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Throughput of Slotted ALOHA with Encoding Rate Optimization and Multipacket Reception

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***Abstract*—This paper considers a slotted ALOHA random access system where users send packets to a common receiver with multipacket reception capability. A collection of m users access the shared medium independently of each other with probability p and, upon access, they choose an encoding rate. A collision occurs when the sum of the rates of all the users exceeds the capacity of the channel. We analytically characterize as a function of m and p the encoding rate which maximizes the expected *global thoughput* of the system. It is shown that for any value of p the throughput converges to one when m tends to infinity, hence there is no loss due to packet collisions. This is in striking contrast with the well known behavior of slotted ALOHA systems in which users cannot adjust the encoding rate. In that case the throughput decreases to zero as the number of users increases. Finally, assuming that users are selfish, we characterize the encoding rate which maximizes the expected *individual throughput* of each user, and show that the corresponding Nash equilibrium is not globally optimum.**

I. INTRODUCTION

Slotted ALOHA systems with multipacket reception capabilities have been well studied in the context of multi-user wireless communications. These systems are composed of a collision channel between multiple transmitters and a single receiver with multipacket reception capabilities. Some classic studies focused on stability and delay performance [7], [12], [10], [13], [18]; while others focused on improving throughput via retransmission diversity [5], [16], [15], [17]. In the latter case, it is assumed that a feedback link between the receiver and the transmitters is available, which can be used to request retransmissions of collided packets. However, the delay due to retransmission can be unacceptable for traffic with stringent delay constraints. Furthermore, there are scenarios where a feedback link might not be available. This naturally motivates the following question: can one exploit multipacket reception capabilities to increase the throughput of random access systems without feedback?

To address the question above, we propose to transmit packets in each slot encoded at different rates. The possibility of decoding packets in the event of a collision depends on the rate at which the packets were encoded. We assume that at most one (normalized) unit of information per slot can be decoded at the receiver, and that colliding packets are correctly decoded when the sum of the rates at which they were encoded does not exceed one. This is a natural generalization of the classic collision model, where packets are always encoded at rate one, so that transmissions are successful only when there is one active user per slot.

In our model, the number of active users in the network is crucial: while transmissions with high encoding rates and a large number of active users cannot be decoded, low encoding rates with a small number of users are throughput inefficient. To visualize this, consider the following example: denote with m the total number of users in our network. If the encoding rate is set to $1/m$, then for any number of active users the sum of the rates is at most one and packets are always successfully received. On the other hand, if there is only one user transmitting per slot, then the shared communication medium is poorly utilized as reliable communication could occur at a rate m times higher.

We assume that users access the channel independently of each other with probability p, and characterize the rate which maximizes the *global throughput*, which is defined as the expected sum of the rates of the active users, where the expectation is taken with respect to users' access probabilities. We discuss some properties of the optimal solution, and show that the global throughput converges to one when the population size tends to infinity, for any value of p. This result is in striking contrast with the case where users are not able to adjust their encoding rate as a function of p and m. In the classic slotted ALOHA protocol, for instance, the global throughput decreases as the population size increases, because collisions become more and more frequent.

The second contribution of this paper is to characterize the throughput when users are selfish agents. We assume that users aim at maximizing their individual expected throughput and we explicitly compute a symmetric Bayesian Nash Equilibrium. We show that this equilibrium results in an inefficient resource allocation, so that the optimal global throughput is not achieved in this case. We recall that game theory has been applied to the study of random access systems in the past [2], [3], [4], [6], [8], [9], [11], [14]. The main difference between our work and those in the related literature is that we allow users to adjust the encoding rate as a function of m and p.

The rest of the paper is organized as follows. The next section formally defines the problem in a game theoretic setup. Section III characterizes the global attainable throughput, Section IV studies the scaling behaviour as the population size grows to infinity. Section V presents the solution in

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the presence of selfish agents and final remarks are made in Section VI.

II. PROBLEM FORMULATION

Consider m users, each of which can be in one of two states, active (1) and not active (0), independently of each other. We denote by $\theta_i \in \{0, 1\}$, the state of user i, and by $p \in (0, 1)$ and $1 - p$ the probabilities that node i is in state 1 and 0, respectively. Users transmit packets to a common receiver on a shared medium that has normalized capacity equal to 1. The number of users m and the probability p are supposed to be common knowledge, moreover each user knows the realization of its own state. Upon transmission, each user chooses the rate at which encoding its own packet. If the sum of the rates chosen by the active users exceeds the channel capacity, then communication between the active users and the receivers is not possible. Conversely, if the sum is below capacity, then the receiver can correctly decode all the transmitted packets. The rate (also referred to as *strategy*) chosen by user i is a mapping $r_i : \theta_i \to [0, \infty)$. The payoff function for user i (also referred to as *utility* function) is defined as follows:

$$u_i(r_i(0), r_{-i}(\theta_{-i})) = 0, \tag{1}$$

if user i is not active, and

$$u_i(r_i(1), r_{-i}(\theta_{-i})) = r_i(1) \quad {}_{\left\{\sum_{a\in A} r_a(1)\le 1\right\}}, \tag{2}$$

if user i active, where $A \subseteq \{1, \cdots, m\}$ denotes the set of active users, and $\{-i\}$ denotes the set of all users minus user i. We can see from (1) and (2) that the payoff function for user i equals the rate chosen upon transmission if the sum of the rates of the active users does not exceed the channel capacity, and it is equal to zero otherwise. The payoff function only depends on the rate chosen by the active users, thus we can assume $r_i(0) = 0$ for all i, and abbreviate without ambiguity $r_i(1) = r_i$. Furthermore, there is no loss of generality in assuming $r_i \in [0, 1]$ for all i, that is, we assume that users are *rational*.

Our first objective is to evaluate the rate that attains the maximum global throughput, which we define as the sum of the expected payoff functions, and where the expectation is taken with respect the joint statistic of the states. Formally, the maximum global throughput is defined as

$$T(p; m) \triangleq \max_{r_1,\cdots,r_m} \sum_{i=1}^{m} \mathbb{E}[u_i(r_i(\theta_i), r_{-i}(\theta_{-i}))], \tag{3}$$

where in (3) we emphasize the dependence on the transmission probability p and the population size m. The fact that each user is active with the same probability p has one important consequence. Although there might be multiple strategies achieving $T(p; m)$, there exists one for which $r_1 = r_2 = \cdots = r_m$. Thus, we consider the case $r_i = r$ for all i, and we denote by $r(p; m)$ the symmetric rate achieving $T(p; m)$, so

$$r(p; m) \triangleq \arg\max_{r} \sum_{i=1}^{m} \mathbb{E}[u_i(r_i(\theta_i), r_{-i}(\theta_{-i}))]. \tag{4}$$

In the sequel, we denote by

$$P_{m,k}(p) \triangleq \binom{m}{k} p^k (1 - p)^{m-k} \tag{5}$$

the probability of getting exactly k successes in m independent trials with success probability p, and we denoted by

$$B_{m,k}(p) \triangleq \sum_{i=0}^{k} P_{m,i}(p) \tag{6}$$

the probability of getting at most k successes.

III. THE MAXIMUM ATTAINABLE THROUGHPUT

The main result of this section is summarized in the following theorem.

Theorem III.1. *Let Π_m represent the partition of the unit interval into the set of m intervals*

$$(\hat{p}_{m,0}, \hat{p}_{m,1}], (\hat{p}_{m,1}, \hat{p}_{m,2}], \cdots, (\hat{p}_{m,m-1}, \hat{p}_{m,m}], \tag{7}$$

where $\hat{p}_{m,0} \triangleq 0$, $\hat{p}_{m,m} \triangleq 1$ and, for $0 < k < m$, $\hat{p}_{m,k}$ is defined as the unique solution in $(0, 1)$ to the following polynomial equation in p

$$\frac{1}{k+1} B_{m-1,k}(p) = \frac{1}{k} B_{m-1,k-1}(p). \tag{8}$$

Then, the following facts hold

1) $\hat{p}_{m,1} = 1/m$ *and* $\hat{p}_{m,k} \in (0, k/m)$ *for* $1 < k < m$.
2) *The maximum global throughput is given by*

$$T(p; m) = mp \sum_{k=1}^{m} \frac{1}{k} B_{m-1,k}(p) \quad {}_{\{p\in(\hat{p}_{m,k-1}, \hat{p}_{m,k}]\}}.$$

3) *The rate which attains the maximum global throughput is given by*

$$r(p; m) = \sum_{k=1}^{m} \frac{1}{k} \quad {}_{\{p\in(\hat{p}_{m,k-1}, \hat{p}_{m,k}]\}}. \tag{9}$$

4) *$T(p; m)$ is a continuous function of p; it is concave and strictly increasing in each interval of the partition Π_m.*

Proof: Omitted for brevity. ■

Inspection of (9) reveals that the optimal choice of rate is a piecewise constant function of p, and the value taken by the function depends on the transmission probability p. If p is in the k-th interval of the partition Π_m, then the optimal rate is $1/k$. The boundary values of the partition, denoted as $\hat{p}_{m,k}$, are given in semi-analytic form as solutions of (8), and closed form expressions are available only for some special values of m and k. Nevertheless, Theorem III.1 provides the upper bound $\hat{p}_{m,k} < k/m$. The structure of the solution is amenable to the following intuitive interpretation. When p is close to zero, the probability of a collision is low, so the optimal strategy consists of encoding at rate 1, i.e. at the maximum rate supported over the communication channel. If there is only one active user, the payoff of that user is the maximum attainable value 1. If more than one user is active, all the payoff functions are zero. The above strategy corresponds to

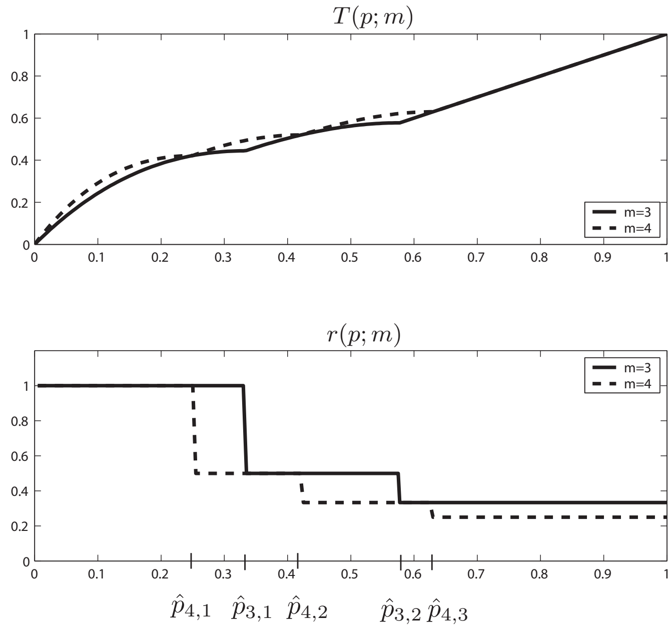

Fig. 1. The maximum expected sum-rate in a random access system with 2,3 and 4 users.

the scheme used in the slotted ALOHA protocol. Notice that since $\hat{p}_{m,1} = 1/m$, it turns out that the strategy used by the ALOHA protocol is optimal when the probability of being active is less that the inverse of the population size in the network. In this case, there is no advantage in exploiting the multi-user capability at the receiver. On the other hand, for $p > 1/m$, the global throughput of an ALOHA system is limited by packet collisions, which become more and more frequent. In this regime, the encoding rate has to decrease in order to accommodate the presence of other potential active users, which become more and more likely as p increases. For values of p next to unity, users are very likely to be simultaneously active. Thus, the optimal strategy is to transmit at rate $1/m$, so that each active user can have nonzero utility even if all users are active. Solving equation (8) for $k = m-1$, we obtain that $\hat{p}_{m,m-1} = m^{-1/(m-1)}$. So, assuming that all users are active is the optimal solution when the probability of being active is greater than $m^{-1/(m-1)}$.

Figure 1 shows the optimal throughput and the corresponding optimal rate function for the case of networks with three and four users. Observe that the optimal throughput is piecewise concave.

An important property stated in the above theorem is that $T(p;m)$ is a *strictly increasing* function of p. It is interesting to notice this property would not hold if users were not able to adjust their encoding rate as a function of p and m. In an ALOHA-type of system, for instance, the throughput decreases to zero as the transmission probability tends to 1, because of the increasing probability of collision (see Figure 2).

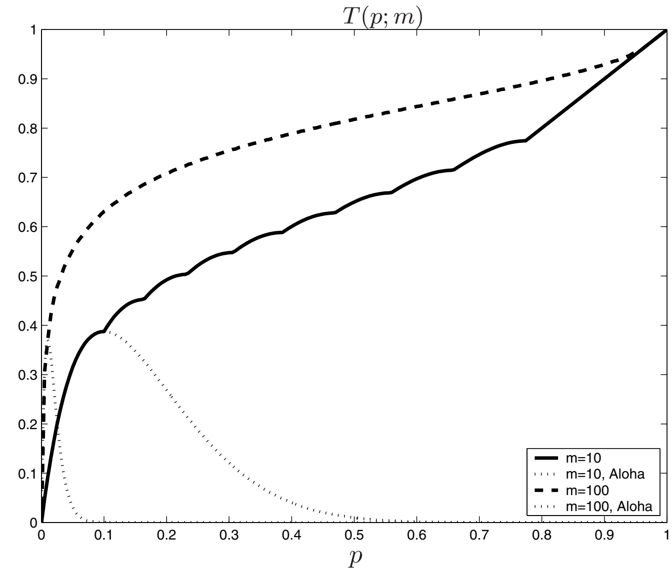

Fig. 2. Comparison between $T(p;m)$ and the throughput of standard slotted-ALOHA for different values of m.

IV. Throughput scaling for increasing values of m

In this section, we investigate how the throughput varies as we let the population size go to infinity. The main result of this section is now stated.

Theorem IV.1. *Let* $p \in (0,1)$. *Then,* $\lim_{m\to\infty} T(p;m) = 1$.

Proof: Omitted for brevity. ■

If we let m grow while keeping p constant, the law of large number implies that number of active users concentrates around mp, so that one would expect that the uncertainty about the number of active users decreases as the population size increases. This intuition is confirmed by Theorem IV.1, which states that the probability of collision tends to zero as m grows to infinity.

Note that the above result is in striking contrast with the throughput scaling with the population size in a slotted ALOHA system. For a fixed value of m, the throughput of slotted ALOHA increases for small p, it reaches a maximum, after which it decreases to zero as p tends to one. It is well known that the maximum is attained at $p = 1/m \to 0$ and, as we let m tends to infinity, the value of the maximum approaches e^{-1}. See Figure 2 for a comparison between $T(p;m)$ and the throughput of standard ALOHA for different values of m.

V. Throughput with selfish users

The optimum rate allocation discussed in the previous sections can be thought of as an optimum *cooperative* strategy, where users cooperate to maximize the the total system utility. In this section, instead, we suppose that users are selfish individuals whose goal is to maximize their individual expected utility function, where the expectation is taken with respect to the distribution of the state of the remaining users. Our goal is to compute non-cooperative strategies that simultaneously maximize users' individual expected payoff. While many equilibria can arise in this non-cooperative game, we restrict our attention to the case where users transmit at the same rate r for comparison with the results developed in the cooperative setup. More precisely, we are interested in characterizing a special type of equilibrium which is defined next.

Definition V.1. *A rate r is a symmetric Bayesian Nash equilibrium iff $r_{-i} = r$ and $p\mathbb{E}_{\theta_{-i}}[u_i(r, r_{-i}(\theta_{-i}))] \geq p\mathbb{E}_{\theta_{-i}}[u_i(r'_i, r_{-i}(\theta_{-i}))]$ $\forall r'_i \neq r$, $1 \leq i \leq m$.*

The above definition says that r is a symmetric Bayesian Nash equilibrium if a user maximizes its own expected utility function by encoding at rate r, given that the remaining users are also encoding at rate r. We abbreviate saying that r is user i's *best response* to $r_{-i} = r$, for all i. For notational simplicity, in defining the variables used in this section we do not indicate the dependence on population size, which is assumed to be fixed and equal to m. The main result of this section is now stated.

Theorem V.1. *Let $0 < k < m$, and $r_{-i} = 1/k$. Let Π'_k represent the partition of the unit interval into the set of k intervals*

$$[\tilde{p}_{k,k}, \tilde{p}_{k,k-1}), [\tilde{p}_{k,k-1}, \tilde{p}_{k,k-2}), \cdots, [\tilde{p}_{k,1}, \tilde{p}_{k,0}),$$

where $\tilde{p}_{k,k} \triangleq 0$, $\tilde{p}_{k,0} \triangleq 1$ and, for $0 < l < k$, $\tilde{p}_{k,l}$ is defined as the unique solution in $(0, 1)$ to the following polynomial equation in p

$$B_{m-1,k-l}(p) = 2B_{m-1,k-l-1}(p). \tag{10}$$

Then, the following facts hold

1) *$\tilde{p}_{k,1} = 1/m$ and $\tilde{p}_{k,l} \in (0, (l+1)(k-l)/(lm+k-2l)$ for $1 < l < m$.*
2) *The rate which maximizes user i's expected payoff is given by*

$$\tilde{r}_{i,k}(p) = \sum_{l=1}^{k} \frac{l}{k} \, {}_{\{p \in [\tilde{p}_{k,l}, \tilde{p}_{k,l-1})\}}, \tag{11}$$

3) *In particular, $r = 1/k$ is a symmetric Bayesian Nash equilibrium when $p \in [\tilde{p}_{k,1}, 1)$.*

Proof: Omitted for brevity. ∎

Theorem V.1 shows that user i's best response to $r_{-i} = 1/k$ can take k possible values, depending on the value of the transmission probability p. If p lies in the $(k-l+1)$-th interval of the partition Π'_k, then user i's best response is l/k and, in particular, user i's expected payoff is maximized by $r = 1/k$ when p is in the last interval of the partition. The boundary values of the partition Π'_k can be computed by solving the polynomial equation (10). While an analytic expression for $\hat{p}_{k,l}$ is available for certain values of l and k, in general we know that $\tilde{p}_{k,l} \in (0, (l+1)(k-l)(lm+k-2l))$.

An interesting question to ask is whether the symmetric Bayesian Nash equilibrium in Theorem V.1 can achieve the optimal throughput computed in Section III. Figure 3 shows the attainable throughput in a network of four selfish users as compared to the optimal throughput $T(p;4)$, and the corresponding rate allocations as a function of the transmission probability p. It can be noticed that not all points on the curve of $T(p;4)$ can be achieved, so that the symmetric Bayesian Nash equilibrium results in an inefficient resource allocation. As one can observe from the lower plot in the figure, the reason for this inefficiency is that $\tilde{p}_{3,1} > \hat{p}_{4,2}$, so that the selfish users transmit at rate $r = 1/2$ when $p \in (\hat{p}_{4,2}, \tilde{p}_{3,1})$, while they should decrease the rate to $1/3$ in order to achieve $T(p;4)$. Similarly, $\tilde{p}_{4,1} > \hat{p}_{4,3}$ and therefore the symmetric Bayesian Nash equilibrium results in an inefficient outcome when p is in the range $(\hat{p}_{4,3}, \tilde{p}_{4,1})$. The next proposition shows that for all $m > 2$ there is always a range of the parameter p for which the symmetric Bayesian Nash equilibrium rate is higher than the optimal value $r(p;m)$.

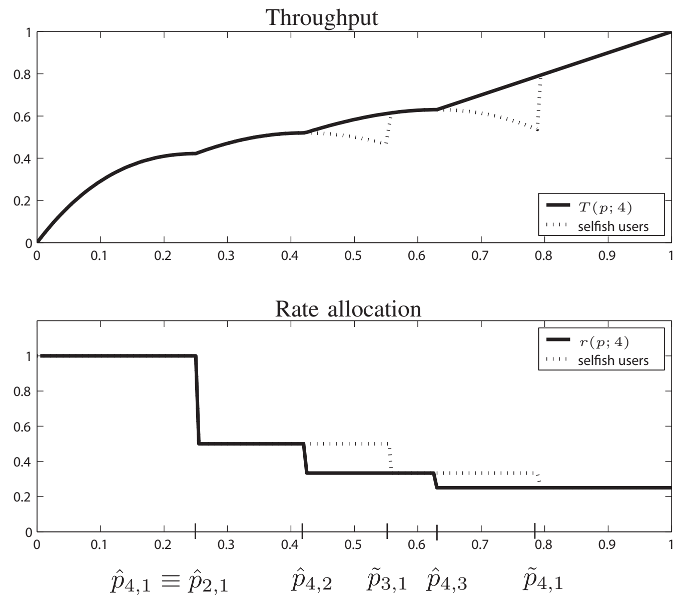

Fig. 3. Comparison between throughput and optimal rate with and without selfish users for a four-user network.

Proposition V.2. *Let $0 < k < m$. Then, $\tilde{p}_{k,1} \geq \hat{p}_{m,k-1}$, with equality iff $k = 1$ and $k = 2$.*

Proof: Omitted for brevity. ∎

Therefore, it immediately follows that

Corollary V.3. *The symmetric Bayesian Nash equilibrium in theorem V.1 is throughput inefficient for all $m > 2$.*

VI. Conclusion

This paper studied random access under the assumption that the receiver has multipacket capabilities. Users access the shared medium independently of each other with probability p and, upon transmission, they choose the rate at which packets are encoded. It was found that by optimization over the probability of packet collision, high throughput is attainable.

This result is in striking contrast to the behavior of random access systems where users cannot adjust the encoding rate. Finally, we considered the problem in a game theoretic set-up. Assuming that users are selfish, we characterized a symmetric encoding rate that maximizes a single user's expected rate, and showed that corresponding Nash equilibrium results in an inefficient resource allocation.

Finally, we want to point out some open research directions that we consider worth of exploration. In this paper, the transmission probability p and the number of users m play a pivotal role in setting the encoding rate, and these quantities are supposed to be known at the transmitters. In practice, the probability p is determined by quality of service requirements, but m usually has to be estimated at the transmitters. It would be interesting to investigate how the number of active users can be estimated and how estimation errors affect the choice of the encoding rate and the corresponding attainable throughput.

VII. Acknowledgment

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