Ernest D. Fasse
Department of Aerospace and Mechanical Engineering, University of Arizona, Tucson, AZ 85721

Peter C. Breedveld
Department of Electrical Engineering, University of Twente, Enschede, The Netherlands

Modeling of Elastically Coupled Bodies: Part I—General Theory and Geometric Potential Function Method

This paper looks at spatio-geometric modeling of elastically coupled rigid bodies. Desirable properties of compliance families are defined (sufficient diversity, parsimony, frame-indifference, and port-indifference). A novel compliance family with the desired properties is defined using geometric potential energy functions. The configuration-dependent wrenches corresponding to these potential functions are derived in a form suitable for automatic computation.

1 Introduction

Modeling, analysis and simulation of flexible, spatial multibody systems is a challenging problem of considerable practical importance. There is significant literature looking at the geometry of compliance. Dimentberg (1965) looks at the geometry of a single rigid body supported by a system of ideal linear springs. Lončarić (1987, 1988) showed that stiffness and compliance matrices could be parameterized in an intuitive geometrical way. Griffis and Duffy (1991) looked at the geometry of compliance using screw theory. Patterson and Lipkin (1993b, 1993a) went on to classify compliance in terms of screw eigenvalues and eigenvectors. Žefran and Kumar (1997) and Howard et al. (1995) have also looked at the geometry of compliance, explaining for example differences in the structure of the stiffness matrix when defined using different implicitly defined affine connections. Huang and Schimmels (1997) look at the realizability of spatial stiffnesses using parallel connections of "simple springs."

2 Problem Statement

Shown in Fig. 1 is a pair of rigid bodies connected by an elastic body. The elastic body need not be an axisymmetric beam. Panel (*a*) depicts the undeformed system in static equilibrium. Panel (*b*) depicts the deformed system. The configuration of a rigid body can be represented by a frame, which in turn can be identified with a homogeneous matrix

$$H = \begin{bmatrix} R & p \\ 0' & 1 \end{bmatrix} \tag{1}$$

where $R = [e_1\ e_2\ e_3]$ is an orthonormal matrix and p is a linear displacement vector.

Six such frames are shown in Fig. 1. Frames "A," "a," and "a′" are attached to one of the rigid bodies, so that they do not move with respect to each other. Frames "B," "b" and "b′" are attached to the other rigid body. Frames A and B are distinguished frames on the rigid bodies, located at any point of interest. Frames a and b are located at the to-be-defined centers of stiffness of the two bodies. It shall be shown that given certain assumptions these centers must coincide in equilibrium, as depicted in Panel (*a*). Frames a and b are used in a later section.

Assume that the elastic body is always in internal equilibrium, so that its potential energy is a function of the configurations of the two rigid bodies. The goal of this research is to identify parameterized families of potential energy functions that satisfy the following criteria: (1) The families should be *sufficiently diverse.* Given any local stiffness behavior there must exist parameters that exactly model this local behavior. (2) The families should be *parsimonious.* There should not be more parameters than are necessary to model arbitrary local behavior. (3) The potential energy functions should be *frame-indifferent* (Marsden and Hughes 1983). If both rigid bodies undergo the same rigid body transformation then the potential energy should be unchanged. Frame-indifference implies that the wrenches acting on the bodies are equal and opposite. (4) The potential energy function should be *port-indifferent.* The potential energy, $U_\pi(H_a, H_b)$, is a function of a number of parameters, π, and the configuration of the two bodies, H_a and H_b. Given any parameters π, then there must exist parameters π' such that $U_\pi(H_a, H_b) = U_{\pi'}(H_b, H_a)$. If a potential function is port-indifferent, then it doesn't matter which body is named A and which is named B. Depending on the name choice the parameters chosen will be either π or π', but the underlying potential energy function will be the same.

3 Notation

Given a Euclidean vector w, let $\tilde{w} = (w)^{\sim}$ denote the cross product matrix, the matrix such that $\tilde{w}v = w \times v$ for any vector v. Let sy(A) denote the symmetric part of matrix A and as(A) denote the antisymmetric part (skew-symmetric part). Let tr(A) denote the trace of matrix A.

The position of the origin of frame α with respect to frame β in the coordinates of frame γ is denoted $p_\alpha^{\gamma,\beta}$. In general, if only one superscript is given then frames β and γ are the same. A 0 or omitted superscript implies that both β and γ are the distinguished inertial frame. Finite and infinitesimal displacements are denoted similarly. Velocities are denoted $v_\alpha^{\gamma,\beta}$.

Matrix $R_\alpha^{\gamma,\beta}$ is the orientation of frame α with respect to frame β in the coordinates of γ. We only use the case that $\beta = \gamma$, in which case the columns of matrix $R_\alpha^\gamma = [e_{1\alpha}^\gamma\ e_{2\alpha}^\gamma\ e_{3\alpha}^\gamma]$ are the three base vectors of frame α in the coordinates of γ. We shall never use "t" to identify a frame, thus to save space R_α^t is equivalent to $(R_\alpha)^t$, the transpose of $R_\alpha = R_\alpha^0$.

Configurations and rigid-body displacements are represented by homogeneous matrices. Matrix $H_\alpha^{\gamma,\beta}$ is the configuration of frame α with respect to frame β in the coordinates of γ.

Contributed by the Dynamic Systems and Control Division for publication in the JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENT, AND CONTROL. Manuscript received by the Dynamic Systems and Control Division December 10, 1997. Associate Technical Editor: J. E. Colgate.

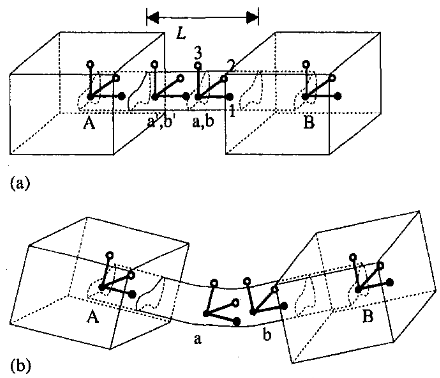

Fig. 1 Two rigid bodies mutually supported by an elastic beam with constant cross section, shown in (*a*) undeformed, relaxed configuration and (*b*) deformed, strained configuration

Finite angular displacements can be represented nonuniquely as a three-tuple $\Delta\theta_\alpha^{\gamma,\beta}$, which is the angular displacement of frame α with respect to frame β in the coordinates of frame γ. Infinitesimal displacements are denoted $\delta\theta_\alpha^{\gamma,\beta}$. Angular velocities are denoted $\omega_\alpha^{\gamma,\beta}$.

The term twist refers in general to an arbitrary rigid body motion involving potentially both translation and rotation. Finite twist displacements are denoted $\Delta T_\alpha^{\gamma,\beta} = [\Delta p_\alpha^{\gamma,\beta};\ \Delta\theta_\alpha^{\gamma,\beta}]$, where the space-saving notation $[x_1; x_2] = [x_1^t, x_2^t]^t$ is used. Infinitesimal twist displacements are denoted $\delta T_\alpha^{\gamma,\beta}$. The corresponding matrix displacement $[\delta T_\alpha^{\gamma,\beta}]$ is

$$[\delta T_\alpha^{\gamma,\beta}] = \begin{bmatrix} \delta\tilde{\theta}_\alpha^{\gamma,\beta} & \delta p_\alpha^{\gamma,\beta} \\ 0^t & 0 \end{bmatrix} \tag{2}$$

To first order, $I + \delta T_\alpha^{\gamma,\beta}$ is a homogeneous matrix, where I is the 4×4 identity matrix. Twist velocities are denoted $T_\alpha^{\gamma,\beta}$.

Forces are denoted $f^{\gamma,\beta}$, which is a force applied at the origin of frame β in the coordinates of frame γ. Moments (torques) are denoted $\tau^{\gamma,\beta}$, which is a moment applied at the origin of frame β in the coordinates of frame γ. Wrenches are denoted $w^{\gamma,\beta} = [f^{\gamma,\beta}; \tau^{\gamma,\beta}]$. Subscripts will be used to identify particular wrenches. For example, $w_b = w_b^0$ will denote the wrench exerted by body B on the elastic element with respect to the distinguished inertial frame.

Given a homogeneous matrix H_α^β, the associated adjoint matrix is

$$\mathrm{Ad}_{H_\alpha^\beta} = \begin{bmatrix} R_\alpha^\beta & \tilde{p}_\alpha^\beta R_\alpha^\beta \\ 0 & R_\alpha^\beta \end{bmatrix} \tag{3}$$

The adjoint matrix relates twists and wrenches in different base frames.

4 Lumped Parameter Modeling of Spatial Compliance

4.1 Canonical Stiffness Parameters. We use the stiffness parameterization of Lončarić (1987), which concerns the compliant behavior near equilibrium. As shown in Fig. 1, attached to body A is a frame A at some point of interest. Attached to body B is a frame B at some point of interest. Temporarily, assume that 'b' is any frame attached to body B. Assume that 'a' is the frame attached to body A that coincides with frame b in equilibrium. Let δT_b^a be an infinitesimal displacement of frame 'b' from the equilibrium frame 'a'. Let w_b^a be the wrench exerted by rigid body B on the elastic body, so that positive work is done on the elastic body when corresponding elements of δT_b^a and w_b^a are positive. For infinitesimal displacements we have approximately

$$w_b^a \approx K\delta T_b^a = \begin{bmatrix} K_t & K_c \\ K_c^t & K_o \end{bmatrix} \delta T_b^a \tag{4}$$

where K is a symmetric stiffness matrix.[1] Matrices K_t and K_o are necessarily symmetric. Lončarić shows that if tr (K_t) is not an eigenvalue of K_t, then K_c is symmetric at a unique point attached to body 'B' known as the center of stiffness. Assuming that the eigenvalues are non-negative, tr (K_t) is an eigenvalue of K_t if and only if at least two of the eigenvalues (principal stiffnesses) are zero.

Assume that the origin of frame 'b' is this point. Let w_a^b be the wrench exerted by rigid body A on the elastic body. Let δT_a^b be the infinitesimal displacement of frame 'a' from frame 'b'. If the potential energy is frame-indifferent then the wrenches exerted by bodies A and B are equal and opposite, $w_a^b = -w_b^b$. Because frames 'a' and 'b' were assumed to coincide in equilibrium we have to first order that $w_a^b = -w_b^b \approx -w_b^a$. It follows that $w_a^b \approx -K\delta T_b^a = K\delta T_a^b$. This shows that the origin of frame 'a' is the unique center of stiffness attached to body A.

Proposition 1. Given two rigid bodies mutually supported by an elastic body with a frame-indifferent potential energy function and corresponding translational stiffness K_t such that tr (K_t) is not an eigenvalue of K_t; then for each body there exists a unique center of stiffness at which K_c is symmetric. Furthermore, the centers of stiffness of the two bodies must coincide in equilibrium.

Although the proof is mathematically trivial given prior results, this was not immediately obvious. Lončarić (1987, 1988) considers elastic multibody systems and defines "*N*-centers of stiffness," where "*N*" is the number of rigid bodies. He does not compute the relative location of the centers of stiffness. The term center of stiffness is usually used in the singular. In fact there is a pair of centers of stiffness. They are naturally identified because they must coincide in equilibrium.

Matrices K_t, K_c, and K_o are determined by their respective sets of orthonormal principal axes and principal stiffnesses. They determine the stiffness in equilibrium, thus their principal axes are expressed equivalently in either coordinates of frame 'a' or 'b'. To emphasize this an unusual ab superscript is used. Matrix $K_t = R_t^{ab}\Gamma_t(R_t^{ab})^t$ is the translational stiffness matrix. Columns of orthonormal matrix $R_t^{ab} = [e_{1t}^{ab}\ e_{2t}^{ab}\ e_{3t}^{ab}]$ are the principal axes of translational stiffness. Matrix $\Gamma_t = \mathrm{diag}\ (\gamma_{1t}, \gamma_{2t}, \gamma_{3t})$ is a matrix of principal translational stiffness. A displacement along any one of the principal axes results in a translational force along the same axis.

Matrix $K_o = R_o^{ab}\Gamma_o(R_o^{ab})^t$ is the rotational stiffness matrix. Columns of orthonormal matrix $R_o^{ab} = [e_{1o}^{ab}\ e_{2o}^{ab}\ e_{3o}^{ab}]$ are the principal axes of rotational stiffness. Matrix $\Gamma_o = \mathrm{diag}\ (\gamma_{1o}, \gamma_{2o}, \gamma_{3o})$ is a matrix of principal rotational stiffnesses. A rotation about any one of the principal axes results in a torque about the same axis.

Matrix $K_c = R_c^{ab}\Gamma_c(R_c^{ab})^t$ is the coupling stiffness matrix. Columns of orthonormal matrix $R_c^{ab} = [e_{1c}^{ab}\ e_{2c}^{ab}\ e_{3c}^{ab}]$ are the principal coupling axes of stiffness. Matrix $\Gamma_c = \mathrm{diag}\ (\gamma_{1c}, \gamma_{2c}, \gamma_{3c})$ is a matrix of principal coupling stiffnesses. A displacement along any one of the principal axes results in a torque about the same axis. A displacement about any one of the principal axes results in a translational force along the same axis.

4.2 Elastic Beam Example. A simple example of considerable practical relevance is a pair of rigid bodies mutually supported by an elastic beam of length L with constant cross section, as depicted in Fig. 1. Let A be the cross-sectional area of the beam. The beam material is assumed to be an ideal

[1] The stiffness matrix can be asymmetric away from equilibrium when it is defined in terms of twist-displacements and wrenches as here, and not in terms of generalized displacements and generalized forces (Žefran and Kumar, 1997).

homogeneous, Hookean material with Young's modulus E and shear modulus G. In equilibrium the centers of stiffness coincide at the centroid of the beam, which necessarily intersects the neutral axes of the beam. Choose the e_1 axis of both frames a and b to lie along the beam axis, as depicted. Choose the e_2 and e_3 axes to be aligned with a pair of orthogonal principal bending axes. If the cross section is not rotationally symmetric then these axes are unique. A bending moment about e_2 results in no strain along axis e_3, so that e_3 is the corresponding neutral axis. Similarly, e_2 is the neutral axis corresponding to a bending moment about e_3.

Let x_1, x_2 and x_3 be rectilinear coordinates associated with axes e_1, e_2 and e_3. The two principal, rectangular moments of inertia are I_2 and I_3:

$$I_2 = \int_{cs} x_3^2 dA \text{ and } I_3 = \int_{cs} x_2^2 dA \tag{5}$$

where $\int_{cs}$ denotes integration over the cross-section of the beam. Moment of inertia I_2 determines the bending behavior given moments about axis e_2; moment of inertia I_3 determines the bending behavior given moments about axis e_3. The polar moment of inertia determines the torsional behavior of the beam given moments about axis e_1. The polar moment of inertia is simply the sum of the rectangular moments of inertia, $I_p = I_2 + I_3$.

The translational and rotational stiffness matrices are then

$$K_t = \begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI_3}{L^3} & 0 \\ 0 & 0 & \frac{12EI_2}{L^3} \end{bmatrix} \tag{6}$$

$$K_o = \begin{bmatrix} \frac{GI_P}{L} & 0 & 0 \\ 0 & \frac{EI_2}{L} & 0 \\ 0 & 0 & \frac{EI_3}{L} \end{bmatrix} \tag{7}$$

The coupling stiffness matrix K_c is zero. Note that moment of inertia I_3 determines the translational stiffness along the e_2 axis; I_2 determines the translational stiffness along the e_3 axis.

Consider, for example, the special case of a circular beam with diameter D. Then $A = \pi D^2/4$, $I_2 = I_3 = \pi D^4/64$, and $I_P = \pi D^4/32$.

4.3 Transformation of Stiffness in Equilibrium. This section shows how the stiffness at the centers of stiffness can be computed given the stiffness at other points. Let frames 'a' and 'a'' be frames attached to body A. Let frames 'b' and 'b'' be frames attached to body B. Frame 'a' coincides with frame 'b' in equilibrium. Frame 'a'' coincides with frame 'b'' in equilibrium. Such a system in equilibrium is shown in Fig. 1. Let $w_{b'}$ be the wrench exerted by body B on the compliant element. Suppose that we know the stiffness K' at 'a'', so that $w_{b'}^{a'} = K'\delta T_{b'}^{a'}$. It is well known that $\delta T_{b'}^{a'} = \mathrm{Ad}_{H_a^{a'}} \delta T_b^a$, where $\mathrm{Ad}_{H_a^{a'}}$ is computed according to (3). Given wrench $w_{b'}^{a'}$, equivalent wrench $w_{b'}^{a}$ is given by $w_{b'}^{a} = \mathrm{Ad}^t_{H_a^{a'}} w_{b'}^{a'}$. We can then conclude that $w_{b'}^{a} = \mathrm{Ad}^t_{H_a^{a'}} K' \mathrm{Ad}_{H_a^{a'}} \delta T_b^a$. This proves that

$$K = \mathrm{Ad}^t_{H_a^{a'}} K' \mathrm{Ad}_{H_a^{a'}} = \mathrm{Ad}^t_{H_b^{b'}} K' \mathrm{Ad}_{H_b^{b'}} \tag{8}$$

It may be that the location of the centers of stiffness, and thus $H_a^{a'}$, is known from material symmetries. If not, $p_a^{a'}$ can be determined from (8) by requiring that the coupling stiffness matrix be symmetric. Matrix $R_a^{a'}$ can always be chosen arbitrarily.

4.4 Properties of Compliance Families. In this section we show that frame-indifferent, port-indifferent, parsimonious compliance families must exert moments on both bodies given finite displacements, excepting the case of isotropic translational stiffness. This moment may be a high order, nonlinear function of displacement, so that it does not affect the coupling stiffness in equilibrium. A compliance family defined by a parameterized set of potential energy functions is parsimonious if there is a bijection between its set of parameters and the set of canonical, local stiffness parameters $\{(K_t, K_c, K_o)\}$. In this case we identify the parameters π with the stiffness matrix K. The wrenches exerted by the two rigid bodies on the elastic element depend on the parameters and the configurations of the two bodies. Let $w_a(\pi, H_a, H_b)$ be the wrench exerted by body 'A' on the elastic element. Let $w_b(\pi, H_a, H_b)$ be the wrench exerted by body 'B' on the elastic element. Port-indifferent, parsimonious compliance families have the following property:

Proposition 2. Port-indifferent, parsimonious compliance families are port symmetric, i.e., given any stiffness matrix K, for arbitrary H_1 and H_2 it follows that (1) $w_a(K, H_1, H_2) = w_b(K, H_2, H_1)$, (2) $w_a^a(K, H_1, H_2) = w_b^b(K, H_2, H_1)$, and (3) $w_a^b(K, H_1, H_2) = w_b^a(K, H_2, H_1)$.

Proof. First assume that $H_a = H_1$ and $H_b = H_2$. Let $\delta T_a = \delta T$ be an infinitesimal, virtual displacement of body 'A' for some arbitrary infinitesimal twist $\delta T = [\delta p; \delta\theta]$. The virtual work associated with this displacement is

$$\begin{aligned} \delta W &= U_K(H_1 + [\delta T]H_1, H_2) - U_K(H_1, H_2) \\ &= (w_a(K, H_1, H_2))^t \delta T \end{aligned} \tag{9}$$

Next assume that $H_a = H_2$ and $H_b = H_1$. Let $\delta T_b = \delta T$ be an infinitesimal, virtual displacement of body B for the same δT. From port-indifference it follows that there exists some K' such that the virtual works are equal

$$\begin{aligned} \delta W &= \delta W' \\ &= U_{K'}(H_2, H_1 + [\delta T]H_1) - U_{K'}(H_2, H_1) \\ &= (w_b(K', H_2, H_1))^t \delta T \end{aligned} \tag{10}$$

From parsimony it follows that $K' = K$. Because δT was arbitrary we conclude that given any K, $w_a(K, H_1, H_2) = w_b(K, H_2, H_1)$.

Using body relative displacements $\delta T_a^a = \delta T$ and $\delta T_b^b = \delta T$ we can conclude from a similar argument that $w_a^a(K, H_1, H_2) = w_b^b(K, H_2, H_1)$.

To prove the third relation note that $w_a^b = \mathrm{Ad}^t_{H_b^a} w_a^a$, where $H_b^a = H_a^{-1} H_b$. Thus

$$w_a^b(K, H_a, H_b) = \begin{bmatrix} R_b^t R_a & 0 \\ -R_b^t(\tilde{p}_b - \tilde{p}_a)R_a & R_b^t R_a \end{bmatrix} w_a^a(K, H_a, H_b) \tag{11}$$

Similarly,

$$w_b^a(K, H_a, H_b) = \begin{bmatrix} R_a^t R_b & 0 \\ -R_a^t(\tilde{p}_a - \tilde{p}_b)R_b & R_a^t R_b \end{bmatrix} w_b^b(K, H_a, H_b) \tag{12}$$

Substituting $H_a = H_1$ and $H_b = H_2$ into (11), substituting $H_a = H_2$ and $H_b = H_1$ into (12) and using the fact that $w_a^a(K, H_1, H_2) = w_b^b(K, H_2, H_1)$ we conclude that $w_a^b(K, H_1, H_2) = w_b^a(K, H_2, H_1)$. □

It is well known in mechanics that frame-indifference implies that w_a and w_b are equal and opposite. It can easily be proven that

Proposition 3. Given a frame-indifferent compliance family with parameters π, then $w_a(\pi, H_a, H_b) = -w_b(\pi, H_a, H_b)$.

An elastic element exerts zero moment with respect to the centers of stiffness of the rigid body pair if and only if the elastic force axis intersects the two centers of stiffness, independently of the rigid body displacement of the two bodies. If this is so then the translational stiffness matrix must be isotropic.

Proposition 4. Given a frame-indifferent, port-indifferent, parsimonious compliance family with stiffness matrix K, then the moments $\tau^b_a(K, H_a, H_b)$ and $\tau^a_b(K, H_a, H_b)$ can be identically zero only if K_t is isotropic.

Proof. Assume that $\tau^b_a(K, H_a, H_b)$ is identically zero. From Proposition 2 it follows that $\tau^a_b(K, H_a, H_b)$ is also identically zero. In general we have that

$$w^a_b = \mathrm{Ad}^t_{H^b_a} w^b_b = -\mathrm{Ad}^t_{H^b_a} w^b_a \tag{13}$$

where the latter equality holds because of frame-indifference. Therefore

$$\begin{bmatrix} f^a_b \\ 0 \end{bmatrix} = -\begin{bmatrix} R^t_a R_b & 0 \\ -R^t_a(\tilde{p}_a - \tilde{p}_b)R_b & R^t_a R_b \end{bmatrix} \begin{bmatrix} f^b_a \\ 0 \end{bmatrix} \tag{14}$$

This can only happen if $(p_a - p_b)^{\sim} f_a$ is identically zero, i.e. if $f_a(K, H_a, H_b)$ is always in the direction of $p_a - p_b$. If this is true then K_t must be isotropic, i.e., $K_t = k_t I$ for some scalar k_t. □

5 Geometric Potential Function Method

5.1 Definition of Potential Energy Functions. In this method constitutive equations are derived via geometric potential energy functions. Other methods are presented in a companion paper. We decompose the potential energy function of the compliant element into translational, rotational and coupling terms. Canonical stiffness parameters were defined in Section 4.1. For analytical purposes it is useful to define some associated parameters. For each $\alpha \in \{t, o, c\}$ let $\Lambda_\alpha = \frac{1}{2}\,\mathrm{tr}\,(\Gamma_\alpha)I - \Gamma_\alpha$. Diagonal matrix Λ_α is the principal co-stiffness matrix associated with principal stiffness matrix Γ_α. There is a bijection between stiffnesses and co-stiffnesses, with $\Gamma_\alpha = \mathrm{tr}\,(\Lambda_\alpha)I - \Lambda_\alpha$. Let matrix $G_\alpha = R^{ab}_\alpha \Lambda_\alpha (R^{ab}_\alpha)^t$ be the co-stiffness matrix associated with stiffness matrix K_α. The co-stiffness matrices turn out to be useful in formulating rotational and coupling energy functions.

A candidate for the translational potential energy is

$$\begin{aligned} U_{K_t} &= \tfrac{1}{4}(p^a_b)^t K_t p^a_b + \tfrac{1}{4}(p^b_a)^t K_t p^b_a \\ &= -\tfrac{1}{4}\,\mathrm{tr}\,(\tilde{p}^a_b G_t \tilde{p}^a_b) - \tfrac{1}{4}\,\mathrm{tr}\,(\tilde{p}^b_a G_t \tilde{p}^b_a) \end{aligned} \tag{15}$$

Effectively there are two conventional, translational springs, each with stiffness $\frac{1}{2} K_t$. The principal axes of one spring are attached to body A. The principal axes of the other spring are attached to body B. The springs act in parallel so that the total stiffness is K_t.

The rotational (orientational) potential energy is defined to be

$$U_{K_o} = -\mathrm{tr}\,(G_o R^a_b) = -\mathrm{tr}\,(G_o R^b_a) \tag{16}$$

where again there is a unique co-stiffness matrix G_o for every stiffness matrix K_o.

This function can be thought of as an index of misalignment of R_a and R_b. To see this look at U_{K_o} for the case that $R^{ab}_o = I$, so that $G_o = \Lambda_o$. In this case

$$U_{K_o} = -\mathrm{tr}\,(\Lambda_o R^t_a R_b) = -\sum_{i=1}^{3} \lambda_{io} e^t_{ia} e_{ib} \tag{17}$$

Each $-\lambda_{io} e^t_{ia} e_{ib}$ term is minimized when vectors e_{ia} and e_{ib} are aligned ($e^t_{ia} e_{ib} = 1$), and is maximized when they are anti-aligned ($e^t_{ia} e_{ib} = -1$). Note that this energy is proportional to the co-stiffness λ_{io}, not the stiffness γ_{io}.

The coupling potential energy is defined to be

$$U_{K_c} = \mathrm{tr}\,(G_c R^b_a \tilde{p}^a_b) = \mathrm{tr}\,(G_c R^a_b \tilde{p}^b_a) \tag{18}$$

The coupling potential energy of (18) is the least transparent of the functions but has a geometrical interpretation. Consider U_{K_c} for the case that $R^{ab}_c = I$, so that $G_c = \Lambda_c$. In this case

$$U_{K_c} = \mathrm{tr}\,(\Lambda_c R^t_b \tilde{p}^{0,a}_b R_a) = \sum_{i=1}^{3} \lambda_{ic} e^t_{ib} \tilde{p}^{0,a}_b e_{ia} \tag{19}$$

Each $\lambda_{ic} e^t_{ib} \tilde{p}^{0,a}_b e_{ia} = \lambda_{ic} e_{ib} \cdot (p^{0,a}_b \times e_{ia}) = \lambda_{ic} p^{0,a}_b \cdot (e_{ia} \times e_{ib})$ term is proportional to the Euclidean, scalar triple-product of vectors $p^{0,a}_b$, e_{ia} and e_{ib}. Again the energy is proportional to the co-stiffness λ_{ic}, not the stiffness γ_{ic}.

The total potential energy is the sum of the translational, rotational and coupling potential energies, $U_K = U_{K_t} + U_{K_o} + U_{K_c}$. We propose this candidate as a model of local compliance because (1) any stiffness K can be achieved, (2) function U_K is frame indifferent, (3) function U_K is port indifferent, (4) the parameter set is parsimonious.

5.2 Dependency of Wrenches on Body Configurations. The wrenches acting on each body are determined by the differential of the energy function. First we compute the wrench exerted by body B on the elastic body. Let $\delta T^b_{b'}$ be an infinitesimal, virtual displacement of body B from a particular, real configuration, H_b to an arbitrary, virtual configuration, $H_{b'}$. For each $\alpha \in \{t, o, c\}$ the infinitesimal, virtual work given $\delta T^b_{b'}$ is

$$\delta W_\alpha = U_{K_\alpha}(H_a, H_b + H_b[\delta T^b_{b'}]) - U_{K_\alpha}(H_a, H_b) \tag{20}$$

Much computation yields

$$\begin{aligned} \delta W_t = &-\tfrac{1}{2}\,\mathrm{tr}\,(\mathrm{as}(G_t \tilde{p}^b_a)^t \delta\tilde{p}^b_{b'}) + \tfrac{1}{2}\,\mathrm{tr}\,((R^b_a \mathrm{as}(G_t \tilde{p}^a_b) R^a_b)^t \delta\tilde{p}^b_{b'}) \\ &+ \tfrac{1}{2}\,\mathrm{tr}\,(\mathrm{as}(G_t \tilde{p}^b_a \tilde{p}^b_a)^t \delta\tilde{\theta}^b_{b'}) \end{aligned} \tag{21}$$

$$\delta W_o = \mathrm{tr}\,(\mathrm{as}(G_o R^a_b)^t \delta\tilde{\theta}^b_{b'}) \tag{22}$$

$$\delta W_c = \mathrm{tr}\,(\mathrm{as}(G_c R^a_b)^t \delta\tilde{p}^b_{b'}) + \mathrm{tr}\,(\mathrm{as}(G_c \tilde{p}^a_b R^a_b)^t \delta\tilde{\theta}^b_{b'}) \tag{23}$$

It is true for each α that

$$\delta W_\alpha = \tfrac{1}{2}\,\mathrm{tr}\,((\tilde{f}^b_{b(\alpha)})^t \delta\tilde{p}^b_{b'}) + \tfrac{1}{2}\,\mathrm{tr}\,((\tilde{\tau}^b_{b(\alpha)})^t \delta\tilde{\theta}^b_{b'}) \tag{24}$$

Because $\delta T^b_{b'}$ was arbitrary we conclude that

$$\begin{aligned} \tilde{f}^b_{b(t)} &= -\mathrm{as}\,(G_t \tilde{p}^b_a) + R^b_a\, \mathrm{as}\,(G_t \tilde{p}^a_b) R^a_b \\ \tilde{\tau}^b_{b(t)} &= \mathrm{as}\,(G_t \tilde{p}^b_a \tilde{p}^b_a) \end{aligned} \tag{25}$$

$$\begin{aligned} \tilde{f}^b_{b(o)} &= \tilde{0} \\ \tilde{\tau}^b_{b(o)} &= 2\,\mathrm{as}\,(G_o R^a_b) \end{aligned} \tag{26}$$

$$\begin{aligned} \tilde{f}^b_{b(c)} &= 2\,\mathrm{as}\,(G_c R^a_b) \\ \tilde{\tau}^b_{b(c)} &= 2\,\mathrm{as}\,(G_c \tilde{p}^a_b R^a_b) \end{aligned} \tag{27}$$

Similar expressions hold for each $\tilde{f}^a_{a(\alpha)}$ and $\tilde{\tau}^a_{a(\alpha)}$, exchanging indices 'a' and 'b'. The total wrench with respect to the distinguished base frame is then

$$w_{\mathrm{b}} = \mathrm{Ad}^{t}_{H_{\mathrm{b}}^{-1}}(w^{\mathrm{b}}_{\mathrm{b}(t)} + w^{\mathrm{b}}_{\mathrm{b}(o)} + w^{\mathrm{b}}_{\mathrm{b}(c)}) \tag{28}$$

Evaluation and simplification yields

$$\tilde{f}_{\mathrm{b}} = \tfrac{1}{2}(R_{\mathrm{a}}K_tR^t_{\mathrm{a}}p^{0,\mathrm{a}}_{\mathrm{b}})^{\sim} + \tfrac{1}{2}(R_{\mathrm{b}}K_tR^t_{\mathrm{b}}p^{0,\mathrm{a}}_{\mathrm{b}})^{\sim} + 2 \text{ as } (R_{\mathrm{b}}G_cR^t_{\mathrm{a}})$$

$$\tilde{\tau}_{\mathrm{b}} = \tfrac{1}{2}(\tilde{p}_{\mathrm{a}}R_{\mathrm{b}}K_tR^t_{\mathrm{b}}p^{0,\mathrm{a}}_{\mathrm{b}})^{\sim} + \tfrac{1}{2}(\tilde{p}_{\mathrm{b}}R_{\mathrm{a}}K_tR^t_{\mathrm{a}}p^{0,\mathrm{a}}_{\mathrm{b}})^{\sim} + 2 \text{ as } (R_{\mathrm{b}}G_oR^t_{\mathrm{a}})$$

$$+ 2 \text{ as } (\tilde{p}_{\mathrm{b}}R_{\mathrm{b}}G_cR^t_{\mathrm{a}}) - 2 \text{ as } (\tilde{p}_{\mathrm{a}}R_{\mathrm{a}}G_cR^t_{\mathrm{b}}) \tag{29}$$

where $p^{0,\mathrm{a}}_{\mathrm{b}} = p_{\mathrm{b}} - p_{\mathrm{a}}$. Wrench w_{b} is computable given H_{a}, H_{b} and all stiffness matrices. Computation requires only multiplication and addition. Wrench w_{a} is equal and opposite, i.e., $w_{\mathrm{a}} = -w_{\mathrm{b}}$.

5.3 Wrenches Resulting From Small Displacements. For small, real relative displacements of the bodies from equilibrium the configuration-wrench map can be approximated by a linear relation. Assume that

$$H^{\mathrm{b}}_{\mathrm{a}} = H_{\mathrm{b}}^{-1}H_{\mathrm{a}} \approx I + [\delta T^{\mathrm{b}}_{\mathrm{a}}] \tag{30}$$

where $\delta T^{\mathrm{b}}_{\mathrm{a}}$ is a small, finite, real twist displacement. The linear approximation is computed by substituting Eq. (30) into Eq.'s (25), (26) and (27), and discarding high-order terms:

$$\tilde{f}^{\mathrm{b}}_{\mathrm{a}(t)} = -\tilde{f}^{\mathrm{b}}_{\mathrm{b}(t)} \approx 2 \text{ as } (G_t\delta\tilde{p}^{\mathrm{b}}_{\mathrm{a}})$$

$$\tilde{\tau}^{\mathrm{b}}_{\mathrm{a}(t)} = -\tilde{\tau}^{\mathrm{b}}_{\mathrm{b}(t)} \approx \tilde{0} \tag{31}$$

$$\tilde{f}^{\mathrm{b}}_{\mathrm{a}(o)} = -\tilde{f}^{\mathrm{b}}_{\mathrm{b}(o)} = \tilde{0}$$

$$\tilde{\tau}^{\mathrm{b}}_{\mathrm{a}(o)} = -\tilde{\tau}^{\mathrm{b}}_{\mathrm{b}(o)} \approx 2 \text{ as } (G_o\delta\tilde{\theta}^{\mathrm{b}}_{\mathrm{a}}) \tag{32}$$

$$\tilde{f}^{\mathrm{b}}_{\mathrm{a}(c)} = -\tilde{f}^{\mathrm{b}}_{\mathrm{b}(c)} \approx 2 \text{ as } (G_c\delta\tilde{\theta}^{\mathrm{b}}_{\mathrm{a}})$$

$$\tilde{\tau}^{\mathrm{b}}_{\mathrm{a}(c)} = -\tilde{\tau}^{\mathrm{b}}_{\mathrm{b}(c)} \approx 2 \text{ as } (G_c\delta\tilde{p}^{\mathrm{b}}_{\mathrm{a}}) \tag{33}$$

In general it is true that $\tilde{v} = A\tilde{w} + \tilde{w}A^t$ if and only if $v = (\mathrm{tr}\,(A)I - A^t)w$. It follows that

$$\begin{bmatrix} f^{\mathrm{b}}_{\mathrm{a}} \\ \tau^{\mathrm{b}}_{\mathrm{a}} \end{bmatrix} \approx K\begin{bmatrix} \delta p^{\mathrm{b}}_{\mathrm{a}} \\ \delta\theta^{\mathrm{b}}_{\mathrm{a}} \end{bmatrix} = \begin{bmatrix} K_t & K_c \\ K_c & K_o \end{bmatrix}\begin{bmatrix} \delta p^{\mathrm{b}}_{\mathrm{a}} \\ \delta\theta^{\mathrm{b}}_{\mathrm{a}} \end{bmatrix} \tag{34}$$

This shows that matrices K_t, K_o, and K_c determine the stiffness of the system at $H_{\mathrm{a}} = H_{\mathrm{b}}$ as claimed. It is also true in equilibrium that $w^{\mathrm{a}}_{\mathrm{b}} \approx K\delta T^{\mathrm{a}}_{\mathrm{b}}$, $w^{\mathrm{a}}_{\mathrm{a}} \approx K\delta T^{\mathrm{a}}_{\mathrm{a}}$, and $w^{\mathrm{b}}_{\mathrm{b}} \approx K\delta T^{\mathrm{b}}_{\mathrm{b}}$.

It is not claimed that the results are valid for large displacements. There is much solid mechanics literature regarding large deformations of elastic materials. Howell and Midha (1994) look at large deflections of planar, compliant mechanisms.

6 Summary

The goal of this research was to define compliance families that (1) were parameterized in an intuitive, geometrical way, (2) had desirable geometric and other properties, and (3) had constitutive equations that could be computed automatically for numerical simulation. The parameterization used was derived from that of Lončarić. Parameters were calculated for an elastic beam. The "desirable geometric and other properties" were sufficient diversity, parsimony, frame-indifference and port-indifference. A novel compliance family that has the desirable properties was presented. The proposed method is demonstrated and compared with alternatives in the companion paper.

Acknowledgments

We would like to thank Barry Goeree, Stefano Stramigioli, Shilong Zhang and the anonymous reviewers for their comments on the manuscript. Analysis of the elastic beam was based in part on work of Hua Li.

References

Dimentberg, F., 1965, *The Screw Calculus and Its Applications in Mechanics*, U.S. Dept. of Commerce. Translation No. AD680993.

Griffis, M., and J. Duffy, 1991, "Kinestatic Control: A Novel Theory for Simultaneously Regulating Force and Displacement," ASME *Journal of Mechanical Design*, Vol. 113, pp. 508–515.

Howard, W., Žefran, M., and Kumar, V., 1995, "On the 6 × 6 Stiffness Matrix for Three Dimensional Motions," *Proc. 9th IFToMM*, pp. 1575–1579.

Howell, L., and Midha, A., 1994, "Evaluation of Equivalent Spring Stiffness for Use in a Pseudo-Rigid-Body Model of Large-Deflection Compliant Mechanisms," ASME Design Technical Conferences, 23rd Biennial Mechanisms Conference, pp. 405–412.

Huang, S., and Schimmels, J., 1997, "The Realizable Space of Spatial Stiffnesses Achieved With a Parallel Connection of Simple Springs," *Proc. of the ASME Dynamic Systems and Control Division*, pp. 519–525.

Lončarić, J., 1987, "Normal Forms of Stiffness and Compliance Matrices," *IEEE Trans. on Robotics and Automation*, Vol. 3, pp. 567–572.

Lončarić, J., 1988, "On Statics of Elastic Systems and Networks of Rigid Bodies," Technical Report SCR TR 88-13, Systems Research Center, University of Maryland.

Marsden, J., and Hughes, T., 1983, *Mathematical Foundations of Elasticity*, Prentice-Hall.

Patterson, T., and Lipkin, H., 1993a, "A Classification of Robot Compliance," ASME *Journal of Mechanical Design*, Vol. 115, pp. 581–584.

Patterson, T., and H. Lipkin, 1993b, "Structure of Robot Compliance," ASME *Journal of Mechanical Design*, Vol. 115, pp. 576–580.

Žefran, M., and Kumar, V., 1997, "Affine Connections for the Cartesian Stiffness Matrix," *Proc. IEEE Int. Conf. on Robotics and Automation*, pp. 1376–1381.