in (6).

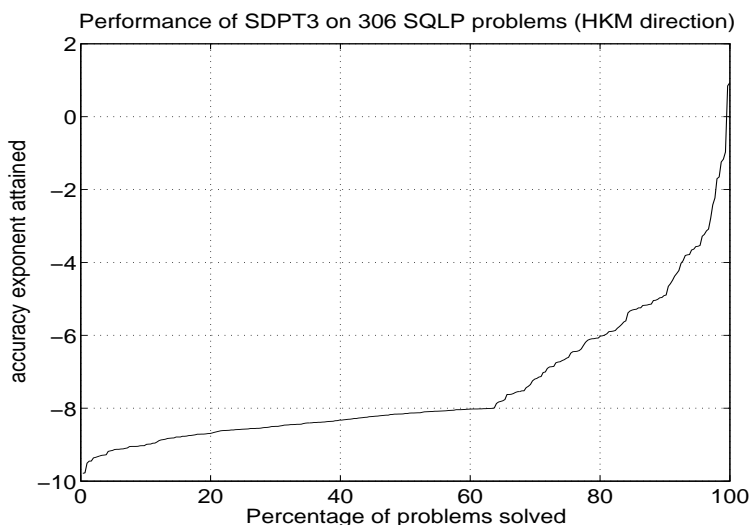


Figure 1: *The accuracy exponent is defined to be  $\log_{10}(\phi)$ , where  $\phi$  is defined as in (6).*

DIMACS error measures

## Appendix: A primal-dual infeasible-interior-point algorithm

Here we give a **pseudo-code** for the algorithm we implemented. Note that this description makes references to earlier sections where details related to the algorithm are explained.

**Algorithm IPC.** Suppose we are given an initial iterate  $(x^0, y^0, z^0)$  with  $x^0, z^0$  strictly satisfying all the conic constraints. Decide on the type of search direction to use. Set  $\gamma^0 = 0.9$ .

**For**  $k = 0, 1, \dots$

(Let the current and the next iterate be  $(x, y, z)$  and  $(x^+, y^+, z^+)$  respectively. Also, let the current and the next step-length parameter be denoted by  $\gamma$  and  $\gamma^+$  respectively.)

- Compute  $\mu(x, z)$  defined in (5), and the accuracy measure  $\phi$  defined in (6). Stop the iteration if  $\phi$  is sufficiently small.
- (Predictor step) Solve the linear system (12) with  $\sigma = 0$  in the right-side vector (14). Denote the solution of (10) by  $(\delta x, \delta y, \delta z)$ . Let  $\alpha_p$  and  $\beta_p$  be the step-lengths defined as in (22) and (23) with  $\Delta x, \Delta z$  replaced by  $\delta x, \delta z$ , respectively.
- Take  $\sigma$  to be

$$\sigma = \min \left( 1, \left[ \frac{\mu(x + \alpha_p \delta x, z + \beta_p \delta z)}{\mu(x, z)} \right]^e \right),$$

where the exponent  $e$  is chosen as follows:

$$e = \begin{cases} \max[1, 3 \min(\alpha_p, \beta_p)^2] & \text{if } \mu(x, z) > 10^{-6}, \\ 1 & \text{if } \mu(x, z) \leq 10^{-6}. \end{cases}$$

- (Corrector step) Solve the linear system (12) with  $R_c$  in the the right-hand side vector (14) replaced by

$$\widehat{R}_c^\tau = R_c^\tau - \text{Mehrotra-corrector term generated from } \delta x^\tau \text{ and } \delta z^\tau, \quad \tau \in \{s, q, l\}.$$

Denote the solution of (10) by  $(\Delta x, \Delta y, \Delta z)$ .

- Update  $(x, y, z)$  to  $(x^+, y^+, z^+)$  by

$$x^+ = x + \alpha \Delta x, \quad y^+ = y + \beta \Delta y, \quad z^+ = z + \beta \Delta z,$$

where  $\alpha$  and  $\beta$  are computed as in (22) and (23) with  $\gamma$  chosen to be  $\gamma = 0.9 + 0.09 \min(\alpha_p, \beta_p)$ .

- Update the step-length parameter by  $\gamma^+ = 0.9 + 0.09 \min(\alpha, \beta)$ .

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problem	$m \mid n_s; n_g; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
arch0	174   161; ; 174;	26	-5.66517269-1	-5.66517274-1	5.5-8	3.9-14	2.2-9	2.3-9	05
arch2	174   161; ; 174;	24	-6.71515401-1	-6.71515409-1	2.8-9	3.7-14	3.5-9	3.5-9	05
arch4	174   161; ; 174;	22	-9.72627407-1	-9.72627420-1	5.4-9	5.0-14	4.4-9	4.4-9	05
arch8	174   161; ; 174;	24	-7.05697992 0	-7.05698005 0	1.4-7	4.7-13	8.9-9	6.0-9	05
control1	21   15; ; ;	17	-1.77846269 1	-1.77846267 1	5.9-9	7.4-11	-4.0-9	2.0-10	00
control2	66   30; ; ;	21	-8.30000364 0	-8.29999999 0	2.0-7	4.1-11	-2.1-7	3.4-10	01
control3	136   45; ; ;	21	-1.36332672 1	-1.36332665 1	4.7-7	1.8-10	-2.2-8	1.8-8	03
control4	231   60; ; ;	21	-1.97942358 1	-1.97942305 1	3.1-7	4.5-10	-1.3-7	2.2-9	06
control5	351   75; ; ;	23	-1.68836048 1	-1.68836008 1	4.3-7	6.1-10	-1.2-7	1.4-7	16
control6	496   90; ; ;	22	-3.73043552 1	-3.73044256 1	<b>1.1-6</b>	1.5-9	9.3-7	4.9-7	31
control7	666   105; ; ;	24	-2.06250645 1	-2.06250748 1	6.6-7	1.7-9	2.4-7	1.6-7	1:05
control8	861   120; ; ;	21	-2.02863293 1	-2.02863711 1	7.7-8	1.9-9	<b>1.0-6</b>	<b>1.0-6</b>	1:39
control9	1081   135; ; ;	21	-1.46754201 1	-1.46754272 1	2.9-7	2.0-9	2.3-7	3.6-7	2:48
control10	1326   150; ; ;	25	-3.85329154 1	-3.85330586 1	7.4-7	5.1-9	<b>1.8-6</b>	<b>1.9-6</b>	1:41
control11	1596   165; ; ;	26	-3.19586089 1	-3.19586886 1	6.4-7	5.0-9	<b>1.2-6</b>	<b>1.2-6</b>	2:45
gpp100	101   100; ; ;	15	4.49435469 1	4.49435488 1	2.9-8	1.7-13	-2.2-8	3.7-10	01
gpp124-1	125   124; ; ;	17	7.34307507 0	7.34307566 0	6.2-9	4.6-13	-3.8-8	6.7-10	01
gpp124-2	125   124; ; ;	15	4.68622922 1	4.68622934 1	3.3-8	2.3-13	-1.3-8	4.1-9	01
gpp124-3	125   124; ; ;	14	1.53014124 2	1.53014125 2	3.8-8	1.3-13	-3.5-9	3.8-9	01
gpp124-4	125   124; ; ;	17	4.18987602 2	4.18987613 2	4.1-8	3.1-13	-1.2-8	3.5-9	01
gpp250-1	251   250; ; ;	17	1.54449168 1	1.54449168 1	1.0-9	2.7-12	-1.4-9	1.5-9	04
gpp250-2	251   250; ; ;	16	8.18689572 1	8.18689580 1	2.2-8	4.7-13	-4.8-9	3.3-10	04
gpp250-3	251   250; ; ;	15	3.03539317 2	3.03539321 2	8.2-8	4.7-12	-5.2-9	7.0-10	04
gpp250-4	251   250; ; ;	14	7.47328311 2	7.47328304 2	8.2-8	2.1-13	4.4-9	8.5-9	04
gpp500-1	501   500; ; ;	21	2.53205446 1	2.53205438 1	9.0-11	2.8-10	1.5-8	1.6-8	29
gpp500-2	501   500; ; ;	16	1.56060388 2	1.56060387 2	4.4-9	8.4-13	2.1-9	3.2-9	23
gpp500-3	501   500; ; ;	15	5.13017612 2	5.13017602 2	2.8-9	9.9-13	9.4-9	9.6-9	22
gpp500-4	501   500; ; ;	17	1.56701880 3	1.56701879 3	1.7-8	3.0-13	1.6-9	1.8-9	24
hinf1	13   14; ; ;	21	-2.03282194 0	-2.03271083 0	5.8-8	1.8-13	<b>-2.2-5</b>	1.0-8	00
hinf2	13   16; ; ;	21	-1.09676614 1	-1.09673573 1	6.2-7	4.4-14	<b>-1.3-5</b>	7.1-9	00

problem	$m$	$n_s; n_q; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
hinf3	13	16; ; ;	21	-5.69615098 1	-5.69511425 1	<b>8.8-6</b>	1.1-13	<b>-9.0-5</b>	5.8-9	00
hinf4	13	16; ; ;	21	-2.74766003 2	-2.74764932 2	1.1-7	8.5-13	<b>-1.9-6</b>	1.8-10	00
hinf5	13	16; ; ;	21	-3.62704607 2	-3.62458662 2	<b>2.9-4</b>	7.1-13	<b>-3.4-4</b>	4.1-8	00
hinf6	13	16; ; ;	21	-4.49067325 2	-4.48998222 2	<b>1.2-5</b>	2.9-13	<b>-7.7-5</b>	<b>1.4-6</b>	00
hinf7	13	16; ; ;	19	-3.90831441 2	-3.90821867 2	<b>5.0-5</b>	1.4-13	<b>-1.2-5</b>	4.2-8	00
hinf8	13	16; ; ;	21	-1.16170056 2	-1.16157996 2	<b>4.5-5</b>	1.7-13	<b>-5.2-5</b>	3.2-7	00
hinf9	13	16; ; ;	21	-2.36249283 2	-2.36249258 2	<b>5.9-6</b>	2.4-14	-5.2-8	3.7-10	00
hinf10	21	18; ; ;	32	-1.08872461 2	-1.08792100 2	<b>1.1-6</b>	3.5-11	<b>-3.7-4</b>	1.4-7	01
hinf11	31	22; ; ;	34	-6.59694072 1	-6.59157089 1	5.0-7	2.7-11	<b>-4.0-4</b>	5.5-8	01
hinf12	43	24; ; ;	46	-1.47406847-1	-7.37049304-2	3.5-10	2.7-9	<b>-6.0-2</b>	2.0-9	01
hinf13	57	30; ; ;	34	-4.45474269 1	-4.44451072 1	<b>7.9-5</b>	1.2-11	<b>-1.1-3</b>	<b>4.5-6</b>	01
hinf14	73	34; ; ;	26	-1.29918409 1	-1.29908452 1	5.2-7	7.2-12	<b>-3.7-5</b>	6.1-8	01
hinf15	91	37; ; ;	29	-2.43319484 1	-2.41545357 1	<b>3.5-5</b>	1.9-11	<b>-3.6-3</b>	<b>5.1-4</b>	02
infd1	10	30; ; ;	5	-5.13280346 0	1.5952217617		primal infeasible			00
infd2	10	30; ; ;	5	5.35505784 0	5.2954130616		primal infeasible			00
infp1	10	30; ; ;	7	-5.64104972 9	-7.62162314 0		dual infeasible			00
infp2	10	30; ; ;	7	-3.85845797 9	-6.90007199 0		dual infeasible			00
mcp100	100	100; ; ;	12	-2.26157350 2	-2.26157352 2	2.3-10	3.0-16	4.3-9	4.3-9	01
mcp124-1	124	124; ; ;	13	-1.41990477 2	-1.41990477 2	2.7-11	2.7-16	4.4-10	4.5-10	01
mcp124-2	124	124; ; ;	13	-2.69880170 2	-2.69880171 2	5.5-12	4.0-16	5.4-10	5.4-10	01
mcp124-3	124	124; ; ;	12	-4.67750108 2	-4.67750115 2	3.7-13	4.6-16	6.7-9	6.7-9	01
mcp124-4	124	124; ; ;	13	-8.64411863 2	-8.64411864 2	2.5-11	4.8-16	6.1-10	6.2-10	01
mcp250-1	250	250; ; ;	14	-3.17264340 2	-3.17264340 2	5.4-12	4.7-16	1.1-9	1.1-9	02
mcp250-2	250	250; ; ;	13	-5.31930080 2	-5.31930084 2	6.3-11	5.1-16	3.4-9	3.4-9	02
mcp250-3	250	250; ; ;	13	-9.81172565 2	-9.81172572 2	5.9-11	6.0-16	3.6-9	3.6-9	02
mcp250-4	250	250; ; ;	14	-1.68196009 3	-1.68196011 3	6.3-12	0.8-15	6.2-9	6.2-9	03
mcp500-1	500	500; ; ;	15	-5.98148516 2	-5.98148517 2	4.5-12	5.2-16	5.9-10	5.9-10	07
mcp500-2	500	500; ; ;	16	-1.07005676 3	-1.07005677 3	9.0-12	6.7-16	1.3-9	1.3-9	11
mcp500-3	500	500; ; ;	15	-1.84797002 3	-1.84797002 3	7.4-12	0.9-15	2.5-10	2.5-10	12
mcp500-4	500	500; ; ;	13	-3.56673799 3	-3.56673805 3	1.9-11	1.0-15	8.5-9	8.5-9	11
qap5	136	26; ; ;	10	4.36000000 2	4.36000000 2	1.2-10	7.3-14	4.6-10	6.1-10	00

problem	$m \mid n_s; n_q; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
qap6	229   37; ; ;	19	3.81432573 2	3.81435498 2	3.3-8	9.6-13	-3.8-6	2.3-9	01
qap7	358   50; ; ;	13	4.24830442 2	4.24810252 2	1.8-8	5.3-13	2.4-5	3.3-5	01
qap8	529   65; ; ;	20	7.56939331 2	7.56947355 2	2.0-7	3.1-12	-5.3-6	7.4-9	05
qap9	748   82; ; ;	20	1.40993093 3	1.40993599 3	3.8-8	1.3-12	-1.8-6	4.4-9	09
qap10	1021   101; ; ;	21	1.09258508 3	1.09259635 3	1.0-7	1.1-12	-5.2-6	2.3-9	17
ss30	132   294; ; 132;	21	-2.02395100 1	-2.02395106 1	4.2-7	2.3-13	1.4-8	1.0-8	22
theta1	104   50; ; ;	11	-2.29999996 1	-2.30000001 1	9.9-12	5.6-15	9.5-9	9.5-9	00
theta2	498   100; ; ;	14	-3.28791689 1	-3.28791690 1	6.2-13	1.2-14	1.4-9	1.4-9	01
theta3	1106   150; ; ;	14	-4.21669813 1	-4.21669815 1	2.6-12	1.4-14	2.1-9	2.1-9	05
theta4	1949   200; ; ;	14	-5.03212213 1	-5.03212220 1	3.4-14	2.1-14	7.8-9	7.8-9	16
theta5	3028   250; ; ;	14	-5.72323069 1	-5.72323073 1	1.9-13	2.9-14	3.9-9	3.9-9	49
theta6	4375   300; ; ;	14	-6.34770870 1	-6.34770872 1	1.7-12	3.2-14	1.9-9	1.9-9	2:10
truss1	6   12; ; 1;	11	8.99999632 0	8.99999631 0	5.8-12	1.0-15	2.7-10	2.7-10	00
truss2	58   132; ; 1;	17	1.23380356 2	1.23380356 2	3.8-10	7.2-15	3.8-10	3.8-10	01
truss3	27   30; ; 1;	12	9.10999622 0	9.10999619 0	9.0-12	5.7-16	1.6-9	1.6-9	00
truss4	12   18; ; 1;	10	9.00999632 0	9.00999626 0	6.0-12	1.2-15	3.1-9	3.1-9	00
truss5	208   330; ; 1;	17	1.32635678 2	1.32635678 2	9.1-11	1.0-14	9.3-10	8.9-10	02
truss6	172   450; ; 1;	28	9.01001420 2	9.01001363 2	5.0-8	2.3-13	3.1-8	4.4-8	02
truss7	86   300; ; 1;	27	9.00001403 2	9.00001391 2	5.3-8	2.4-13	6.9-9	2.2-8	02
truss8	496   627; ; 1;	16	1.33114589 2	1.33114588 2	1.4-10	9.3-15	3.5-9	3.5-9	04
maxG11	800   800; ; ;	15	-6.29164777 2	-6.29164783 2	2.8-11	7.4-16	5.2-9	5.2-9	26
maxG32	2000   2000; ; ;	15	-1.56763962 3	-1.56763964 3	1.3-10	1.1-15	9.1-9	9.1-9	4:21
maxG51	1000   1000; ; ;	17	-4.00625552 3	-4.00625552 3	2.6-12	3.6-16	2.8-10	2.8-10	1:06
qpG11	800   1600; ; ;	15	-2.44865909 3	-2.44865913 3	3.3-11	0.0-16	8.6-9	8.6-9	26
qpG51	1000   2000; ; ;	17	-1.18179999 4	-1.18180000 4	2.3-10	0.0-16	2.2-9	2.2-9	1:03
thetaG11	2401   801; ; ;	19	-3.99999995 2	-4.00000000 2	5.0-9	1.7-13	6.8-9	6.8-9	1:33
thetaG51	6910   1001; ; ;	34	-3.48999315 2	-3.49000166 2	1.4-10	6.8-13	1.2-6	1.2-6	37:14
equalG11	801   801; ; ;	16	-6.29155280 2	-6.29155293 2	8.7-10	1.0-15	9.9-9	9.9-9	1:17
equalG51	1001   1001; ; ;	18	-4.00560126 3	-4.00560132 3	6.5-8	1.3-15	7.4-9	7.5-9	2:38
bm1	883   882; ; ;	20	2.34398777 1	2.34398186 1	2.2-6	1.8-11	1.2-6	1.2-6	2:04
copo14	1275   196; ; 364;	17	2.62856497-10	-6.99780924-10	2.5-11	6.2-15	9.6-10	9.6-10	08

problem	$m \mid n_s; n_q; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
copo23	5820   529; ; 1771;	21	2.56114272-9	-6.38325819-9	2.2-12	8.1-15	8.9-9	8.9-9	18:20
hamming-7-	1793   128; ; ;	9	-4.26666666 1	-4.26666667 1	6.8-11	0.0-16	2.7-10	2.0-10	10
hamming-9-	2305   512; ; ;	9	-2.23999997 2	-2.24000001 2	3.4-13	8.4-14	8.3-9	8.3-9	24
minphase	48   48; ; ;	29	5.97872628 0	5.98119359 0	1.3-8	5.7-13	<b>-1.9-4</b>	4.8-7	01
torusg3-8	512   512; ; ;	15	-4.83409459 7	-4.83409459 7	3.4-12	7.8-16	9.5-10	9.5-10	11
toruspm3-8	512   512; ; ;	14	-5.27808660 2	-5.27808663 2	6.9-11	6.0-16	2.8-9	2.8-9	10
torusg3-15	3375   3375; ; ;	16	-6.37621845 3	-6.37621855 3	1.2-9	1.5-15	7.6-9	7.6-9	23:41
toruspm3-1	3375   3375; ; ;	16	-3.47513185 3	-3.47513186 3	2.1-11	1.5-15	2.0-9	2.0-9	23:46
filter48	969   48; 49; 931;	43	1.41612972 0	1.41612914 0	2.9-7	8.9-14	1.5-7	1.2-8	1:01
filtinfl	983   49; 49; 945;	33	0.00000000-16	2.98905416 1		primal infeasible			48
nb	123   ; 2379; 4;	22	-5.07030865-2	-5.07030948-2	3.1-13	6.1-16	7.5-9	7.5-9	07
nb-L1	915   ; 2379; 797;	23	-1.30122706 1	-1.30122709 1	1.6-10	9.7-14	9.0-9	9.0-9	13
nb-L2	123   ; 4191; 4;	17	-1.62897195 0	-1.62897198 0	1.7-10	1.9-15	7.2-9	7.2-9	10
nb-L2-bess	123   ; 2637; 4;	19	-1.02569502-1	-1.02569511-1	1.0-13	1.1-15	7.4-9	7.4-9	07
nql30	3680   ; 2700; 3602;	35	-9.46028502-1	-9.46028517-1	5.2-11	2.6-10	4.9-9	5.7-9	04
nql60	14560   ; 10800; 14402;	33	-9.35052951-1	-9.35052975-1	5.0-11	7.4-11	8.2-9	9.4-9	17
nql180	130080   ; 97200; 129602;	39	-9.27728621-1	-9.27728643-1	7.8-10	2.0-11	7.8-9	8.8-9	4:38
qssp30	3691   ; 7564; 2;	17	-6.49667573 0	-6.49667575 0	1.9-10	1.1-13	1.5-9	1.5-9	03
qssp60	14581   ; 29524; 2;	23	-6.56270608 0	-6.56270649 0	2.9-8	4.0-15	2.8-8	3.4-9	17
qssp180	130141   ; 261364; 2;	28	-6.63960843 0	-6.63961086 0	3.1-7	1.0-14	1.7-7	4.2-9	6:31
sched-50-5	2527   ; 2477; 2502;	31	2.66753929 4	2.66728864 4	<b>1.5-5</b>	1.5-7	<b>4.7-5</b>	<b>4.7-5</b>	03
sched-100-	4844   ; 4744; 5002;	34	1.81900808 5	1.81887534 5	<b>3.4-4</b>	1.1-9	<b>3.6-5</b>	<b>3.6-5</b>	07
sched-100-	8338   ; 8238; 10002;	29	7.17597254 5	7.17350508 5	<b>1.4-2</b>	1.0-10	<b>1.7-4</b>	<b>1.7-4</b>	13
sched-200-	18087   ; 17887; 20002;	35	1.41946190 5	1.41176207 5	<b>1.3-4</b>	5.6-7	<b>2.7-3</b>	<b>2.7-3</b>	46
sched-50-5	2526   ; 2475; 2502;	27	7.85203874 0	7.85203843 0	2.3-7	4.3-15	1.9-8	1.9-8	03
sched-100-	4843   ; 4742; 5002;	29	6.71650648 1	6.71650261 1	1.3-7	8.6-14	2.9-7	2.9-7	06
sched-100-	8337   ; 8236; 10002;	29	2.73307879 1	2.73307855 1	1.1-7	2.8-14	4.3-8	4.3-8	12
sched-200-	18086   ; 17885; 20002;	37	5.18119615 1	5.18119610 1	1.0-7	1.2-13	4.5-9	4.5-9	44
biggs	1819   702; ; ;	53	-1.41425775 3	-1.41425841 3	3.4-9	1.9-11	2.3-7	4.7-9	1:30
buck1	36   49; ; 36;	17	-1.46419152 2	-1.46419152 2	8.2-10	3.2-14	7.7-10	8.1-10	01
buck2	144   193; ; 144;	21	-2.92368313 2	-2.92368295 2	8.3-7	7.0-14	-3.1-8	2.0-8	04

problem	$m \mid n_s; n_g; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
buck3	544   641; ; 544;	28	-6.07589498 2	-6.07608699 2	<b>1.2-4</b>	2.1-13	<b>1.6-5</b>	<b>1.7-5</b>	20
buck4	1200   1345; ; 1200;	33	-4.86141048 2	-4.86143830 2	<b>9.9-6</b>	4.0-13	<b>2.9-6</b>	<b>2.8-6</b>	2:12
buck5	3280   3521; ; 3280;	37	-4.36170859 2	-4.36283022 2	<b>2.2-5</b>	6.5-13	<b>1.3-4</b>	<b>1.3-4</b>	25:33
cnhil10	5005   220; ; ;	27	0.00000000-16	-1.65482516-4	6.4-8	3.2-12	<b>1.7-4</b>	2.9-7	55
cnhil8	1716   120; ; ;	25	0.00000000-16	-1.78946733-5	1.8-8	1.1-12	<b>1.8-5</b>	5.0-8	07
cphil10	5005   220; ; ;	9	0.00000000-16	-8.11544501-9	1.1-14	0.0-16	8.1-9	8.1-9	31
cphil12	12376   364; ; ;	10	0.00000000-16	-2.93810832-10	1.5-14	0.0-16	2.9-10	2.9-10	4:01
G40-mb	2001   2000; ; ;	22	-2.86432310 3	-2.86432323 3	9.1-10	2.1-11	2.1-8	2.1-8	21:04
G40mc	2000   2000; ; ;	18	-5.72957909 3	-5.72957911 3	1.1-10	5.8-16	1.1-9	1.1-9	7:07
G48mc	3000   3000; ; ;	12	-1.20000000 4	-1.20000000 4	5.6-12	5.7-15	4.2-10	4.2-10	10:06
G55mc	5000   5000; ; ;	16	-2.20789206 4	-2.20789208 4	9.6-10	2.2-15	4.7-9	4.7-9	1:14:47
G59mc	5000   5000; ; ;	19	-1.46246530 4	-1.46246531 4	6.6-10	1.0-15	4.7-9	4.7-9	1:36:36
mater-1	103   220; ; 2;	21	1.43465439 2	1.43465438 2	1.4-10	1.6-14	3.5-9	3.5-9	01
mater-2	423   1012; ; 2;	20	1.41591867 2	1.41591866 2	6.3-12	7.2-14	1.1-9	1.1-9	04
mater-3	1439   3586; ; 2;	24	1.33916258 2	1.33916256 2	7.3-11	2.5-13	9.6-9	9.7-9	16
mater-4	4807   12496; ; 2;	29	1.34262716 2	1.34262716 2	1.6-9	8.8-13	2.6-9	3.8-9	1:11
mater-5	10143   26818; ; 2;	30	1.33801638 2	1.33801640 2	9.9-9	1.8-12	-6.4-9	4.4-9	3:03
mater-6	20463   54626; ; 2;	32	1.33538673 2	1.33538714 2	6.2-8	3.1-12	-1.5-7	8.0-9	8:21
neosfbr12	1441   122; ; ;	17	5.29319164 2	5.29319158 2	3.5-11	0.8-15	5.7-9	5.7-9	18
neosfbr20									
neosfbr21									
neosfbr22									
neosfbr24									
neosfbr25									
neosfbr30e									
neu1	3003   252; ; 2;	31	4.36376962-7	-1.34195143-5	7.3-10	2.3-11	<b>1.4-5</b>	<b>3.6-5</b>	8:21
neu2	3003   252; ; 2;	40	-1.61121339-4	-2.36860767-4	1.6-8	6.8-11	<b>7.6-5</b>	<b>6.5-4</b>	10:21
neu2c	3002   1253; ; 2;	59	3.41771574 4	3.40921240 4	<b>3.8-3</b>	2.5-11	<b>1.2-3</b>	<b>2.0-4</b>	32:28
neu2g	3002   252; ; ;	38	3.40999994 4	3.40999997 4	3.0-9	1.1-10	-5.3-9	1.4-8	9:12
neu3	7364   418; ; 2;	46	1.33597463-7	-1.96267080-4	2.4-9	8.7-10	<b>2.0-4</b>	<b>2.7-6</b>	8:01
neu3g	8007   462; ; ;	47	9.25315729-9	-1.56570129-4	1.6-9	7.7-10	<b>1.6-4</b>	2.0-7	11:02

problem	$m \mid n_s; n_g; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
r1-6-0	601   600; ; ;	21	-5.57968708 4	-5.57968706 4	6.5-7	5.9-11	-1.6-9	1.3-10	41
r1-6-1	601   600; ; ;	14	-5.58043922 4	-5.58043924 4	2.8-10	2.1-14	2.2-9	2.2-9	29
r1-6-1e-6	601   600; ; ;	14	-5.57968722 4	-5.57968731 4	1.1-7	6.7-12	8.3-9	7.9-9	29
rose13	2379   105; ; ;	31	1.20000012 1	1.19999998 1	8.4-9	5.9-16	5.6-8	7.4-9	1:08
rose15	3860   135; ; 2;	38	1.82090663-5	-1.26745042-5	7.3-8	1.7-13	<b>3.1-5</b>	5.7-8	5:44
sdmint3	5255   379; 5255; ;	29	-4.78135763 3	-4.78131886 3	<b>2.5-4</b>	9.4-14	<b>-4.1-6</b>	<b>9.3-6</b>	47:47
shmup1	16   81; ; 32;	17	-1.88414831 2	-1.88414832 2	1.1-9	3.0-15	2.8-9	2.8-9	01
shmup2	200   881; ; 400;	39	-3.46242668 3	-3.46242684 3	<b>1.0-6</b>	7.3-15	2.3-8	2.6-8	57
shmup3	420   1801; ; 840;	31	-2.09883698 3	-2.09883796 3	<b>2.3-6</b>	6.5-15	2.3-7	2.2-7	4:06
shmup4	800   3361; ; 1600;	38	-7.99254514 3	-7.99255230 3	<b>7.8-6</b>	7.4-15	4.5-7	4.5-7	20:54
shmup5	1800   7441; ; 3600;	36	-2.38576762 4	-2.38591658 4	<b>6.5-6</b>	3.1-14	<b>3.1-5</b>	<b>3.1-5</b>	2:36:00
tahala	3002   1680; ; ;	27	-9.31533327-1	-1.25589295 0	<b>9.2-2</b>	4.7-12	<b>1.0-1</b>	<b>9.0-2</b>	15:27
tahalb	8007   1606; ; 3;	37	-7.73313007-1	-7.73313020-1	2.5-11	3.2-15	4.9-9	5.9-9	47:32
trto1	36   25; ; 36;	21	-1.10450000 3	-1.10450000 3	4.4-8	4.5-14	6.4-10	3.5-10	01
trto2	144   97; ; 144;	21	-1.27999978 4	-1.28000007 4	3.7-7	1.2-12	1.1-7	1.1-7	01
trto3	544   321; ; 544;	24	-1.27999344 4	-1.28000857 4	<b>3.7-4</b>	3.1-12	<b>5.9-6</b>	<b>5.9-6</b>	09
trto4	1200   673; ; 1200;	27	-1.27606346 4	-1.27701954 4	<b>1.4-3</b>	7.4-12	<b>3.7-4</b>	<b>3.7-4</b>	52
trto5	3280   1761; ; 3280;	29	-1.27928272 4	-1.28021188 4	<b>5.7-4</b>	1.7-11	<b>3.6-4</b>	<b>3.6-4</b>	10:43
vibra1	36   49; ; 36;	13	-4.08190124 1	-4.08190124 1	3.3-10	3.2-15	4.7-10	4.8-10	01
vibra2	144   193; ; 144;	23	-1.66015338 2	-1.66015364 2	<b>1.2-5</b>	2.0-14	8.0-8	8.8-8	05
vibra3	544   641; ; 544;	29	-1.72605378 2	-1.72614934 2	<b>1.1-4</b>	8.0-14	<b>2.8-5</b>	<b>2.8-5</b>	20
vibra4	1200   1345; ; 1200;	29	-1.27636319 4	-1.27676506 4	<b>1.5-3</b>	9.4-12	<b>1.6-4</b>	<b>1.6-4</b>	1:50
vibra5	3280   3521; ; 3280;	57	-1.65803004 2	-1.65935491 2	<b>7.2-5</b>	2.6-13	<b>4.0-4</b>	<b>4.0-4</b>	39:13
yalsdp	5051   300; ; ;	13	-1.79212601 0	-1.79212676 0	1.5-11	1.6-13	1.6-7	1.6-7	10:33
cancer-100	10469   569; ; ;	14	-2.76091613 4	-2.76233961 4	<b>6.0-6</b>	9.8-12	<b>2.6-4</b>	<b>3.0-4</b>	26:16
checker-1.	3970   3970; ; ;	24	3.30388458 3	3.30388454 3	3.6-10	0.1-16	6.4-9	6.4-9	45:04
foot	2209   2208; ; ;	20	-5.85252481 5	-5.85298195 5	<b>5.8-5</b>	2.0-8	<b>3.9-5</b>	<b>3.9-5</b>	24:57
hand	1297   1296; ; ;	21	-2.47477792 4	-2.47477791 4	6.0-7	2.4-10	-1.4-9	1.1-10	5:31
inc-600	2515   600; ; 2514;	25	-6.60104091-1	-6.68915553-1	<b>2.1-5</b>	1.5-9	<b>3.8-3</b>	<b>4.1-3</b>	2:07
inc-1200	5175   1200; ; 5174;	15	2.12231915 1	-1.41967770 0	<b>2.0-5</b>	1.1-9	<b>9.6-1</b>	<b>9.6-1</b>	7:16
swissroll	3380   800; ; ;	31	-5.48284431 5	-5.59816094 5	<b>2.7-3</b>	6.4-9	<b>1.0-2</b>	<b>1.1-1</b>	5:30



problem	$m \mid n_s; n_q; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
tiger-text	1802   1801; ; ;	25	3.48084179 2	3.44294211 2	<b>2.1-5</b>	1.7-8	<b>5.5-3</b>	<b>1.5-2</b>	6:22
diamond-pa	5478   5477; ; ;	31	3.22638853 1	1.56694179 1	<b>1.6-5</b>	1.6-7	<b>3.4-1</b>	<b>1.2 0</b>	2:45:18
ice-2.0	8113   8113; ; ;	28	6.80838634 3	6.80838622 3	1.1-10	0.0-16	9.1-9	9.1-9	7:13:27
p-auss2-3.									
butcher	6434   330; ; 22512;	43	-1.41958589 1	-1.40018649 1	<b>3.7-2</b>	<b>1.1-6</b>	<b>-6.6-3</b>	<b>1.7-3</b>	45:05
rabmo	5004   220; ; 6606;	47	-3.72725362 0	-3.72725497 0	<b>6.2-6</b>	5.1-8	1.6-7	<b>1.7-5</b>	15:53
reimer5	6187   462; ; 102144;	24	-1.51824956 1	-1.51835452 1	<b>1.0-3</b>	1.0-9	<b>3.3-5</b>	<b>1.6-4</b>	59:43
chs-500	9974   4980; ; ;	23	9.32785242-10	-8.36219073-9	8.2-14	0.0-16	9.3-9	9.3-9	30
nonc-500	4990   2998; ; ;	22	6.25761159-2	6.25597999-2	1.3-8	2.2-15	<b>1.5-5</b>	<b>1.4-5</b>	10
ros-500	4988   2992; ; ;	17	2.49499943 0	2.49499945 0	1.2-8	1.8-15	-3.5-9	3.1-9	08
fp210	1000   176; ; ; 66	25	3.74999982-1	3.74999999-1	1.3-7	3.9-10	-10.0-9	4.4-9	09
fp22	14   15; ; ;	12	-7.99999999 0	-8.00000007 0	4.2-10	1.4-15	4.7-9	6.9-9	00
fp23	209   119; ; ;	29	2.13000000 2	2.12999999 2	2.0-11	1.2-13	1.3-9	2.0-9	02
fp24	2379   595; ; ;	22	1.95000000 2	1.95000000 2	1.0-11	3.5-13	1.1-9	1.4-9	56
fp25	209   133; ; ;	18	1.10000000 1	1.10000000 1	1.2-12	1.2-14	8.7-10	1.1-9	01
fp26	1000   407; ; ;	22	2.68014631 2	2.68014631 2	9.5-10	3.0-15	4.5-10	7.7-10	10
fp27	1000   341; ; ;	21	3.90000000 1	3.89999997 1	4.2-10	4.7-13	4.5-9	6.4-9	09
fp32	3002   1155; ; ;	44	-7.04227560 0	-7.05278292 0	<b>4.4-3</b>	9.7-12	<b>7.0-4</b>	<b>2.9-4</b>	7:05
fp33	125   117; ; ;	39	-1.01265940 4	-1.01266024 4	2.6-10	4.3-11	4.1-7	2.5-9	02
fp34	209   140; ; ;	22	1.72000000 2	1.72000000 2	1.2-12	1.0-13	-2.9-10	6.2-10	01
fp35	164   195; ; ;	18	3.99999993 0	3.99999965 0	4.2-9	1.1-13	3.0-8	5.3-8	03
fp410	14   18; ; ; 1	16	1.67388932 1	1.67388931 1	1.6-11	4.7-16	3.3-9	3.6-9	00
fp42	6   10; ; ;	10	7.58731237 0	7.58731235 0	6.5-12	6.4-16	1.1-9	1.1-9	00
fp43	50   76; ; ;	16	6.54776263 2	6.63152380 2	<b>2.5-4</b>	2.1-10	<b>-6.4-3</b>	<b>1.7-6</b>	01
fp44	6   10; ; ;	22	4.43671691 2	4.43671704 2	2.7-8	4.5-14	-1.5-8	1.9-9	00
fp45	4   7; ; ;	13	1.24858128-9	-5.52323520-10	1.1-9	5.0-16	1.8-9	2.4-9	00
fp46	27   22; ; ;	21	9.94242098-10	-4.64240829-9	1.3-10	2.1-16	5.6-9	7.3-9	00
fp47	6   10; ; ;	16	2.43000000 2	2.43000000 2	5.3-11	1.9-14	2.1-10	1.6-10	00
fp48	4   7; ; ;	9	7.50000000 0	7.50000000 0	2.1-11	1.5-16	3.1-10	3.0-10	00
fp49	14   18; ; ; 1	16	1.67388932 1	1.67388931 1	1.6-11	4.7-16	3.3-9	3.6-9	00
l1	14   6; ; ;	8	4.92634657-1	4.92634644-1	4.3-12	0.2-16	6.5-9	6.5-9	00

problem	$m \mid n_s; n_g; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
l2	14   6; ; ;	9	1.14580631 1	1.14580630 1	5.1-11	2.2-16	1.5-9	1.5-9	00
l4	152   45; ; ;	19	3.70370379-2	3.70371914-2	4.0-10	1.4-14	-1.4-7	8.6-9	01
l5	14   15; ; ;	12	-7.99999999 0	-8.00000007 0	4.2-10	1.4-15	4.7-9	6.9-9	00
5n	31   26; ; ;	9	2.24000000 0	2.23999999 0	6.8-12	2.6-16	2.8-9	2.8-9	00
a12	793   79; ; ;	12	2.10000000 1	2.10000000 1	3.8-11	1.6-15	9.3-10	9.3-10	03
aw29	465   130; ; ;	11	3.00000000 0	3.00000000 0	6.8-11	4.9-16	1.1-9	1.2-9	06
c5	31   26; ; ;	8	1.50000001 0	1.49999999 0	1.1-11	1.1-16	3.9-9	4.0-9	00
fp1131	847   176; ; ;	11	4.50000001 0	4.49999996 0	1.4-12	1.6-15	4.9-9	4.9-9	22
fp1132	847   176; ; ;	12	1.55000000 1	1.54999999 1	5.9-14	5.2-15	3.8-9	3.8-9	24
fp1133	847   176; ; ;	12	1.75000000 1	1.75000000 1	5.4-14	5.4-15	1.7-9	1.7-9	24
fp1134	847   176; ; ;	12	1.95000000 1	1.94999999 1	4.8-14	5.1-15	3.0-9	3.0-9	24
fp1135	847   176; ; ;	12	2.20000000 1	2.20000000 1	1.5-11	1.7-15	8.8-10	8.8-10	24
fp1136	847   176; ; ;	12	1.45000000 1	1.44999999 1	5.9-14	5.1-15	4.8-9	4.8-9	24
fp1137	847   176; ; ;	12	1.65000000 1	1.64999999 1	8.3-14	4.1-15	2.5-9	2.5-9	24
fp1138	847   176; ; ;	13	1.75000000 1	1.75000000 1	5.9-14	5.1-15	6.8-10	6.8-10	26
fp1139	847   176; ; ;	12	2.30000000 1	2.29999999 1	2.4-12	2.6-15	3.1-9	3.1-9	24
k5	31   31; ; ;	8	1.00000001 0	9.99999995-1	3.7-11	2.1-16	4.5-9	4.5-9	00
p10	847   176; ; ;	11	4.50000001 0	4.49999996 0	2.9-12	1.3-15	4.9-9	4.9-9	22
bifur	454   84; ; ; 1661	21	-3.37301697-1	-3.37301697-1	2.1-9	2.1-10	-3.4-11	1.7-9	04
boom	3002   210; ; ; 8764	29	-3.23707245 2	-3.23707248 2	1.1-8	<b>1.2-6</b>	4.6-9	8.5-7	4:00
brown	461   56; ; ; 925	34	3.04680725-11	0.00000000-16	1.7-11	3.0-11	3.0-11	3.8-9	04
butcher	6434   330; ; ; 11256	48	-1.39810186 1	-1.39324497 1	<b>4.4-6</b>	<b>6.1-4</b>	<b>-1.7-3</b>	<b>2.2-2</b>	47:32
camera1s	209   28; ; ; 168	57	-1.78688795 4	-1.78686202 4	<b>2.0-4</b>	6.0-8	<b>-7.3-6</b>	<b>8.3-6</b>	02
caprasse	209   35; ; ; 60	17	-2.36780178-1	-2.36780183-1	9.3-11	2.1-9	3.8-9	6.8-9	01
cassou	4844   495; ; ; 35126	27	-3.8317999911	-6.15988262 6			dual infeasible		1:17:04
cdpm5	125   21; ; ; 5	11	4.81891306-10	-7.57760884-9	5.4-15	0.0-16	8.1-9	8.1-9	00
chemequ	461   56; ; ; 525	75	-7.55021050 8	-4.87753741 7			dual infeasible		07
chemequs	125   21; ; ; 45	15	-1.46594733 7	-1.25336541 5			dual infeasible		00
cohn2	209   35; ; ; 4	41	2.44989647-10	-1.17751717-8	2.7-7	1.1-8	1.2-8	1.2-8	01
cohn3	209   35; ; ; 4	15	1.34558839-11	-4.75678015-10	2.9-13	4.1-15	4.9-10	4.9-10	01
comb3000	1000   66; ; ; 595	19	-4.74520651-10	-6.87418379-10	1.6-11	4.4-10	2.1-10	2.2-9	05

problem	$m \mid n_s; n_g; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
conform1	83   20; ; ; 30	13	-6.04355528 8	-6.64897488 3		dual infeasible			00
conform3	285   56; ; ; 630	22	-9.45599155-12	0.00000000-16	1.8-9	5.0-11	-9.5-12	6.7-10	01
conform4	454   84; ; ; 1890	21	3.66213726-11	0.00000000-16	2.6-9	2.0-11	3.7-11	3.0-9	04
des18-3	12869   495; ; ; 58920	82	-3.23787125 6	-2.76403795 6		dual infeasible			8:58:52
des22-24	1000   66; ; ; 660	23	-6.74166365 3	-6.74166365 3	1.6-10	6.9-9	4.5-11	1.0-9	06
discret3	44   9; ; ; 8	11	-3.70033135 1	-3.70033140 1	3.2-11	3.4-15	6.7-9	6.7-9	00
eco5	461   56; ; ; 525	19	-1.20463311 3	-1.20463311 3	2.5-9	2.6-9	-2.2-12	8.1-10	02
eco6	923   84; ; ; 924	26	-1.00281559 4	-1.00281559 4	4.7-11	3.1-9	5.8-11	3.7-10	09
eco7	1715   120; ; ; 1512	24	-3.91531048 3	-3.91531047 3	7.3-9	2.0-9	-4.0-10	4.8-10	32
eco8	3002   165; ; ; 2340	26	-5.82038418 3	-5.82038418 3	2.7-10	8.1-9	-4.2-12	1.5-10	2:15
fourbar	69   15; ; ; 4	11	1.16235821-10	-4.24153800-9	4.7-12	1.8-12	4.4-9	4.5-9	00
geneig	923   84; ; ; 546	16	-2.52663014 0	-2.52663014 0	5.6-12	1.5-9	9.6-11	7.9-9	05
heart	3002   165; ; ; 4320	17	-8.70927422 1	-8.70927422 1	2.5-11	8.1-9	2.2-11	4.6-9	1:29
il	1000   66; ; ; 10	12	-1.66775273 0	-1.66775273 0	2.3-13	0.9-16	8.6-10	8.6-10	04
ipp	494   45; ; ; 360	30	-1.31158853 1	-1.31158853 1	9.9-10	1.5-9	-6.0-11	2.1-9	02
katsura5	209   28; ; ; 168	22	-8.16044579-2	-8.16044579-2	3.3-9	1.3-10	3.6-11	2.0-9	01
kinema	714   55; ; ; 495	28	-4.19683963 4	-4.19683963 4	2.8-8	4.6-9	-1.0-10	1.2-9	04
ku10	1000   66; ; ; 660	24	-7.13900000 3	-7.13900000 3	2.0-10	2.9-9	9.0-11	2.0-9	07
lorentz	69   15; ; ; 60	16	-5.00000000 0	-5.00000000 0	5.9-12	2.7-9	6.1-11	2.0-9	00
manocha	90   28; ; ; 42	27	-2.45943538-1	-2.45956385-1	4.0-8	<b>1.6-5</b>	<b>8.6-6</b>	<b>8.9-4</b>	01
noon3	83   20; ; ; 30	11	-2.08695033 1	-2.08695034 1	3.6-13	1.6-9	1.2-9	2.6-9	00
noon4	209   35; ; ; 60	15	-1.71283759 1	-1.71283761 1	1.9-11	1.8-9	5.7-9	8.1-9	01
noon5	461   56; ; ; 105	17	-1.58524243 1	-1.58524243 1	1.7-12	4.9-9	3.3-10	2.1-9	02
proddeco	69   15; ; ; 4	11	1.84597568-11	-3.37004061-10	4.3-14	0.0-16	3.6-10	3.6-10	00
puma	3002   165; ; ; 8280	37	-3.05299490 1	-3.05299490 1	2.4-8	2.4-9	-7.9-10	1.7-9	3:16
quadfor2	209   35; ; ; 270	19	-6.18518518 0	-6.18518519 0	2.8-12	4.3-9	6.7-10	4.5-9	01
quadgrid	461   56; ; ; 505	82	-2.96150529 7	-2.95257544 7		dual infeasible			11
rabmo	5004   220; ; ; 3303	59	-3.72725312 0	-3.72725312 0	8.6-8	3.9-10	-3.9-10	2.3-9	19:16
rbpl	923   84; ; ; 546	18	-7.94063377 0	-7.94063378 0	5.1-10	9.8-10	5.2-10	9.8-9	06
redeco5	20   6; ; ; 5	8	-2.53906250-1	-2.53906251-1	1.6-15	0.4-16	1.2-9	1.2-9	00
redeco6	27   7; ; ; 6	8	-2.01600000-1	-2.01600002-1	3.4-14	1.1-16	1.6-9	1.6-9	00

problem	$m \mid n_s; n_q; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
redeco7	35   8; ; ; 7	8	-1.67438271-1	-1.67438274-1	1.5-14	0.5-16	2.0-9	2.0-9	00
redeco8	44   9; ; ; 8	8	-1.43273636-1	-1.43273639-1	9.2-15	0.7-16	2.4-9	2.4-9	00
rediff3	9   4; ; ; 3	9	4.04469621-10	-1.21340922-9	1.0-15	0.0-16	1.6-9	1.6-9	00
rose	679   120; ; ; 2281	31	-1.72122696 0	-1.70726881 0	<b>1.0-6</b>	<b>3.4-4</b>	<b>-3.2-3</b>	<b>3.7-2</b>	18
s9-1	494   45; ; ; 360	21	-4.27369565 0	-4.27369565 0	5.3-11	5.8-10	2.0-10	5.8-9	02
sendra	65   21; ; ; 12	11	-2.37687542 1	-2.37687543 1	2.4-10	3.4-12	2.5-9	2.8-9	00
solotarev	69   15; ; ; 32	14	-5.88961333 0	-5.88961337 0	2.3-12	1.3-9	2.6-9	6.6-9	00
stewart1	714   55; ; ; 495	35	-8.76585278 0	-8.76585278 0	1.3-10	3.4-9	-9.1-11	6.7-9	05
stewart2	1819   91; ; ; 910	32	-1.27531388 1	-1.27531386 1	1.0-7	8.1-10	-1.0-8	1.0-9	39
trinks	209   28; ; ; 141	31	-2.43523491-1	-2.43523492-1	3.3-9	1.3-11	9.4-11	9.1-10	01
visasoro	44   9; ; ; 8	11	6.97297058-10	-4.26654223-9	1.9-15	0.0-16	5.0-9	5.0-9	00
wood	69   15; ; ; 32	20	-6.64233343-2	-6.64233383-2	1.0-9	3.3-10	3.5-9	8.4-9	00
wright	20   6; ; ; 5	9	-2.00000000 1	-2.00000002 1	5.9-13	2.1-16	6.5-9	6.5-9	00
nql30o	3680   ; 2700; 3602;	35	-9.46028502-1	-9.46028517-1	7.0-12	2.6-10	4.9-9	5.7-9	04
nql60o	14560   ; 10800; 14402;	33	-9.35052951-1	-9.35052975-1	3.9-10	7.4-11	8.1-9	9.4-9	15
nql90o	32640   ; 24300; 32402;	36	-9.31383164-1	-9.31383188-1	7.0-11	4.2-11	8.1-9	9.2-9	43
nql120o	57920   ; 43200; 57602;	36	-9.29550235-1	-9.29550250-1	7.1-10	2.0-11	5.1-9	5.9-9	1:29
nql180o	130080   ; 97200; 129602;	39	-9.27728621-1	-9.27728643-1	4.4-10	2.0-11	7.8-9	8.8-9	4:31
qs30o	1861   ; 3844; 2;	15	-6.29531550 0	-6.29531557 0	1.0-9	2.4-15	5.0-9	4.3-9	01
qs60o	7321   ; 14884; 2;	17	-6.38210348 0	-6.38210354 0	1.8-9	6.2-14	4.5-9	5.0-9	05
qs90o	16381   ; 33124; 2;	20	-6.42377345 0	-6.42377320 0	1.2-8	6.7-15	-1.8-8	1.3-9	16
qs120o	29041   ; 58564; 2;	21	-6.45014299 0	-6.45014275 0	9.4-9	8.9-15	-1.7-8	1.9-10	38
qs180o	65161   ; 131044; 2;	22	-6.48350884 0	-6.48350889 0	9.3-10	1.3-14	3.5-9	6.2-9	2:04
qt30o	3924   ; 2883; 3846;	43	-3.43399895-1	-3.43399908-1	2.2-11	9.0-11	7.8-9	8.1-9	17
qt60o	15044   ; 11163; 14886;	52	-3.90343193-1	-3.90343204-1	4.4-11	3.8-11	6.4-9	6.1-9	2:43
qt90o	33364   ; 24843; 33126;	54	-4.05801683-1	-4.05801698-1	9.8-11	3.1-11	8.0-9	7.6-9	10:18
qt120o	58884   ; 43923; 58566;	53	-4.12995497-1	-4.12995510-1	1.5-10	2.0-11	7.3-9	7.1-9	27:55
qt180o	131524   ; 98283; ; 65523	38	-4.19731365-1	-4.19960022-1	8.3-10	<b>3.5-6</b>	<b>1.2-4</b>	<b>1.1-3</b>	1:13:29
q30o	7482   ; 11163; 2;	32	-9.36405103-1	-9.36405121-1	2.8-10	2.2-15	6.4-9	6.6-9	40
q60o	29362   ; 43923; 2;	23	-8.73587717-7	-1.74652326-3	<b>6.7-6</b>	8.5-11	<b>1.7-3</b>	<b>4.2-4</b>	30:17
dsNRL	406   ; 15897; ;	36	-5.57436155-5	-5.57492818-5	4.4-12	3.6-14	5.7-9	5.7-9	11:02

problem	$m \mid n_s; n_g; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
firL1Linfa	3074   ; 17532; ;	26	-3.06731077-3	-3.06731718-3	3.7-11	2.5-14	6.4-9	6.4-9	2:26
firL1Linfe	7088   ; 13932; 1;	30	-2.71128527-3	-2.71129475-3	8.2-9	1.4-14	9.4-9	9.4-9	2:05
firL1	6223   ; 17766; ;	22	-2.92573392-4	-2.92582151-4	1.5-11	3.1-14	8.8-9	8.8-9	7:47
firL2a	1002   ; 1003; ;	7	-7.14574800-4	-7.14579339-4	0.8-15	0.8-16	4.5-9	4.5-9	22
firL2L1alp	5868   ; 9611; 1;	18	-5.76340548-5	-5.76372249-5	1.6-12	1.3-14	3.2-9	3.2-9	1:09
firL2L1eps	4124   ; 11969; ;	18	-8.44806167-4	-8.44815323-4	9.5-12	2.2-14	9.1-9	9.1-9	2:13
firL2Linfa	203   ; 9029; ;	28	-7.05910969-3	-7.05911673-3	1.1-11	1.6-14	6.9-9	6.9-9	1:21
firL2Linfe	6086   ; 14711; ;	16	-1.48919889-3	-1.48920536-3	1.2-10	1.6-14	6.5-9	6.5-9	3:51
firL2	102   ; 103; ;	7	-3.11866437-3	-3.11866484-3	7.5-16	1.1-16	4.7-10	4.7-10	00
firLinf	402   ; 11886; ;	24	-1.00681685-2	-1.00681770-2	1.4-9	2.5-14	8.3-9	8.2-9	5:23
BeH-2Sigma	948   1406; ; ;	31	1.66935641 1	1.66935639 1	3.7-12	5.9-16	7.0-9	7.0-9	2:11
BH-1Sigma+	948   1406; ; ;	30	2.72063377 1	2.72063375 1	2.3-11	2.4-16	3.9-9	3.9-9	5:53
BH2-2A1-ST	1743   2166; ; ;	30	3.04301171 1	3.04301166 1	3.0-9	3.8-16	7.5-9	7.5-9	24:18
BH+-2Sigma	948   1406; ; ;	30	2.69796662 1	2.69796657 1	3.9-12	2.8-16	9.6-9	9.6-9	2:09
CH+-1Sigma	948   1406; ; ;	29	4.06927879 1	4.06927873 1	3.6-11	6.9-16	7.4-9	7.4-9	2:06
CH2-1A1-ST	1743   2166; ; ;	29	4.48537628 1	4.48537626 1	1.1-9	2.6-16	2.7-9	2.8-9	23:36
CH2-3B1-ST	1743   2166; ; ;	30	4.50291330 1	4.50291327 1	1.1-9	1.0-15	2.8-9	2.8-9	24:15
CH-2Pi-STO	948   1406; ; ;	29	4.10222179 1	4.10222176 1	6.5-11	4.7-16	3.7-9	3.7-9	5:39
CH-3Sigma	948   1406; ; ;	30	4.09070914 1	4.09070909 1	1.9-11	4.9-16	6.5-9	6.5-9	5:52
H2O-1A1-ST	1743   2166; ; ;	28	8.49236908 1	8.49236900 1	9.9-12	2.9-16	4.7-9	4.7-9	22:41
H2O+-2B1-S	1743   2166; ; ;	30	8.42163768 1	8.42163759 1	9.9-11	6.7-16	5.5-9	5.5-9	24:16
HF-1Sigma+	948   1406; ; ;	27	1.04720454 2	1.04720452 2	4.2-12	4.9-16	7.6-9	7.6-9	5:15
HF+-2Pi-ST	948   1406; ; ;	27	1.03885668 2	1.03885666 2	2.6-11	5.2-16	8.5-9	8.5-9	5:17
LiH-1Sigma	948   1406; ; ;	31	8.96721196 0	8.96721180 0	3.0-11	7.5-16	8.4-9	8.4-9	2:10
NH2-2B1-ST	1743   2166; ; ;	30	6.29798021 1	6.29798015 1	1.5-10	0.9-15	4.6-9	4.6-9	24:17
NH+-2Pi-ST	948   1406; ; ;	29	5.78593622 1	5.78593619 1	3.1-11	7.3-16	3.0-9	3.0-9	5:41
NH-2Pi-ST	948   1406; ; ;	29	5.80546397 1	5.80546388 1	9.1-12	2.3-16	7.4-9	7.4-9	2:09
NH-3Sigma-	948   1406; ; ;	29	5.83910025 1	5.83910016 1	4.4-11	3.6-16	8.2-9	8.2-9	2:08
OH-1Sigma	948   1406; ; ;	28	7.91680600 1	7.91680587 1	2.2-12	4.3-16	8.0-9	8.0-9	5:32
OH-2Pi-STO	948   1406; ; ;	28	7.94670773 1	7.94670763 1	3.2-11	4.1-16	6.5-9	6.5-9	5:28
OH+-3Sigma	948   1406; ; ;	29	7.88863800 1	7.88863788 1	1.5-11	0.8-15	7.4-9	7.4-9	2:04

problem	$m \mid n_s; n_g; n_l; n_u$	it.	primal obj	dual obj	err <sub>1</sub>	err <sub>3</sub>	err <sub>5</sub>	err <sub>6</sub>	time
H3O+-1-A1-	2964   3008; ; 154;	30	9.01065676 1	9.01065664 1	2.2-11	1.0-15	6.9-9	6.9-9	1:36:21
NH3-1-A1-S	2964   3008; ; 154;	30	6.79248736 1	6.79248731 1	2.9-11	7.5-16	3.4-9	3.4-9	1:36:10
Li.2S.STO6	465   780; ; ; 35	36	7.40023835 0	7.40023833 0	1.2-8	2.3-10	1.4-9	5.3-9	1:08
Be.1S.STO6	465   780; ; ; 35	34	1.45560894 1	1.45560861 1	7.8-8	1.4-9	1.1-7	1.3-7	1:04
BeH+.1Sigm	948   1312; ; ; 47	45	1.64575093 1	1.64575041 1	1.4-7	1.4-9	1.5-7	1.7-7	7:44
H3.2A1.DZ.	948   1312; ; ; 47	40	3.36464947 0	3.36464790 0	2.6-7	1.1-9	2.0-7	2.5-7	6:53
FH2+.1A1.S	1743   2044; ; ; 61	42	1.09990444 2	1.09990363 2	2.1-7	2.1-9	3.6-7	5.4-7	34:00
NH2-.1A1.S	1743   2044; ; ; 61	46	6.27062185 1	6.27062092 1	1.5-7	9.0-10	7.3-8	9.2-8	37:18
CH3+.1E.ST	2964   3008; ; ; 77	40	4.87507287 1	4.87505865 1	8.0-7	6.6-9	<b>1.4-6</b>	<b>2.3-6</b>	2:08:23
NH3+.2A2.S	2964   3008; ; ; 77	41	6.75058463 1	6.75055336 1	3.7-7	1.1-8	<b>2.3-6</b>	<b>3.2-6</b>	2:11:36
CH3.2A2.ST	2964   3008; ; ; 77	40	4.92214627 1	4.92211215 1	2.4-7	1.9-8	<b>3.4-6</b>	<b>4.9-6</b>	2:08:28
CH4.1A1.ST	4743   4236; ; ; 95	37	5.36627612 1	5.36621129 1	<b>1.4-6</b>	1.8-8	<b>6.0-6</b>	<b>7.6-6</b>	6:47:01
Na.2S.STO6	4743   4236; ; ; 95	43	1.61077125 2	1.61076921 2	3.9-7	6.4-9	6.3-7	7.8-7	7:52:40
NH4+.1A1.S	4743   4236; ; ; 95	38	7.27767126 1	7.27761528 1	<b>1.6-6</b>	1.4-8	<b>3.8-6</b>	<b>4.8-6</b>	6:58:57