PROBABILITY DISTRIBUTIONS FOR THE NUMBER OF RADIO TRANSCEIVERS IN A HOT SPOT WITH AN APPLICATION TO THE EVALUATION OF BLOCKING PROBABILITIES

J. Orriss*, A. Zanella^o, R. Verdone^o, S. K. Barton*
^{o} CSITE-CNR, University of Bologna, I
* Department of Computer Science, University of Manchester, UK

ABSTRACT

This paper considers two sets of points which can communicate with one another and which are uniformly and randomly distributed, one set within a confined area, a hot spot, on the plane, and the other with a different density over the rest of the plane: an inverse power law with lognormal shadowing is assumed for attenuation. Communication is assumed to be possible if the attenuation of a signal transmitted from one point, say a mobile phone, does not exceed some specified value by the time it reaches the other, say a base station. The probability distribution of the number of base stations which can hear such a signal from a given mobile is found, taking account of the effect of the hot spot. The conditional distribution of the number of such base stations within the hot spot, given the total number over the whole plane, is deduced, and this result is applied in an example to finding the blocking probability for a Directed Retry algorithm. The results also apply to the number of mobiles within range of a base station and, in an ad hoc network, to the number of nodes within range of a given node.

INTRODUCTION

Mobiles and base stations are both assumed to be uniformly and randomly distributed on the (infinite) plane, but with different densities, ρ_M and ρ_B: more precisely, the probability that there is one node in an area of size δA is $\rho\delta A$ (ρ being equal to ρ_M or ρ_B as appropriate) to first order in δA, and the probability of more than one is of higher order than δA. The loss in decibels at (random) distance R from a node is

$$L = k_0 + k_1 \ln R + S, \tag{1}$$

where k_0 and k_1 are constants, and S is a shadowing effect, normally distributed with mean zero, variance σ^2. Here natural logarithms are used: using logarithms to base 10, the loss may be written

$$L = k_0 + k_d \log_{10} R + S. \tag{2}$$

Let mobile M and base station B be within communication range. This is possible if the loss from M to B does not exceed some specified value l_1. Suppose therefore that

$$L = k_0 + k_1 \ln R + S \le l_1. \tag{3}$$

If this condition is satisfied then M and B will be referred to as l_1-audible or, when there is no ambiguity, simply as audible to one another.

THE NUMBER OF AUDIBLE BASE STATIONS WITHIN A RANGE OF DISTANCES

Define $P_n(r)$ to be the probability that there are exactly n stations between distances r_1 and r from the mobile ($r_1 \le r$) and with loss $\le l_1$, and define a probability generating function

$$\Pi(r,z) = \sum_{n=0}^{\infty} P_n(r) z^n. \tag{4}$$

In [4] it is shown that $\Pi(r,z)$ satisfies the partial differential equation,

$$\frac{\partial}{\partial r}(\ln \Pi) = 2\pi(z-1)\rho r \Phi(a - b\ln r), \tag{5}$$

whose general solution is

$$\ln \Pi = \pi(z-1)\rho\left[\Psi(a,b;r) + G(z)\right], \tag{6}$$

where $G(z)$ is an arbitrary function of z and

$$\Psi(a,b;r) = r^2\Phi(a - b\ln r) - e^{\frac{2a}{b}+\frac{2}{b^2}}\Phi(a - b\ln r + 2/b), \tag{7}$$

(which is an indefinite integral of $2r\Phi(a - b\ln r)$) with $a = [l_1 - k_0 - 2\sigma^2/k_1]/\sigma$, $b = k_1/\sigma$, and $\Phi(x) = \int_{-\infty}^{x}(1/\sqrt{2\pi})e^{-u^2/2}du$.
The boundary condition is $\Pi(z) \equiv 1$ when $r = r_1$, leading to the solution

$$\ln \Pi = \pi(z-1)\rho\left[\Psi(a,b;r) - \Psi(a,b;r_1)\right]. \tag{8}$$

This is the probability generating function of a Poisson distribution whose mean is

$$\mu_{r_1,r} = \pi\rho\left[\Psi(a,b;r) - \Psi(a,b;r_1)\right]. \tag{9}$$

THE NUMBER OF AUDIBLE BASE STATIONS WITHIN A HOT SPOT

The result (9) for the number of audible base stations within an annulus can be modified simply by replacing π by θ to show that the number within a sector of an annulus between r_1 and r and subtending an angle 2θ at the mobile has a Poisson distribution with mean

$$\mu_{r_1,r;\theta} = \theta\rho\left[\Psi(a,b;r) - \Psi(a,b;r_1)\right], \qquad 0 \le \theta \le \pi. \tag{10}$$

If the annulus extends from r to $r+\delta r$ (see figure 1), and $\theta = \theta(r)$, then this mean becomes

$$\mu_{r,r+\delta r;\theta} = \theta(r)\rho\left[\Psi(a,b;r+\delta r) - \Psi(a,b;r)\right], \tag{11}$$

for $0 \le \theta \le \pi$, which can be written

$$\mu_{r,r+\delta r;\theta} = \theta(r)\rho\frac{\delta\Psi(a,b;r)}{\delta r}\delta r, \qquad 0 \le \theta \le \pi. \tag{12}$$

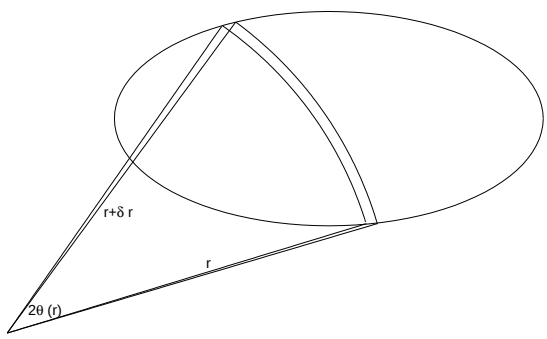

Figure 1: General Hot Spot

This result makes it possible to consider a specified area over which the density of base stations, while remaining constant, differs from that over the rest of the plane. The former density may be the greater (resulting from the operator's response to an increased intensity of mobile traffic in the same area), and the area may then be called a hot spot. Consequently, if a hot spot lies between radii r_1 and r_2 with its boundary at distance r defined by $\theta(r)$ (which, for a convex region, merely requires that the boundary can be expressed in terms of polar coordinates so that $2\theta(r)$ is the angular distance between the two boundary points at radius r), then the number of audible base stations within the hot spot has a Poisson distribution with mean

$$\mu_{r_1,r_2;\theta(r)} = \int_{r_1}^{r_2} \theta(r)\rho\frac{d\Psi(a,b;r)}{dr}dr. \tag{13}$$

Remembering from (7) that $\Psi(a,b;r)$ is an indefinite integral of $2r\Phi(a-b\ln r)$,

$$\mu_{r_1,r_2;\theta(r)} = \int_{r_1}^{r_2} 2\theta(r)\rho r\Phi(a-b\ln r)dr. \tag{14}$$

These methods can be applied to the case of the two-slope model (in which the power of the attenuation law changes at some specified distance). The distribution remains Poisson, and its mean can be found.
This integral is likely to require numerical integration, as the example illustrated in Figure 2 indicates. A circular hot spot of radius r_h is centred at distance r_c from the receiver.

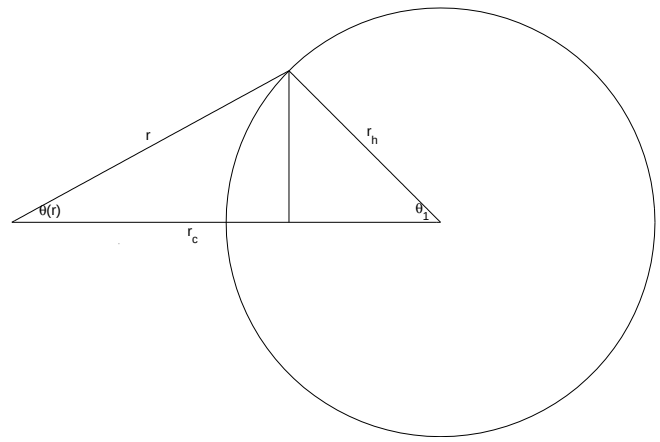

Figure 2: Hot Spot Example

From the diagram it can be seen that $r\sin\theta(r) = r_h\sin\theta_1$ and $r_c = r\cos\theta(r) + r_h\cos\theta_1$. Elimination of θ_1 leads to a quadratic equation in $\sin\theta(r)$. This leads to a formal expression for $\theta(r)$, but substitution of this into (14) does not give an integral which can be evaluated in closed form.

CONDITIONAL DISTRIBUTION

Now define random variables N_1, the total number of base stations which can hear a given mobile, and N, the number within a hot spot. Denote the mean of N, which is given by (14), by μ_2. Further, setting $r_1 = 0$ and letting $r \to \infty$ in (9), the mean value of N_1 is

$$\pi\rho e^{2(\sigma^2/k_1^2 - k_0/k_1)}e^{2l_1/k_1}. \tag{15}$$

Consequently the mean number of audible base stations excluding those in the hot spot, $N_1 - N$, to be denoted by μ_1, is given by subtracting μ_2 from (15). The situation now is that there are two independent Poissons with means μ_1 and μ_2 for the numbers respectively outside and inside the hot spot. The probability that these numbers are n and $n_1 - n$ is therefore

$$\frac{\mu_1^n e^{-\mu_1}}{n!}\frac{\mu_2^{n_1-n}e^{-\mu_2}}{(n_1-n)!} \tag{16}$$

The sum of the two independent Poissons is itself Poisson with mean $\mu_1 + \mu_2$, so the probability that the total number of stations involved is n_1 is

$$\frac{(\mu_1 + \mu_2)^{n_1} e^{-(\mu_1+\mu_2)}}{n_1!}. \tag{17}$$

The conditional probability that $N = n$ given $N_1 = n_1$ is then obtained by dividing (16) by (17). It is

$$\frac{n_1!}{n!(n_1 - n)!} \left(\frac{\mu_1}{\mu_1 + \mu_2}\right)^n \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^{n_1 - n}. \tag{18}$$

This is, of course, a binomial distribution.

DIRECTED RETRY BLOCKING PROBABILITY EXAMPLE

Let us denote by R the maximum number of cells that can be considered at call set up by the DR algorithm (regardless of the actual number of base stations that can be heard by the mobile). A call attempt is blocked if all channels are occupied in the M cells ($M = \min[N_1, R]$) which provide the user with the strongest received power, which must be larger than a threshold P_{rth}. The blocking probability decreases when M increases: however, the assessment of the effectiveness of DR requires knowledge of the statistics of N.

Denote by (x_m, y_m) the position of the generic mobile at call set up. A circular area H with centre (x_H, y_H) and radius r_H is defined (see Fig. 3), where the user distribution is uniform in L=S-H with given traffic offered by new calls, and in H with a different value of offered traffic. If the latter is larger than the former, H represents a hot spot. We also assume that each base inside H has the same number, n_{cH}, of available channels, and each base outside H has the same number, n_{cL}, of available channels.

At call set up, a base station can be considered as a candidate for the assignment of a channel to the mobile user provided that the received power at (x_m, y_m) from the base is larger than the fixed threshold P_{rth}. Therefore, the candidate bases (cells) are those whose signal has a power loss P smaller than $L_p = P_t - P_{rth}$ at (x_m, y_m), where P_t is the power of transmission. Under these assumptions, the blocking probability $P_b(x_m, y_m)$ is generally a function of the position (x_m, y_m) of the mobile inside S,

$$\sum_{n_1=0}^{\infty} \sum_{n=0}^{n_1} P_{b|n,n_1}(x_m, y_m) \cdot Q_{n|n_1}(x_m, y_m) \cdot Q_{n_1}(x_m, y_m), \tag{19}$$

where $P_{b|n,n_1}(x_m, y_m)$ is the blocking probability for the specified values of n and n_1, $Q_{n_1}(x_m, y_m) = P(N_1 = n_1)$, and $Q_{n|n_1}(x_m, y_m) = P(N = n|N_1 = n_1)$.

$Q_{n_1}(x_m, y_m) = e^{-\mu} \frac{\mu^{n_1}}{n_1!}$, where μ is given by (15), and $Q_{n|n_1}(x_m, y_m)$ is given by (18). Finding $P_{b|n,n_1}(x_m, y_m)$ is much more complex, and here only the particular case of the mobile at (x_H, y_H), that is, situated in the centre of the circular area H (see Fig. 3) is considered. Thus we can rewrite expression (19) as

$$P_b = \sum_{N=0}^{\infty} Q_{n_1} \sum_{n_1=0}^{N} P_{b|n,n_1} \cdot Q_{n|n_1}. \tag{20}$$

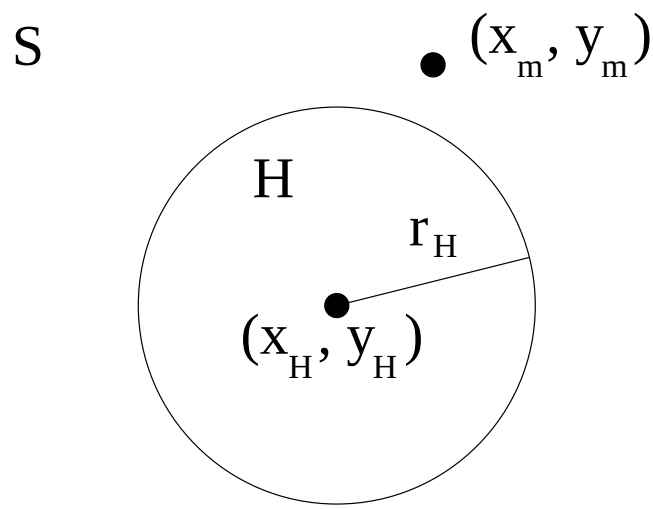

Figure 3: Hot Spot Scenario

Now, n represents the number of base stations belonging to a circle of radius r_H that can be heard by a mobile user located in the centre of the circle. This case is considered in [6], where the probability $P_{b|n,n_1}$ is found in two cases,

i) the whole re-try traffic is generated by the cells belonging to H;

ii) the whole re-try traffic is generated by the cells belonging to L.

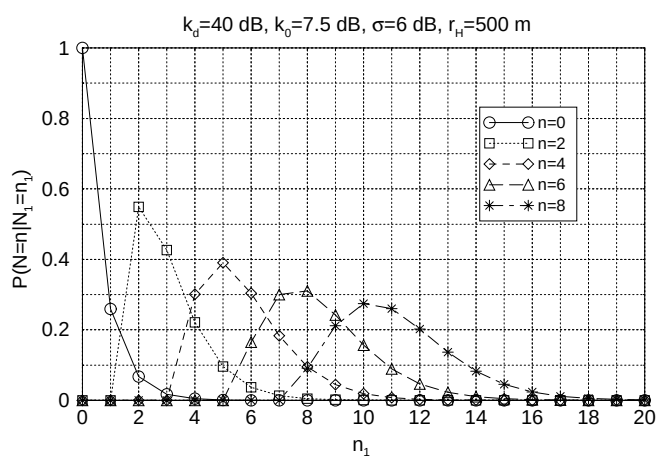

Figure 4: Conditional Probabilities

This approach provides two different approximations of the blocking probability, P_{bH} and P_{bL}, based on cases i) and ii), one of them an upper, the other a lower bound to the blocking probability, though which of the two cases gives the upper bound is not apparent a priori. In [6] graphs of the blocking probability are presented and discussed: here a brief extract is given. E_L and E_H represent offered traffic in regions L and H. Values of r_H range from 0 representing a uniform distribution with offered traffic E_L to 2000 which is numerically equivalent to a hot spot of infinite radius and hence to a uniform distribution with offered traffic E_H.

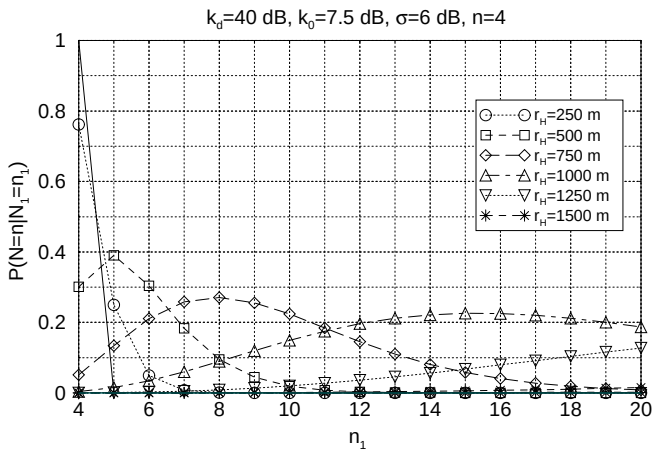

Figure 5: Conditional Probabilities

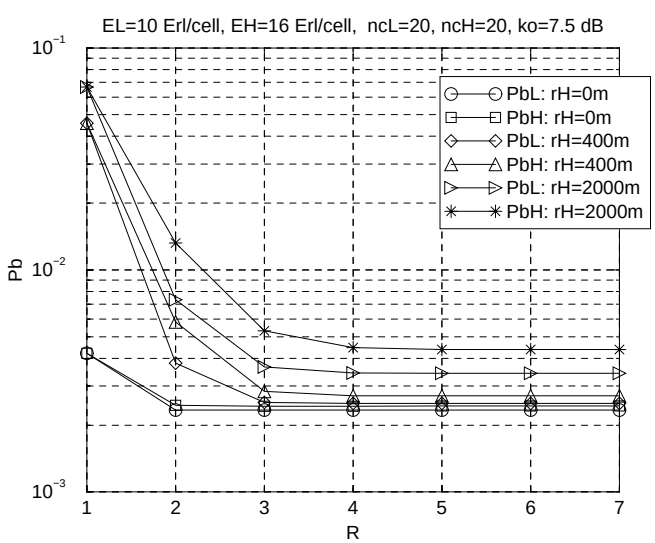

Figure 6: Blocking Probabilities

In the analytical results we have set $\rho = 2.183 \cdot 10^{-6}\text{m}^{-2}$, $L_p = 125$ dB, $k_d = 40$ dB, $E_L = 10$ Erl/cell and $n_{cL} = 20$ channels per cell; σ and k_0 are varied in order to have different values of μ, the mean of N_1, ranging from about 4 to 11 and representing different situations characterised by various degrees of overlap between cells.

Figure 4 shows $P(N = n \mid N_1 = n_1)$, the conditional probability, for $r_H = 500$, and Figure 5 shows it for $n = 4$ and varying r_H. Figure 6 displays pairs of curves related to the two approximations, P_{bH} and P_{bL} and giving the blocking probability as a function of R for three different values of the radius of the hot spot r_H, $k_0 = 7.5$dB, and $\sigma = 5$ dB. The figure shows that the presence of a hot spot increases the blocking probability, and that the retry mechanism reduces P_b as R moves from 1 to 4. In this case the mean of N_1 is approximately equal to 6.

Note that the two methods of approximation to the blocking probability P_{bH} and P_{bL} give very similar results whatever the value of r_H.

CONCLUSION

A previously reported procedure for obtaining the probability distribution of the number of base stations audible to a mobile can be modified to cater for the presence of a hot spot. This result in turn leads to a conditional distribution for the number within the hot spot when the total number is known. An example shows how this second result can be applied to the evaluation of blocking probabilities in a directed retry algorithm.

REFERENCES

[1] J. Orriss, A. R. Phillips, S. K. Barton. *"A Statistical Model for the Spatial Distribution of Mobiles and Base Stations"*. VTC 1999-Fall, Amsterdam, 1999, paper number 2.2.4.

[2] R. Verdone, A. Zanella *"Performance of Directed Retry in a Mobile Radio System: a Comparison between Analytical and Simulation Results"*. PIMRC2000, London, 2000.

[3] J. Orriss, S. K. Barton. *"On the Capacity of Cellular CDMA System Up-links with Multiple Base Station Diversity"*. PIMRC2000, London, 2000.

[4] J. Orriss, S. K. Barton. *"Probability Distributions for the Number of Radio Transceivers Which Can Communicate with One Another"*, provisionally accepted by IEEE Transactions on Communications.

[5] R. Verdone, A. Zanella, *"Analytical Evaluation of Blocking Probability in a Mobile Radio System with Directed Retry"*. IEEE Journ. on Select. Areas in Comm., Vol. 19, No. 2, Pp. 322-331, Feb. 2001.

[6] A. Zanella, J. Orriss, S. K. Barton, R. Verdone, *"Performance of Directed Retry in Non-Uniform Cellular Environments"*, submitted.