CONTINUOUS-TIME SIGNAL PROCESSING BASED ON POLYNOMIAL APPROXIMATION

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ABSTRACT

A new approach for continuous-time processing of a discrete-time signal is proposed. This approach is based on simple modifications of the Farrow structure, which is a polynomial-based interpolation technique employing discrete-time filters for enabling arbitrary resampling of a time series. In this paper, an extension of the Farrow structure for other time-domain operations used for signal analysis than just interpolation is presented. Filter optimization and design examples corresponding to each application using the minimax optimization criterion are provided. Also, an illustrative example of the time-domain performance of this approach is given.

1. INTRODUCTION

This paper introduces an efficient approach for processing sampled signals in the time domain using polynomial interpolation. The main idea is to form a polynomial approximation (or estimation) to a continuous time signal using its discrete-time samples. The approximation is a piecewise polynomial, i.e., there is a different polynomial approximation for every sample interval. Different operations can then be applied to this approximation to analyze or process the signal.

The advantage of this approach is that we can process a continuous time representation of a sampled signal using discrete-time filters and operations. For example, the derivative of the signal can be calculated at any point between the samples, not just at the discrete points [1]. This makes it possible to find the local maximum or minimum of the signal even if it is lying between its discrete-time samples.

Examples of operations which can be used to analyze signals are given. These include interpolation, as well as the derivative and integral of the signal. Here the interpolation means that the polynomial approximation is just evaluated at the desired point. Other operations can also be used.

The polynomial approximation can be calculated using $L + 1$ parallel FIR branch filters, where L is the degree of the piecewise polynomials. This idea is based on the use of the Farrow structure [2], which can be exploited to interpolate new sample values between the existing discrete-time samples. It is also discussed how the FIR filters can be optimized for interpolation, derivative, and integral operations.

The paper is organized as follows. In Section 2, a generalization of the Farrow structure for other time-domain operations than just interpolation is established. Section 3 introduces some signal processing applications based on the use of the general Farrow structure. Filter optimization and design examples using the minimax method with an illustrative example performed on an ECG signal are provided in Section 4. Conclusions are drawn in Section 5.

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2. POLYNOMIAL APPROXIMATION FOR CONTINUOUS-TIME SIGNALS

The Farrow structure has been used to form a piecewise polynomial approximation for the sampled signals and to evaluate this polynomial at the desired point. In other words, the Farrow structure can be used to interpolate new sample values between the existing discrete-time samples.

The Farrow structure consists of $L + 1$ parallel FIR branch filters, where L is the degree of the piecewise polynomials. The output samples of these branch filters are directly coefficients for the approximating polynomial.

In this paper, we will use the branch filters of the Farrow structure to form a piecewise polynomial approximation for the signal. Then we can apply different time-domain operations to this approximation. Therefore, this idea is a generalization of the idea in the original paper [2] to cover also other operations than just interpolation.

The block diagram of the proposed general Farrow structure is shown in Fig. 1. The input sample sequence $x(n)$ is filtered using FIR filters $G_l(z)$ for $l = 0, 1, \cdots, L$ to obtain the coefficients $v_l(n)$ for the approximating polynomial $p_n(t)$.

Because there is a different approximating polynomial for every sample interval, $p_n(t)$ is only used for $nT \leq t < (n+1)T$, where T is the sampling interval. Therefore, we can replace variable t by a new variable μ. This new variable, referred as the fractional interval, can be given by

$$\mu = \frac{t}{T} - n. \tag{1}$$

Therefore, the approximating polynomial for the continuous-time signal $x(t)$ in the interval $nT \leq t < (n+1)T$ is defined by

$$x(t) \approx p_n(t)_{|t=(n+\mu)T} \equiv p(n, \mu) = \sum_{l=0}^{L} v_l(n)\mu^l, \tag{2}$$

where $\mu \in [0, 1)$ is the fractional interval, n is the index of the interval, and L is the degree of the polynomial. The polynomial

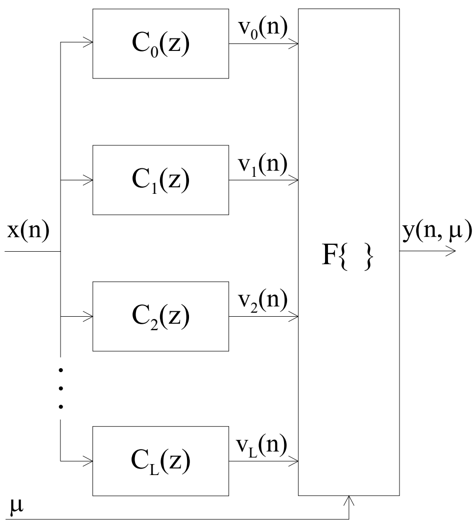

Figure 1: The block diagram of the general Farrow structure

coefficients can be given by

$$v_l(n) = \sum_{k=n-\frac{N}{2}+1}^{n+\frac{N}{2}} x(k)c_l(n-k), \tag{3}$$

where N is the length of the FIR filters. The transfer functions of these FIR filters are given by

$$C_l(z) = \sum_{k=0}^{N-1} c_l(k - \frac{N}{2})z^{-k}, \; for \; l = 0, \cdots, L. \tag{4}$$

When the approximating polynomial $p(n,\mu)$ is calculated, we can apply a time-domain operation, denoted by $F\{\}$ for this polynomial. As a result, we get a new polynomial which is denoted by

$$p^*(n,\mu) = F\{p(n,\mu)\}. \tag{5}$$

$F\{\}$ can be any operation which is valid for L^{th} order polynomial $p(n,\mu)$ for $\mu \in [0,1)$ and which gives K^{th} order polynomial $p^*(n,\mu)$ as a result. It should be noted that different operations $F\{\}$ can be applied to the same polynomial coefficients $v_l(n)$ with small additional computational cost.

After applying the operation $F\{\}$, the output sample $y(n,\mu)$ is calculated by evaluating the value of the polynomial $p^*(n,\mu)$ at any given time instant. In Fig. 1, the fractional interval μ is used to determine this time instant $t = (n+\mu)T$.

3. SIGNAL ANALYSIS

In this section, we will consider three time domain operations which are widely used in signal analysis and which can be calculated using the general Farrow structure of Fig. 1. These operations are, the interpolation, derivative, and integral.

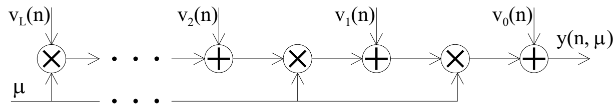

Figure 2: Block diagram of $F\{\}$ for interpolation.

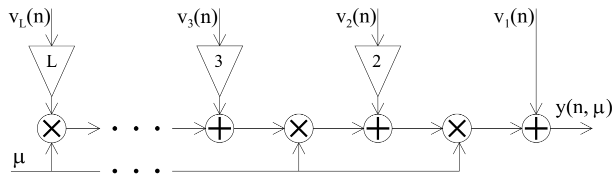

Figure 3: Block diagram of $F\{\}$ for derivative.

3.1. Interpolation

The polynomial interpolation is utilized in numerous applications. These include, e.g., timing adjustment in digital receivers [3]-[5], arbitrary sampling rate conversion [6], simulation of continuous-time systems [7], and echo cancellation [2].

The idea of the interpolation is to form some approximating function by using discrete-time samples $x(n)$. In the polynomial interpolation, the approximating function is a polynomial of degree L. This polynomial can be used to approximate the value of the original continuous-time signal $x(t)$ at any point between the discrete-time samples.

The general Farrow structure of Fig. 1 can be used to perform interpolation. This can be done by evaluating the value of the approximating polynomial $p(n,\mu)$ at the given point $t = (n+\mu)T$. The corresponding block diagram of the function $F\{\}$ is shown in Fig. 2. Actually, this is the original form of the Farrow structure introduced in [2].

3.2. Derivative

The need to obtain the time derivative of a measured or observed signal appears in many fields. For example, it can be used to locate the maximum or minimum of the signal. In biomechanical investigations, estimation of the second order derivatives from position data is required [8].

The derivative of the continuous-time signal $x(t)$ can be approximated at any point $t = (n+\mu)T$ by taking the derivative of the approximating polynomial $p(n,\mu)$. Therefore, differentiating Eq. (2) with respect to t, we obtain

$$\frac{dx(t)}{dt}\Big|_{t=(n+\mu)T} \approx \frac{dp(n,\mu)}{d\mu} = \sum_{l=1}^{L} l v_l(n)\mu^{l-1} \tag{6}$$

for $\mu \in [0,1)$. The block diagram of the derivative function is shown in Fig. 3. It is also possible to calculate higher order derivatives in the similar way.

3.3. Integral

Integration via interpolation is an old concept used to approximate a numerical integral of a continuous-time function [9]. Recent system functions of digital integrators are designed using simple linear

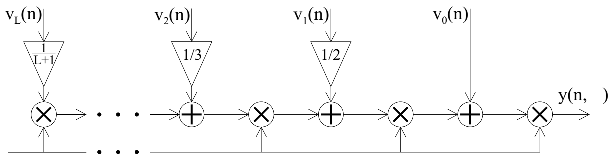

Figure 4: Block diagram of $F\{\}$ for integral.

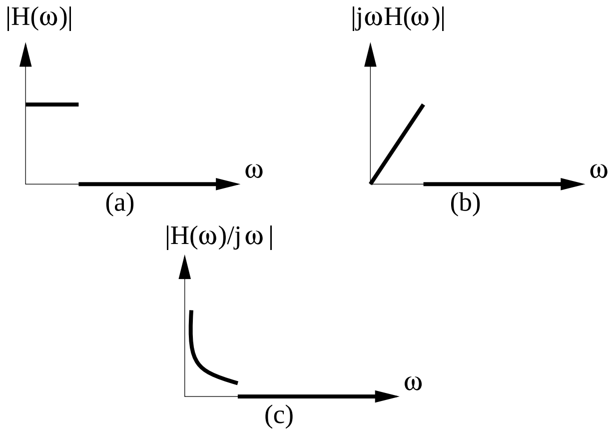

Figure 5: Ideal frequency responses for (a) interpolator, (b) differentiator, and (c) integrator.

interpolation between the rectangular, trapezoidal and Simpson integrators [9]-[10]. Our proposed method estimates the continuous-time integral of any discrete-time signal via polynomial interpolation.

The integral function of the continuous-time signal $x(t)$ is approximated by taking the integral of the approximating polynomial $p(n, \mu)$ with respect to μ. Integrating Eq. (2), we obtain

$$\int x(t)dt \Big|_{t=(n+\mu)T} \approx \int p(n, \mu)d\mu = \sum_{l=0}^{L} \frac{v_l(n)}{l+1}\mu^{l+1} \qquad (7)$$

for $\mu \in [0, 1)$. The integral of the signal $x(t)$ approximated in some interval, let us say in $(n + \mu_1)T \le t \le (n + \mu_2)T$, can be given by

$$\int_{(n+\mu_1)T}^{(n+\mu_2)T} x(t)dt \approx p^*(n, \mu_2) - p^*(n, \mu_1), \qquad (8)$$

where $p^*(n, \mu)$ is the integral function in Eq. (7).

It is also possible to calculate the integral in longer interval that includes more than one sample segment, i.e., for $(n_1+\mu_1)T \le t \le (n_2 + \mu_2)T$. However, this has to be done in such a way that the integral is first calculated in each sample segment, and then, these subintegrals are added together to obtain the overall integral. The block diagram of the integral function is shown in Fig. 4.

4. FILTER OPTIMIZATION

Interpolation filters (also the term "interpolator" is used) can be designed and analyzed by using the underlying continuous-time impulse response $h(t)$ [3]. If this impulse response is piecewise polynomial, then the interpolator can be implemented using the Farrow structure. In the Farrow structure, the coefficients $c_l(k)$ of the FIR filters are the l^{th} order polynomial coefficients of the impulse response $h(t)$.

The ideal frequency response of the interpolation filter is an ideal lowpass filter with passband covering the frequency band of the useful signal, as shown in Fig. 5(a). The calculation of the coefficients for the Lagrange interpolators can be found in [4]. However, the Lagrange interpolators do not offer a good stopband attenuation and they are not optimal in any sense. A better and more general optimization method has been introduced in [11].

The ideal frequency response is different when the general Farrow structure of Fig. 1 is used as a differentiator or integrator. Consequently, the existing design methods for interpolators can not be directly used.

The ideal frequency response of the differentiator is $j\omega H(\omega)$ (Fig. 5(b)), where $H(\omega)$ is the ideal response of the interpolator. Therefore, the weight of ω should be used in the interpolator design in order to obtain the desired passband and stopband shapes for the differentiator. In other words, highest frequencies in the passband and stopband are quite critical for the differentiator performance.

The ideal frequency response of the integrator is $H(\omega)/j\omega$ (Fig. 5(c)). Now the lowest frequencies in the passband and stopband are quite critical, and therefore, $1/\omega$ should be used as a passband and stopband weight.

The optimization of the integrators is more difficult at low frequencies, because the value of the frequency response increases without limit towards the zero-frequency, and at $\omega = 0$, the frequency response is not defined.

When different functions are applied to the same polynomial coefficients, then the optimization has to be done using some kind of compromise between each function. For example, if it is desired to calculate both interpolation and derivative approximations, then the frequency response specifications could be some combination of constant and ω-type of weights.

4.1. Design examples

Next we will design interpolator, differentiator, and integrator by using the minimax optimization method described in [11]. The design parameters are:

- the length of the filter is $N = 8$.
- the degree of the piecewise polynomials is $L = 5$.
- passband edge is $\omega_p = 0.35F_s$.
- stopband edge is $\omega_s = 0.65F_s$.

The passband and stopband weights are constant for the interpolator. For the differentiator and integrator the passband and stopband weights are ω and $1/\omega$, respectively.

The frequency responses for the optimized differentiator and integrator in the linear scale are shown in Figures 6 and 7. The ideal responses in the passband are also shown (dashed line).

The frequency responses for the interpolator, differentiator, and integrator in the dB-scale are shown in Figures 8, 9 and 10. Note that after using the weight of ω and $1/\omega$ for differentiator and integrator, respectively, the corresponding stopbands have equiripple behaviors as in the case of the interpolator with a constant weight.

Figure 10 illustrates an exmample of the time domain performance of the optimized interpolator and differentiator when applied to a discrete-time ECG signal.

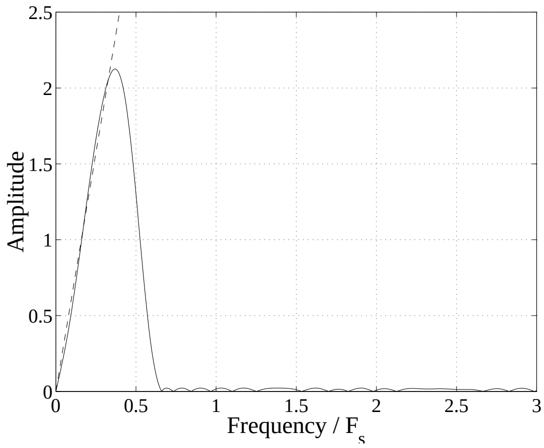

Figure 6: The frequency response for the optimized differentiator. The ideal response is given by a dashed line.

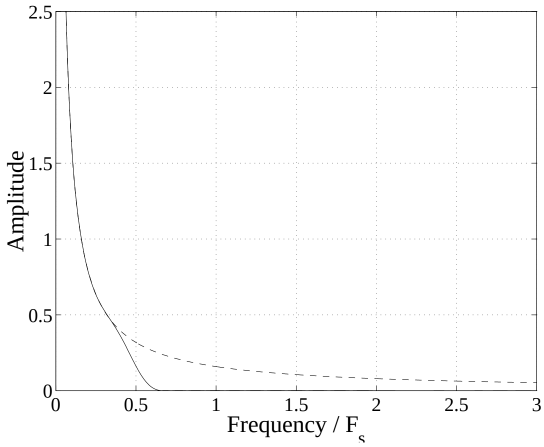

Figure 7: The frequency response for the optimized integrator. The ideal response in the passband is given by a dashed line.

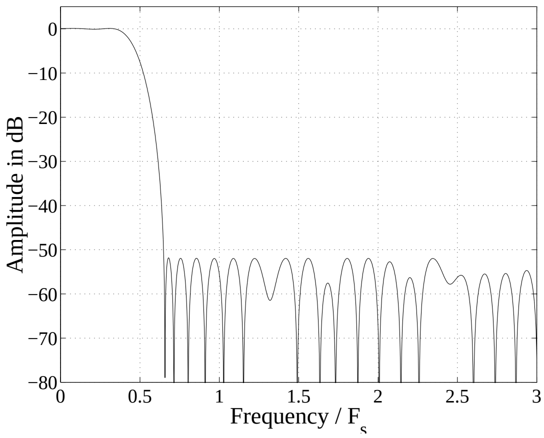

Figure 8: The frequency response for the optimized interpolator.

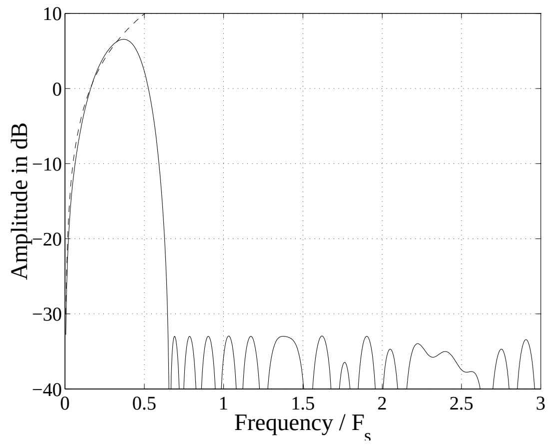

Figure 9: The frequency response for the optimized differentiator. Dashed line is the ideal response in the passband.

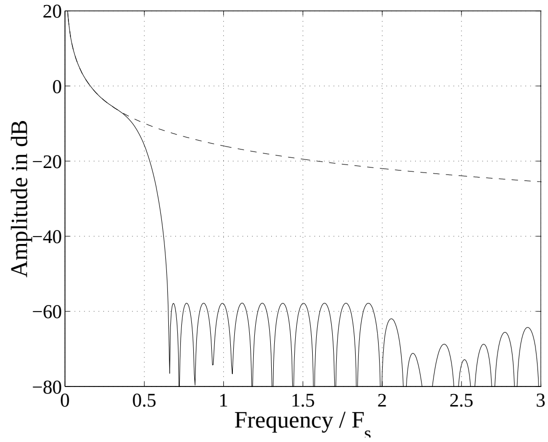

Figure 10: The frequency response for the optimized integrator. Dashed line is the ideal response in the passband.

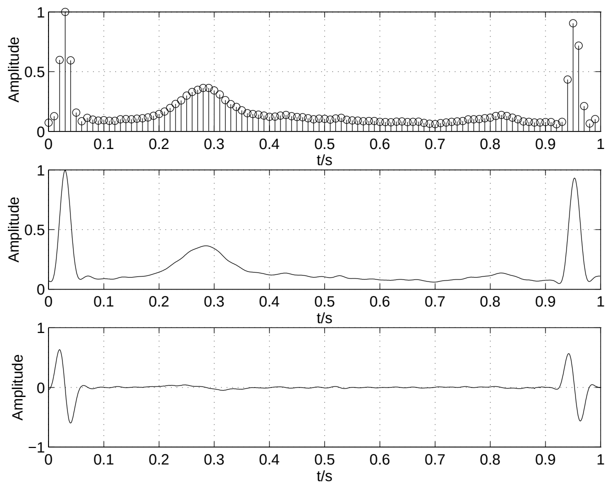

Figure 11: Discrete-time ECG signal, its approximated continuous-time waveform, and continuous-time derivative.

5. CONCLUSIONS

A new approach for continuous-time processing of discrete-time signals based on an extension of the Farrow structure has been proposed. Using this general Farrow structure, different time-domain operations for signal analysis have been presented. Filter optimization and design examples corresponding to each application have also been provided. The good performance of this time-domain approach is illustrated via an example. It remains a topic for future work to consider other applications, like for example correlation using this new general structure.

6. REFERENCES

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