ON THE GEOMETRIC AND THE ALGEBRAIC RANK OF GRAPH MANIFOLDS

JENNIFER SCHULTENS AND RICHARD WEIDMANN

Abstract. For any $n \in \mathbb{N}$ we construct graph manifolds of genus $4n$ that have $3n$-generated fundamental group.

1. introduction

A *Heegaard surface* of an orientable closed 3-manifold M is an embedded orientable surface S such that $\overline{M - S}$ consists of 2 handlebodies V_1 and V_2. This decomposition of M is called a *Heegaard splitting* and denoted by $M = V_1 \cup_S V_2$. We say that the splitting is of *genus g* if S is of genus g. It is not difficult to see that any orientable closed 3-manifold admits a Heegaard splitting. If M admits a Heegaard splitting of genus g but no Heegaard splitting of smaller genus then we say that M has *Heegaard genus g* and write $g(M) = g$.

Clearly any curve in a handlebody can be homotoped to its boundary. It follows that for any Heegaard splitting $M = V_1 \cup_S V_2$ every curve in M can be homotoped into V_1. Thus the map induced by the inclusion of V_1 into M maps a generating set of $\pi_1(V_1)$ to a generating set of $\pi_1(M)$. As $\pi_1(V_1)$ is generated by g elements it follows that $\pi_1(M)$ is also generated by g elements. Thus $g(M) \geq r(M)$ where $r(M)$ denotes the minimal number of generators of $\pi_1(M)$. Sometimes we will refer to $g(M)$ as the *geometric rank* and to $r(M)$ as the *algebraic rank* of M.

F. Waldhausen [12] asked whether the converse inequality also holds, i.e., whether $g(M) = r(M)$. A positive answer would have implied the Poincaré conjecture. First counterexamples however were found by M. Boileau and H. Zieschang [1]. These examples were Seifert fibered manifolds with $g(M) = 3$ and $r(M) = 2$. The work of Y. Moriah and J. Schultens [4] further shows that this class extends to higher genus examples, i.e. Seifert manifolds with $g(M) = n + 1$ and $r(M) = n$. In [13] a class of graph manifolds was found for which $g(M) = 3$ and $r(M) = 2$. The original Boileau-Zieschang examples can be interpreted as a special case of these graph manifolds.

We here show how the phenomenon observed in [13] generalizes and how it can occur multiple times within a single graph manifold. This yields graph manifolds where the difference between the algebraic and the geometric rank is arbitrarily high.

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2. Formulation of the main results

Let M be a closed graph manifold. We will always assume that M comes equipped with its characteristic tori $\mathcal{T} = \mathcal{T}_M$ and a fixed Seifert fibration on every component of $\overline{M - \mathcal{T}}$. Recall that the Seifert fibrations are unique up to isotopy except for components homeomorphic to Q, the Seifert space with base orbifold the disk with two cone points of order 2. The space Q can also be fibered as the orientable circle bundle over the Möbius band. We will refer to the components of $\overline{M - \mathcal{T}}$ as the *Seifert pieces* of M. Recall that the Seifert pieces of M are up to isotopy precisely the maximal Seifert submanifolds of M. We will mostly work with *totally orientable* graph manifolds, i.e. orientable graph manifolds whose Seifert pieces have orientable base orbifold. This makes the Seifert fibrations unique up to isotopy on all Seifert pieces.

Let N be a Seifert piece of M. Denote the fiber of N by f. Let $T_1, \ldots, T_n$ be the boundary components of N and let $\gamma_i \subset T_i$ be the curve corresponding to the fiber of the Seifert piece L_i where L_i is the Seifert piece reached by travelling from N transversely through T_i. Note that we possibly have $N = L_i$. The maximality of the Seifert piece N guarantees that for all i the intersection number of f with γ_i does not vanish.

We then define $\hat{N}$ to be the manifold $N(\gamma_1, \ldots, \gamma_n)$ obtained from N by performing a Dehn filling with slope γ_i at each boundary component T_i. It is clear that the Seifert fibration of N can be extended to a Seifert fibration of $\hat{N}$ as f has non-trivial intersection number with all γ_i.

In the following we will denote the base orbifold of a Seifert piece N by $\mathcal{O}(N)$. We will denote an orbifold by its topological type with a list of the orders of cone points, where ∞ stands for a boundary component. We will denote the disc by D, the sphere by S^2, the annulus by A, the orientable surface of genus g by F_g and the projective plane by P^2.

Theorem 1. *Let M be a closed graph manifold consisting of two Seifert pieces N_1 and N_2 glued along T, where $\mathcal{O}(N_1) = F_g(r, \infty)$, $\mathcal{O}(N_2) = D(p, q)$ with $(p, q) = 1$ and $\min(p, q) \leq 2g+1$ such that the intersection number of the fibers of N_1 and N_2 equals 1.*

Then $\pi_1(M)$ is generated by $2g+1$ elements. Furthermore M admits a Heegaard splitting of genus $2g+1$ if and only if one of the following holds:

1. *N_2 is the exterior of a s-bridge knot with $s \leq 2g+1$ and the fiber of N_1 is identified with the meridian of N_2, i.e. $\hat{N}_2 = S^3$.*
2. *$\hat{N}_1$ admits a horizontal Heegaard splitting of genus $2g$.*

We will further see that all manifolds of this type admit a Heegaard splitting of genus $2g + 2$. Furthermore, most of these manifolds do not admit a Heegaard splitting of genus $2g + 1$ as for any given pair of such manifolds N_1 and N_2 there are at most three glueing maps that yield a graph manifold of genus $2g + 1$. It is also possible to show that $\pi_1(M)$ cannot be generated by less than $2g+1$ elements, the argument however is complicated.

A careful analysis of the above examples shows that the phenomenon is of a local nature, it can therefore be reproduced multiple times within a graph manifold with a more complex underlying graph. This yields the following:

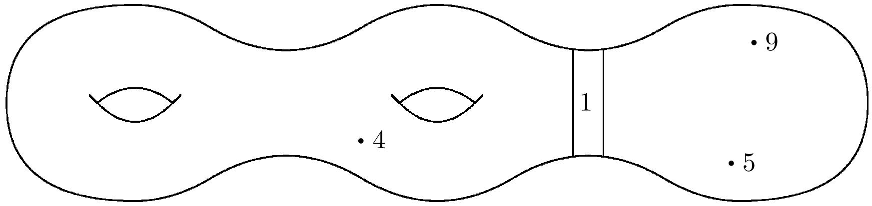

Figure 1. A graph manifold with 5-generated fundamental group

Theorem 2. *For any $n \in \mathbb{N}$ there exists a graph manifold M_n with $3n$-generated fundamental group that has Heegaard genus $4n$.*

This paper is organized as follows. In Section 3 we review the structure theorem for Heegaard splittings of totally orientable graph manifolds as proven in [9]. Then we study in more detail how Heegaard surfaces can intersect the Seifert pieces that are the building blocks of our examples. In Section 5 and Section 6 we will then give the proofs of Theorem 1 and Theorem 2. We conclude by describing a class orientable Seifert manifolds with $2n$-generated fundamental group which we believe to be of Heegaard genus $3n$. These manifolds are however not totally orientable.

3. Heegaard splittings of totally orientable graph manifolds

A graph manifold M is *totally orientable* if every Seifert piece N of M fibers over an orientable base space and if M itself is orientable. In [9] it is shown that the Heegaard splittings of totally orientable graph manifolds have a structure that can be completely described. To do so, one considers a decomposition of M into edge manifolds and vertex manifolds. The edge manifolds are the submanifolds of the form $T \times I$, where T is one of the characteristic tori, $\mathcal{T}$, of M. The vertex manifolds are the components of the complement of the edge manifolds. Note that each vertex manifold is homeomorphic to a component of $M - \mathcal{T}$.

Heegaard splittings themselves are rather unwieldy. Instead we work with the surfaces arising in what is called a "strongly irreducible untelescoping" of a Heegaard splitting. We use the terms pseudohorizontal, horizontal, pseudovertical and vertical to describe the possible structure for the restriction of such a surface to the vertex manifolds. The restriction of such a surface to the edge manifolds takes three possible forms. It too plays a nontrivial role in the structure of the Heegaard splitting of a graph manifold.

A 2-sided surface F in a 3-manifold M is said to be *weakly reducible* if there are disjoint essential curves a, b in F that bound disks D_a, D_b whose interior is disjoint from F and such that near their boundary D_a, D_b lie on opposite sides of F. A 2-sided surface F in a 3-manifold M is said to be *strongly irreducible* if it is not weakly reducible.

Heegaard splittings correspond to handle decompositions. Given a 3-manifold M and a decomposition $M = V \cup_S W$ into two handlebodies, one handlebody, say V, provides the 0-handles and 1-handles and the other, W, provides the 2-handles and 3-handles. Without loss of generality, there is only one 0-handle and one 3-handle. Corresponding to $M = V \cup_S W$ we then have a handle decomposition in which all 1-handles are attached before any of the 2-handles. An *untelescoping* of a Heegaard splitting is a rearrangement of the order in which the 1-handles and 2-handles are attached. In the handle decomposition obtained we first attach the 0-handle, then some 1-handles, then some 2-handles, then some 1-handles, then some 2-handles, etc and finally, the 3-handle. We specify an untelescoping by a collection of surfaces $S_1, F_1, S_2, F_2, \dots, F_{n-1}, S_n$. These surfaces are obtained as follows: S_1 is the boundary of the submanifold of M obtained by attaching the 0-handle and the first batch of 1-handles. F_1 is the boundary of the submanifold of M obtained by attaching the 0-handle, the first batch of 1-handles and the first batch of 2-handles. S_2 is the boundary of the submanifold of M obtained by attaching the 0-handle, the first batch of 1-handles, the first batch of 2-handles and the second batch of 1-handles. F_2 is the boundary of the submanifold of M obtained by attaching the 0-handle, the first batch of 1-handles, the first batch of 2-handles, the second batch of 1-handles and the second batch of 2-handles. Etc. An untelescoping $S_1, F_1, S_2, F_2, \dots, S_n$ is said to be strongly irreducible if each S_i is a strongly irreducible surface in M and each F_i is an incompressible surface in M. Note that a Heegaard splitting can be considered a trivial untelescoping S. If it is strongly irreducible, then it is its own strongly irreducible untelescoping.

For the discussion here it will be useful to note the following: 1) Each of the S_i and each of the F_i is separating; 2) Each pair S_i and F_i cobound a submanifold homeomorphic to $S_i \times I$ with 2-handles attached to $S_i \times \{1\}$. In particular, $\chi(S_i) < \chi(F_i)$. Similarly for F_i and S_{i+1}. The following theorem summarizes the discussion in [7], [8] and [6, Lemma 2].

Theorem 3. *Let M be a 3-manifold and $M = V \cup_S W$ a Heegaard splitting. Then $M = V \cup_S W$ has a strongly irreducible untelescoping $S_1, F_1, S_2, F_2, \dots, S_n$. Furthermore,*

$$-\chi(S) = \sum_{i=1}^{n} (\chi(F_{i-1}) - \chi(S_i)).$$

A surface in a Seifert fibered space is *horizontal* if it is everywhere transverse to the fibration. It is *pseudohorizontal* if it is horizontal away from a fiber e and intersects a regular neighborhood $N(e)$ of e in an annulus that is a bicollar of e. Note that in [4] the Heegaard splittings of a Seifert fibered space with pseudohorizontal splitting surface are called *horizontal Heegaard splittings.*

Let F be a surface in a 3-manifold M and α an arc with interior in $M \backslash F$ and endpoints on F. Let $C(\alpha)$ be a collar of α in M. The boundary of $C(\alpha)$ consists of an annulus A together with two disks D_1, D_2, which we may assume to lie in F. We call the process of replacing F by $(F \backslash (D_1 \cup D_2)) \cup A$ *performing ambient* 1*-surgery on F along α.*

A surface S in a Seifert fibered space is *vertical* if it consists of regular fibers. It is *pseudovertical* if there is a vertical surface $\mathcal{V}$ and a collection of arcs Γ with interior disjoint from $\mathcal{V}$ that projects to an embedded collection of arcs such that S is obtained from $\mathcal{V}$ by ambient 1-surgery along Γ.

The definition of a standard Heegaard splitting for a graph manifold is rather lengthy. Let M be a graph manifold. A strongly irreducible untelescoping $S_1, F_1, S_2, F_2, \dots, S_n$ of a Heegaard splitting $M = V \cup_S W$ is standard if it is as follows: 1) Each F_i intersects each vertex manifold either in a horizontal or in a vertical surface (or $\emptyset$); 2) Each F_i is either a torus entirely contained in an edge manifold or intersects an edge manifold in spanning annuli (or $\emptyset$); 3) Each S_i intersects each vertex manifold in either a horizontal, pseudohorizontal, vertical or pseudovertical surface (or $\emptyset$); 4) Each S_i intersects each edge manifold $M_e = (torus) \times [0,1]$ in one of three possible ways: a) $S_i \cap M_e$ consists of incompressible annuli (or $\emptyset$); or b) $S_i \cap M_e$ can be obtained from a collection of incompressible annuli by ambient 1-surgery along an arc that is isotopic to an embedded arc in the boundary of the edge manifold; or c) there is a pair of simple closed curves $c, c' \subset (torus)$ such that $c \cap c'$ consists of a single point p and $S_i \cap M_e$ is the portion of the boundary of a collar of $c \times \{0\} \cup p \times [0,1] \cup c' \times \{1\}$ that lies in the interior of M_e. Furthermore, each edge manifold must be met by at least one of the S_i.

Recall that for each i, F_i and S_i are separating. Thus if F_i or S_i intersects an edge manifold M_e in spanning annuli, then it must do so in an even number of spanning annuli. It is a non trivial fact that if $S_1, F_1, S_2, F_2, \dots, S_n$ meets M_e in spanning annuli, then between any two components of $F_i \cap M_e$ there must be two components of either $S_i \cap M_e$ or $S_{i+1} \cap M_e$.

The Heegaard splitting $M = V \cup_S W$ is *standard* if every strongly irreducible untelescoping $S_1, F_1, S_2, F_2, \dots, S_n$ of $M = V \cup_S W$, $(\cup_i F_i) \cup (\cup_i S_i)$ can be isotoped to be standard.

The main theorem in [9] is the following:

Theorem 4. *Let $M = V \cup_S W$ be an irreducible Heegaard splitting of a totally orientable graph manifold. Then $M = V \cup_S W$ is standard.*

This theorem has many consequences some of which will be used in the following. We assume that M is a totally orientable graph manifold, $M = V \cup_S W$ a Heegaard splitting and $S_1, F_1, \dots, F_{n-1}, S_n$ a strongly irreducible untelescoping of $M = V \cup_S W$ that is standard. Then:

Fact 1: For N a vertex or edge manifold of M,

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq 0.$$

Fact 2: Suppose e is an edge that abuts v. And suppose N_e, N_v, respectively, are the edge and vertex manifolds corresponding to e, v, respectively. Further suppose that $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_v$ is vertical and pseudovertical and a component $\tilde{S}$ of $(\cup_i S_i) \cap N_e$ is as in c). Then any annuli in $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_e$ that are parallel into ∂N_v can be isotoped to lie entirely in N_v. After this isotopy, $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_v$ is still vertical and pseudovertical.

Fact 3: Suppose e is an edge that abuts v. And suppose N_e, N_v, respectively, are the edge and vertex manifolds corresponding to e, v, respectively. Further suppose that $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_v$ is horizontal. Then $(\cup_i F_i) \cap N_e$ does not contain a torus.

Fact 4: Suppose N_v is a vertex manifold and that a component $\tilde{S}$ of $(\cup_i S_i)) \cap N_v$ is pseudohorizontal. Then $(((\cup_i F_i) \cup (\cup_i S_i)) \cap N_v) = \tilde{S}$.

Consider the following: Suppose that M is a closed totally orientable graph manifold and that $S_1, F_1, S_2, F_2, \dots, S_n$ is a strongly irreducible untelescoping of a Heegaard splitting $M = V \cup_S W$. Suppose further that $S_1, F_1, S_2, F_2, \dots, S_n$ has been isotoped to be standard. This implies in particular that for any vertex manifold N, $(\cup_i F_i) \cup (\cup_i S_i)$ meets ∂N in parallel simple closed curves. Thus to any vertex manifold N of M we associate the manifold N_S, which is the manifold obtained from N by performing a Dehn filling at every component of ∂N along a slope represented by the curves $((\cup_i F_i) \cup (\cup_i S_i)) \cap \partial N$. Here N_S is not canonical.

It depends on a specific (not necessarily unique) positioning of an (not necessarily unique) untelescoping. But we merely introduce this notation to discuss consequences of the existence of certain setups. Note that N_S is a Seifert manifold if N contains a horizontal or pseudohorizontal component of $((\cup_i F_i) \cup (\cup_i S_i)) \cap N$, as $((\cup_i F_i) \cup (\cup_i S_i)) \cap \partial N$ then consist of curves that have non-trivial intersection number with the fibre of N. We have the following observation:

Lemma 5. *Suppose that for some i, $S_i \cap N$ is pseudohorizontal. Then the Seifert manifold N_S has a Heegaard surface S' such that $S' \cap N = S_i \cap N$. The corresponding Heegaard splitting is a horizontal Heegaard splitting of N_S. If $S_i \cap N$ is planar then S' is homeomorphic to S^2.*

Proof. Recall Fact 4 above, it tells us that if $S_i \cap N$ is pseudohorizontal, then $((\cup_i F_i) \cup (\cup_i S_i)) \cap N$ consists of a single component which we denote by $\tilde{S}$.

We may extend $\tilde{S}$ to a Heegaard surface of N_S by gluing meridional discs of the glued in solid tori to the boundary components of $\tilde{S}$. The corresponding Heegaard splitting for N_S is horizontal. If $\tilde{S}$ is planar then all boundary components get capped off which results in S^2. The assertion follows. □

4. Some lemmata

The following lemmata will enable us to compute the Heegaard genus of certain graph manifolds in the next section. We start by discussing the possible pseudohorizontal surfaces in the relevant Seifert manifolds. Some proofs rely on the theory of 2-dimensional orbifolds and their covering theory as discussed in [11]. These lemmata will be used in our discussion of Heegaard splittings and their untelescopings. But many of these results are more general. We do not necessarily require S to be the splitting surface of a Heegaard splitting or to be a surface in an untelescoping. Lemma 13 concerns vertical and pseudovertical surfaces.

Lemma 6. *Let M be a graph manifold and N be a Seifert piece with $\mathcal{O}(N) = D(p,q)$ and $(p,q) = 1$. Suppose $S \cap N$ is a planar surface that is pseudohorizontal. Then the following hold:*

1. *N_S is homeomorphic to S^3.*
2. *$S \cap \partial N$ contains exactly $2p$ or $2q$ components.*

It should be noted that N_S being homeomorphic to S^3 is equivalent to N being the exterior of an r-bridge knot with meridian μ parallel to $\partial N \cap S$ where $r = \min(p,q)$.

Proof. Possibly after exchanging p and q we can assume that S is horizontal in the space $\bar{N}$ obtained from N after removing a regular neighborhood of the exceptional fiber corresponding to the cone point of order q or by removing a neighborhood of a regular fiber. Clearly $\bar{N}$ is a Seifert space with $\mathcal{O}(\bar{N}) = A(p)$ or $\mathcal{O}(\bar{N}) = A(p,q)$. Let T_1 be the boundary component of $\bar{N}$ that bounds the drilled out solid torus and T_2 be the boundary of N. Let $\bar{S}$ be a component of $S \cap \bar{N}$. Clearly $\bar{S}$ is planar as it is a subsurface of a planar surface.

As we assume that S is pseudohorizontal in N it follows that $\bar{S} \cap T_1$ consists of a single loop α. Let γ be one component of $\bar{S} \cap T_2$ and let g be an element of $\pi_1(\bar{N})$ corresponding to γ. Recall that all other components of $\bar{S} \cap T_2$ are parallel to γ. Let n be the intersection number of γ with the fiber.

As $\bar{S}$ is horizontal in $\bar{N}$ it follows that there exists a finite sheeted orbifold covering $p : \bar{S} \to O(\bar{N})$, in particular $p_*(\pi_1(\bar{S}))$ is of finite index in $\pi_1(O(\bar{N}))$. We distinguish the cases $O(\bar{N}) = A(p)$ and $O(\bar{N}) = A(p, q)$.

Case 1: $O(\bar{N}) = A(p)$**.** We have $\pi_1(A(p)) = \langle x, y | x^p \rangle$ where the generator y corresponds to the boundary curve corresponding to T_2. This implies in particular that $p_*(g)$ is conjugate to y^n.

As $\bar{S}$ is planar this implies that $\pi_1(\bar{S})$ is generated by homotopy classes that correspond to the components of $\bar{S} \cap T_2$, i.e. $p_*(\pi_1(\bar{S}))$ is generated by conjugates of the element y^n. Let $N(y^n)$ be the normal closure of y in $\pi_1(A(p))$. Clearly $\pi_1(A(p))/N(y^n) \cong \mathbb{Z}_n * \mathbb{Z}_p$ is infinite unless $n = 1$. As $p_*(\pi_1(\bar{S})) \subset N(y^n)$ this implies that $n = 1$ as otherwise $p_*(\pi_1(\bar{S}))$ is contained in a subgroup of infinite index in $\pi_1(A(p))$ and is therefore of infinite index itself. Thus we can assume that $n = 1$ and that $p_*(\pi_1(\bar{S})) \subset N(y)$.

Note first that the orbifold covering space $\tilde{S}$ corresponding to $N(y)$ is a orbifold without cone points and is homeomorphic to the $(p+1)$-punctured sphere. Denote the corresponding covering map by $\tilde{p}$.

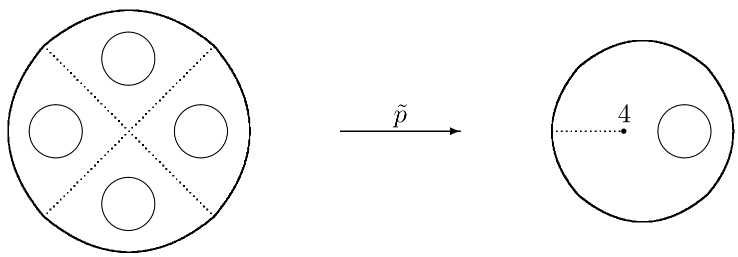

FIGURE 2. The 4-sheeted covering of $A(4)$ by a 5-punctured sphere

As $p_*(\pi_1(\bar{S})) \subset N(y)$ it follows that there exists a covering $p' : \bar{S} \to \tilde{S}$ such that $p = \tilde{p} \circ p'$.

Claim p' is a homeomorphism. As for both $\bar{S}$ and $\tilde{S}$ all but one boundary component map onto a curve corresponding to the element y it follows that p' is a homeomorphism when restricted to any of these boundary components. In particular p' extends to a covering $p'_{\#} : \bar{S}_{\#} \to \tilde{S}_{\#}$ where $\bar{S}_{\#}$ and $\tilde{S}_{\#}$ are the spaces obtained from $\bar{S}$ and $\tilde{S}$ by gluing discs to these boundary components. As $\bar{S}_{\#}$ and $\tilde{S}_{\#}$ are discs it follows that the obtained map is a homeomorphism. Thus the original p' was a homeomorphism which proves the claim.

The second assertion is now immediate as $S \cap \bar{N}$ is obtained from 2 copies of $\bar{S}$ and identifying two boundary components. All resulting boundary components lie in T_2. The first assertion follows from Lemma 5.

Case 2: $O(\bar{N}) = A(p, q)$**.** We have $\pi_1(A(p, q)) = \langle x, y, z | y^p, z^q \rangle$ where the generator x corresponds to the boundary curve corresponding to T_2. We see as in the first case that $p_*(O(\bar{N}))$ lies in the kernel of the map $\phi : \pi_1(A(p, q)) \to \pi_1(A(p, q))/N(x^n)$. As $\pi_1(A(p, q))/N(y^n) \cong \mathbb{Z}_n * \mathbb{Z}_p * \mathbb{Z}_q$ is infinite for all $n \in \mathbb{N}$ this implies that $p_*(O(\bar{N}))$ is of infinite index in $\pi_1(A(p, q))$ which contradicts our assumption. $\square$

Lemma 7. *Let M be a graph manifold and let N be a Seifert piece with $\mathcal{O}(N) = F_g(p,\infty)$ or $\mathcal{O}(N) = F_g(p,\infty,\infty)$. Suppose that $S \cap N$ is pseudohorizontal and $\chi(S \cap N) > -8g$ or $\chi(S \cap N) > -8g - 4$, respectively. Then the following hold*

(1) *$S \cap T$ has two components for every component T of ∂N.*
(2) *$S \cap N$ extends to the splitting surface of a horizontal Heegaard surface of genus $2g$ of N_S.*

Proof. We only deal with the case $\mathcal{O}(N) = F_g(p,\infty)$ the other case is analogous.

Suppose that $S \cap N$ is pseudohorizontal with respect to the exceptional fiber or a regular fiber and let $\bar{N}$ be the space obtained by drilling out the neighborhood of this fiber. Let $\bar{S}$ be a component of $\bar{N} \cap S$. Recall that $S \cap N$ is obtained from two copies of $\bar{S}$ by identifying them along a boundary component. In particular we have that $\chi(S \cap N) = 2\chi(\bar{S})$.

Now $\bar{S}$ is a finite sheeted covering of $\mathcal{O}(\bar{N})$ where $\mathcal{O}(\bar{N}) = F_g(\infty,\infty)$ or $\mathcal{O}(\bar{N}) = F_g(p,\infty,\infty)$ depending on what kind of fiber was drilled out. Suppose that the covering is n-sheeted. Note that in the case $\mathcal{O}(\bar{N}) = F_g(p,\infty,\infty)$ it must hold that $n \geq p$ as otherwise the covering space must be a orbifold with singularities. Thus we have

$$\chi(S \cap N) = 2\chi(\bar{S}) = 2n\chi(\mathcal{O}(\bar{N})).$$

As $\chi(\mathcal{O}(\bar{N})) = -2g$ or $\chi(\mathcal{O}(\bar{N})) = -2g - 1 + 1/p$ it follows immediately from the hypothesis on the Euler characteristic that $n = 1$. Thus $\mathcal{O}(\bar{N}) = F_g(\infty,\infty)$, i.e. the exceptional fiber was drilled out. Assertion (1) is now immediate and (2) follows from the proof of Lemma 5. □

It will be important that many Seifert manifolds do not admit a pseudohorizontal surface of small genus indiscriminately of what graph manifold they belong to.

Lemma 8. *Let N be a Seifert manifold with $\mathcal{O}(N) = F_g(p,\infty)$ such that the exceptional fiber has invariant (α,β) with $1 \leq \beta < \alpha$. Then the following hold:*

(1) *If $\alpha = 2$ then there exist two slopes γ on ∂N such that $N(\gamma)$ admits a horizontal Heegaard splitting of genus $2g$.*
(2) *If $\alpha \neq 2$ and $\beta \in \{1, \alpha - 1\}$ then exists one slope γ on ∂N such that $N(\gamma)$ admits a horizontal Heegaard splitting of genus $2g$.*
(3) *In all other cases $N(\gamma)$ has no Heegaard splitting of genus $2g$ if $\gamma \neq f$.*

Proof. If γ is the fiber then $N(\gamma)$ is not a Seifert manifold. In particular $N(\gamma)$ admits no horizontal Heegaard splitting as those are only defined for Seifert manifolds. If the intersection number m of γ with the fiber is greater than 1 then $M(\gamma)$ is a Seifert space with base orbifold $F_g(p,m)$ which has no Heegaard splitting of genus $2g$ by (i) of Proposition 1.4 of [1]. Suppose now that $m = 1$. Let $e \in \mathbb{Z}$ be the Euler class of the Seifert space. By (iii) of Proposition 1.4 of [1] it follows that $N(\gamma)$ admits no Heegaard splitting of genus $2g$ unless $\beta - e\alpha = \pm 1$. It is clear that there exists two values for e such that the equation holds if $\beta = 1$ and $\alpha = 2$, that there exists one solution if $\beta \in \{1, \alpha - 1\}$ and none otherwise. The corresponding Heegaard splittings are constructed in Section 1.10 of [1]. This proves the assertion. □

Lemma 9. *Let N be a Seifert manifold with $\mathcal{O}(N) = D(p,q)$ and $(p,q) = 1$. Then N contains no compact planar horizontal surface.*

Proof. Suppose that S is a compact planar horizontal surface in N. Then there exists a finite sheeted orbifold covering $p : S \to D(p,q)$. As all components of ∂S are parallel on ∂N it follows that there exists a number $n \in \mathbb{N}$ such the restriction of p to any component of ∂S is a n-sheeted covering. This implies that we can extend p to a orbifold covering $p : S^2 \to S^2(p,q,n)$ by gluing a disc to any component of ∂S and a disc with a cone point of order n to $D(p,q)$. If $n = 1$ this yields a contradiction as $S^2(p,q,1) = S^2(p,q)$ is a bad orbifold which admits no covering by a manifold. If $n \neq 1$, then $S^2(p,q,n)$ must be a spherical orbifold with universal cover the sphere. Moreover, N_S is a Seifert manifold with $\mathcal{O}(N_S) = S(p,q,n)$. As such it is irreducible. This yields a contradiction, as $S \subset N$ extends to a horizontal, hence incompressible, sphere in N_S. □

Lemma 10. *Let M be a graph manifold and let N be a Seifert piece with $\mathcal{O}(N) = F_g(p,\infty)$ or $\mathcal{O}(N) = F_g(p,\infty,\infty)$. If $S \cap N$ is horizontal, then $\chi(S \cap N) \leq -4g+1$ or $\chi(S \cap N) \leq -4g-p+1$, respectively.*

Proof. Suppose that S is a horizontal incompressible surface in N that covers regular points of $F_g(p,\infty)$ k times. Note that here $k \geq p \geq 2$. By the Riemann-Hurwitz formula, $\chi(S) = k(-2g+\frac{1}{p}) \leq p(-2g+\frac{1}{p}) = -2pg+1 \leq -4g+1$ or $\chi(S) = k(-2g-1+\frac{1}{p}) \leq p(-2g-1+\frac{1}{p}) = -2pg-p+1 \leq -4g-p+1$, respectively. □

Lemma 11. *Let M be a graph manifold and let N be a Seifert piece with $\mathcal{O}(N) = F_g(p,\infty)$ or $\mathcal{O}(N) = F_g(p,\infty,\infty)$. Let $M = V \cup_S W$ be a Heegaard splitting and $S_1, F_1, \ldots, F_{n-1}, S_n$ an untelescoping. If $S_1, F_1, \ldots, F_{n-1}, S_n$ meets N in such a way that $F_i \cap N, S_i \cap N$ are horizontal for each i, then*

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq 8g-2$$

or

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq 8g+2p-2,$$

respectively.

Proof. Note that since the surfaces $S_1 \cap N, F_1 \cap N, \ldots, F_{n-1} \cap N, S_n \cap N$ are disjoint and horizontal, they must be parallel. Recall that for each i, $F_i \cap N$ and $S_i \cap N$ is separating. A connected horizontal incompressible surface is non separating, thus each $F_i \cap N$ and each $S_i \cap N$ consists of an even number of parallel horizontal surfaces. Further note that between any two components of $F_i \cap N$ there must be two components of $S_i \cap N$ or of $S_{i+1} \cap N$. In other words, unless $\cup_i F_i \cap N = \emptyset$, there will be twice as many components of $\cup_i S_i \cap N$ as of $\cup_i F_i \cap N$. The lemma then follows from Lemma 10. □

Lemma 12. *Let N be a Seifert manifold with $\mathcal{O}(N) = D(p,q)$ with $(p,q) = 1$ and S be a properly embedded surface.*

1. *If $S \cap N$ is horizontal, then there is an $l \geq 1$ such that $|S \cap N| \leq l$, $\chi(S \cap N) \leq l \cdot p \cdot q \cdot (-1 + \frac{1}{p} + \frac{1}{q})$ and genus$(S \cap N) \geq 1$.*
2. *If $S \cap N$ is pseudohorizontal, then $\chi(S \cap N) \leq -2\min(p,q)+2$. Furthermore, either $S \cap N$ is as in Lemma 6, or genus$(S \cap N) \geq 2$.*

Proof. (1) Clearly $S \cap N$ is a finite sheeted cover of $D(p,q)$. The degree of this covering musts be a positive multiple of $p \cdot q$, say $l \cdot p \cdot q$. It is clear that $S \cap N$ has at most l components. The second assertion follows from the Riemann-Hurwitz formula as $\chi(D(p,q)) = -1 + \frac{1}{q} + \frac{1}{q}$. The last assertion holds as by Lemma 9, S is non-planar, so $genus(S \cap N) \geq 1$.

(2) Suppose first that $S \cap N$ is pseudohorizontal with respect to the fiber e. Let $N' = N - \eta(e)$ and S' be a component of $S \cap N'$. Recall that S' is horizontal by the definition of a pseudohorizonal surface.

If e is a regular fiber then S' must cover $A(p,q)$ at least pq times, i.e. we have $\chi(S') \leq pq(-2 + \frac{1}{p} + \frac{1}{q}) = -2pq + p + q$ and therefore $\chi(S) = 2\chi(S') \leq -4pq + 2p + 2q \leq -2\min(p,q) + 2$. The remaining assertion follows from the proof of Lemma 6 which implies that S' cannot be planar.

Thus we can assume that e is an exceptional fiber. Suppose that e is the exceptional fiber of index q and let $N' = N - \eta(e)$. Suppose that H' is a horizontal incompressible surface in N' that covers regular points k times. Clearly $k \geq p$. Then $\chi(H') = k(-1 + \frac{1}{p}) \leq p(-1 + \frac{1}{p}) = -p + 1$. Thus if $S \cap N$ is pseudohorizontal with respect to e, then $\chi(S \cap N) \leq 2\chi(H') \leq -2p + 2$. The first assertion now follows as this argument is symmetric in p and q; the last comment follows immediately from Lemma 6. □

Lemma 13. *Let M be a graph manifold and let N be a vertex manifold. Let $M = V \cup_S W$ be a Heegaard splitting and $S_1, F_1, \ldots, F_{n-1}, S_n$ an untelescoping. Suppose that $F_i \cap N$ is vertical for each i and $S_i \cap N$ is vertical or pseudovertical for each i. Then*

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq -2\chi(H) + 2s + \sum_i (|F_{i-1} \cap \partial N| - |S_i \cap \partial N|)$$

Where H is the underlying surface of $\mathcal{O}(N)$ and s the number of exceptional fiberes.

Moreover, if If $\mathcal{O}(N) = F_g(p, \infty)$ and $(\cup_i S_i) \cap \partial N \neq \emptyset$, then

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq 4g + 2 + \sum_i (|F_{i-1} \cap \partial N| - |S_i \cap \partial N|).$$

If $\mathcal{O}(N) = F_g(p, \infty, \infty)$, denote the components of ∂N by ∂N_1 and ∂N_2. If $(\cup_i S_i) \cap \partial N_j \neq \emptyset$ for $j = 1, 2$, then

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq 4g + 4 + \sum_i (|F_{i-1} \cap \partial N| - |S_i \cap \partial N|).$$

Proof. We denote $\mathcal{O}(N)$ by F so long as we need not distinguish between the cases. Since $F_i \cap N$ is vertical, $F_i \cap N$ consists of saturated annuli and tori. Since $S_i \cap N$ is vertical or pseudovertical, $S_i \cap N$ is obtained from saturated annuli $A^i_1, \ldots, A^i_{n_i}$ and tori $T^i_1, \ldots, T^i_{k_i}$ (some of them parallel to components of $F_{i-1} \cap N$) by performing ambient 1-surgery along arcs $\beta^i_1, \ldots, \beta^i_{m_i}$ that project to disjoint imbedded arcs $b^i_1, \ldots, b^i_{m_i}$ disjoint from the projection of $A^i_1, \ldots, A^i_{n_i}$ and $T^i_1, \ldots, T^i_{k_i}$ except at their endpoints.

For the purposes of the computation in this lemma, we may amalgamate $((\cup_i F_i)\cup(\cup_i S_i))\cap N$. Though it may not be possible to amalgamate $(\cup_i F_i)\cup(\cup_i S_i)$ without destroying its simultaneous structure on all vertex and edge manifolds, it is possible to perform an amalgamation without destroying the structure in a given vertex manifold. Said differently, a partial amalgamation in a given vertex manifold extends to a partial amalgamation in the graph manifold (though nothing can be said, for instance, about the structure of the resulting non strongly irreducible untelescoping of $M = V \cup_S W$ in edge manifolds adjacent to the given vertex manifold). Here the result of such an amalagamation with respect to N is a surface $\tilde{S}$ such that $\tilde{S}\cap N$ is pseudovertical. (For details on amalgamation involving vertical and pseudovertical surfaces see [10, Proposition 2.10], though note the difference in terminology.)

Since $\tilde{S}\cap N$ is pseudovertical, it is obtained from saturated annuli $A_1,\dots,A_{\tilde{n}}$ and tori $T_1,\dots,T_{\tilde{k}}$ by performing ambient 1-surgery along arcs $\beta_1,\dots,\beta_{\tilde{m}}$ that project to disjoint imbedded arcs $b_1,\dots,b_{\tilde{m}}$. These arcs are disjoint from the projections $a_1,\dots,a_{\tilde{n}}$ of $A_1,\dots,A_{\tilde{n}}$ and $t_1,\dots,t_{\tilde{k}}$ of $T_1,\dots,T_{\tilde{k}}$ except at their endpoints. Here each b_j corresponds either to b^i_l or to an arc dual to b^i_l for some l, i, and conversely. Furthermore,

$$-\chi(\tilde{S}\cap N) = 2\tilde{m} = 2\sum_{i,j} m^i_j = \sum_i (\chi(F_{i-1}\cap N) - \chi(S_i\cap N))$$

and

$$|\tilde{S}\cap\partial N| = 2\tilde{n} = \sum_i (|S_i\cap\partial N| - |F_{i-1}\cap\partial N|)$$

Recall that $\tilde{S}$ cuts a submanifold of M that contains N into two compression bodies. Thus the (not necessarily connected) submanifolds into which $\tilde{S}\cap N$ cuts N can be analyzed from two perspectives: On the one hand, they result from cutting compression bodies along essential annuli. On the other hand, they contain Seifert fibered submanifolds of N; specifically, the Seifert fibered submanifolds of N that project to the appropriate components of the complement of the graph $\Gamma = (\cup_j a_j)\cup(\cup_i t_i)\cup(\cup_l b_l)\cup\partial F$ in F. This is impossible unless the Seifert fibered spaces in question are fibered over a disk with at most one cone point (i.e., solid tori) or fibered over an annulus with no cone point. Each such solid torus or $(annulus)\times S^1$ must meet $\tilde{S}$. Furthermore, exactly one of the boundary components of any such $(annulus)\times S^1$ must lie in ∂N.

We denote the set of vertices of Γ by $V\Gamma$ and the set of edges by $E\Gamma$. We may assume that each vertex of Γ is either of valence two or of valence three. Each vertex on a circular component (corresponding either to a boundary component without attached b_i or to some t_i without attached b_i) is of valence two and each endpoint of an arc a_j and each endpoint of an arc b_l is a vertex of valence three. Then $\#V\Gamma = 2\tilde{n} + 2\tilde{m} + k$ and $\#E\Gamma = 3\tilde{n} + 3\tilde{m} + k$ where k is the number of circular components of Γ.

Denote the underlying surface of F by H. Now Γ induces a decomposition of H into 0-cells, 1-cells, 2-cells and annuli. Denote the union of the 2-cells and annuli by $D\Gamma$. Note that each such annulus must be cobounded by a component of ∂H. Let l be the number of annuli.

This implies that

$$\chi(H) = \#V\Gamma - (\#E\Gamma) + (\#D\Gamma - l)$$

Combining these insights we obtain the following:

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) + \sum_i (|S_i \cap \partial N| - |F_{i-1} \cap \partial N|) =$$

$$2\tilde{m} + 2\tilde{n} =$$

$$-4\tilde{n} - 4\tilde{m} + 6\tilde{n} + 6\tilde{m} - 2(\#D\Gamma - l) + 2(\#D\Gamma - l) =$$

$$-2\chi(H) + 2(\#D\Gamma - l)$$

Thus we have

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq$$

$$-2\chi(H) + 2(\#D\Gamma - l) + \sum_i (|F_{i-1} \cap \partial N| - |S_i \cap \partial N|)$$

Hence

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq -2\chi(H) + 2s + \sum_i (|F_{i-1} \cap \partial N| - |S_i \cap \partial N|)$$

as every cone point must lie in a disk component. Now note that $\tilde{S}$ induces a bicoloring on the components of the complement of Γ in F according to which side of $\tilde{S}$ the Seifert fibered space that projects to that component lies. Thus $\#D\Gamma \geq 2$.

In the cases $F = F_g(p, \infty,)$ or $F = F_g(p, \infty, \infty)$, $\#D\Gamma - l \geq 1$ because there must be a disk containing the cone point. Furthermore, if $l > 0$, then the result of cutting H along Γ yields annuli cobounded by boundary components of ∂H. This is impossible if $F = F_g(p, \infty)$ and $(\cup_i S_i) \cap \partial N \neq \emptyset$ or if $F = F_g(p, \infty, \infty)$ and $(\cup_i S_i) \cap \partial N_j \neq \emptyset$, for $j = 1, 2$, where N_1 and N_2 are the boundary components of N. Thus the additional formulas hold. □

Lemma 14. *Let M be a graph manifold and N a Seifert fibered submanifold with $\mathcal{O}(N) = D(p, q)$. Let $M = V \cup_S W$ be a Heegaard splitting and $S_1, F_1, \ldots, F_{n-1}, S_n$ and untelescoping. If $F_i \cap N$ is vertical for each i and $S_i \cap N$ is vertical or pseudovertical for each i, then*

$$\sum_i (\chi(F_{i-1} \cap N) - \chi(S_i \cap N)) \geq 2 + \sum_i (|F_{i-1} \cap \partial N| - |S_i \cap \partial N|)).$$

Proof. The argument is analogous to that in Lemma 13, with one minor difference: Here $D\Gamma$ must contain at least two 2-cells containing one cone point. Thus $D\Gamma - l \geq 2$ in any case. □

5. The proof of Theorem 1

In order to give the proof of Theorem 1 we will first show that the fundamental groups can in fact be generated by $2g+1$ elements and then that only the manifolds listed admit a Heegaard splitting of genus $2g + 1$.

Lemma 15. *The manifolds described in Theorem 1 have $2g + 1$-generated fundamental groups.*

Proof. We first recall the presentations of the fundamental groups of N_1 and N_2:

$$\pi_1(N_1) = \langle a_1, b_1, \ldots, a_g, b_g, s, t, f_1 \mid R\rangle \text{ with}$$
$$R = \{[a_1, f_1], \ldots, [a_g, f_1], [b_1, f_1], \ldots, [b_g, f_1], [s, f_1], [t, f_1],$$
$$s^r = f_1^{\beta}, [a_1, b_1] \cdot \ldots \cdot [a_g, b_g]st = f_1^e\}$$

and

$$\pi_1(N_2) = \langle x, y, f_2 \mid [x, f_2], [y, f_2], x^p = f_2^{\beta_1}, y^q = f_2^{\beta_2}\rangle.$$

As the manifold M is obtained from the manifold N_1 and N_2 by identifying their boundary it follows from van Kampen's theorem that

$$\pi_1(M) = \pi_1(N_1) *_C \pi_1(N_2) \text{ with } C \cong \mathbb{Z}^2.$$

Note that $f_1 = xyf_2^l$ for some $l \in \mathbb{Z}$ as we assume that the intersection number between f_1 and f_2 is 1. A simple calculation (see [5]) shows that $n = \min(p, q)$ conjugates of f_1 generate a subgroup of $\pi_1(N_2)$ that maps surjectively onto the orbifold group $\pi_1(D(p, q))$. We do however need something slightly stronger:

Claim: We can choose elements $h_2, \ldots, h_n \in \pi_1(N_2)$ such that

$$U = \langle f_1, h_2 f_1 h_2^{-1}, \ldots, h_n f_1 h_n^{-1}\rangle$$

maps surjectively onto the base group and that additionally $h_i \in U$ for $2 \le i \le n$.

Choose k_i such that $\langle f_1, k_2 f_1 k_2^{-1}, \ldots, k_n f_1 k_n^{-1}\rangle$ maps surjectively. For any k_i choose $h_i \in U$ and $z_i \in \mathbb{Z}$ such that $k_i = h_i f_2^{z_i}$. Clearly such h_i and z_i exist as we assume that U maps surjectively one $\pi_1(D(p, q))$ and as the kernel is generated by f_2. As f_1 and f_2 commute it follows that $k_i f_1 k_i^{-1} = h_i f_2^{z_i} f_1 f_2^{-z_i} h_i^{-1} = h_i f_1 h_i^{-1}$. This clearly implies that $U = \langle f_1, h_2 f_1 h_2^{-1}, \ldots, h_n f_1 h_n^{-1}\rangle$. The claim follows.

Note that U is a subgroup of finite index in $\pi_1(N_2)$ and that we can choose the elements h_i such that $\pi_1(N_2) = U$ if and only if N_2 is the exterior of a torus knot with meridian f_1. It is however always true that $\pi_1(N_2) = \langle U, C\rangle$ as $f_2 \in C$.

Note further that the subgroup $\langle s, f_1\rangle$ of $\pi_1(N_1)$ is generated by a single element g_0 which corresponds to the core of the solid torus corresponding to the exceptional fiber of N_1. It follows that $g_0^k = f_1$ for some $k \in \mathbb{Z}$. In order to prove the lemma we describe elements $g_1, \ldots, g_{2g} \in \pi_1(M)$ such that $\pi_1(M) = \langle g_0, \ldots, g_{2g}\rangle$.

Recall that by assumption $n + 1 \le 2g$. Put $h_i = 1$ for $n + 1 \le i \le 2g$. We define

- $g_i = h_i a_i$ for $1 \le i \le g$
- $g_i = h_i b_{i-g}$ for $g + 1 \le i \le 2g$

Claim: $U \subset \langle g_0, \ldots, g_{2g}\rangle$.

To see this it clearly suffices to show that f_1 and the elements $h_i f_1 h_i^{-1}$ lie in $\langle g_0, \ldots, g_{2g}\rangle$ for $1 \le i \le 2g$. Clearly $f_1 \in \langle g_0, \ldots, g_{2g}\rangle$ as $f_1 = g_0^k$. Furthermore $h_i f_1 h_i^{-1} \in \langle g_0, \ldots, g_{2g}\rangle$ for $1 \le i \le g$ as $g_i g_0^k g_i^{-1} = h_i a_i f_1 a_i^{-1} h_i^{-1} = h_i f_1 h_i^{-1}$. The same argument shows that $g_i g_0^k g_i^{-1} = h_i f_1 h_i^{-1}$ for $g + 1 \le i \le 2g$. Thus the claim is proven.

As $h_i \in U$ for $1 \le i \le 2g$ this implies that $h_i \in \langle g_0, \ldots, g_{2g}\rangle$ for $1 \le i \le 2g$ and therefore $h_i^{-1} g_i \in \langle g_0, \ldots, g_{2g}\rangle$ for $1 \le i \le 2g$. As $h_i^{-1} g_i = a_i$ for $1 \le i \le g$ and $h_i^{-1} g_i = b_{i-g}$ for $g + 1 \le i \le 2g$ it follows that all a_i and b_i lie in $\langle g_0, \ldots, g_{2g}\rangle$. Furthermore both f_1 and s are powers of g_0 and lie in $\langle g_0, \ldots, g_{2g}\rangle$. The last generator t can be written as a product in the remaining generators by the last relation. Thus all generators of $\pi_1(N_1)$ lie in $\langle g_0, \ldots, g_{2g}\rangle$ which shows that

$\pi_1(N_1) \subset \langle g_0, \dots, g_{2g}\rangle$. Thus $C \subset \langle g_0, \dots, g_{2g}\rangle$ and therefore $\pi_1(N_2) = \langle U, C\rangle \subset \langle g_0, \dots, g_{2g}\rangle$. This shows that $\pi_1(M) = \langle g_0, \dots, g_{2g}\rangle$. □

Lemma 16. *Let M be a manifold as described in Theorem 1 and let $M = V \cup_S W$ be a Heegaard splitting. Then one of the following holds:*

1. *$S \cap N_1$ is vertical, $S \cap N_2$ is planar and pseudohorizontal with respect to the exceptional fiber e of index p as in Lemma 6 and $q \leq 2g + 1$.*
2. *$S \cap N_1$ is as in Lemma 7, $S \cap N_2$ consists of a single annulus and genus$(S) = 2g + 1$.*
3. *genus$(S) \geq 2g + 2$.*

Proof. Let M be a manifold as described in Theorem 1 and let $M = V \cup_S W$ be a Heegaard splitting. Furthermore, let $S_1, F_1, \dots, F_{n-1}, S_n$ be a strongly irreducible untelescoping of $M = V \cup_S W$ that is standard.

Case 1: $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_1$ and $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_2$ are vertical or pseudovertical.

If $(\cup_i F_i) \cup (\cup_i S_i)$ meets the edge manifold N_e between N_1 and N_2 in annuli including spanning annuli, then M must be a Seifert fibered space. It then follows the main theorem of [4] that $g(S) \geq 2g + 2$. The same is true if $(\cup_i F_i) \cup (\cup_i S_i)$ meets N_e in annuli and a component obtained by ambient 1-surgery on spanning annuli.

If $(\cup_i F_i)$ meets the edge manifold N_e in a torus, then we may assume that $\cup(\cup_i S_i)$ is disjoint from N_e. (Annuli that are boundary parallel in N_e can be isotoped into the vertex manifolds.) Then Lemma 13 tells us that

$$\sum_i (\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1)) \geq$$
$$4g + 2 + \sum_i (|F_{i-1} \cap \partial N_1| - |S_i \cap \partial N_1|) \geq 4g$$

and Lemma 14 tells us that

$$\sum_i (\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2)) \geq 2 + \sum_i (|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|) = 2$$

Hence by Theorem 3, $2genus(S) - 2 = -\chi(S) \geq 4g + 2$; thus $genus(S) \geq 2g + 2$.

Otherwise $(\cup_i F_i) \cup (\cup_i S_i)$ meets the edge manifold between N_1 and N_2 in boundary parallel annuli and one component of Euler characteristic -2 contained in $(\cup_i S_i) \cap N_e$. Any boundary parallel annuli in $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_e$ can be isotoped into N_1 or N_2. It then follows from Lemma 13 and Lemma 14 that

$$\sum_i (\chi(F_{i-1}) - \chi(S_i)) =$$
$$\sum_i [(\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1)] + \sum_i [\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2)] +$$
$$+ \sum_i [\chi(F_{i-1} \cap N_e) - \chi(S_i \cap N_e)] \geq$$
$$(4g + 2 - 2) + (2 - 2) + 2 = 4g + 2.$$

Hence by Theorem 3, $2genus(S) - 2 = -\chi(S) \geq 4g + 2$, whence $genus(S) \geq 2g + 2$.

Case 2: $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_1$ is horizontal.

Recall Fact 1 following Theorem 4, it tells us that for any vertex or edge manifold N we have. $\sum_i((\chi(F_{i-1} \cap N) - \chi(S_i \cap N) \geq 0$. It follows that

$$\sum_i(\chi(F_{i-1}) - \chi(S_i)) \geq \sum_i((\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1).$$

By Lemma 11,

$$\sum_i((\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1)) \geq 8g - 2.$$

Thus

$$\sum_i(\chi(F_{i-1}) - \chi(S_i)) \geq 8g - 2.$$

Hence by Theorem 3, $2genus(S) - 2 = -\chi(S) \geq 8g - 2$, whence $genus(S) \geq 4g \geq 2g + 2$.

Case 3: A component of $\cup_i S_i \cap N_1$ is pseudohorizontal.

Denote the pseudohorizontal component of $(\cup_i S_i) \cap N_1$ by $\tilde{S}$. Then by Lemma 7, either $\tilde{S}$ is as in Lemma 7 and $(\cup_i S_i) \cap N_2$ consists of a single annulus or $genus(S) \geq 2g + 2$. This puts us into situation (2) or (3), respectively.

Case 4: $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_2$ is horizontal.

If $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_1$ is horizontal, then the result follows by Case 2. If a component of $(\cup_i S_i) \cap N_1$ is pseudohorizontal, then the result follows by Case 3. Thus we may assume that $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_1$ is vertical.

Note that the components of $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_2$ are all parallel. Let H be such a component, then

$$\chi(H) = 2 - 2genus(H) - |H \cap \partial N_2|.$$

Recall that by Lemma 9, $genus(H) \geq 1$.

Thus

$$\sum_i((\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2)) =$$

$$(2genus(H) - 2)(|S_i \cap N_2| - |F_{i-1} \cap N_2|) - \sum_i(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|) \geq$$

$$-\sum_i(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|.$$

Fact 4 following Theorem 4 tells us that $(\cup_i F_i) \cap N_e$ does not contain a torus. It follows that $\cup_i S_i \cap \partial N_1 \neq \emptyset$.

Now

$$\sum_i(\chi(F_{i-1}) - \chi(S_i)) =$$

$$\sum_i((\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1)) + \sum_i((\chi(F_{i-1} \cap N_e) - \chi(S_i \cap N_e)) +$$

$$+ \sum_i(\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2))$$

If we denote the edge manifold by N_e, then Fact 1 following Theorem 4 tells us that $\sum_i((\chi(F_{i-1} \cap N_e) - \chi(S_i \cap N_e)) \geq 0$. So

$$\sum_i(\chi(F_{i-1}) - \chi(S_i)) \geq$$
$$\sum_i((\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1)) + (\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2))$$

Thus by Lemma 13,

$$\sum_i(\chi(F_{i-1}) - \chi(S_i)) \geq$$
$$4g + 2 + \sum_i(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|) - \sum_i(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|) =$$
$$4g + 2.$$

Therefore by Theorem 3, $2genus(S) - 2 = -\chi(S) \geq 4g + 2$, whence $genus(S) \geq 2g + 2$.

Case 5: A component of $(\cup_i S_i) \cap N_2$ is pseudohorizontal.

Here too, note that if $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_1$ is horizontal, then the result follows by Case 2. And if a component of $(\cup_i S_i) \cap N_1$ is pseudohorizontal, then the result follows by Case 3. Thus we may assume that $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_1$ is vertical. Denote the pseudohorizontal component of $(\cup_i S_i) \cap N_2$ by $\tilde{S}$ and note that here $(((\cup_i F_i) \cup (\cup_i S_i)) - \tilde{S}) \cap N_2 = \emptyset$.

Here $\chi(\tilde{S}) = 2 - 2genus(\tilde{S}) - |S \cap \partial N_2|$. By Lemma 12, $\tilde{S}$ is either as in Lemma 6 or $genus(\tilde{S}) \geq 2$.

Thus

$$\sum_i((\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2)) = -\chi(\tilde{S}) =$$
$$2genus(\tilde{S}) - 2 + |S \cap \partial N_2| \geq |S \cap \partial N_2|.$$

Any boundary parallel annuli in $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_e$ must be parallel into N_1 and can be isotoped into N_1. We may then assume that

$$\sum_i(|F_{i-1} \cap \partial N_1| - |S_i \cap \partial N_1|) = -2.$$

Thus

$$\sum_i((\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1)) \geq 4g + 2 + \sum_i(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|)$$

Hence arguing as in Case 4, we obtain

$$\sum_i(\chi(F_{i-1}) - \chi(S_i)) \geq$$
$$\sum_i((\chi(F_{i-1} \cap N_1) - \chi(S_i \cap N_1)) + (\chi(F_{i-1} \cap N_2) - \chi(S_i \cap N_2)) \geq$$
$$4g + 2 + \sum_i(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|) - \sum_i(|F_{i-1} \cap \partial N_2| - |S_i \cap \partial N_2|) =$$
$$4g + 2 - 2 + 2 = 4g + 2.$$

Again, by Theorem 3, $2genus(S) - 2 = -\chi(S) \geq 4g + 2$, whence $genus(S) \geq 2g + 2$.

$\square$

Proof of Theorem 1 If option 1 occurs in Lemma 16, then Lemma 6 implies that N_2 is a q-bridge knot complement and the fiber of N_1 is identified with the meridian of N_2. This puts us into situation 1 of Theorem 1. If option 2 occurs in Lemma 16, then $\hat{N}_1$ admits a horizontal Heegaard splitting of genus $2g$ by Lemma 7 and we are in situation 2 of Theorem 1. If option 3 occurs there is nothing to show. □

6. The proof of Theorem 2

In this section we construct for any $n \in \mathbb{N}$ such that $n \geq 3$ a graph manifold M_n such that $\pi_1(M_n)$ is $3n$-generated but that the Heegaard genus of M_n is $4n$. We denote the graph underlying M_n by Γ_n. Γ_n is a tree on $2n+1$ vertices $z, c_1, \ldots, c_n, d_1, \ldots, d_n$ and $2n$ edges $e_1, \ldots, e_n, f_1, \ldots, f_n$ such that c_i and d_i are the endpoints of e_i and that d_i and z are the endpoints of f_i.

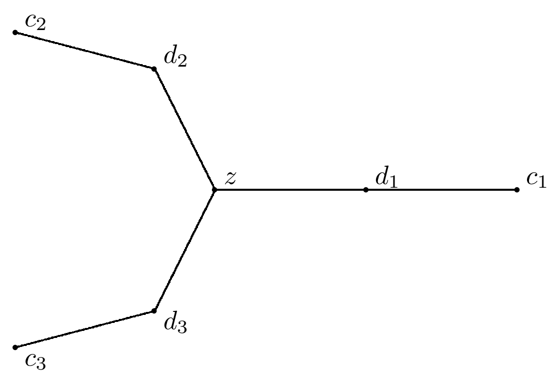

Figure 3. The tree Γ_3

The closed graph manifold M_n is then constructed as follows, where we denote the Seifert piece corresponding to a vertex v by N_v.

(1) The intersection number between the fibers of the adjacent Seifert spaces is 1 at any torus of the JSJ decomposition.
(2) $\mathcal{O}(N_z)$ is a n-punctured sphere with one cone point of order $20n$ and $\hat{N}_z = S^3$.
(3) $\mathcal{O}(N_{d_i}) = T^2(\infty, \infty, 20n)$ and N_{d_i} admits no pseudohorizontal surface that has genus 2.
(4) $\mathcal{O}(N_{c_i})$ is of type $D(3,q)$ with $q \geq 20n$ and $(3,q) = 1$ but N_{c_i} is not homeomorphic to the exterior of a 2-bridge knot in S^3.

Remark 17. Note that (2) is equivalent to stating that N_z is the exterior of a Seifert fibered n component n-bridge link in S^3, in particular $\pi_1(N_z)$ is generated by the fibers of the N_{d_i}. The existence of the spaces N_{d_i} satisfying (3) is an immediate consequence of Lemma 8.

The first part of the proof of Theorem 2 is again a simple calculation:

Lemma 18. $\pi_1(M_n)$ *can be generated by* $3n$ *elements.*

Proof. The proof is almost identical to the proof of Lemma 15 and we frequently omit explicit calculations if they are identical. Recall that

$$\pi_1(N_{d_i}) = \langle a_i, b_i, s_i, t_{i1}, t_{i2}, f_i \mid R_i \rangle \text{ with}$$

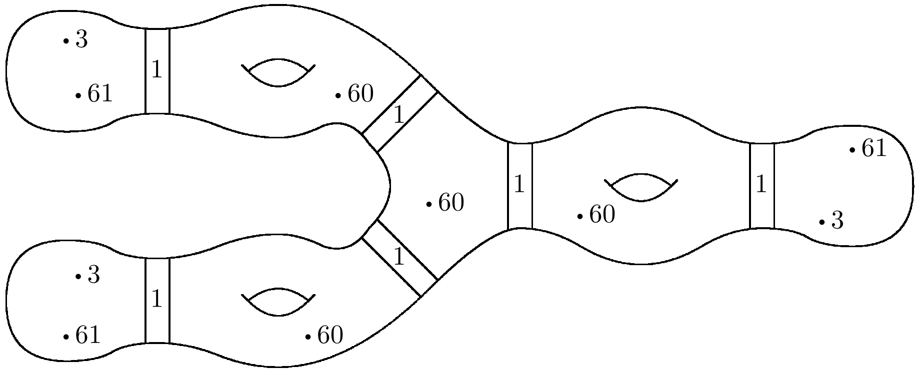

FIGURE 4. A graph manifold M with $g(M) = 12$ and $r(M) \leq 9$

$$R_i = \{[a_i, f_i], [b_i, f_i], [s_i, f_i], [t_{i1}, f_i], [t_{i2}, f_i], s_i^{5n} = f_i^{\beta_i}, [a_i b_i] t_{i1} t_{i2} s_i = f_i^{e_i}\}$$

where t_{i1} corresponds the the boundary component between N_{d_i} and N_z and t_{i2} corresponds to the boundary component between N_{d_i} and N_{c_i}.

Recall from the proof of Lemma 15 that there exist elements $h_{i1}, h_{i2} \in \pi_1(N_{c_i})$ such that $U_i = \langle f_i, h_{i1} f_i h_{i1}^{-1}, h_{i2} f_i h_{i2}^{-1} \rangle$ is a subgroup of finite index in $\pi_1(N_{c_i})$ that maps surjectively onto the fundamental group of $\mathcal{O}(N_{c_i})$ and that $h_{i1}, h_{i2} \in U_i$.

We will show that $\pi_1(M_n)$ is generated by the generators $g_1, \ldots, g_{3n}$ defined as follows:

(1) g_i is the generator of the cyclic group $\langle f_i, s_i \rangle$ for $1 \leq i \leq n$.
(2) $g_{n+i} = h_{i1} a_i$ for $1 \leq i \leq n$.
(3) $g_{2n+i} = h_{i2} b_i$ for $1 \leq i \leq n$.

Let $H = \langle g_1, \ldots, g_{3n} \rangle$. We show that $H = \pi_1(M_n)$.

Note first that $\pi_1(N_z) \subset H$ as $g_i \in H$ implies $f_i \in H$ for $1 \leq i \leq n$ and $\pi_1(N_z)$ is generated by the f_i. This implies that $t_{i1} \in H$ for $1 \leq i \leq n$.

The same calculation as in the proof of Lemma 15 further shows that $U_i \subset H$ for $1 \leq i \leq n$. It follows that $a_i, b_i \in H$ for $1 \leq i \leq n$. Thus $\pi_1(N_{d_i}) \subset H$ as $\pi_1(N_{d_i})$ is generated by $a_i, b_i, s_i, f_i, t_{i_1}$ and s_i and f_i are powers of g_i.

It now further follows that $\pi_1(N_{c_i}) \subset H$ as $\pi_1(N_{c_i})$ is generated by U_i and C_i where $C_i = \pi_1(N_{c_i}) \cap \pi_1(N_{d_i})$. □

To conclude it clearly suffices to establish the following:

Proposition 19. *The Heegaard genus of M_n is at least $4n$.*

Before we proceed with the proof of Proposition 19 we show that small genus Heegaard splittings have very special untelescopings.

Lemma 20. *Let $M_n = V \cup_S W$ be a Heegaard splitting of M_n. Then either $g(S) \geq 4n$ or there is a strongly irreducible untelescoping $S_1, F_1, \ldots, F_{k-1}, S_k$ of $M_N = V \cup_S W$ such that for any vertex manifold N no component of $S_i \cap N$ or $F_i \cap N$ is horizontal. In particular all F_i are vertical incompressible tori.*

Proof. Suppose that some component F of $S_i \cap N$ or $F_i \cap N$ is horizontal for some i and some vertex manifold N. Note first that no component of ∂F bounds a disk as any component is an essential curve in an incompressible torus. It follows that $\chi(F) \geq \chi(F_i)$ (or $\chi(F) \geq \chi(S_i)$) where F_i (or S_i) is the surface containing F.

Note first that $F \cap N$ is a covering of the base space $\mathcal{O}$ of N of degree at least $20n$. It is furthermore easy to see that we have $\chi(\mathcal{O}) \leq -\frac{1}{2}$ for any choice of N. If follows that $\chi(F \cap N) \leq -10n$ and therefore $\chi(F_i) \leq -10n$ (or $\chi(S_i) \leq -10n$). This however implies that the genus of F_i (or S_i) is greater than $5n$ which implies that the Heegaard surface S is of genus at least $5n$. This proves the assertion. □

Proof of Proposition 19 To see that M_n admits no Heegaard splitting of genus less than $4n$, proceed along the same lines as in the proof of Lemma 16. Let $M = V \cup_S W$ be a Heegaard splitting and let $S_1, F_1, \ldots, F_{k-1}, S_k$ be a strongly irreducible untelescoping of $M = V \cup_S W$. We consider the various possible cases for $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_{c_j}$ and $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_{d_j}$.

Case 1: Fix j and suppose that $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_{c_j}$ and $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_{d_j}$ are vertical or pseudovertical.

Note that in this case it is impossible for $((\cup_i F_i) \cup (\cup_i S_i))$ to meet the edge manifold N_{e_j} between N_{c_j} and N_{d_j} in spanning annuli. Thus either $\cup_i F_i$ meets N_{e_j} in an essential torus, or $\cup_i S_i$ meets N_{e_j} in the only other possible configuration. In the first case, we may assume that

$$\sum_i (-|S_i \cap \partial N_{c_j}| + |F_{i-1} \cap \partial N_{c_j}|) = 0$$

and

$$\sum_i (-|S_i \cap \partial N_{d_j}| + |F_{i-1} \cap \partial N_{d_j}|) = \sum_i (-|S_i \cap \partial N_z^j| + |F_{i-1} \cap \partial N_z^j|)$$

where ∂N_z^j is the component of ∂N_z that meets the edge manifold N_{g_j} between N_z and N_{d_j}. In the second case we may assume that

$$\sum_i (-|S_i \cap \partial N_{c_j}| + |F_{i-1} \cap \partial N_{c_j}|) = 2$$

and

$$\sum_i (-|S_i \cap \partial N_{d_j}| + |F_{i-1} \cap \partial N_{d_j}|) = 2 + \sum_i (-|S_i \cap \partial N_z^j| + |F_{i-1} \cap \partial N_z^j|)$$

We further distinguish the cases in which $\cup_i F_i$ meets or does not meet the edge manifold N_{g_j} in an essential torus.

Subcase 1.1: $\cup_i F_i$ meets N_{e_j} in an essential torus.

Here, by Lemma 13 and Lemma 14, we have

$$\sum_i ((\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j})) \geq$$

$$4 + 2 + \sum_i (|F_{i-1} \cap \partial N_{d_j}| - |S_i \cap \partial N_{d_j}|) + 2 + \sum_i (|F_{i-1} \cap \partial N_{c_j}| - |S_i \cap \partial N_{c_j}|) \geq$$

$$4 + 2 + \sum_i (|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|) + 2 \geq$$

$$8 + \sum_i (|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|).$$

Subcase 1.2: $\cup_i F_i$ meets neither N_{g_j} nor N_{e_j} in an essential torus.

Here, by Lemma 13 and Lemma 14, we have

$$\sum_i ((\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) +$$
$$+ (\chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})) \geq$$
$$4 + 4 + \sum_i (|F_{i-1} \cap \partial N_{d_j}| - |S_i \cap \partial N_{d_j}|) + 2 + \sum_i (|F_{i-1} \cap \partial N_{c_j}| - |S_i \cap \partial N_{c_j}|) + 2 \geq$$
$$4 + 4 - 2 + \sum_i (|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|) + 2 - 2 + 2 \geq$$
$$8 + \sum_i (|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|).$$

Subcase 1.3: $\cup_i F_i$ meets N_{g_j} in an essential torus but does not meet N_{e_j} in an essential torus.

Here Lemma 13 and Lemma 14 yield only the following:

$$\sum_i ((\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) +$$
$$+ (\chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})) \geq$$
$$4 + 2 + \sum_i (|F_{i-1} \cap \partial N_{d_j}| - |S_i \cap \partial N_{d_j}|) + 2 + \sum_i (|F_{i-1} \cap \partial N_{c_j}| - |S_i \cap \partial N_{c_j}|) + 2 \geq$$
$$4 + 2 - 2 + 2 - 2 + 2 \geq 6$$

It is important to note that in this case $((\cup_i F_i) \cup (\cup_i S_i)) \cap N_z$ must be vertical or pseudovertical.

Note that in all cases we have

$$\sum_i (\chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) + \chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})) \geq 2.$$

Case 2: Fix j and suppose a component of $\cup_i S_i \cap N_{d_j}$ is pseudohorizontal.

Recall that in this case $((\cup_i S_i) \cup (\cup_i F_i)) \cap N_{d_j}$ is connected. In particular, $\cup_i F_i \cap N_{d_j} = \emptyset$.

By construction, the genus of a pseudohorizontal surface is even. Thus it follows from the assumption that N_{d_j} admits no pseudohorizontal surface of genus 2 that the genus of $(\cup_i S_i) \cap N_{d_j}$ is at least 4. Hence

$$\sum_i ((\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j})) = \sum_i (0 - \chi(S_i \cap N_{d_j})) \geq 6 + b$$

where b is the number of boundary components of the connected pseudohorizontal surface $(\cup_i S_i) \cap N_{d_j}$.

Thus

$$\sum_i ((\chi(F_{i-1} \cap N_{d_j}) - \chi(S_i \cap N_{d_j}) + \chi(F_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j})) \geq$$
$$6 + b + 2 + \sum_i (|F_{i-1} \cap \partial N_{c_j}| - |S_i \cap \partial N_{c_j}|) =$$

$$8-\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)$$

Case 3: Fix j and suppose a component of $(\cup_i S_i)\cap N_{c_j}$ is pseudohorizontal.

It will suffice to consider the case in which $((\cup_i F_i)\cup(\cup_i S_i))\cap N_{d_j}$ is vertical or pseudovertical. Denote the pseudohorizontal component of $(\cup_i S_i)\cap N_{c_j}$ by $\tilde{S}$ and note that here $(((\cup_i F_i)\cup(\cup_i S_i))-\tilde{S})\cap N_{c_j}=\emptyset$. Note that here we may assume that $\sum_i(|F_{i-1}\cap\partial N_{c_j}|-|S_i\cap\partial N_{c_j}|)=-2$ and $\sum_i(|F_{i-1}\cap\partial N_{d_j}|-|S_i\cap\partial N_{d_j}|)=-2+\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)$.

Thus by Lemma 13 and Lemma 12, and since $min(p,q)\geq 3$,

$$\sum_i((\chi(F_{i-1}\cap N_{d_j})-\chi(S_i\cap N_{d_j})+\chi(F_{i-1}\cap N_{c_j})-\chi(S_i\cap N_{c_j}))\geq$$

$$4+2+\sum_i(|F_{i-1}\cap\partial N_{d_j}|-|S_i\cap\partial N_{d_j}|)+2min(p,q)-2\geq$$

$$4+2-2+\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)+2min(p,q)-2\geq$$

$$8+\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|).$$

Note that here

$$\sum_i(\chi(F_{i-1}\cap N_{c_j})-\chi(S_i\cap N_{c_j}))=-\chi(\tilde{S})\geq 4.$$

Putting these computations together we must consider the various options for $((\cup_i F_i)\cup(\cup_i S_i))\cap N_z$:

Case A: $((\cup_i F_i)\cup(\cup_i S_i))\cap N_z$ is vertical and pseudovertical.

Note that in this case the options for $((\cup_i F_i)\cup(\cup_i S_i))\cap N_{g_j}$ are severely limited. If $((\cup_i F_i)\cup(\cup_i S_i))\cap N_{d_j}$ is vertical and pseudovertical, then $((\cup_i F_i)\cup(\cup_i S_i))\cap N_{g_j}$ cannot consist of spanning annuli. So either $\cup_i F_i$ meets N_{g_j} in an essential torus, or $\cup_i S_i$ meets N_{g_j} in the only other possible configuration. If $((\cup_i F_i)\cup(\cup_i S_i))\cap N_{d_j}$ is pseudohorizontal, then N_{g_j} cannot meet a toral component of $((\cup_i F_i)$. So it must consist either of spanning annuli or the only other possible configuration.

Denote the set of j such that $\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)=0$ by J_0. Then $\sum_i((\chi(F_{i-1}\cap N_{g_j})-\chi(S_i\cap N_{g_j}))=0$ for $j\in J_0$.

Denote the set of j not in J_0 such that $((\cup_i F_i)\cup(\cup_i S_i))\cap N_{d_j}$ are vertical or pseudovertical by J_1. Then $\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)=-2$, and $\sum_i((\chi(F_{i-1}\cap N_{g_j})-\chi(S_i\cap N_{g_j})=2$ for $j\in J_1$.

Denote the set of j such that $((\cup_i F_i)\cup(\cup_i S_i))\cap N_{d_j}$ is pseudohorizontal by J_2. We clearly have $J=J_0\dot{\cup}J_1\dot{\cup}J_2$.

By Lemma 13,

$$\sum_i\sum_j((\chi(F_{i-1}\cap N_z)-\chi(S_i\cap N_z))\geq -2(2-n)+2+\sum_i(|F_{i-1}\cap\partial N_z|-|S_i\cap\partial N_z|)$$

Thus,

$$-\chi(S)=\sum_i(\chi(F_{i-1})-\chi(S_i))\geq$$

$$\sum_i((\chi(F_{i-1}\cap N_z)-\chi(S_i\cap N_z)+\sum_j(\chi(F_{i-1}\cap N_{c_j})-\chi(S_i\cap N_{c_j})+$$
$$+\chi(F_{i-1}\cap N_{d_j})-\chi(S_i\cap N_{d_j})+\chi(F_{i-1}\cap N_{e_j})-\chi(S_i\cap N_{e_j})+$$
$$+(\chi(F_{i-1}\cap N_{g_j})-\chi(S_i\cap N_{g_j})))=$$

$$\sum_i((\chi(F_{i-1}\cap N_z)-\chi(S_i\cap N_z))+$$
$$+\sum_j\sum_i(\chi(F_{i-1}\cap N_{c_j})-\chi(S_i\cap N_{c_j})+\chi(F_{i-1}\cap N_{d_j})-\chi(S_i\cap N_{d_j})+$$
$$+\chi(F_{i-1}\cap N_{e_j})-\chi(S_i\cap N_{e_j})+\chi(F_{i-1}\cap N_{g_j})-\chi(S_i\cap N_{g_j}))\geq$$

$$-2(2-n-1)+\sum_i(|F_{i-1}\cap\partial N_z|-|S_i\cap\partial N_z|)+$$
$$+\sum_{j\in J_0}\sum_i(\chi(F_{i-1}\cap N_{c_j})-\chi(S_i\cap N_{c_j})+\chi(F_{i-1}\cap N_{d_j})-\chi(S_i\cap N_{d_j})+$$
$$+\chi(F_{i-1}\cap N_{e_j})-\chi(S_i\cap N_{e_j})+\chi(F_{i-1}\cap N_{g_j})-\chi(S_i\cap N_{g_j}))+$$
$$+\sum_{j\in J_1}\sum_i(\chi(F_{i-1}\cap N_{c_j})-\chi(S_i\cap N_{c_j})+\chi(F_{i-1}\cap N_{d_j})-\chi(S_i\cap N_{d_j})+$$
$$+\chi(F_{i-1}\cap N_{e_j})-\chi(S_i\cap N_{e_j})+\chi(F_{i-1}\cap N_{g_j})-\chi(S_i\cap N_{g_j}))+$$
$$+\sum_{j\in J_2}\sum_i(\chi(F_{i-1}\cap N_{c_j})-\chi(S_i\cap N_{c_j})+\chi(F_{i-1}\cap N_{d_j})-\chi(S_i\cap N_{d_j})+$$
$$+(\chi(F_{i-1}\cap N_{e_j})-\chi(S_i\cap N_{e_j})+(\chi(F_{i-1}\cap N_{g_j})-\chi(S_i\cap N_{g_j}))\geq$$

$$-2(1-n)+\sum_i\sum_j(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)+$$
$$\sum_{j\in J_0}6+\sum_{j\in J_1}(8-2+2)+$$
$$+\sum_{j\in J_2}(8-\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|))=$$

$$-2(1-n)+\sum_{j\in J_0}\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)+$$
$$+\sum_{j\in J_1}\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)+\sum_{j\in J_2}\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)+$$
$$\sum_{j\in J_0}6+\sum_{j\in J_1}(8-2+2)+\sum_{j\in J_2}(8-\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|))=$$

$$-2+2n+0+\sum_{j\in J_1}(-2)+\sum_{j\in J_2}\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|)+$$
$$\sum_{j\in J_0}6+\sum_{j\in J_1}(8-2+2)+\sum_{j\in J_2}8-\sum_{j\in J_2}\sum_i(|F_{i-1}\cap\partial N_z^j|-|S_i\cap\partial N_z^j|))=$$

$$-2 + 2n + \sum_{j\in J_0} 6 + \sum_{j\in J_1} 6 + \sum_{j\in J_2} 8 \geq -2 + 2n + 6n = 8n - 2$$

Hence $genus(S) \geq 4n$.

Case B: A component of $(\cup_i S_i) \cap N_z$ is pseudohorizontal.

Denote the pseudohorizontal component of $(\cup_i S_i) \cap N_z$ by $\tilde{S}$ and note that here $(((\cup_i F_i) \cup (\cup_i S_i)) - \tilde{S}) \cap N_z = \emptyset$. Now $\chi(\tilde{S}) = 2 - 2genus(\tilde{S}) - |\partial\tilde{S}|$ and

$$\sum_i (|F_{i-1} \cap \partial N_z| - |S_i \cap \partial N_z| = -|\partial\tilde{S}|.$$

Thus here,

$$-\chi(S) = \sum_i (\chi(F_{i-1}) - \chi(S_i)) \geq$$

$$\sum_i (\chi(F_{i-1} \cap N_z) - \chi(S_i \cap N_z)) + \sum_i \sum_j ((\chi(f_{i-1} \cap N_{c_j}) - \chi(S_i \cap N_{c_j}) + (\chi(F_{i-1} \cap N_{d_j}) -$$

$$\chi(S_i \cap N_{d_j}) + (\chi(F_{i-1} \cap N_{e_j}) - \chi(S_i \cap N_{e_j})) =$$

$$-2 + 2genus(\tilde{S}) + |\partial\tilde{S}| + \sum_j \sum_i (8 - (|F_{i-1} \cap \partial N_z^j| - |S_i \cap \partial N_z^j|)) =$$

$$-2 + 2genus(\tilde{S}) + \sum_j 8 \geq -2 + 8n.$$

Whence, $genus(S) \geq 4n$. □

7. Some comments on non totally orientable graph manifolds

During the proofs of Theorem 1 and Theorem 2 we make extensive use of the structure theorem for Heegaard splittings of totally orientable graph manifolds [9]. We believe however that similar statements are true for graph manifolds in general. This suggests a more straightforward generalization of the examples provided in [13] which are not totally orientable.

It should be noted that the verification that the manifolds constructed in [13] are not of Heegaard genus 2 relies on the classification of 3-manifolds with non-empty characteristic submanifold that have a genus 2 Heegaard splitting as given by T. Kobayashi [3].

Thus we conjecture that the manifolds M_n constructed below are of Heegaard genus $3n$, the same argument as above shows that they can be generated by $2n$ elements.

The graph underlying the manifold M_n is again Γ_n, the Seifert piece corresponding to the vertex v is again denoted by N_v and the following hold:

(1) The intersection number between the fibers of the adjacent Seifert spaces is 1 at any torus of the JSJ decomposition.
(2) $\mathcal{O}(N_z)$ is a n-punctured sphere with at most one cone point and $\hat{N}_z = S^3$.
(3) $\mathcal{O}(N_{d_i}) = P^2(\infty, \infty, 5n)$ and N_{d_i} admits no pseudohorizontal surface that has genus 2.
(4) $\mathcal{O}(N_{c_i})$ is of type $D(2, q)$ with odd q but N_{c_i} is not homeomorphic to the exterior of a 2-bridge knot in S^3.

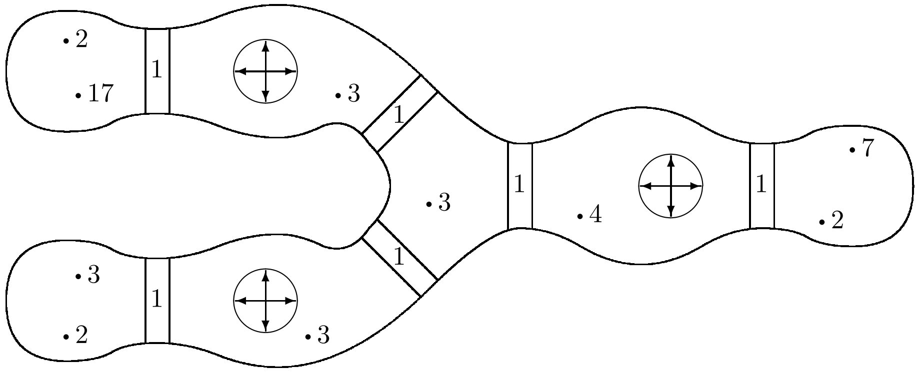

FIGURE 5. A graph manifold M with $g(M) = 9$ and $r(M) \leq 6$?

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DEPARTMENT OF MATHEMATICS, 1 SHIELDS AVE, UNIVERSITY OF CALIFORNIA, DAVIS, CA 95616, USA

Current address: MPIM-Bonn, Vivatsgasse 7, D-53111 Bonn, Germany

E-mail address: jcs@math.ucdavis.edu

FAKULTÄT FÜR MATHEMATIK, RUHR-UNIVERSITÄT BOCHUM, 44780 BOCHUM, GERMANY

E-mail address: Richard.Weidmann@ruhr-uni-bochum.de