Privacy Preserving Data Mining*

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Abstract

In this paper we address the issue of privacy preserving data mining. Specifically, we consider a scenario in which two parties owning confidential databases wish to run a data mining algorithm on the union of their databases, without revealing any unnecessary information. Our work is motivated by the need to both protect privileged information and enable its use for research or other purposes.

The above problem is a specific example of secure multi-party computation and as such, can be solved using known generic protocols. However, data mining algorithms are typically complex and, furthermore, the input usually consists of massive data sets. The generic protocols in such a case are of no practical use and therefore more efficient protocols are required. We focus on the problem of decision tree learning with the popular ID3 algorithm. Our protocol is considerably more efficient than generic solutions and demands both very few rounds of communication and reasonable bandwidth.

Key words: Secure two-party computation, Oblivious transfer, Oblivious polynomial evaluation, Data mining, Decision trees.

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1 Introduction

We consider a scenario where two parties having private databases wish to cooperate by computing a data mining algorithm on the union of their databases. Since the databases are confidential, neither party is willing to divulge any of the contents to the other. We show how the involved data mining problem of decision tree learning can be efficiently computed, with no party learning anything other than the output itself. We demonstrate this on ID3, a well-known and influential algorithm for the task of decision tree learning. We note that extensions of ID3 are widely used in real market applications.

Data mining. Data mining is a recently emerging field, connecting the three worlds of Databases, Artificial Intelligence and Statistics. The information age has enabled many organizations to gather large volumes of data. However, the usefulness of this data is negligible if "meaningful information" or "knowledge" cannot be extracted from it. Data mining, otherwise known as *knowledge discovery*, attempts to answer this need. In contrast to standard statistical methods, data mining techniques search for *interesting* information without demanding a priori hypotheses. As a field, it has introduced new concepts and algorithms such as association rule learning. It has also applied known machine-learning algorithms such as inductive-rule learning (e.g., by decision trees) to the setting where very large databases are involved. Data mining techniques are used in business and research and are becoming more and more popular with time.

Confidentiality issues in data mining. A key problem that arises in any en masse collection of data is that of *confidentiality*. The need for privacy is sometimes due to law (e.g., for medical databases) or can be motivated by business interests. However, there are situations where the *sharing* of data can lead to mutual gain. A key utility of large databases today is research, whether it be scientific, or economic and market oriented. Thus, for example, the medical field has much to gain by pooling data for research; as can even competing businesses with mutual interests. Despite the potential gain, this is often not possible due to the confidentiality issues which arise.

We address this question and show that highly efficient solutions are possible. Our scenario is the following:

> Let P_1 and P_2 be parties owning (large) private databases D_1 and D_2. The parties wish to apply a data-mining algorithm to the joint database $D_1 \cup D_2$ without revealing any unnecessary information about their individual databases. That is, the only information learned by P_1 about D_2 is that which can be learned from the output of the data mining algorithm, and vice versa. We do not assume any "trusted" third party who computes the joint output.

Very large databases and efficient secure computation. We have described a model which is exactly that of multi-party computation. Therefore, there exists a secure protocol for *any* probabilistic polynomial-time functionality [10, 17]. However, as we discuss in Section 1.1, these generic solutions are very inefficient, especially when large inputs and complex algorithms are involved. Thus, in the case of private data mining, more efficient solutions are required.

It is clear that any reasonable solution must have the individual parties do the majority of the computation independently. Our solution is based on this guiding principle and in fact, the number of bits communicated is dependent on the number of transactions by a logarithmic factor only. We remark that a necessary condition for obtaining such a private protocol is the existence of a (non-private) distributed protocol with low communication complexity.

Semi-honest adversaries. In any multi-party computation setting, a *malicious* adversary can always alter its input. In the data-mining setting, this fact can be very damaging since the adversary can define its input to be the empty database. Then, the output obtained is the result of the algorithm on the other party's database alone. Although this attack cannot be prevented, we would like to prevent a malicious party from executing any other attack. However, for this initial work we assume that the adversary is *semi-honest* (also known as *passive*). That is, it correctly follows the protocol specification, yet attempts to learn additional information by analyzing the transcript of messages received during the execution. We remark that although the semi-honest adversarial model is far weaker than the malicious model (where a party may arbitrarily deviate from the protocol specification), it is often a realistic one. This is because deviating from a specified program which may be buried in a complex application is a non-trivial task. Semi-honest adversarial behavior also models a scenario in which both parties that participate in the protocol are honest. However, following the protocol execution, an adversary may obtain a transcript of the protocol execution by breaking into one of the parties' machines.

1.1 Related Work

Secure two party computation was first investigated by Yao [17], and was later generalized to multi-party computation in [10, 1, 4]. These works all use a similar methodology: the functionality f to be computed is first represented as a combinatorial circuit, and then the parties run a short protocol for every gate in the circuit. While this approach is appealing in its generality and simplicity, the protocols it generates depend on the size of the circuit. This size depends on the size of the input (which might be huge as in a data mining application), and on the complexity of expressing f as a circuit (for example, a naive multiplication circuit is quadratic in the size of its inputs). We stress that secure two-party computation of small circuits with small inputs may be *practical* using the [17] protocol.[1]

Due to the inefficiency of generic protocols, some research has focused on finding efficient protocols for *specific* (interesting) problems of secure computation. See [2, 5, 7, 13] for just a few examples. In this paper, we continue this direction of research for the specific problem of distributed ID3.

1.2 Organization

In the next section we describe the problem of classification and a widely used solution to it, the ID3 algorithm for decision tree learning. Then, the definition of security is presented in Section 3 followed by a description of the cryptographic tools used in Section 4. Section 5 contains the protocol for private distributed ID3 and in Section 6 we describe the main subprotocol that privately computes random shares of $f(v_1, v_2) \stackrel{\text{def}}{=} (v_1+v_2)\ln(v_1+v_2)$. Finally, in Section 7 we discuss practical considerations and the efficiency of our protocol.

2 Classification by Decision Tree Learning

This section briefly describes the machine learning and data mining problem of *classification* and ID3, a well-known algorithm for it. The presentation here is rather simplistic and very brief and we refer the reader to Mitchell [12] for an in-depth treatment of the subject. The ID3 algorithm

[1]The [17] protocol requires only two rounds of communication. Furthermore, since the circuit and inputs are small, the bandwidth is not too great and only a reasonable number of oblivious transfers need be executed.

for generating decision trees was first introduced by Quinlan in [15] and has since become a very popular learning tool.

2.1 The Classification Problem

The aim of a classification problem is to classify transactions into one of a discrete set of possible categories. The input is a structured database comprised of attribute-value pairs. Each row of the database is a *transaction* and each column is an *attribute* taking on different values. One of the attributes in the database is designated as the *class* attribute; the set of possible values for this attribute being the classes. We wish to predict the class of a transaction by viewing only the non-class attributes. This can then be used to predict the class of new transactions for which the class is unknown.

For example, a bank may wish to conduct credit risk analysis in an attempt to identify non-profitable customers before giving a loan. The bank then defines "Profitable-customer" (obtaining values "yes" or "no") to be the class attribute. Other database attributes may include: Home-Owner, Income, Years-of-Credit, Other-Delinquent-Accounts and other relevant information. The bank is interested in learning rules such as:

If (Other-Delinquent-Accounts = 0) and (Income > 30k or Years-of-Credit > 3)
then Profitable-customer = YES [accept credit-card application]

A collection of such rules covering all possible transactions can then be used to classify a *new* customer as potentially profitable or not. The classification may also be accompanied with a probability of error. Not all classification techniques output a set of meaningful rules, we have brought just one example here.

Another example application is to attempt to predict whether a woman is at high risk for an Emergency Caesarean Section, based on data gathered during the pregnancy. There are many useful examples and it is not hard to see why this type of learning or mining task has become so popular.

The success of an algorithm on a given data set is measured by the percentage of new transactions correctly classified. Although this is an important data mining (and machine learning) issue, we do not go into it here.

2.2 Decision Trees and the ID3 Algorithm

A decision tree is a rooted tree containing nodes and edges. Each internal node is a test node and corresponds to an attribute; the edges leaving a node correspond to the possible values taken on by that attribute. For example, the attribute "Home-Owner" would have two edges leaving it, one for "Yes" and one for "No". Finally, the leaves of the tree contain the *expected* class value for transactions matching the path from the root to that leaf.

Given a decision tree, one can predict the class of a new transaction t as follows. Let the attribute of a given node v (initially the root) be A, where A obtains possible values $a_1, ..., a_m$. Then, as described, the m edges leaving v are labeled $a_1, ..., a_m$ respectively. If the value of A in t equals a_i, we simply go to the son pointed to by a_i. We then continue recursively until we reach a leaf. The class found in the leaf is then assigned to the transaction.

We use the following notation:

- R: the set of *attributes*
- C: the *class* attribute

- T: the set of *transactions*

The ID3 algorithm assumes that each attribute is categorical, that is containing discrete data only, in contrast to continuous data such as age, height etc.

The principle of the ID3 algorithm is as follows. The tree is constructed top-down in a recursive fashion. At the root, each attribute is tested to determine how well it alone classifies the transactions. The "best" attribute (to be discussed below) is then chosen and the remaining transactions are partitioned by it. ID3 is then recursively called on each partition (which is a smaller database containing only the appropriate transactions and without the splitting attribute). See Figure 1 for a description of the ID3 algorithm.

ID3(R, C, T)

1. If R is empty, return a leaf-node with the class value assigned to the most transactions in T.
2. If T consists of transactions which all have the same value c for the class attribute, return a leaf-node with the value c (finished classification path).
3. Otherwise,
   (a) Determine the attribute that *best* classifies the transactions in T, let it be A.
   (b) Let $a_1, ..., a_m$ be the values of attribute A and let $T(a_1), ..., T(a_m)$ be a partition of T such that every transaction in $T(a_i)$ has the attribute value a_i.
   (c) Return a tree whose root is labeled A (this is the test attribute) and has edges labeled $a_1, ..., a_m$ such that for every i, the edge a_i goes to the tree ID3$(R - \{A\}, C, T(a_i))$.

Figure 1: The ID3 Algorithm for Decision Tree Learning

What remains is to explain how the *best* predicting attribute is chosen. This is the central principle of ID3 and is based on information theory. The entropy of the class attribute clearly expresses the difficulty of prediction. We know the class of a set of transactions when the class entropy for them equals zero. The idea is therefore to check which attribute reduces the information of the class-attribute to the greatest degree. This results in a greedy algorithm which searches for a small decision tree consistent with the database. The bias favoring short descriptions of a hypothesis is based on Occam's razor. As a result of this, decision trees are usually relatively small, even for large databases.[2]

The exact test for determining the best attribute is defined as follows. Let $c_1, ..., c_\ell$ be the class-attribute values and let $T(c_i)$ denote the set of transactions with class c_i. Then the information needed to identify the class of a transaction in T is the entropy, given by:

$$H_C(T) = \sum_{i=1}^{\ell} -\frac{|T(c_i)|}{|T|} \log \frac{|T(c_i)|}{|T|}$$

[2] We note that the resulting decision tree is not *guaranteed* to be small. A large tree may result in situations where the entropy reduction at many of the nodes is very small. Intuitively, this means that no attribute classifies the remaining transaction in a meaningful way (this occurs, for example, in a random database but is also common close to the leaves of the tree of a well-structured database). In such a case, continuing to develop the decision tree is unlikely to yield a better classification (and may actually make it worse). One relatively simple solution to this problem is simply to cease the development of the tree (outputting the majority class of the remaining transactions) if the information gain is below some predetermined threshold. This ensures that the resulting decision tree is usually small, in accordance with Occam's razor.

Let T be a set of transactions, C the class attribute and A some non-class attribute. We wish to quantify the information needed to identify the class of a transaction in T *given* that the value of A has been obtained. Let A obtain values $a_1, ..., a_m$ and let $T(a_j)$ be the transactions obtaining value a_j for A. Then, the conditional information of T given A, equals:

$$H_C(T|A) = \sum_{j=1}^{m} \frac{|T(a_j)|}{|T|} H_C(T(a_j))$$

Now, for each attribute A the information-gain is defined by,

$$\text{Gain}(A) \stackrel{\text{def}}{=} H_C(T) - H_C(T|A)$$

The attribute A which has the maximum gain (or equivalently minimum $H_C(T|A)$) over all attributes in R is then chosen.

Extensions of ID3. Since its inception there have been many extensions to the original algorithm, the most well-known being C4.5. We now briefly describe some of these extensions. One of the immediate shortcomings of ID3 is that it only works on discrete data, whereas most databases contain continuous data. A number of methods enable the incorporation of continuous-value attributes, even as the class attribute. Other extensions include handling missing attribute values, alternative measures for selecting attributes and reducing the problems of overfitting by pruning. (The strategy described in footnote 2 also addresses the problem of overfitting.)

See Appendix A for a short example of a database and its resulting decision tree.

2.3 The ID3_δ Approximation

The ID3 algorithm chooses the "best" predicting attribute by comparing entropies that are given as real numbers. If at a given point, two entropies are very close together, then the two (different) trees resulting from choosing one attribute or the other are expected to have almost the same predicting capability. Formally stated, let δ be some small value. Then, for a pair of attributes A_1 and A_2, we say that A_1 and A_2 have *δ-close information gains* if

$$|H_C(T|A_1) - H_C(T|A_2)| \leq \delta$$

This definition gives rise to an approximation of ID3 as follows. Let A be the attribute for which $H_C(T|A)$ is minimum (over all attributes). Then, let A_δ equal the set of all attributes A', for which A and A' have δ-close information gains. Now, denote by $\mathcal{ID}3_\delta$ the set of all possible trees which are generated by running the ID3 algorithm with the following modification to Step 3(a). Let A be the best predicting attribute for the remaining subset of transactions. Then, the algorithm can choose *any* attribute from A_δ as the best predicting attribute (instead of A itself). Thus, any tree taken from $\mathcal{ID}3_\delta$ approximates ID3 in that the difference in information gain at any given node is at most δ. We actually present a protocol for the secure computation of a specific algorithm $\text{ID3}_\delta \in \mathcal{ID}3_\delta$, in which the choice of A' from A_δ is implicit by an approximation that is used instead of the log function. The value of δ influences the efficiency, but only by a logarithmic factor.

Note that any naive implementation of ID3 that computes the logarithm function to a predefined precision level has an approximation error, and therefore essentially computes a tree from $\mathcal{ID}3_\delta$. However, a more elaborate implementation of ID3 can resolve this problem as follows. First, the information gain for each attribute is computed using a predefined precision level that results in a log approximation of error at most δ. Then, if two information gains compared are found to be within δ' of each other (where δ' is the precision error of the information gain resulting from a δ error in the log function), then the information gains are recomputed using a higher precision level for the log function. This is continued until it is ensured that the attribute with the maximal information gain is found. We do not know how to achieve similar accuracy in a privacy preserving implementation.

3 Definitions

3.1 Private Two-Party Protocols

The model for this work is that of two-party computation where the adversarial party may be *semi-honest*. The definitions presented here are according to Goldreich in [9].

Two-party computation. A two-party protocol problem is cast by specifying a random process that maps pairs of inputs to pairs of outputs (one for each party). We refer to such a process as a functionality and denote it $f : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^* \times \{0,1\}^*$, where $f = (f_1, f_2)$. That is, for every pair of inputs (x, y), the output-pair is a random variable $(f_1(x,y), f_2(x,y))$ ranging over pairs of strings. The first party (with input x) wishes to obtain $f_1(x,y)$ and the second party (with input y) wishes to obtain $f_2(x,y)$. We often denote such a functionality by $(x,y) \mapsto (f_1(x,y), f_2(x,y))$. Thus, for example, the problem of distributed ID3 is denoted by $(D_1, D_2) \mapsto (\mathrm{ID3}(D_1 \cup D_2), \mathrm{ID3}(D_1 \cup D_2))$.

Privacy by simulation. Intuitively, a protocol is private if whatever can be computed by a party participating in the protocol can be computed based on its input and output only. This is formalized according to the simulation paradigm. Loosely speaking, we require that a party's *view* in a protocol execution be simulatable given only its input and output.[3] This then implies that the parties learn nothing from the protocol *execution* itself, as desired.

Definition of security. We begin with the following notations:

- Let $f = (f_1, f_2)$ be a probabilistic, polynomial-time functionality and let Π be a two-party protocol for computing f.
- The view of the first (*resp., second*) party during an execution of Π on (x, y), denoted $\mathrm{view}_1^{\Pi}(x,y)$ (*resp.,* $\mathrm{view}_2^{\Pi}(x,y)$), is $(x, r^1, m_1^1, ..., m_t^1)$ (*resp.,* $(y, r^2, m_1^2, ..., m_t^2)$) where r^1 (*resp.,* r^2) represents the outcome of the first (*resp., second*) party's *internal* coin tosses, and m_i^1 (*resp.,* m_i^2) represents the i'th message it has received.
- The output of the first (*resp., second*) party during an execution of Π on (x, y) is denoted $\mathrm{output}_1^{\Pi}(x,y)$ (*resp.,* $\mathrm{output}_2^{\Pi}(x,y)$), and is implicit in the party's view of the execution.

[3] A different definition of security for multiparty computation compares the output of a real protocol execution to the output of an *ideal* computation involving an incorruptible trusted third party. This trusted party receives the parties' inputs, computes the functionality on these inputs and returns to each their respective output. Loosely speaking, a protocol is secure if any real-model adversary can be converted into an ideal-model adversary such that the output distributions are computationally indistinguishable. We remark that in the case of semi-honest adversaries, this definition is equivalent to the (simpler) simulation-based definition presented here.

Definition 1 (privacy w.r.t. semi-honest behavior): *For a functionality f, we say that* Π privately computes f *if there exist probabilistic polynomial time algorithms, denoted S_1 and S_2, such that*

$$\{(S_1(x, f_1(x,y)), f_2(x,y))\}_{x,y\in\{0,1\}^*} \stackrel{c}{\equiv} \left\{(\mathrm{view}_1^\Pi(x,y), \mathrm{output}_2^\Pi(x,y))\right\}_{x,y\in\{0,1\}^*} \quad (1)$$

$$\{(f_1(x,y), S_2(y, f_2(x,y)))\}_{x,y\in\{0,1\}^*} \stackrel{c}{\equiv} \left\{(\mathrm{output}_1^\Pi(x,y), \mathrm{view}_2^\Pi(x,y))\right\}_{x,y\in\{0,1\}^*} \quad (2)$$

where $\stackrel{c}{\equiv}$ denotes computational indistinguishability.

Equations (1) and (2) state that the view of a party can be simulated by a probabilistic polynomial-time algorithm given access to the party's *input and output only.* We emphasize that the adversary here is semi-honest and therefore the view is exactly according to the protocol definition. We note that it is not enough for the simulator S_1 to generate a string indistinguishable from $\mathrm{view}_1^\Pi(x,y)$. Rather, the *joint distribution* of the simulator's output and $f_2(x,y)$ must be indistinguishable from $(\mathrm{view}_1^\Pi(x,y), \mathrm{output}_2^\Pi(x,y))$. This is necessary for probabilistic functionalities; see [3, 9] for a full discussion.

Private data mining. We now discuss issues specific to the case of two-party computation where the inputs x and y are databases. Denote the two parties P_1 and P_2 and their respective private databases D_1 and D_2. First, we assume that D_1 and D_2 have the same structure and that the attribute names are public. This is essential for carrying out any joint computation in this setting. There is a somewhat delicate issue when it comes to the names of the possible values for each attribute. On the one hand, universal names must clearly be agreed upon in order to compute any joint function. On the other hand, even the *existence* of a certain attribute value in a database can be sensitive information. This problem can be solved by a pre-processing phase in which random value names are assigned to the values such that they are consistent in both databases. Doing this efficiently is in itself a non-trivial problem. However, in our work we assume that the attribute-value names are also public (as would be *after* the above-described random mapping stage). Next, as we have discussed, each party should receive the output of some data mining algorithm on the union of their databases, $D_1 \cup D_2$. We note that in actuality we consider a *merging* of the two databases so that if the same transaction appears in both databases, then it appears twice in the merged database. Finally, we assume that an upper-bound on the size of $|D_1 \cup D_2|$ is known and public.

3.2 Composition of Private Protocols

In this section, we briefly discuss the composition of private protocols. The theorem and its corollary brought here are used a number of times throughout the paper.

The protocol for privately computing $\mathrm{ID3}_\delta$ is composed of many invocations of smaller private computations. In particular, we reduce the problem to that of privately computing smaller subproblems and show how to compose them together in order to obtain a complete $\mathrm{ID3}_\delta$ solution. Although intuitively clear, this composition requires formal justification. We present a brief, informal discussion and refer the reader to Goldreich [9] for a complete, formal treatment.

Informally, consider oracle-aided protocols, where the queries are supplied by both parties. The oracle-answer may be different for each party depending on its definition, and may also be probabilistic. An oracle-aided protocol is said to privately reduce g to f if it privately computes g when using the oracle functionality f. The security of our solution relies heavily on the following intuitive theorem.

Theorem 2 (composition theorem for the semi-honest model, two parties): *Suppose that g is privately reducible to f and that there exists a protocol for privately computing f. Then, the protocol defined by replacing each oracle-call to f by a protocol that privately computes f, is a protocol for privately computing g.*

Since the adversary considered here is semi-honest, this theorem is easily obtained by plugging in the simulator for the private computation of the oracle functionality. Furthermore, it is easily generalized to the case where a number of oracle-functionalities $f_1, f_2, ...$ are used in privately computing g.

Many of the protocols presented in this paper involve the sequential invocation of two private subprotocols, where the parties' outputs of the first subprotocol are random shares which are then input into the second subprotocol. The following corollary to Theorem 2 states that such a composed protocol is private.

Corollary 3 *Let Π_g and Π_h be two protocols that privately compute probabilistic polynomial-time functionalities g and h respectively. Furthermore, let g be such that the parties' outputs, when viewed independently of each other, are uniformly distributed (in some finite field). Then, the protocol Π comprised of first running Π_g and then using the output of Π_g as input into Π_h, is a private protocol for computing the functionality $f(x,y) = h(g_1(x,y), g_2(x,y))$, where $g = (g_1, g_2)$.*

Proof: By Theorem 2 it is enough to show that the oracle-aided protocol is private. However, this is immediate because, apart from the final output, the parties' views consist only of uniformly distributed shares that can be generated by the required simulators. ■

3.3 Private Computation of Approximations and of ID3_δ

Our work takes ID3_δ as the starting point and privacy is guaranteed relative to the approximated algorithm, rather than to ID3 itself. That is, we present a private protocol for computing ID3_δ. This means that P_1's view can be simulated given D_1 and $\text{ID3}_\delta(D_1 \cup D_2)$ only (and likewise for P_2's view). However, this does *not* mean that $\text{ID3}_\delta(D_1 \cup D_2)$ reveals the "same" (or less) information as $\text{ID3}(D_1 \cup D_2)$ does (in particular, given D_1 and $\text{ID3}(D_1 \cup D_2)$ it may not be possible to compute $\text{ID3}_\delta(D_1 \cup D_2)$). In fact, it is clear that although the computation of ID3_δ is private, the resulting tree may be different from the tree output by the exact ID3 algorithm itself (intuitively though, no "more" information is revealed).

The problem of secure distributed computation of approximations was introduced by Feigenbaum et. al. [8]. Their motivation was a scenario in which the computation of an approximation to a function f might be considerably more efficient than the computation of f itself. According to their definition, a protocol constitutes a private approximation of f if the approximation reveals no more about the inputs than f itself does. Thus, our protocol is not a private approximation of ID3, but rather a private protocol for computing ID3_δ.[4]

4 Cryptographic Tools

Oblivious transfer. The notion of 1-out-2 oblivious transfer (OT_1^2) was suggested by Even, Goldreich and Lempel [6] as a generalization of Rabin's "oblivious transfer" [16]. This protocol

[4] We note that our protocol uses many invocations of an approximation of the natural logarithm function. However, none of these approximations are revealed (they constitute intermediate values which are hidden from the parties). The only approximation which becomes known to the parties is the final ID3_δ decision tree.

involves two parties, the *sender* and the *receiver*. The sender's input is a pair (x_0, x_1) and the receiver's input is a bit $\sigma \in \{0, 1\}$. The protocol is such that the receiver learns x_σ (and nothing else) and the sender learns nothing. In other words, the oblivious transfer functionality is denoted by $((x_0, x_1), \sigma) \mapsto (\lambda, x_\sigma)$. In the case of semi-honest adversaries, there exist simple and efficient protocols for oblivious transfer [6, 9].

Oblivious polynomial evaluation. The problem of "oblivious polynomial evaluation" was first considered in [13]. As with oblivious transfer, this problem involves a sender and a receiver. The sender's input is a polynomial Q of degree k over some finite field $\mathcal{F}$ and the receiver's input is an element $z \in \mathcal{F}$ (the degree k of Q is public). The protocol is such that the receiver obtains $Q(z)$ without learning anything else about the polynomial Q, and the sender learns nothing. That is, the problem considered is the private computation of the following functionality: $(Q, z) \mapsto (\lambda, Q(z))$. An efficient solution to this problem was presented in [13]. The overhead of that protocol is $O(k)$ exponentiations (using the method suggested in [14] for doing a 1-out-of-N oblivious transfer with $O(1)$ exponentiations). (Note that the protocol suggested there maintains privacy in the face of a malicious adversary, while our scenario only requires a simpler protocol that provides security against semi-honest adversaries. Such a protocol can be designed based on any homomorphic encryption scheme, with an overhead of $O(k)$ computation and $O(k|\mathcal{F}|)$ communication.)

Yao's two-party protocol. In [17], Yao presented a constant-round protocol for privately computing any probabilistic polynomial-time functionality (where the adversary may be semi-honest). Denote Party 1 and Party 2's respective inputs by x and y and let f be the functionality that they wish to compute (for simplicity, assume that both parties wish to receive the same value $f(x, y)$). Loosely speaking, Yao's protocol works by having one of the parties (say Party 1) first generate an "encrypted" or "garbled" circuit computing $f(x, \cdot)$ and send it to Party 2. The circuit is such that it reveals nothing in its encrypted form and therefore Party 2 learns nothing from this stage. However, Party 2 can obtain the output $f(x, y)$ by "decrypting" the circuit. In order to ensure that nothing is learned beyond the output itself, this decryption must be "partial" and must reveal $f(x, y)$ only. Without going into details here, this is accomplished by Party 2 obtaining a series of keys corresponding to its input y such that given these keys and the circuit, the output value $f(x, y)$ (and only this value) may be obtained. Of course, Party 2 must obtain these keys from Party 1 without revealing anything about y and this can be done by running $|y|$ instances of a private 1-out-of-2 Oblivious Transfer protocol. See Appendix B for a more detailed description of Yao's protocol.

The overhead of the protocol involves: (1) Party 1 sending Party 2 tables of size linear in the size of the circuit (each node is assigned a table of keys for the decryption process), (2) Party 1 and Party 2 engaging in an oblivious transfer protocol for every input wire of the circuit, and (3) Party 2 computing a pseudo-random function a constant number of times for every gate (this is the cost incurred in decrypting the circuit). Therefore, the number of rounds of the protocol is constant (namely, two rounds using the oblivious transfer of [6, 9]), and if the circuit is small (e.g., linear in the size of the input) then the main computational overhead is that of running the oblivious transfers.

5 The Protocol

In our protocol, we use the paradigm that all intermediate values of the computation seen by the players are pseudorandom. That is, at each stage, the players obtain random shares v_1 and v_2, such that their sum equals an appropriate intermediate value. Efficiency is achieved by having the parties do most of the computation independently. Recall that there is a known upper bound on the size of the union of the databases, and that the attribute-value names are public.

5.1 A Closer Look at $\mathrm{ID3}_\delta$

Distributed ID3 (the non-private case). First, consider the problem of computing distributed ID3 in a *non-private* setting. In such a scenario, it is always possible for one party to send the other its entire database. However, ID3 yields a solution with far lower communication complexity (with respect to bandwidth). As with the non-distributed version of the algorithm, the parties recursively compute each node of the decision tree based on the remaining transactions. At each node, the parties first compute the value $H_C(T|A)$ for every attribute A. Then, the node is labeled with the attribute A for which $H_C(T|A)$ is *minimum* (as this is the attribute causing the largest information gain).

We now show that it is possible for the parties to determine the attribute with the highest information gain with very little communication. We begin by describing a simple way for the parties to jointly compute $H_C(T|A)$ for a given attribute A. Let A have m possible values $a_1, \ldots, a_m$, and let the class attribute C have ℓ possible values $c_1, \ldots, c_\ell$. Denote by $T(a_j)$ the set of transactions with attribute A set to a_j, and by $T(a_j, c_i)$ the set of transactions with attribute A set to a_j and with class c_i. Then,

$$
\begin{aligned}
H_C(T|A) &= \sum_{j=1}^{m} \frac{|T(a_j)|}{|T|} H_C(T(a_j)) \\
&= \frac{1}{|T|} \sum_{j=1}^{m} |T(a_j)| \sum_{i=1}^{\ell} -\frac{|T(a_j, c_i)|}{|T(a_j)|} \cdot \log(\frac{|T(a_j, c_i)|}{|T(a_j)|}) \\
&= \frac{1}{|T|} \left(-\sum_{j=1}^{m} \sum_{i=1}^{\ell} |T(a_j, c_i)| \log(|T(a_j, c_i)|) + \sum_{j=1}^{m} |T(a_j)| \log(|T(a_j)|) \right) \qquad (3)
\end{aligned}
$$

Therefore, it is enough for the parties to jointly compute all the values $T(a_j)$ and $T(a_j, c_i)$ in order to compute $H_C(T|A)$. Recall that the database for which ID3 is being computed is a union of two databases: party P_1 has database D_1 and party P_2 has database D_2. The number of transactions for which attribute A has value a_j can therefore be written as $|T(a_j)| = |T_1(a_j)| + |T_2(a_j)|$, where $T_b(a_j)$ equals the set of transactions with attribute A set to a_j *in database* D_b. Therefore, Eq. (3) is easily computed by party P_1 sending P_2 all of the values $|T_1(a_j)|$ and $|T_1(a_j, c_i)|$ from its database. Party P_2 then sums these together with the values $|T_2(a_j)|$ and $|T_2(a_j, c_i)|$ from its database and completes the computation. The communication complexity required here is only *logarithmic* in the number of transactions and linear in the number of attributes and attribute-values/class pairs. Specifically, the number of bits sent for each attribute is at most $O(m \cdot \ell \cdot \log |T|)$ (where the $\log |T|$ factor is due to the number of bits required to represent the values $|T(a_j)|$ and $|T(a_j, c_i)|$). This is repeated for each attribute and thus $O(|R| m\ell \log |T|)$ bits are sent overall for each node of the decision tree output by ID3.

Private distributed ID3. Our aim is to privately compute ID3 such that the communication complexity is close to that of the non-private protocol described above. A key observation enabling us to achieve this is that each node of the tree can be computed separately, with the output made public, before continuing to the next node. In general, private protocols have the property that intermediate values remain hidden. However, in this specific case, some of these intermediate values (specifically, the assignments of attributes to nodes) are actually part of the output and may therefore be revealed. We stress that although the name of the attribute with the highest information gain *is* revealed, nothing is learned of the actual $H_C(T|A)$ values themselves. Once the attribute of a given node has been found, both parties can separately partition their remaining transactions accordingly for the coming recursive calls. We therefore conclude that private distributed ID3 can be reduced to privately finding the attribute with the highest information gain. (We note that this is slightly simplified as the other steps of ID3 must also be carefully dealt with. However, the main issues arise within this step.)

As we have mentioned, our aim is to privately find the attribute A for which $H_C(T|A)$ is minimum. We do this by computing random shares of $H_C(T|A)$ for every attribute A. That is, Parties P_1 and P_2 receive random values $S_{A,1}$ and $S_{A,2}$ respectively, such that $S_{A,1} + S_{A,2} = H_C(T|A)$. Thus, neither party learns anything of these intermediate values, yet given shares of all these values, it is easy to privately find the attribute with the smallest $H_C(T|A)$.

Now, notice that the algorithm needs only to find the name of the attribute A which minimizes $H_C(T|A)$; the actual value is irrelevant. Therefore, in Eq. (3), the coefficient $1/|T|$ can be ignored (it is the same for every attribute). Furthermore, natural logarithms can be used instead of logarithms to base 2. As in the non-private case the values $|T_1(a_j)|$ and $|T_1(a_j, c_i)|$ can be computed by party P_1 independently, and the same holds for P_2. Therefore the value $H_C(T|A)$ can be written as a sum of expressions of the form

$$(v_1 + v_2) \cdot \ln(v_1 + v_2)$$

where v_1 is known to P_1 and v_2 is known to P_2 (e.g., $v_1 = |T_1(a_j)|,\ v_2 = |T_2(a_j)|$). The main task is therefore to privately compute $x \ln x$ using a protocol that receives private inputs x_1 and x_2 such that $x_1 + x_2 = x$ and outputs random shares of an approximation of $x \ln x$. In Section 6 a protocol for this task is presented. In the remainder of this section, we show how the private $x \ln x$ protocol can be used in order to privately compute ID3_δ.

5.2 Finding the Attribute with Maximum Gain

Given the above described protocol for privately computing shares of $x \ln x$, the attribute with the maximum information gain can be determined. This is done in two stages: first, the parties obtain shares of $H_C(T|A) \cdot |T| \cdot \ln 2$ for all attributes A and second, the shares are input into a small circuit which outputs the appropriate attribute. In this section we refer to a field $\mathcal{F}$ which is defined so that $|\mathcal{F}| > H_C(T|A) \cdot |T| \cdot \ln 2$.

Stage 1 (computing shares): For every attribute A, every attribute-value $a_j \in A$ and every class $c_i \in C$, parties P_1 and P_2 use the private $x \ln x$ protocol in order to obtain random shares $w_{A,1}(a_j)$, $w_{A,2}(a_j)$, $w_{A,1}(a_j, c_i)$ and $w_{A,2}(a_j, c_i) \in_R \mathcal{F}$ such that

$$w_{A,1}(a_j) + w_{A,2}(a_j) \approx |T(a_j)| \cdot \ln(|T(a_j)|) \mod |\mathcal{F}|$$
$$w_{A,1}(a_j, c_i) + w_{A,2}(a_j, c_i) \approx |T(a_j, c_i)| \cdot \ln(|T(a_j, c_i)|) \mod |\mathcal{F}|$$

where the quality of the approximation can be determined by the parties. Specifically, the approximation factor is set so that the resulting approximation to $H_C(T|A)$ ensures that the output tree is from $\mathcal{ID}3_\delta$. The choice of the approximation level required is discussed in detail in Section 6.4.

Now, define $\hat{H}_C(T|A) \stackrel{\text{def}}{=} H_C(T|A) \cdot |T| \cdot \ln 2$. Then,

$$\hat{H}_C(T|A) = -\sum_{j=1}^{m}\sum_{i=1}^{\ell} |T(a_j, c_i)| \cdot \ln(|T(a_j, c_i)|) + \sum_{j=1}^{m} |T(a_j)| \cdot \ln(|T(a_j)|)$$

Therefore, given the above shares, P_1 (and likewise P_2) can compute its own share in $\hat{H}_C(T|A)$ as follows:

$$S_{A,1} = -\sum_{j=1}^{m}\sum_{i=1}^{\ell} w_{A,1}(a_j, c_i) + \sum_{j=1}^{m} w_{A,1}(a_j) \mod |\mathcal{F}|$$

It follows that $S_{A,1} + S_{A,2} \approx \hat{H}_C(T|A) \mod |\mathcal{F}|$ and we therefore have that for every attribute A, parties P_1 and P_2 obtain (approximate) shares of $\hat{H}_C(T|A)$ (with this last step involving local computation only).

Stage 2 (finding the attribute): It remains to find the attribute minimizing $\hat{H}_C(T|A)$ (and therefore $H_C(T|A)$). This is done using Yao's protocol for two-party computation [17]. The functionality to be computed is defined as follows:

- **Input:** The parties' input consists of their respective shares $S_{A,1}$ and $S_{A,2}$ for every attribute A.
- **Output:** The name of the attribute A for which $S_{A,1} + S_{A,2} \mod |\mathcal{F}|$ is minimum (recall that $S_{A,1} + S_{A,2} \approx \hat{H}_C(T|A) \mod |\mathcal{F}|$).

The above functionality can be computed by a small circuit. First notice that since $\hat{H}_C(T|A) < |\mathcal{F}|$, it holds that either $S_{A,1}+S_{A,2} \approx \hat{H}_C(T|A)$ or $S_{A,1}+S_{A,2} \approx \hat{H}_C(T|A)+|\mathcal{F}|$. Therefore, the modular addition can be computed by first summing $S_{A,1}$ and $S_{A,2}$ and then subtracting $|\mathcal{F}|$ if the sum of the shares is larger than $|\mathcal{F}|-1$, or leaving it otherwise. The circuit computes this value for every attribute and then outputs the attribute name with the minimum value. This circuit has $2|R|$ inputs of size $\log|\mathcal{F}|$ and its size is $O(|R|\log|\mathcal{F}|)$. Note that $|R|\log|\mathcal{F}|$ is a small number and thus this circuit evaluation is efficient.

Privacy: The above protocol for finding the attribute with the smallest $H_C(T|A)$ involves invoking two private subprotocols. The parties' outputs of the first subprotocol are random shares which are then input into the second subprotocol. Therefore, the privacy of the protocol is obtained directly from Corollary 3. (We note that Stage 1 actually contains the parallel composition of many private $x \ln x$ protocols. However, in the semi-honest case, parallel composition holds. Therefore, we can view Stage 1 as a single private protocol for computing many $x \ln x$ values simultaneously.)

Efficiency: Note the efficiency achieved by the above protocol. Each party has to compute the same set of values $|T(a_j, c_i)|$ as it computes in the non-private distributed version of ID3. For each of these values it engages in the $x \ln x$ protocol. (We stress that the number of values here does not depend on the number of transactions, but rather on the number of different possible values for each attribute, which is usually smaller by orders of magnitude.) The party then locally sums the results of these protocols together and runs Yao's protocol on a circuit whose size is only linear in the number of attributes.

5.3 The Private $\mathrm{ID3}_\delta$ Protocol

In the previous subsection we showed how each node can be privately computed. The complete protocol for privately computing $\mathrm{ID3}_\delta$ can be seen in Figure 2. The steps of the protocol correspond to those in the original algorithm (see Figure 1).

Privacy-Preserving Protocol for ID3:

Step 1: *If R is empty, return a leaf-node with the class value assigned to the most transactions in T.*

Since the set of attributes is known to both parties, they both publicly know if R is empty. If yes, the parties run Yao's protocol for the following functionality: Parties P_1 and P_2 input $(|T_1(c_1)|, \ldots, |T_1(c_\ell)|)$ and $(|T_2(c_1)|, \ldots, |T_2(c_\ell)|)$ respectively. The output is the class index i for which $|T_1(c_i)| + |T_2(c_i)|$ is largest. The size of the circuit computing the above functionality is linear in ℓ and $\log |T|$.

Step 2: *If T consists of transactions which all have the same value c for the class attribute, return a leaf-node with the value c.*

In order to compute this step privately, we must determine whether both parties remain with the same single class or not. We define a fixed symbol $-$ symbolizing the fact that a party has more than one remaining class. A party's input to this step is then $-$, or c_i if it is its one remaining class. All that remains to do is check *equality* of the two inputs. The value causing the equality can then be publicly announced as c_i (halting the tree on this path) or $-$ (to continue growing the tree from the current point). For efficient secure protocols for checking equality, see [7, 13] or simply run Yao's protocol with a circuit for testing equality.

Step 3: *(a) Determine the attribute that best classifies the transactions in T, let it be A.*

For every value a_j of every attribute A, and for every value c_i of the class attribute C, the parties run the $x \ln x$ protocol of Section 6 for $|T(a_j)|$ and $|T(a_j, c_i)|$. They then continue as described in Section 5.2 by computing independent additions, and inputting the results into Yao's protocol for a small circuit computing the attribute with the highest information gain. This attribute is public knowledge as it becomes part of the output.

(b,c) Recursively call $\mathrm{ID3}_\delta$ for the remaining attributes on the transaction sets $T(a_1), \ldots, T(a_m)$ (where $a_1, \ldots, a_m$ are the values of attribute A).

The result of 3(a) and the attribute values of A are public and therefore both parties can individually partition the database and prepare their input for the recursive calls.

Figure 2: Protocol for Privately Computing $\mathrm{ID3}_\delta$

Although each individual step of the complete protocol is private, we must show that the composition is also private. Recall that the composition theorem (Theorem 2) only states that if the *oracle-aided* protocol is private, then so too is the protocol for which we use private protocols instead of oracles. Here we prove that the oracle-aided $\mathrm{ID3}_\delta$ protocol is indeed private.

The central issue in the proof involves showing that despite the fact that the control flow *depends* on the input (and is not predetermined), a simulator can exactly predict the control flow of the protocol from the output. This is non-trivial and in fact, as we remark below, were we to switch Steps (1) and (2) in the protocol (as the algorithm is in fact presented in [12]) the protocol would no longer be private. Formally, of course, we show how the simulator generates a party's view based solely on the input and output.

Theorem 4 *The protocol for computing $\mathrm{ID3}_\delta$ is private.*

Proof: In this proof the simulator is described in generic terms as it is identical for P_1 and P_2. Furthermore, we skip details which are straightforward. Recall that the simulator is given the output decision tree.

We need to show that a party's view can be correctly simulated based solely on its input and output. Recall that the computation of the tree is recursive beginning at the root. For each node, a "splitting" class is chosen (due to it having the highest information gain) developing the tree to the next level. Any implementation defines the order of developing the tree and this order is the one followed by the simulator as well. According to this specified order, at any given step the computation is based on finding the highest information gain for a *known* node (for the proof we ignore optimizations which find the gain for more than one node in parallel, although this is easily dealt with). We now describe the simulator for each node.

We differentiate between two cases: (1) a given node is a leaf node and (2) a given node is not a leaf.

1. *The current node in the computation is a leaf-node:* The simulator checks, by looking at the input, if the set of attributes R at this point is empty or not. If it is *not* empty (this can be deduced from the tree and the attribute-list which is public), then the computation proceeds to Step (2). In this case, the simulator writes that the oracle-answer from the equality call in Step (2) is equal (or else it would not be a leaf). On the other hand, if the list of attributes *is* empty, the computation is executed in Step (1) and the simulator writes the output of the majority evaluation to be the class appearing in the leaf.

2. *The current node in the computation is not a leaf-node:* In this case Step (1) is skipped and the oracle-answer of Step (2) must be not-equal; this is therefore what the simulator writes. The computation then proceeds to Step (3) which involves many invocations of the $x \ln x$ protocol, returning values uniformly distributed in $\mathcal{F}$. Therefore, the simulator simply chooses the correct number of random values (based on the public list of attribute names, values and class values) and writes them. The next step of the algorithm is a local computation (not included in the view) and a private protocol for finding the best attribute. The simulator simply looks to see which attribute is written in the tree at this node and writes the attribute name as the oracle-answer for this functionality query.

We have therefore shown that for each party there exists a simulator that given the party's input and the output decision tree, generates a string that is computationally indistinguishable from the party's view in a real execution. (In fact, in the *oracle-aided* protocol, the view generated by the simulator is identically distributed to that of a real execution.) This completes the proof. ■

Remark. It is interesting to note that if Steps (1) and (2) of the protocol are switched (as the algorithm is in fact presented in [12]), then it is no longer private. This is due to the equality evaluation in Step (2), which may leak information about the other party's input. Consider the case of a computation in which at a certain point the list of attributes is *empty* and P_1 has only one class c left in its remaining transactions. The output of the tree at this point is a leaf with a class, assume that the class is c. From the output it is impossible for P_1 to know if P_2's transactions also have only one remaining class or if the result is because the majority of the transactions of both databases together have the class c. The majority circuit of Step (1) covers both cases and therefore does not reveal this information. However, if P_1 and P_2 first execute the equality evaluation, this information is revealed.

Extending the ID3_δ protocol. In footnote 2 we discussed the problem of decision trees which may be very large. As we mentioned, one strategy employed to prevent this problem is to halt in the case that *no* attributes have information gain above some predetermined threshold. Such an extension can be included by modifying Step 2 of the private ID3_δ protocol as follows. In the new Step 2, the parties privately check whether or not there exists an attribute with information gain above the threshold. If there is no such attribute, then the output is defined to be the class assigned to the most transactions in T. (Notice that this replaces Step 2 because in the case that all the transactions have the same class, the information gain for every attribute equals zero.) As in Step 3, most of the work involves computing shares of $H_C(T|A)$. These shares (along with shares of $H_C(T)$) are then input into a circuit that outputs the desired functionality. Of course, in order to improve efficiency, Steps 2 and 3 should then be combined together.

5.4 Complexity

The complexity (measuring both communication and computational complexity) for each node is as follows (recall that R denotes the set of attributes and T the set of transactions):

- The $x \ln x$ protocol is repeated $m(\ell + 1)$ times for each attribute where m and ℓ are the number of attribute and class values respectively (see Eq. (3)). For all $|R|$ attributes we thus have $O(m \cdot \ell \cdot |R|)$ invocations of the $x \ln x$ protocol. The complexity of the $x \ln x$ protocol can be found in Section 6.3. In short, the computational overhead of the $x \ln x$ protocol is dominated by $O(\log |T|)$ oblivious transfers and the bandwidth is $O(k \cdot \log |T| \cdot |S|)$ bits, where k is a parameter depending logarithmically on δ that determines the accuracy of the $x \ln x$ approximation and $|S|$ is the length of the key for a pseudorandom function (say 128 bits).

- As we have mentioned, the size of the circuit computing the attribute with the minimum conditional entropy is $O(|R| \log |\mathcal{F}|)$ where $|\mathcal{F}| = O(|T|)$.[5] The *bandwidth* involved in sending the garbled circuit of Yao's protocol is thus $O(|R| \log |T| \cdot |S|)$ where $|S|$ is the length of the key for a pseudorandom function. (This factor is explained in the paragraph titled "overhead" in Appendix B.)

  The *computational complexity* of the above circuit evaluation involves $|R| \log |\mathcal{F}| = O(|R| \log |T|)$ oblivious transfers (one for each bit of the circuit input) and $O(|R| \log |T|)$ pseudorandom function evaluations.

- The number of *rounds* needed for each node is constant (the $x \ln x$ protocol also requires only a constant number of rounds, see Section 6).

The overhead of the $x \ln x$ invocations far outweighs the circuit evaluation that completes the computation. We thus consider only these invocations in the overall complexity. The analysis is completed by multiplying the above complexity by the number of nodes in the resulting decision tree (expected to be quite small).[6] We note that by computing nodes on the same level of the tree in parallel, the number of rounds of communication can be reduced to the order of the *depth* of the tree (which is bounded by $|R|$ but is expected to be far smaller).

[5] We note that the size of the field $\mathcal{F}$ needed for the $x \ln x$ protocol is actually larger than that required for this part of the protocol. As is described in Section 6, $\log |\mathcal{F}| = O(k \log |T|)$ where k is a parameter depending on δ as described above. However, $k \approx 12$ provides high accuracy and therefore this does not make a significant difference.

[6] Note that the overhead is actually even smaller since the effective number of attributes in a node of depth d' is $|R| - d'$. Since most nodes are at lower levels of the tree and since this is a multiplicative factor in the expression of the overhead, the overhead is decreased substantially. The effective value of $|R|$ for the overall overhead can be reduced to about $|R|$ minus the depth of the resulting decision tree.

Comparison to non-private distributed ID3. We conclude by comparing the communication complexity to that of the non-private distributed ID3 protocol (see Section 5.1). In the non-private case, the bandwidth for each node is exactly $|R|m\ell \log |T|$ bits. On the other hand, in order to achieve a private protocol, an additional multiplicative factor of $k \cdot |S|$ is incurred (plus the constants incurred by the $x \ln x$ and Yao protocols). Thus, the communication complexity of the private protocol is reasonably close to that of its non-private counterpart.

6 A Private Protocol for Approximating $x \ln x$

This section describes an efficient protocol for privately computing an approximation of the $x \ln x$ function, as defined in Figure 3.

- **Input:** P_1's input is a value v_1; P_2's input is v_2.
- **Auxiliary input:** A large enough field $\mathcal{F}$, the size of which will be discussed later.
- **Output:** P_1 obtains $w_1 \in \mathcal{F}$ and P_2 obtains $w_2 \in \mathcal{F}$ such that:
  1. $w_1 + w_2 \approx (v_1 + v_2) \cdot \ln(v_1 + v_2) \bmod |\mathcal{F}|$ (where the quality of the approximation can be determined by the protocol specification),
  2. w_1 and w_2 are uniformly distributed in $\mathcal{F}$ when viewed independently of one another.

Figure 3: Definition of the $x \ln x$ protocol.

The protocol for approximating $x \ln x$ involves two distinct stages. In the first stage, random shares of $\ln x$ are computed. This is the main challenge of this section and conceptually involves the following two steps:

1. Yao's protocol is used to obtain a very rough approximation to $\ln x$. Loosely speaking, the outputs from this step are (random shares) of the values n and ε such that $x = 2^n(1+\varepsilon)$ and $-1/2 \leq \varepsilon \leq 1/2$. Thus, $n \ln 2$ is a rough estimate on $\ln x$ and $\ln(1+\varepsilon)$ is the "remainder". (As we will see, the circuit required for computing such a function is very small.)

2. The value ε output from the previous step is used to privately compute the Taylor series for $\ln(1+\varepsilon)$ in order to refine the approximation. This computation involves a private polynomial evaluation of an integer polynomial.

Next, we provide a simple and efficient protocol for private, distributed multiplication. Thus, given random shares of x and of $\ln x$, we are able to efficiently obtain random shares of $x \ln x$.

6.1 Computing Shares of $\ln x$

We now show how to compute random shares u_1 and u_2 such that $u_1 + u_2 \approx \ln x$ (assume for now that $x \geq 1$). The starting point for the solution is the Taylor series of the natural logarithm, namely:

$$\ln(1+\varepsilon) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}\varepsilon^i}{i} = \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} + \cdots \qquad \text{for} \quad -1 < \varepsilon < 1$$

It is easy to verify that the error for a partial evaluation of the series is as follows:

$$\left| \ln(1+\varepsilon) - \sum_{i=1}^{k} \frac{(-1)^{i-1}\varepsilon^i}{i} \right| < \frac{|\varepsilon|^{k+1}}{k+1} \cdot \frac{1}{1-|\varepsilon|} \tag{4}$$

Thus, the error shrinks exponentially as k grows (see Section 6.4 for an analysis of the cumulative effect of this error in computing ID3_δ).

Given an input x, let 2^n be the power of 2 which is closest to x (in the ID3_δ application, note that $n < \log|T|$). Therefore, $x = 2^n(1+\varepsilon)$ where $-1/2 \le \varepsilon \le 1/2$. Consequently,

$$\ln(x) = \ln(2^n(1+\varepsilon)) = n\ln 2 + \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} + \cdots$$

Our aim is to compute this Taylor series to the k'th place. Let N be a predetermined (public) upper-bound on the value of n ($N > n$ always). In order to do this, we first use Yao's protocol to privately evaluate a small circuit that receives as input v_1 and v_2 such that $v_1 + v_2 = x$ (the value of N is hardwired into the circuit), and outputs random shares of the following values:

- $2^N \cdot n\ln 2$ (for computing the first element in the series of $\ln x$)
- $\varepsilon \cdot 2^N$ (for computing the remainder of the series).

This circuit is easily constructed: notice that $\varepsilon \cdot 2^n = x - 2^n$, where n can be determined by looking at the two most significant bits of x, and $\varepsilon \cdot 2^N$ is obtained simply by shifting the result by $N - n$ bits to the left. The possible values of $2^N n \ln 2$ are hardwired into the circuit. (Actually, the values here are also approximations. However, they may be made arbitrarily close to the true values and we therefore ignore this factor from here on.) Therefore, following this step the parties have shares α_1, β_1 and α_2, β_2 such that,

$$\alpha_1 + \alpha_2 = \varepsilon 2^N \quad \text{and} \quad \beta_1 + \beta_2 = 2^N n \ln 2$$

and the shares α_i and β_i are uniformly distributed in the finite field $\mathcal{F}$ (unless otherwise specified, all arithmetic is in this field). The above is correct for the case of $x \ge 1$. However, if $x = 0$, then x cannot be written as $2^n(1+\varepsilon)$ for $-1/2 \le \varepsilon \le 1/2$. Therefore, the circuit is modified to simply output shares of zero for both values in the case of $x = 0$ (i.e., $\alpha_1 + \alpha_2 = 0$ and $\beta_1 + \beta_2 = 0$).

The second step of the protocol involves computing shares of the Taylor series approximation. In fact, it computes shares of

$$\text{lcm}(2,...k) \cdot 2^N \left(n\ln 2 + \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \cdots \frac{\varepsilon^k}{k} \right) \approx \text{lcm}(2,...k) \cdot 2^N \cdot \ln x \tag{5}$$

(where $\text{lcm}(2,...,k)$ is the lowest common multiple of $\{2,\ldots,k\}$, and we multiply by it to ensure that there are no fractions). In order to do this P_1 defines the following polynomial:

$$Q(z) = \text{lcm}(2,\ldots,k) \cdot \sum_{i=1}^{k} \frac{(-1)^{i-1}}{2^{N(i-1)}} \frac{(\alpha_1 + z)^i}{i} - z_1$$

where $z_1 \in_R \mathcal{F}$ is randomly chosen. It is easy to see that

$$z_2 \stackrel{\text{def}}{=} Q(\alpha_2) = \text{lcm}(2,...,k) \cdot 2^N \cdot \left(\sum_{i=1}^{k} \frac{(-1)^{i-1}\varepsilon^i}{i} \right) - z_1$$

Therefore after a single private polynomial evaluation of the k-degree polynomial $Q(\cdot)$, parties P_1 and P_2 obtain random shares z_1 and z_2 to the approximation in Eq. (5). Namely P_1 defines $u_1 = z_1 + \text{lcm}(2,\ldots,k)\beta_1$ and likewise P_2. We conclude that

$$u_1 + u_2 \approx \text{lcm}(2,\ldots,k) \cdot 2^N \cdot \ln x$$

This equation is accurate up to an approximation error which depends on k, and the shares are random as required. Since N and k are known to both parties, the additional multiplicative factor of $2^N \cdot \mathrm{lcm}(2,\ldots,k)$ is public and can be removed at the end (if desired). Notice that *all* the values in the computation are integers (except for $2^N n \ln 2$ which is given as the closest integer).

The size of the field $\mathcal{F}$. It is necessary that the field be chosen large enough so that the initial inputs in each evaluation and the final output be between 0 and $|\mathcal{F}|-1$. Notice that all computation is based on $\varepsilon 2^N$. This value is raised to powers up to k and multiplied by $\mathrm{lcm}(2,\ldots,k)$. Therefore a field of size 2^{Nk+2k} is large enough, and requires $Nk+2k$ bits for representation. (This calculation is based on bounding $\mathrm{lcm}(2,\ldots,k)$ by $e^k < 2^{2k}$.)

We now summarize the $\ln x$ protocol (recall that N is a public upper bound on $\log |T|$):

Protocol 1 (Protocol $\ln x$)

- **Input:** P_1 and P_2 have respective inputs v_1 and v_2 such that $v_1+v_2 = x$. Denote $x = 2^n(1+\varepsilon)$ for n and ε as described above.
- **The protocol:**
  1. P_1 and P_2, upon input v_1 and v_2 respectively, run Yao's protocol for a circuit that outputs the following: (1) Random shares α_1 and α_2 such that $\alpha_1 + \alpha_2 = \varepsilon 2^N \bmod |\mathcal{F}|$, and (2) Random shares β_1,β_2 such that $\beta_1 + \beta_2 = 2^N \cdot n \ln 2 \bmod |\mathcal{F}|$.
  2. P_1 chooses $z_1 \in_R \mathcal{F}$ and defines the following polynomial
  $$Q(z) = \mathrm{lcm}(2,\ldots,k) \cdot \sum_{i=1}^{k} \frac{(-1)^{i-1}}{2^{N(i-1)}} \frac{(\alpha_1 + z)^i}{i} - z_1$$
  3. P_1 and P_2 then execute a private polynomial evaluation with P_1 inputting $Q(\cdot)$ and P_2 inputting α_2, in which P_2 obtains $z_2 = Q(\alpha_2)$.
  4. P_1 and P_2 define $u_1 = \mathrm{lcm}(2,\ldots,k)\beta_1 + z_1$ and $u_2 = \mathrm{lcm}(2,\ldots,k)\beta_2 + z_2$, respectively. We have that $u_1 + u_2 \approx \mathrm{lcm}(2,\ldots,k) \cdot 2^N \cdot \ln x$

We now prove that the $\ln x$ protocol is correct and secure. We prove correctness by showing that the field and intermediate values are such that the output shares uniquely define the result. On the other hand, privacy is derived directly from Corollary 3.

Before beginning the proof, we introduce notation for measuring the accuracy of the approximation. That is, we say that $\tilde{x}$ is a Δ-approximation of x if $|x - \tilde{x}| \le \Delta$.

Proposition 5 *Protocol* 1 *constitutes a private protocol for computing random shares of a* $\frac{c}{2^k(k+1)}$*-approximation of* $c \cdot \ln x$ *in* $\mathcal{F}$*, where* $c = \mathrm{lcm}(2,\ldots,k) \cdot 2^N$.

Proof: We begin by showing that the protocol correctly computes shares of an approximation of $c \ln x$. In order to do this, we must show that the computation over $\mathcal{F}$ results in a correct result over the reals. We first note that *all* the intermediate values are integers. In particular, $\varepsilon 2^n$ equals $x - 2^n$ and is therefore an integer as is $\varepsilon 2^N$ (since $N > n$). Furthermore, every division by i ($2 \le i \le k$) is counteracted by a multiplication by $\mathrm{lcm}(2,\ldots,k)$. The only exception is $2^N n \ln 2$. However, this is taken care of by having the original circuit output the closest integer to $2^N n \ln 2$.

Secondly, the field $\mathcal{F}$ is defined to be large enough so that all intermediate values (i.e. the sum of shares) and the final output (as a real number times $\text{lcm}(2,\ldots,k)\cdot 2^N$) are between 0 and $|\mathcal{F}|-1$. Therefore the two shares uniquely identify the result, which equals the sum (over the integers) of the two random shares if it is less than $|\mathcal{F}|-1$, or the sum minus $|\mathcal{F}|$ otherwise.

Finally, we show that the accuracy of the approximation is as desired. As we have mentioned in Eq. (4), the $\ln(1+\varepsilon)$ error is bounded by $\frac{|\varepsilon|^{k+1}}{k+1}\frac{1}{1-|\varepsilon|}$. Since $-\frac{1}{2} \leq \varepsilon \leq \frac{1}{2}$, we have that this error rate is maximum at $|\varepsilon| = \frac{1}{2}$. We therefore have that $\left|\widetilde{\ln}(1+\varepsilon) - \ln(1+\varepsilon)\right| \leq \frac{1}{2^k(k+1)}$, where $\widetilde{\ln}(1+\varepsilon)$ denotes the approximation of the $\ln x$ protocol. Now, $c\ln x = cn\ln 2 + c\ln(1+\varepsilon)$ and therefore by adding $cn\ln 2$ to both $c\ln(1+\varepsilon)$ and $c\widetilde{\ln}(1+\varepsilon)$ we have that this has no effect on the error. (We note that we actually add an *approximation* of $cn\ln 2$ to $c\widetilde{\ln}(1+\varepsilon)$ in the protocol. Nevertheless, the error of this approximation can be made to be much smaller than $\frac{c}{2^k(k+1)}$. This is because the approximation of $2^N n\ln 2$ is hardwired into the circuit as the closest integer, and thus by increasing N the error can be made as small as desired.) Therefore, the total error of $c\widetilde{\ln}x$ is $\frac{c}{2^k(k+1)}$, which means that the effective error of the approximation of $\ln x$ is only $\frac{1}{2^k(k+1)}$.

Privacy: The fact that the $\ln x$ protocol is private is derived directly from Corollary 3. Recall that this lemma states that a protocol composed of two private protocols, where the first one outputs random shares only, is private. The $\ln x$ protocol is constructed in exactly this way and thus the privacy follows from the lemma. ■

6.2 Computing Shares of $x\ln x$

We begin by describing a multiplication protocol that on private inputs a_1 and a_2 outputs random shares b_1 and b_2 (in some finite field $\mathcal{F}$) such that $b_1 + b_2 = a_1 \cdot a_2$. The protocol is very simple and is based on a single private evaluation of a linear polynomial.

Protocol 2 (Protocol $Mult(a_1, a_2)$**)**

1. P_1 chooses a random value $b_1 \in_R \mathcal{F}$ and defines the linear polynomial $Q(z) = a_1 z - b_1$.
2. P_1 and P_2 engage in a private evaluation of Q, in which P_2 obtains $b_2 = Q(a_2) = a_1 \cdot a_2 - b_1$.
3. The respective outputs of P_1 and P_2 are defined as b_1 and b_2, giving us that $b_1 + b_2 = a_1 \cdot a_2$.

The correctness of the protocol (i.e., that b_1 and b_2 are uniformly distributed in $\mathcal{F}$ and sum up to $a_1 \cdot a_2$) is immediate from the definition of Q. Furthermore, the privacy follows from the privacy of the polynomial evaluation. We thus have the following proposition:

Proposition 6 *Protocol* 2 *constitutes a private protocol for computing* Mult *as defined above.*

We are now ready to present the complete $x\ln x$ protocol (recall that P_1 and P_2's respective inputs are v_1 and v_2 where $v_1 + v_2 = x$):

Protocol 3 (Protocol $x\ln x$**)**

1. P_1 and P_2 run Protocol 1 for privately computing shares of $\ln x$ and obtain random shares u_1 and u_2 such that $u_1 + u_2 \approx \ln x$.
2. P_1 and P_2 use two invocations of Protocol 2 in order to obtain shares of $u_1 \cdot v_2$ and $u_2 \cdot v_1$.

3. P_1 (resp., P_2) then defines his output w_1 (resp., w_2) to be the sum of the two *Mult* shares and $u_1 \cdot v_1$ (resp., $u_2 \cdot v_2$).

4. We have that $w_1 + w_2 = u_1v_1 + u_1v_2 + u_2v_1 + u_2v_2 = (u_1 + u_2)(v_1 + v_2) \approx x \ln x$ as required.

Applying Corollary 3 we obtain the following theorem:

Theorem 7 *Protocol* 3 *is a private protocol for computing random shares of* $x \ln x$*.*

6.3 Complexity

The $\ln x$ **Protocol (Protocol 1):**

1. Step 1 of the protocol (computing random shares $\alpha_1, \alpha_2, \beta_1$ and β_2) involves running Yao's protocol on a circuit that is linear in the size of v_1 and v_2 (these values are of size at most $\log |T|$). The *bandwidth* involved in sending the garbled circuit in Yao's protocol is $O(\log |\mathcal{F}||S|) = O(k \log |T| \cdot |S|)$ communication bits where $|S|$ is the length of the key for a pseudorandom function. (This is explained in Appendix B.)

   The *computational complexity* is dominated by the oblivious transfers that are required for every bit of the circuit input. Since the size of the circuit input is at most $2 \log |T|$, this upper bounds the number of oblivious transfers required.

2. Steps 2 and 3 of the protocol (computing the Taylor series) involve the private evaluation of a polynomial of degree k over the field $\mathcal{F}$. This private evaluation can be computed using $O(k)$ exponentiations and $O(k)$ messages of total length $O(k|E|)$ where $|E|$ is the length of an element in the group in which the oblivious transfers and exponentiations are implemented.

The overall computation overhead of the protocol is thus $O(\max\{\log |T|, k\})$ exponentiations. Since $|T|$ is usually large (e.g. $\log |T| = 20$), and on the other hand k can be set to small values (e.g. $k = 12$, see below), the computational overhead can be defined as $O(\log |T|)$ oblivious transfers. The main communication overhead is incurred by Step 1, and is $O(k \log |T| \cdot |S|)$ bits.

The $Mult$ **Protocol (Protocol 2):** This protocol involves a single oblivious evaluation of a linear polynomial by the players.

The $x \ln x$ **Protocol (Protocol 3):** This step involves one invocation of Protocol 1, and two invocations of Protocol 2. Its overhead is therefore dominated by Protocol 1. We conclude that the overall computational complexity is $O(\log |T|)$ oblivious transfers and that the bandwidth is $O(k \log |T| \cdot |S|)$ bits.

6.4 Choosing the Parameter k for the Approximation

Recall that the parameter k defines the accuracy of the Taylor approximation of the "ln" function. Given δ, we analyze the value of k needed in order to ensure that the defined δ-approximation is correctly estimated. From here on we denote an approximation of a value z by $\widetilde{z}$. The approximation definition of ID3$_\delta$ requires that if an attribute A' is chosen for any given node, then $|H_C(T|A') - H_C(T|A)| \leq \delta$, where A denotes the attribute with the maximum information gain. In order to ensure that only attributes that are δ-close to A are chosen, it is sufficient to have that for all pairs of attributes A and A'

$$H_C(T|A') > H_C(T|A) + \delta \Rightarrow \widetilde{H_C}(T|A') > \widetilde{H_C}(T|A) \tag{6}$$

This is enough because the attribute A' chosen by our specific protocol is that which has the smallest $\widetilde{H_C}(T|A')$. If Eq. (6) holds, then we are guaranteed that $H_C(T|A') - H_C(T|A) \leq \delta$ as required (because otherwise we would have that $\widetilde{H_C}(T|A') > \widetilde{H_C}(T|A)$ and then A' would not have been chosen). Eq. (6) is fulfilled if the approximation is such that for every attribute A,

$$\left|H_C(T|A) - \widetilde{H_C}(T|A)\right| < \frac{\delta}{2}$$

We now bound the difference on each $|\ln x - \widetilde{\ln x}|$ that suffices for the above condition to be fulfilled. By replacing $\log x$ by $\frac{1}{\ln 2}|\ln x - \widetilde{\ln x}|$ in Eq. (3) computing $H_C(T|A)$ (see Section 5.1), we obtain a bound on the error of $\left|H_C(T|A) - \widetilde{H_C}(T|A)\right|$. A straightforward algebraic manipulation gives us that if $\frac{1}{\ln 2}|\ln x - \widetilde{\ln x}| < \frac{\delta}{4}$, then the error is less than $\frac{\delta}{2}$ as required.[7] By Proposition 5, we have that the $\ln x$ error is bounded by $\frac{1}{2^k(k+1)}$ (the multiplicative factor of c is common to all attributes and can therefore be ignored). Therefore, given δ, we set $\frac{1}{2^k k+1} < \frac{\delta}{4} \cdot \ln 2$ or $k + \log(k+1) > \log\left[\frac{4}{\delta \ln 2}\right]$ (for $\delta = 0.0001$, it is enough to take $k > 12$). Notice that the value of k is *not* dependent on the input database.

7 Practical Considerations and Protocol Efficiency

A detailed analysis of the complexity of the $x \ln x$ protocol can be found in Section 6.3 and the overall $\mathrm{ID3}_\delta$ complexity is analyzed in Section 5.4. In this section we demonstrate the efficiency of our protocol with a concrete analysis based on example parameters for an input database. Furthermore, a comparison of the efficiency of our protocol to that of generic solutions is presented.

A Concrete Example. Assume that there are a million transactions (namely $|T| = 2^{20}$), $|R| = 15$ attributes, each attribute has $m = 10$ possible values, the class attribute has $\ell = 4$ values, and $k = 10$ suffices to have the desired accuracy. Say that the depth of the tree is $d = 7$, and that the length of a key for the pseudorandom function is $|S| = 80$ bits.

As is described in Section 5.4 there are at most $m \cdot \ell \cdot |R| = 600$ invocations of the $x \ln x$ protocol for each node and these dominate the overall complexity. (In fact, a node of depth d' in the tree requires only $m \cdot \ell \cdot (|R| - d')$ invocations.)

- **Bandwidth:** Each invocation has a communication overhead of $O(k \cdot \log|T| \cdot |S|)$ bits, where the constant in the "O" notation is fairly small. We conclude that the communication overhead of evaluating for each node can be transmitted in a matter of seconds using a fast communication network (e.g. a T1 line with 1.5Mbps bandwidth, or a T3 line with 35Mbps).

[7] The full calculation is as follows:

$$\begin{aligned}
&\left|H_C(T|A) - \widetilde{H_C}(T|A)\right| \\
&\leq \frac{1}{|T|}\left(\sum_{j=1}^{m}\sum_{i=1}^{\ell}|T(a_j,c_i)| \cdot \left|\log|T(a_j,c_i)| - \log|\widetilde{T(a_j,c_i)}|\right| + \sum_{j=1}^{m}|T(a_j)| \cdot \left|\log|T(a_j)| - \log|\widetilde{T(a_j)}|\right|\right) \\
&< \frac{1}{|T|}\left(\sum_{j=1}^{m}\sum_{i=1}^{\ell}|T(a_j,c_i)| \cdot \frac{\delta}{4} + \sum_{j=1}^{m}|T(a_j)| \cdot \frac{\delta}{4}\right) = \frac{1}{|T|}\left(\frac{\delta}{4} \cdot |T| + \frac{\delta}{4} \cdot |T|\right) = \frac{\delta}{2}
\end{aligned}$$

- **Computation:** The computation overhead for each $x \ln x$ protocol is $O(\log |T|)$ oblivious transfers (and thus exponentiations). In our example $\log |T| = 20$, and each node requires several hundred evaluations of the $x \ln x$ protocol. We can therefore assume that each node requires several tens of thousands of oblivious transfers (and therefore exponentiations). Assuming that a modern PC can compute 50 exponentiations per second, we conclude that the computation per node can be completed in a matter of minutes. The protocol can further benefit from the computation/communication tradeoff for oblivious transfer suggested in [14], which can reduce the number of exponentiations by a factor of c at the price of increasing the communication by a factor of 2^c. Since the latency incurred by the computation overhead is much greater than that incurred by the communication overhead it may make sense to use this tradeoff to balance the two.

A Comparison to Generic Solutions. Consider a generic solution for the entire ID3$_\delta$ task using Yao's two party protocol. Such a solution would require a total of $|R| \cdot |T| \cdot \lceil \log m \rceil$ oblivious transfers (one for every input bit). For the above example parameters, we have a total $60,000,000$ overall oblivious transfers. Furthermore, as the number of transactions $|T|$ grows, the gap between the complexity of the generic protocol and the complexity of our protocol grows rapidly, since the overhead of our protocol is only logarithmic in $|T|$. The size of the circuit sent in the generic protocol is also at least $O(|R| \cdot |T| \cdot |S|)$ (a very optimistic estimate) which is once again much larger than in our protocol.

Consider now a *semi-generic* solution for which the protocol is exactly as described in Figure 2. However, a generic (circuit-based) solution is used for the $x \ln x$ protocol instead of the protocol of Section 6. This generic protocol should compute the Taylor series, namely k multiplications in $\mathcal{F}$, with a communication overhead of $O(k \log^2 |\mathcal{F}||S|) = O(k^3 \log^2 |T||S|)$ (circuit multiplication is *quadratic* in the length of the inputs). This is larger by a factor of $k^2 \log |T|$ than the communication overhead of our protocol. On the other hand, the number of oblivious transfers would remain much the same in both cases, and this overhead dominates the computation overhead of both protocols.

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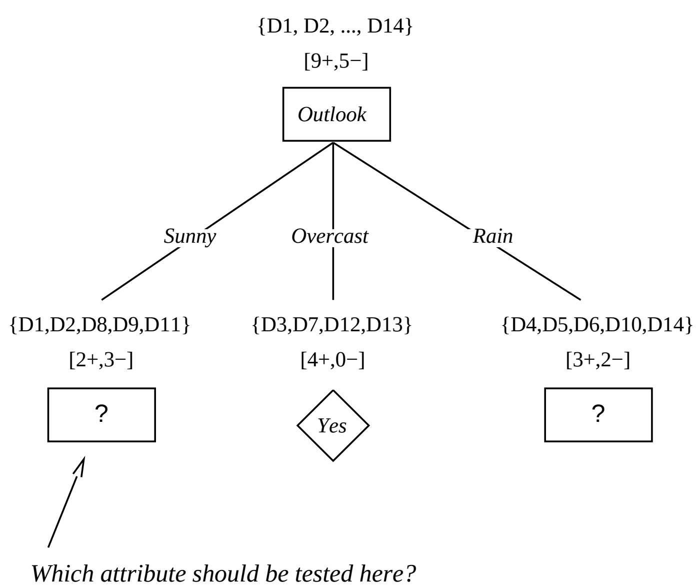
A A Decision Tree Example

In this appendix we give an example of a data set and the resulting decision tree. (The examples are taken from Chapter 3 of Tom Mitchell's book *Machine Learning*, see [12].) The aim of the task here is to learn the weather conditions suitable for playing tennis.

Day	Outlook	Temperature	Humidity	Wind	Play Tennis
D1	Sunny	Hot	High	Weak	No
D2	Sunny	Hot	High	Strong	No
D3	Overcast	Hot	High	Weak	Yes
D4	Rain	Mild	High	Weak	Yes
D5	Rain	Cool	Normal	Weak	Yes
D6	Rain	Cool	Normal	Strong	No
D7	Overcast	Cool	Normal	Strong	Yes
D8	Sunny	Mild	High	Weak	No
D9	Sunny	Cool	Normal	Weak	Yes
D10	Rain	Mild	Normal	Weak	Yes
D11	Sunny	Mild	Normal	Strong	Yes
D12	Overcast	Mild	High	Strong	Yes
D13	Overcast	Hot	Normal	Weak	Yes
D14	Rain	Mild	High	Strong	No

The first attribute chosen in the tree for the above database is *Outlook*. This is seen by a quick computation of the *Gain*. By a quick calculation one can confirm that $Gain(T, Outlook) = 0.246$ which is maximum over the gain of all other attributes. We can see the entropy *gain* calculation for one of the lower nodes in the tree below.



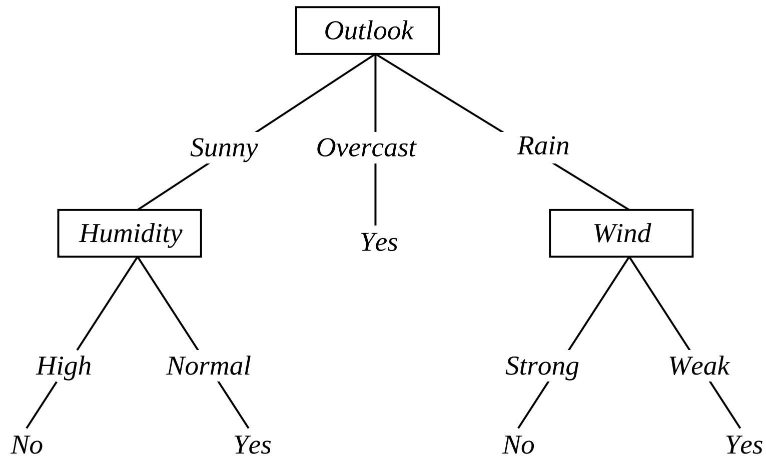
S_{sunny} = {D1,D2,D8,D9,D11}

$Gain(S_{sunny}, Humidity) = .970 - (3/5)\,0.0 - (2/5)\,0.0 = .970$

$Gain(S_{sunny}, Temperature) = .970 - (2/5)\,0.0 - (2/5)\,1.0 - (1/5)\,0.0 = .570$

$Gain(S_{sunny}, Wind) = .970 - (2/5)\,1.0 - (3/5)\,.918 = .019$

It is clear that "Humidity" is the best choice by its *Gain* value. Intuitively, this is logical as it completes the classification in the most concise manner (there is no need for any further tests). Below, we can see the final decision tree.



A new transaction is then classified by traversing the tree according to the attribute/values. For example, the transaction:

$$(\mathsf{Outlook} = \mathsf{Sunny}, \mathsf{Temperature} = \mathsf{Hot}, \mathsf{Humidity} = \mathsf{High}, \mathsf{Wind} = \mathsf{Strong})$$

is classified as *No* by seeing that it is a sunny and humid day. This example demonstrates ID3's bias in favor of short hypotheses (at most two out of the four attributes are tested).

Our aim is to output a tree such as this one, such that *nothing* can be learned about the other party's database that cannot be learned from the output itself.

B Yao's Protocol for Private Two-Party Computation

We now describe Yao's protocol for private two-party computation in the case that the adversary is semi-honest. Let A and B be parties with respective inputs x and y and let f be the functionality to be privately computed by the two parties. We note that f may be a probabilistic functionality. In this brief description, we assume for simplicity that both A and B are to obtain the same value $f(x,y)$.

The protocol is based on expressing f as a combinatorial circuit with gates defined over some fixed base $\mathcal{B}$ (e.g., $\mathcal{B}$ can include all the functions $g : \{0,1\} \times \{0,1\} \mapsto \{0,1\}$). The bits of the input are entered into input wires and are propagated through the gates.

Protocol for two-party secure function evaluation

Input: A and B have respective input values x and y.

Output: A and B both receive the value $f(x,y)$.

The Protocol:

- **Encrypting the circuit:** A constructs an "encrypted" version of the circuit computing f as follows. First, A hardwires its input into the circuit, thus obtaining a circuit computing $f(x,\cdot)$. Then, A assigns to each wire i of the circuit two random values (W_i^0, W_i^1) corresponding to values 0 and 1 of the wire (the random values should be long enough to be used as keys to a pseudo-random function). Denote the value of the wire by $b_i \in \{0,1\}$. Party A also assigns to the wire a random permutation over $\{0,1\}$, $\pi_i : b_i \mapsto c_i$. Denote $\langle W_i^{b_i}, c_i \rangle$ as the 'garbled value' of wire i.

Consider a gate g which computes the value of the wire k as a function of wires i and j, $b_k = g(b_i, b_j)$. A prepares a table T_g which enables computation of the *garbled* output of g, $\langle W_k^{b_k}, c_k \rangle$, from the garbled inputs to g, namely the values $\langle W_i^{b_i}, c_i \rangle, \langle W_j^{b_j}, c_j \rangle$. Given the two garbled inputs to g, the table does not disclose information about the output of g for any other inputs, nor does it reveal the values of the bits b_i, b_j, b_k. The table essentially encrypts the garbled value of the output wire using the output of a pseudo-random function F keyed by the garbled values of the input wires.

The construction of T_g uses a pseudo-random function F whose output length is $|W_k^{b_k}| + 1$ bits. Assume that the fan out of each gate is one. The table contains four entries of the form

$$c_i, c_j : \langle (W_k^{g(b_i,b_j)}, c_k) \oplus F_{W_i^{b_i}}(c_j) \oplus F_{W_j^{b_j}}(c_i) \rangle$$

for $0 \le i, j \le 1$, where $c_i = \pi(b_i), c_j = \pi(b_j)$, and $c_k = \pi_k(b_k) = \pi_k(g(b_i, b_j))$. (The entry does not include its index c_i, c_j explicitly, as it can be deduced from the location.)

To verify that the table enables B to compute the garbled output value given the garbled input values, assume that B knows $\langle W_i^{b_i}, c_i \rangle, \langle W_j^{b_j}, c_j \rangle$. B should find the entry (c_i, c_j) in the table T_k, and compute its exclusive-or with $(F_{W_i^{b_i}}(c_j) \oplus F_{W_j^{b_j}}(c_i))$. The result is $\langle W_k^{b_k} = W_k^{g(b_i,b_j)}, c_k \rangle$.

If the fan out of a a gate is greater than 1, care should be taken to ensure that same value is not used to mask values in different gates that have the same input. This can be done by assigning a different id for every gate and using this id as an input to the pseudo-random function F. Namely, a gate with id g has table entries of the form $c_i, c_j : \langle (W_k^{g(b_i,b_j)}, c_k) \oplus F_{W_i^{b_i}}(g, c_j) \oplus F_{W_j^{b_j}}(g, c_i) \rangle$.

- **Coding the input:** The tables described above enables the computation of the garbled output of every gate from its garbled inputs. Therefore given these tables and the garbled values $\langle W_i^{b_i}, c_i \rangle$ of the input wires of the circuit, it is possible to compute the garbled values of its output wires. Party B should therefore obtain the garbled values of the input wires.

  For each input wire, A and B engage in a 1-out-of-2 oblivious transfer protocol in which A is the sender, whose inputs are the two garbled values of this wire, and B is the receiver, whose input is an input bit. As a result of the oblivious transfer protocol B learns the garbled value of its input bit (and nothing about the garbled value of the other bit), and A learns nothing.

  A sends to B the tables that encode the circuit gates and a translation table from the garbled values of the output wires to output bits.

- **Computing the circuit:** At the end of the oblivious transfer protocols, party B has sufficient information to compute the output of the circuit $f(x, y)$ by its own. After receiving $f(x, y)$, party B sends A this value so that both parties obtain the output.

To show that the protocol is secure it should be proved that the view of the parties can be simulated based on the input and output only. The main observation regarding the security of each gate is that every masking value (e.g. $F_{W_i^{b_i}}(c_j)$) is used only once, and that the pseudo-randomness of F ensures that without knowledge of the correct key these values look random.

Therefore knowledge of one garbled value of each of the input wires discloses only a single garbled output value of the gate; the other output values are indistinguishable from random to B.

As for the security of the complete circuit, the oblivious transfer protocol ensures that B learns only a single garbled value for each input wire, and A does not learn which value it was. Inductively, B can compute only a single garbled output value of each gate, and in particular of the circuit. The use of permuted bit values c_k hides the values of intermediate results (i.e. of gates inside the circuit).

It is possible to adapt the protocol for circuits in which gates have more than two inputs, and even for wires with more than two possible values. The size of the table for a gate with ℓ inputs which each can have d values is d^ℓ.

Overhead

Note that the communication between the two parties can be done in two rounds (assuming the use of a two-round oblivious transfer protocol [6, 9]).

Consider a circuit with n inputs and m gates. The protocol requires A to prepare m tables and send them to B. This is the major communication overhead of the protocol. In the case of binary gates, the communication overhead is $4m$ times the length of the output of the pseudo-random function (typically 64 to 128 bits long).

The main computational overhead of the protocol is the computation of the n oblivious transfers. Afterwards party B computes the output of the circuit, and this stage involves m applications of a pseudo-random function. For small circuits, the overhead of this stage is typically negligible compared to the oblivious transfer stage.