Optimal Choice of Threshold in Two Level Processor Sharing

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Abstract

We analyze the Two Level Processor Sharing (TLPS) scheduling discipline with the hyper-exponential job size distribution and with the Poisson arrival process. TLPS is a convenient model to study the benefit of the file size based differentiation in TCP/IP networks. In the case of the hyper-exponential job size distribution with two phases, we find a closed form analytic expression for the expected sojourn time and an approximation for the optimal value of the threshold that minimizes the expected sojourn time. In the case of the hyper-exponential job size distribution with more than two phases, we derive a tight upper bound for the expected sojourn time conditioned on the job size. We show that when the variance of the job size distribution increases, the gain in system performance increases and the sensitivity to the choice of the threshold near its optimal value decreases.

1 Introduction

It has been known for a long time that a clever scheduling of tasks can significantly improve system performance. For instance, Shortest Remaining Processing Time (SRPT) scheduling discipline minimizes the expected sojourn time [15]. However, SRPT requires to keep track of all jobs in the system and also requires the knowledge of the remaining processing times. These requirements are often not feasible in applications. The examples of such applications are file size based differentiation in TCP/IP networks [3, 9] and Web server request differentiation [10, 11].

The Two Level Processor Sharing (TLPS) scheduling discipline [12] helps to overcome the above mentioned requirements. It uses the differentiation of jobs according to a threshold on the attained service and gives priority to the jobs with small sizes. A detail description of the TLPS discipline is presented in the ensuing section. Of course, TLPS provides a sub-optimal mechanism in comparison with SRPT. Nevertheless, as was shown in [1], when the job size distribution has a decreasing hazard rate, the performance of TLPS with appropriate choice of threshold is very close to optimal. It turns out that the distribution of file sizes in the Internet indeed has a decreasing hazard rate and often could be modeled with a heavy-tailed distributions. It is known, that the heavy-tailed distribution could be approximated with a hyper-exponential distribution with a significant number of phases [5, 8]. Also in [7], it was shown that the hyper-exponential distribution models well the file size distribution in the Internet. Therefore, in the present work we analyze the TLPS system with hyper-exponential job size distribution.

The paper organization and main results are as follows. In Section 2 we provide the model formulation, main definitions and equations. In Section 3 we study the TLPS discipline in the case of the hyper-exponential job size distribution with two phases. It is known that the Internet connections belong to two distinct classes with very different sizes of transfer. The

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first class is composed of short HTTP connections and P2P signaling connections. The second class corresponds to downloads (PDF files, MP3 files, MPEG files, etc.). This fact provides motivation to consider first the hyper-exponential job size distribution with two phases.

We find an analytical expression for the expected sojourn time in the TLPS system. Then, we present the approximation of the optimal threshold which minimizes the expected sojourn time. We show that the approximated value of the threshold tends to the optimal threshold when the second moment of the job size distribution function goes to infinity.

We show that the use of the TLPS scheduling discipline can lead to 45% gain in the expected sojourn time in comparison with the standard Processor Sharing. We also show that the system performance is not too sensitive to the choice of the threshold around its optimal value.

In Section 4 we analyze the TLPS discipline when the job size distribution is hyper-exponential with many phases. We provide an expression of the expected conditional sojourn time as the solution of a system of linear equations. Also we apply an iteration method to find the expression of the expected conditional sojourn time and using the resulting expression obtain an explicit and tight upper bound for the expected sojourn time function. In the experimental results we show that the relative error of the latter upper bound with respect to the expected sojourn time function is 6-7%.

We study the properties of the expected sojourn time function when the parameters of the job size distribution function are selected in a such a way that with the increasing number of phases the variance increases. We show numerically that with the increasing number of phases the relative error of the found upper bound decreases. We also show that when the variance of the job size distribution increases the gain in system performance increases and the sensitivity of the system to the selection of the optimal threshold value decreases.

We put some technical proofs in the Appendix.

2 Model description

2.1 Main definitions

We study the Two Level Processor Sharing (TLPS) scheduling discipline with the hyper-exponential job size distribution. Let us describe the model in detail.

Jobs arrive to the system according to a Poisson process with rate λ. We measure the job size in time units. Specifically, as the job size we define the time which would be spent by the server to treat the job if there were no other jobs in the system.

Let θ be a given threshold. The jobs in the system that attained a service less than θ are assigned to the high priority queue. If in addition there are jobs with attained service greater than θ, such a job is separated into two parts. The first part of size θ is assigned to the high priority queue and the second part of size $x - \theta$ waits in the lower priority queue. The low priority queue is served when the high priority queue is empty. Both queues are served according to the Processor Sharing (PS) discipline.

Let us denote the job size distribution by $F(x)$. By $\overline{F}(x) = 1 - F(x)$ we denote the complementary distribution function. The mean job size is given by $m = \int_0^\infty x dF(x)$ and the system load is $\rho = \lambda m$. We assume that the system is stable ($\rho < 1$) and is in steady state.

It is known that many important probability distributions associated with network traffic are heavy-tailed. In particular, the file size distribution in the Internet is heavy-tailed.

A distribution function has a heavy tail if $e^{\epsilon x}(1 - F(x)) \to \infty$ as $x \to \infty$, $\forall \epsilon > 0$. The heavy-tailed distributions are not only important and prevalent, but also difficult to analyze. Often it is helpful to have the Laplace transform of the job size distribution. However, there is evidently no convenient analytic expression for the Laplace transforms of the Pareto and Weibull distributions, the most common examples of heavy-tailed distributions. In [5, 8] it was shown that it is possible to approximate heavy-tailed distributions by hyper-exponential distribution with a significant number of phases.

A hyper-exponential distribution $F_N(x)$ is a convex combination of N exponents, $1 \le N \le \infty$, namely,

$$F_N(x) = 1 - \sum_{i=1}^{N} p_i e^{-\mu_i x}, \quad \mu_i > 0,\ p_i \ge 0, \quad i = 1, ..., N, \quad \text{and} \quad \sum_{i=1}^{N} p_i = 1. \qquad (1)$$

In particular, we can construct a sequence of hyper-exponential distributions such that it converges to a heavy-tailed distribution [5]. For instance, if we select

$$p_i = \frac{\nu}{i^{\gamma_1}}, \quad \mu_i = \frac{\eta}{i^{\gamma_2}}, \quad i = 1, ..., N,$$
$$\gamma_1 > 1, \quad \frac{\gamma_1 - 1}{2} < \gamma_2 < \gamma_1 - 1,$$

where $\nu = 1/\sum_{i=1,..,N} i^{-\gamma_1}$, $\quad \eta = \nu/m \sum_{i=1,...,N} i^{\gamma_2 - \gamma_1}$, then the first moment of the job size distribution is finite, but the second moment is infinite when $N \to \infty$. Namely, the first and the second moments m and d for the hyper-exponential distribution are given by:

$$m = \int_0^\infty x\, dF(x) = \sum_{i=1}^{N} \frac{p_i}{\mu_i}, \quad d = \int_0^\infty x^2\, dF(x) = 2 \sum_{i=1}^{N} \frac{p_i}{\mu_i^2}. \qquad (2)$$

Let us denote

$$\overline{F_\theta^i} = p_i e^{-\mu_i \theta}, \quad i = 1, ..., N. \qquad (3)$$

We note that $\sum_i \overline{F_\theta^i} = \overline{F}(\theta)$. The hyper-exponential distribution has a simple Laplace transform:

$$L_{\overline{F(x)}}(s) = \sum_{i=1}^{N} \frac{p_i \mu_i}{s + \mu_i}.$$

We would like to note that the hyper-exponential distribution has a decreasing hazard rate. In [1] it was shown, that when a job size distribution has a decreasing hazard rate, then with the selection of the threshold the expected sojourn time of the TLPS system could be reduced in comparison to standard PS system.

Thus, in our work we use hyper-exponential distributions to represent job size distribution functions. In particular, the application of the hyper-exponential job size distribution with two phases is motivated by the fact that in the Internet connections belong to two distinct classes with very different sizes of transfer. The first class is composed of short HTTP connections and P2P signaling connections. The second class corresponds to downloads (PDF files, MP3 files, MPEG files, etc.). So, in the first part of our paper we look at the case of the hyper-exponential job size distribution with two phases and in the second part of the paper we study the case of more than two phases.

2.2 The expected sojourn time in TLPS system

Let us denote by $\overline{T}^{TLPS}(x)$ the expected conditional sojourn time in the TLPS system for a job of size x. Of course, $\overline{T}^{TLPS}(x)$ also depends on θ, but for expected conditional sojourn time we only emphasize the dependence on the job size. On the other hand, we denote by $\overline{T}(\theta)$ the overall expected sojourn time in the TLPS system. Here we emphasize the dependence on θ as later we shall optimize the overall expected sojourn time with respect to the threshold value.

To calculate the expected sojourn time in the TLPS system we need to calculate the time spent by a job of size x in the first high priority queue and in the second low priority queue. For the jobs with size $x \le \theta$ the system will behave as the standard PS system where the service time distribution is truncated at θ. Let us denote by

$$\overline{X_\theta^n} = \int_0^\theta n y^{n-1} \overline{F}(y) dy \tag{4}$$

the n-th moment of the distribution truncated at θ. In the following sections we will need

$$\overline{X_\theta^1} = m - \sum_{i=1}^N \frac{\overline{F_\theta^i}}{\mu_i}, \quad \overline{X_\theta^2} = 2\sum_{i=1}^N \frac{p_i}{\mu_i^2} - 2\theta\left(m - \sum_{i=1}^N \frac{\overline{F_\theta^i}}{\mu_i}\right) - 2\sum_{i=1}^N \frac{\overline{F_\theta^i}}{\mu_i^2}. \tag{5}$$

The utilization factor for the truncated distribution is

$$\rho_\theta = \lambda \overline{X_\theta^1} = \rho - \lambda \sum_{i=1}^N \frac{\overline{F_\theta^i}}{\mu_i}. \tag{6}$$

Then, the expected conditional response time is given by

$$\overline{T}^{TLPS}(x) = \begin{cases} \dfrac{x}{1-\rho_\theta}, & x \in [0, \theta], \\ \dfrac{\overline{W}(\theta) + \theta + \alpha(x-\theta)}{1-\rho_\theta}, & x \in (\theta, \infty). \end{cases}$$

According to [12], here $(\overline{W}(\theta)+\theta)/(1-\rho_\theta)$ expresses the time needed to reach the low priority queue. This time consists of the time $\theta/(1-\rho_\theta)$ spent in the high priority queue, where the flow is served up to the threshold θ, plus the time $\overline{W}(\theta)/(1-\rho_\theta)$ which is spent waiting for the high priority queue to empty. Here $\overline{W}(\theta) = \lambda \overline{X_\theta^2}/(2(1-\rho_\theta))$.

The remaining term $\alpha(x-\theta)/(1-\rho_\theta)$ is the time spent in the low priority queue. To find $\alpha(x)$ we can use the interpretation of the lower priority queue as a PS system with batch arrivals [4, 14]. As was shown in [12], $\alpha'(x) = d\alpha/dx$ is the solution of the following integral equation

$$\alpha'(x) = \lambda \overline{n} \int_0^\infty \alpha'(y) \overline{B}(x+y) dy + \lambda \overline{n} \int_0^x \alpha'(y) \overline{B}(x-y) dy + b\overline{B}(x) + 1. \tag{7}$$

Here $\overline{n} = \overline{F}(\theta)/(1-\rho_\theta)$ is the average batch size, $\overline{B}(x) = \overline{F}(\theta+x)/\overline{F}(\theta)$ is the complementary truncated distribution and $b = b(\theta) = 2\lambda \overline{F}(\theta)(\overline{W}(\theta)+\theta)/(1-\rho_\theta)$ is the average number of jobs that arrive to the low priority queue in addition to the tagged job.

The expected sojourn time in the system is given by the following equations:

$$\overline{T}(\theta) = \int_0^\infty \overline{T}^{TLPS}(x)dF(x),$$

$$\overline{T}(\theta) = \frac{\overline{X_\theta^1} + \overline{W}(\theta)\overline{F}(\theta)}{1-\rho_\theta} + \frac{1}{1-\rho_\theta}\overline{T}^{BPS}(\theta), \tag{8}$$

$$\overline{T}^{BPS}(\theta) = \int_\theta^\infty \alpha(x-\theta)dF(x) = \int_0^\infty \alpha'(x)\overline{F}(x+\theta)dx. \tag{9}$$

3 Hyper-exponential job size distribution with two phases

3.1 Notations

In the first part of our work we consider the hyper-exponential job size distribution with two phases. Namely, according to (1) the cumulative distribution function $F(x)$ for $N=2$ is given by

$$F(x) = 1 - p_1 e^{-\mu_1 x} - p_2 e^{-\mu_2 x},$$

where $p_1 + p_2 = 1$ and $p_1, p_2 > 0$.

The mean job size m, the second moment d, the parameters $\overline{F_\theta^i}$, $\overline{X_\theta^1}$, $\overline{X_\theta^2}$ and ρ_θ are defined as in Section 2.1 and Section 2.2 by formulas (2),(3),(5), (6) with $N=2$.

We note that the system has four free parameters. In particular, if we fix μ_1, $\epsilon = \mu_2/\mu_1$, m, and ρ, the other parameters μ_2, p_1, p_2 and λ will be functions of the former parameters.

3.2 Explicit form for the expected sojourn time

To find $\overline{T}^{TLPS}(x)$ we need to solve the integral equation (7). To solve (7) we use the Laplace transform based method described in [6].

Theorem 1. *The expected sojourn time in the TLPS system with the hyper-exponential job size distribution with two phases is given by*

$$\overline{T}(\theta) = \frac{\overline{X_\theta^1} + \overline{W}(\theta)\overline{F}(\theta)}{1-\rho_\theta} + \frac{m - \overline{X_\theta^1}}{1-\rho} + \frac{b(\theta)\left(\mu_1\mu_2(m-\overline{X_\theta^1})^2 + \delta_\rho(\theta)\overline{F}^2(\theta)\right)}{2(1-\rho)\overline{F}(\theta)\left(\mu_1+\mu_2-\gamma(\theta)\overline{F}(\theta)\right)}, \tag{10}$$

where $\delta_\rho(\theta) = 1 - \gamma(\theta)(m - \overline{X_\theta^1}) = (1-\rho)/(1-\rho_\theta)$ *and* $\gamma(\theta) = \lambda/(1-\rho_\theta)$.

Proof. We can rewrite integral equation (7) in the following way

$$\alpha'(x) = \gamma(\theta)\int_0^\infty \alpha'(y)\overline{F}(x+y+\theta)dy + \gamma(\theta)\int_0^x \alpha'(y)\overline{F}(x-y+\theta)dy + b(\theta)\overline{B}(x) + 1,$$

$$\alpha'(x) = \gamma(\theta)\sum_{i=1,2}\overline{F_\theta^i}e^{-\mu_i x}\int_0^\infty \alpha'(y)e^{-\mu_i y}dy + \gamma(\theta)\int_0^x \alpha'(y)\overline{F}(x-y+\theta)dy + b(\theta)\overline{B}(x) + 1.$$

We note that in the latter equation $\int_0^\infty \alpha'(y)e^{-\mu_i y}dy$, $i=1,2$ are the Laplace transforms of $\alpha'(y)$ evaluated at $\mu_i, i=1,2$. Denote

$$L_i = \int_0^\infty \alpha'(y)e^{-\mu_i y}dy, \qquad i=1,2.$$

Then, we have

$$\alpha'(x) = \gamma(\theta) \sum_{i=1,2} \overline{F^i_\theta} L_i e^{-\mu_i x} + \gamma(\theta) \int_0^x \alpha'(y) \overline{F}(x-y+\theta) dy + b(\theta)\overline{B}(x) + 1.$$

Now taking the Laplace transform of the above equation and using the convolution property, we get

$$L_{\alpha'}(s) = \gamma(\theta) \sum_{i=1,2} \frac{\overline{F^i_\theta} L_i}{s+\mu_i} + \gamma(\theta) \sum_{i=1,2} \frac{\overline{F^i_\theta} L_{\alpha'}(s)}{s+\mu_i} + \frac{b(\theta)}{\overline{F}(\theta)} \sum_{i=1,2} \frac{\overline{F^i_\theta}}{s+\mu_i} + \frac{1}{s},$$

$$\Longrightarrow \quad L_{\alpha'}(s) \left(1 - \gamma(\theta) \sum_{i=1,2} \frac{\overline{F^i_\theta}}{s+\mu_i} \right) = \gamma(\theta) \sum_{i=1,2} \frac{\overline{F^i_\theta} L_i}{s+\mu_i} + \frac{b(\theta)}{\overline{F}(\theta)} \sum_{i=1,2} \frac{\overline{F^i_\theta}}{s+\mu_i} + \frac{1}{s}.$$

Here $L_{\alpha'}(s) = \int_0^\infty \alpha'(x) e^{-sx} dx$ is the Laplace transform of $\alpha'(x)$. Let us note that $L_{\alpha'}(\mu_i) = L_i, \; i = 1,2$. Then, if we substitute into the above equation $s = \mu_1$ and $s = \mu_2$, we can get L_1 and L_2 as a solution of the linear system

$$L_1 = \frac{1}{\left(\mu_1 + \mu_2 - \gamma(\theta)\overline{F}(\theta)\right) \delta_\rho(\theta)} \left(\frac{b(\theta)}{2\overline{F}(\theta)} \left(\mu_2 (m - \overline{X^1_\theta}) + \delta_\rho(\theta) \overline{F}(\theta) \right) \right) + \frac{1}{\mu_1 \delta_\rho(\theta)},$$

$$L_2 = \frac{1}{\left(\mu_1 + \mu_2 - \gamma(\theta)\overline{F}(\theta)\right) \delta_\rho(\theta)} \left(\frac{b(\theta)}{2\overline{F}(\theta)} \left(\mu_1 (m - \overline{X^1_\theta}) + \delta_\rho(\theta) \overline{F}(\theta) \right) \right) + \frac{1}{\mu_2 \delta_\rho(\theta)}.$$

Next we need to calculate $\overline{T}^{BPS}(\theta)$.

$$\overline{T}^{BPS}(\theta) = \int_0^\infty \alpha'(x) \overline{F}(x+\theta) dx = \int_0^\infty \alpha'(x) \sum_{i=1,2} \overline{F^i_\theta} e^{-\mu_i x} dx = \sum_{i=1,2} \overline{F^i_\theta} L_i,$$

$$\overline{T}^{BPS}(\theta) = \frac{1-\rho_\theta}{1-\rho} \left(m - \overline{X^1_\theta} + \frac{b(\theta) \left(\mu_1 \mu_2 (m - \overline{X^1_\theta})^2 + \delta_\rho(\theta) \overline{F}^{\,2}(\theta) \right)}{2\overline{F}(\theta) \left(\mu_1 + \mu_2 - \gamma(\theta) \overline{F}(\theta) \right)} \right).$$

Finally, by (8) we have (10). □

3.3 Optimal threshold approximation

We are interested in the minimization of the expected sojourn time $\overline{T}(\theta)$ with respect to θ. Of course, one can differentiate the exact analytic expression provided in Theorem 1 and set the result of the differentiation to zero. However, this will give a transcendental equation for the optimal value of the threshold.

In order to find an approximate solution of $\overline{T}'(\theta) = d\overline{T}(\theta)/d\theta = 0$, we shall approximate the derivative $\overline{T}'(\theta)$ by some function $\widetilde{\overline{T}}'(\theta)$ and obtain a solution to $\widetilde{\overline{T}}'(\tilde{\theta}_{opt}) = 0$.

Since in the Internet connections belong to two distinct classes with very different sizes of transfer (see Section 2.1), then to find the approximation of $\overline{T}'(\theta)$ we consider a particular case when $\mu_2 << \mu_1$. Let us introduce a small parameter ϵ such that

$$\mu_2 = \epsilon \mu_1, \quad \epsilon \to 0, \quad p_1 = 1 - \frac{\epsilon \; (m\mu_1 - 1)}{1-\epsilon}, \quad p_2 = \frac{\epsilon \; (m\mu_1 - 1)}{1-\epsilon}.$$

We note that when $\epsilon \to 0$ the second moment of the job size distribution goes to infinity.

We then verify that $\tilde{\theta}_{opt}$ indeed converges to the minimum of $\overline{T}(\theta)$ when $\epsilon \to 0$.

Lemma 2. *The following inequality holds:* $\mu_1\rho > \lambda$.

Proof. Since $p_1 > 0$ and $p_2 > 0$, we have the following inequality $m\mu_1 > 1$. Then, $m > \frac{1}{\mu_1}$. Taking into account that $\lambda m = \rho$ we get $\frac{\rho}{\lambda} > \frac{1}{\mu_1}$. Consequently, we have that $\mu_1\rho > \lambda$. □

Proposition 3. *The derivative of* $\overline{T}(\theta)$ *can be approximated by the following function:*

$$\tilde{\overline{T}}'(\theta) = -e^{-\mu_1\theta}\mu_1 c_1 + e^{-\mu_2\theta}\mu_2 c_2,$$

where

$$c_1 = \frac{\lambda(m\mu_1 - 1)}{\mu_1(\mu_1 - \lambda)(1-\rho)}, \quad c_2 = \frac{\lambda(m\mu_1 - 1)}{(\mu_1 - \lambda)^2}. \tag{11}$$

Namely,

$$\overline{T}'(\theta) - \tilde{\overline{T}}'(\theta) = O(\mu_2/\mu_1).$$

Proof. Using the analytical expression for both $\overline{T}'(\theta)$ and $\tilde{\overline{T}}'(\theta)$, we get the Taylor series for $\overline{T}'(\theta) - \tilde{\overline{T}}'(\theta)$ with respect to ϵ, which shows that indeed

$$\overline{T}'(\theta) - \tilde{\overline{T}}'(\theta) = O(\epsilon).$$

□

Thus we have found an approximation of the derivative of $\overline{T}(\theta)$. Now we can find an approximation of the optimal threshold by solving the equation $\tilde{\overline{T}}'(\theta) = 0$.

Theorem 4. *Let* θ_{opt} *denote the optimal value of the threshold. Namely,* $\theta_{opt} = \arg\min \overline{T}(\theta)$. *The value* $\tilde{\theta}_{opt}$ *given by*

$$\tilde{\theta}_{opt} = \frac{1}{\mu_1 - \mu_2} \ln\left(\frac{(\mu_1 - \lambda)}{\mu_2(1-\rho)}\right)$$

approximates θ_{opt} *so that* $\overline{T}'(\tilde{\theta}_{opt}) = \mathrm{o}(\mu_2/\mu_1)$.

Proof. Solving the equation

$$\tilde{\overline{T}}'(\theta) = 0,$$

we get an analytic expression for the approximation of the optimal threshold:

$$\tilde{\theta}_{opt} = -\frac{1}{\mu_1(1-\epsilon)} \ln\left(\epsilon\frac{\mu_1(1-\rho)}{(\mu_1 - \lambda)}\right) = \frac{1}{\mu_1 - \mu_2} \ln\left(\frac{(\mu_1 - \lambda)}{\mu_2(1-\rho)}\right).$$

Let us show that the above threshold approximation is greater than zero. We have to show that $\frac{(\mu_1-\lambda)}{\mu_2(1-\rho)} > 1$. Since $\mu_1 > \mu_2$ and $\mu_1\rho > \lambda$ (see Lemma 2), we have

$$\begin{aligned}
& \mu_1 > \mu_2 \\
\Longrightarrow \quad & \mu_1(1-\rho) > \mu_2(1-\rho) \\
\Longrightarrow \quad & \lambda < \mu_1\rho < \mu_1 - \mu_2(1-\rho) \\
\Longrightarrow \quad & (\mu_1 - \lambda) > \mu_2(1-\rho).
\end{aligned}$$

Expanding $\overline{T}'(\widetilde{\theta}_{opt})$ as a power series with respect to ϵ gives:

$$\overline{T}'(\widetilde{\theta}_{opt}) = \epsilon^2(const_0 + const_1 \ln \epsilon + const_2 \ln^2 \epsilon),$$

where $const_i,\ i = 1, 2$ are some constant values[2] with respect to ϵ. Thus,

$$\overline{T}'(\widetilde{\theta}_{opt}) = o(\epsilon) = o(\mu_2/\mu_1),$$

which completes the proof. □

In the next proposition we characterize the limiting behavior of $\overline{T}(\theta_{opt})$ and $\overline{T}(\widetilde{\theta}_{opt})$ as $\epsilon \to 0$. In particular, we show that $\overline{T}(\widetilde{\theta}_{opt})$ tends to the exact minimum of $\overline{T}(\theta)$ when $\epsilon \to 0$.

Proposition 5.

$$\lim_{\epsilon \to 0} \overline{T}(\theta_{opt}) = \lim_{\epsilon \to 0} \overline{T}(\widetilde{\theta}_{opt}) = \frac{m}{1-\rho} - c_1,$$

where c_1 is given by (11).

Proof. We find the following limit, when $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \overline{T}(\theta) = \frac{m}{1-\rho} - \frac{\lambda(m\mu_1 - 1)}{\mu_1(\mu_1 - \lambda)(1-\rho)} + \frac{\lambda(m\mu_1 - 1)e^{-\mu_1\theta}}{\mu_1(\mu_1 - \lambda)(1-\rho)},$$
$$\lim_{\epsilon \to 0} \overline{T}(\theta) = \frac{m}{1-\rho} - c_1 + c_1 e^{-\mu_1\theta},$$

where c_1 is given by (11). Since the function $\lim_{\epsilon \to 0} \overline{T}(\theta)$ is a decreasing function, the optimal threshold for it is $\theta_{opt} = \infty$. Thus,

$$\lim_{\epsilon \to 0} \overline{T}(\theta_{opt}) = \lim_{\theta \to \infty} \lim_{\epsilon \to 0} \overline{T}(\theta) = \frac{m}{1-\rho} - c_1.$$

On the other hand, we obtain

$$\lim_{\epsilon \to 0} \overline{T}(\widetilde{\theta}_{opt}) = \frac{m}{1-\rho} - c_1,$$

which proves the proposition. □

3.4 Experimental results

In Figure 1-2 we show the plots for the following parameters: $\rho = 10/11$ (default value), $m = 20/11,\ \mu_1 = 1,\ \mu_2 = 1/10,$ so $\lambda = 1/2$ and $\epsilon = \mu_2/\mu_1 = 1/10$. Then, $p_1 = 10/11$ and $p_2 = 1/11$.

In Figure 1 we plot $\overline{T}(\theta)$, $\overline{T}^{PS}$ and $\overline{T}(\widetilde{\theta}_{opt})$. We note, that the expected sojourn time in the standard PS system $\overline{T}^{PS}$ is equal to $\overline{T}(0)$. We observe that $\overline{T}(\widetilde{\theta}_{opt})$ corresponds well to the optimum even though $\epsilon = 1/10$ is not too small.

Let us now study the gain that we obtain using TLPS, by setting $\theta = \widetilde{\theta}_{opt}$, in comparison with the standard PS. To this end, we plot the ratio $g(\rho) = \frac{\overline{T}^{PS} - \overline{T}(\widetilde{\theta}_{opt})}{\overline{T}^{PS}}$ in Figure 2. The gain in the system performance with TLPS in comparison with PS strongly depends on ρ, the load of the system. One can see, that the gain of the TLPS system with respect to the standard PS system goes up to 45% when the load of the system increases.

[2]The expressions for the constants $const_i$ are cumbersome and can be found using Maple command “series”.

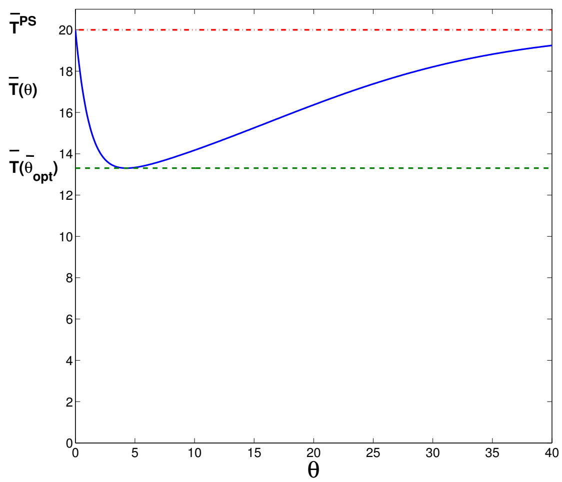

Figure 1: $\overline{T}(\theta)$ - solid line, $\overline{T}^{PS}(\theta)$ - dash dot line, $\overline{T}(\widetilde{\theta}_{opt})$ - dash line

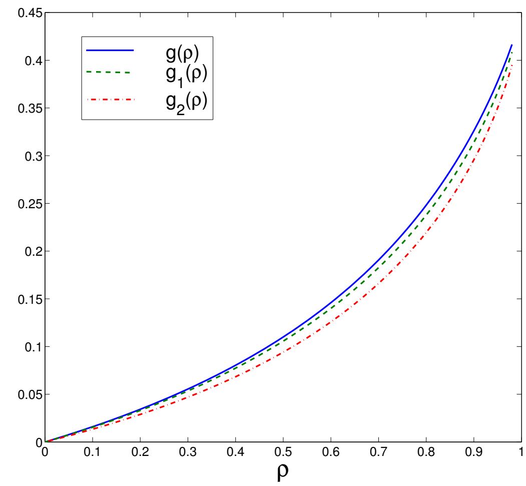

Figure 2: $g(\rho)$ - solid line, $g_1(\rho)$ - dash line, $g_2(\rho)$ - dash dot line

To study the sensitivity of the TLPS system with respect to θ, we find the gain of the TLPS system with respect to the standard PS system, we plot in Figure 2 $g_1(\rho) = \frac{\overline{T}^{PS} - \overline{T}(\frac{3}{2}\widetilde{\theta}_{opt})}{\overline{T}^{PS}}$ and $g_2(\rho) = \frac{\overline{T}^{PS} - \overline{T}(\frac{1}{2}\widetilde{\theta}_{opt})}{\overline{T}^{PS}}$. Thus, even with the 50% error of the $\widetilde{\theta}_{opt}$ value, the system performance is close to optimal.

One can see that it is beneficial to use TLPS instead of PS in the case of heavy and moderately heavy loads. We also observe that the optimal TLPS system is not too sensitive to the choice of the threshold near its optimal value, when the job size distribution is hyper-exponential with two phases. Nevertheless, it is better to choose larger rather than smaller values of the threshold.

4 Hyper-exponential job size distribution with more than two phases

4.1 Notations

In the second part of the presented work we analyze the TLPS discipline with the hyper-exponential job size distribution with more than two phases. As was shown in [5, 7, 8], the hyper-exponential distribution with a significant number of phases models well the file size distribution in the Internet. Thus, in this section as the job size distribution we take the hyper-exponential function with many phases. Namely, according to (1),

$$F(x) = 1 - \sum_{i=1}^{N} p_i\, e^{-\mu_i\, x}, \quad \sum_{i=1}^{N} p_i = 1, \;\; \mu_i > 0, \;\; p_i \geq 0, \;\; i = 1, ..., N, \;\; 1 < N \leq \infty.$$

In the following we shall write simply $\sum_i$ instead of $\sum_{i=1}^{N}$.
The mean job size m, the second moment d, the parameters $\overline{F_\theta^i}$, $\overline{X_\theta^1}$, $\overline{X_\theta^2}$ and ρ_θ are defined as in Section 2.1 and Section 2.2 by formulas (2),(3),(5), (6) for any $1 \leq N \leq \infty$. The formulas

presented in Section 2.2 can still be used to calculate $b(\theta)$, $\overline{B}(x)$, $\overline{W}(\theta)$, $\gamma(\theta)$, $\delta_\rho(\theta)$, $\overline{T}^{TLPS}(x)$, $\overline{T}(\theta)$.

We shall also need the following operator notations:

$$\begin{aligned} \Phi_1(\beta(x)) &= \gamma(\theta)\int_0^\infty \beta(y)\overline{F}(x+y+\theta)dy + \gamma(\theta)\int_0^x \beta(y)\overline{F}(x-y+\theta)dy, && (12)\\ \Phi_2(\beta(x)) &= \int_0^\infty \beta(y)\overline{F}(y+\theta)dy && (13) \end{aligned}$$

for any function $\beta(x)$. In particular, for some given constant c, we have

$$\begin{aligned} \Phi_1(c) &= c\,\gamma(\theta)(m-\overline{X^1_\theta}) = c\,q, && (14)\\ \Phi_2(c) &= c\,(m-\overline{X^1_\theta}), && (15) \end{aligned}$$

where

$$q = \gamma(\theta)(m-\overline{X^1_\theta}) = \frac{\lambda(m-\overline{X^1_\theta})}{1-\rho_\theta} = \frac{\rho-\rho_\theta}{1-\rho_\theta} < 1. \tag{16}$$

The integral equation (7) can now be rewritten in the form

$$\alpha'(x) = \Phi_1(\alpha'(y)) + \frac{b(\theta)}{\overline{F}(\theta)}\overline{F}(x+\theta) + 1. \tag{17}$$

and equation (9) for $\overline{T}^{BPS}(\theta)$ takes the form

$$\overline{T}^{BPS}(\theta) = \Phi_2(\alpha'(x)). \tag{18}$$

4.2 Linear system based solution

Similarly to the first part of the proof of Theorem 1, we obtain the following proposition.

Proposition 6.

$$\overline{T}^{BPS}(\theta) = \sum_i \overline{F^i_\theta} L_i,$$

with

$$L_i = L_i^* + \frac{1}{\delta_\rho(\theta)\mu_i},$$

where the L_i^* are the solution of the linear system

$$L_p^*\left(1-\gamma(\theta)\sum_i \frac{\overline{F^i_\theta}}{\lambda_p+\mu_i}\right) = \gamma(\theta)\sum_i \frac{\overline{F^i_\theta}L_i^*}{\lambda_p+\mu_i} + \frac{b(\theta)}{\overline{F}(\theta)}\sum_i \frac{\overline{F^i_\theta}}{\lambda_p+\mu_i}, \quad p=1,...,N. \tag{19}$$

Unfortunately, the system (19) does not seem to have a tractable finite form analytic solution. Therefore, in the ensuing subsections we proposed an alternative solution based on an operator series and construct a tight upper bound.

4.3 Operator series form for the expected sojourn time

Since the operator Φ_1 is a contraction [3, 4], we can iterate equation (17) starting from some initial point α'_0. The initial point could be simply a constant. As shown in [3, 4] the iterations will converge to the unique solution of (17). Specifically, we make iterations in the following way:

$$\alpha'_{n+1}(x) = \Phi_1(\alpha'_n(x)) + \frac{b(\theta)}{\overline{F}(\theta)}\overline{F}(x+\theta) + 1, \quad n = 0, 1, 2, ... \tag{20}$$

At every iteration step we construct the following approximation of $\overline{T}^{BPS}(\theta)$ according to (18):

$$\overline{T}^{BPS}_{n+1}(\theta) = \Phi_2(\alpha'_{n+1}(x)). \tag{21}$$

Using (20) and (21), we can construct the operator series expression for the expected sojourn time in the TLPS system.

Theorem 7. *The expected sojourn time $\overline{T}(\theta)$ in the TLPS system with the hyper-exponential job size distribution is given by*

$$\overline{T}(\theta) = \frac{\overline{X^1_\theta} + \overline{W}(\theta)\overline{F}(\theta)}{1-\rho_\theta} + \frac{m - \overline{X^1_\theta}}{1-\rho} + \frac{b(\theta)}{\overline{F}(\theta)(1-\rho_\theta)}\left(\sum_{i=0}^{\infty}\Phi_2\left(\Phi_1^i(\overline{F}(x+\theta))\right)\right). \tag{22}$$

Proof. From (20) we have

$$\alpha'_n = q^n\alpha'_0 + \sum_{i=1}^{n-1} q^i + \frac{b(\theta)}{\overline{F}(\theta)}\sum_{i=1}^{n-1}\Phi_1^i(\overline{F}(x+\theta)) + \frac{b(\theta)}{\overline{F}(\theta)}\overline{F}(x+\theta) + 1,$$

and then from (21) and (14) it follows, that

$$\overline{T}^{BPS}_n(\theta) = (m - \overline{X^1_\theta})\left(q^n\alpha'_0 + \sum_{i=0}^{n-1} q^i\right) + \frac{b(\theta)}{\overline{F}(\theta)}\left(\Phi_2\left(\sum_{i=0}^{n-1}\Phi_1^i(\overline{F}(x+\theta))\right)\right).$$

Using the facts (see (16)):

1. $q < \rho < 1 \Longrightarrow q^n \to 0$ as $n \to \infty$,
2. $\sum_{i=0}^{\infty} q^i = \frac{1}{1-q} = \frac{1-\rho_\theta}{1-\rho}$,

we conclude that

$$\overline{T}^{BPS}(\theta) = \lim_{n\to\infty}\overline{T}^{BPS}_n(\theta) = (m - \overline{X^1_\theta})\left(\frac{1-\rho_\theta}{1-\rho}\right) + \frac{b(\theta)}{\overline{F}(\theta)}\left(\sum_{i=0}^{\infty}\Phi_2\left(\Phi_1^i(\overline{F}(x+\theta))\right)\right).$$

Finally, using (8) we obtain (22). □

The resulting formula (22) is difficult to analyze and does not have a clear analytic expression. Using this result in the next subsection we find an approximation,which is also an upper bound, of the expected sojourn time function in a more explicit form.

4.4 Upper bound for the expected sojourn time

Let us start with auxiliary results.

Lemma 8. *For any function $\beta(x) \geq 0$ with $\beta_j = \int_0^\infty \beta(x) e^{-\mu_i x} dx$,*

$$\text{if} \quad \frac{d(\beta_j \mu_j)}{d\mu_j} \geq 0, \quad j = 1, ..., N \quad \text{it follows, that} \quad \Phi_2\left(\Phi_1(\beta(x))\right) \leq q\Phi_2\left(\beta(x)\right).$$

Proof. See Appendix. □

Lemma 9. *For the TLPS system with the hyper-exponential job size distribution the following statement holds:*

$$\Phi_2\left(\Phi_1(\alpha'(x))\right) \leq q\Phi_2\left(\alpha'(x)\right). \tag{23}$$

Proof. We define $\alpha'_j = \int_0^\infty \alpha'(x) e^{-\mu_j x} dx, \ j = 1, ...N$. As was shown in [14], $\alpha'(x)$ has the following structure:

$$\alpha'(x) = a_0 + \sum_k a_k e^{-b_k x}, \quad a_0 \geq 0,\, a_k \geq 0,\, b_k > 0, \quad k = 1, ..., N.$$

Then, we have that $\alpha'(x) \geq 0$ and

$$\alpha'_j = \frac{a_0}{\mu_j} + \sum_k \frac{a_k}{b_k + \mu_j}, \quad j = 1, ..., N,$$

$$\Longrightarrow \quad \frac{d(\alpha'_j \mu_j)}{d\mu_j} = \sum_k \frac{a_k}{b_k + \mu_j} - \sum_k \frac{a_k \mu_j}{(b_k + \mu_j)^2} = \sum_k \frac{a_k b_k}{(b_k + \mu_j)^2} \geq 0, \quad j = 1, ..., N,$$

as $a_k \geq 0,\ b_k > 0, \quad k = 1, ..., N$. So, then, according to Lemma 8 we have (23). □

Let us state the following Theorem:

Theorem 10. *An upper bound for the expected sojourn time $\overline{T}(\theta)$ in the TLPS system with the hyper-exponential job size distribution function with many phases is given by $\overline{\Upsilon}(\theta)$:*

$$\overline{T}(\theta) \leq \overline{\Upsilon}(\theta) = \frac{\overline{X^1_\theta} + \overline{W}(\theta)\overline{F}(\theta)}{1 - \rho_\theta} + \frac{m - \overline{X^1_\theta}}{1 - \rho} + \frac{b(\theta)}{\overline{F}(\theta)(1 - \rho)} \sum_{i,j} \frac{\overline{F^i_\theta}\,\overline{F^j_\theta}}{\mu_i + \mu_j}. \tag{24}$$

Proof. According to the recursion (20) we have for $\alpha'_n(x)$ we approximate $\alpha'(x)$ with the function $\widetilde{\alpha}'(x)$, which satisfies the following equation:

$$\widetilde{\alpha}'(x) = \widetilde{\alpha}'(x)\Phi_1(1) + \frac{b(\theta)}{\overline{F}(\theta)}\overline{F}(x + \theta) + 1.$$

Then, according to (14) we can find the analytical expression for $\widetilde{\alpha}'(x)$:

$$\widetilde{\alpha}'(x) = q\widetilde{\alpha}'(x) + \frac{b(\theta)}{\overline{F}(\theta)}\overline{F}(x + \theta) + 1,$$

$$\Longrightarrow \quad \widetilde{\alpha}'(x) = \frac{1}{1 - q}\left(\frac{b(\theta)}{\overline{F}(\theta)}\overline{F}(x + \theta) + 1\right).$$

We take $\overline{\Upsilon}^{BPS}(\theta) = \Phi_2(\widetilde{\alpha}'(x))$ as an approximation for $\overline{T}^{BPS}(\theta) = \Phi_2(\alpha'(x))$. Then

$$\overline{\Upsilon}^{BPS}(\theta) = \Phi_2(\widetilde{\alpha}'(x)) = \frac{(m - \overline{X_\theta^1})}{1-q} + \frac{b(\theta)}{\overline{F}(\theta)}\Phi_2(\overline{F}(x+\theta)) = \frac{(m - \overline{X_\theta^1})}{1-q} + \frac{b(\theta)}{\overline{F}(\theta)}\sum_{i,j}\frac{\overline{F_\theta^i}\,\overline{F_\theta^j}}{\mu_i + \mu_j}.$$

Let us prove, that

$$\overline{T}^{BPS}(\theta) \le \overline{\Upsilon}^{BPS}(\theta),$$

or equivalently

$$\overline{T}^{BPS}(\theta) - \overline{\Upsilon}^{BPS}(\theta) = \Phi_2(\alpha'(x)) - \Phi_2(\widetilde{\alpha}'(x)) \le 0.$$

Let us look at

$$\begin{aligned}
&\Phi_2(\alpha'(x)) - \Phi_2(\widetilde{\alpha}'(x)) = \\
&= \Phi_2(\Phi_1(\alpha'(x))) + \Phi_2\left(\frac{b(\theta)}{\overline{F}(\theta)}\overline{F}(x+\theta) + 1\right) - \left(q\Phi_2(\widetilde{\alpha}'(x)) + \Phi_2\left(\frac{b(\theta)}{\overline{F}(\theta)}\overline{F}(x+\theta) + 1\right)\right) \\
&= \Phi_2(\Phi_1(\alpha'(x))) - q\Phi_2(\alpha'(x)) + q\left(\Phi_2(\alpha'(x)) - \Phi_2(\widetilde{\alpha}'(x))\right) \\
&\Longrightarrow \\
&\Phi_2(\alpha'(x)) - \Phi_2(\widetilde{\alpha}'(x)) = \frac{1}{1-q}\left(\Phi_2(\Phi_1(\alpha'(x))) - q\Phi_2(\alpha'(x))\right).
\end{aligned}$$

And from Lemma 9 and formula (8) we conclude that (24) is true. □

In this subsection we found the analytical expression of the upper bound of the expected sojourn time in the case when the job size distribution is a hyper-exponential function with many phases. In the experimental results of the following subsection we show that the obtained upper bound is also a close approximation. The analytic expression of the upper bound which we obtained is more clear and easier to analyze then the expression of the expected sojourn time. It could be used in the future research on TLPS model.

4.5 Experimental results

We calculate $\overline{T}(\theta)$ and $\overline{\Upsilon}(\theta)$ for different numbers of phases N of the job size distribution function. We take $N = 10, 100, 500, 1000$. To calculate $\overline{T}(\theta)$ we find the numerical solution of the system of linear equations (19) using the Gauss method. Then using the result of Proposition 6 we find $\overline{T}(\theta)$. For $\overline{\Upsilon}(\theta)$ we use equation (24).

As was mentioned in Subsection 2.1, by using the hyper-exponential distribution with many phases, one can approximate a heavy-tailed distribution. In our numerical experiments, we fix ρ, m, and select p_i and μ_i in a such a way, that by increasing the number of phases we let the second moment d (see (2)) increase as well. Here we take

$$\rho = 10/11,\ \lambda = 0.5,\ p_i = \frac{\nu}{i^{2.5}},\ \mu_i = \frac{\eta}{i^{1.2}}, \quad i = 1, ..., N.$$

In particular, we have

$$\begin{aligned}
&\sum_i p_i = 1, \quad \Longrightarrow \nu = \frac{1}{\sum_i i^{-2.5}}, \\
&\sum_i \frac{p_i}{\mu_i} = m, \ \Longrightarrow \eta = \frac{\nu}{m}\sum_i i^{-1.3}.
\end{aligned}$$

In Figure 3 one can see the plots of the expected sojourn time and its upper bound as functions of θ when N varies from 10 up to 1000. In Figure 4 we plot the relative error of the upper bound

$$\Delta(\theta) = \frac{\overline{\Upsilon}(\theta) - \overline{T}(\theta)}{\overline{T}(\theta)},$$

when N varies from 10 up to 1000. As one can see, the upper bound (24) is very tight.

We find the maximum gain of the expected sojourn time of the TLPS system with respect to the standard PS system. The gain is given by $g(\theta) = \frac{\overline{T}^{PS} - \overline{T}(\theta)}{\overline{T}^{PS}}$. Here $\overline{T}^{PS}$ is an expected sojourn time in the standard PS system. Let us notice, that $\overline{T}^{PS} = \overline{T}(0)$.

The data and results are summarized in Table 1.

N	η	d	θ_{opt}	$\max_\theta g(\theta)$	$\max_\theta \Delta(\theta)$
10	0.95	7.20	5	32.98%	0.0640
100	1.26	32.28	12	45.75%	0.0807
500	1.40	113.31	21	49.26%	0.0766
1000	1.44	200.04	26	50.12%	0.0743

Table 1: Increasing the number of phases

With the increasing number of phases we observe that

1. the second moment d increases;
2. the maximum gain $\max_\theta g(\theta)$ in expected sojourn time in comparison with PS increases;
3. the relative error of the upper bound $\Delta(\theta)$ with the expected sojourn time decreases after the number of phases becomes sufficiently large;
4. the sensitivity of the system performance with respect to the selection of the sub-optimal threshold value decreases.

Thus the TLPS system produces better and more robust performance as the variance of the job size distribution increases.

5 Conclusion

We analyze the TLPS scheduling mechanism with the hyper-exponential job size distribution function.

In Section 3 we analyze the system when the job size distribution function has two phases and find the analytical expressions of the expected conditional sojourn time and the expected sojourn time of the TLPS system.

Connections in the Internet belong to two distinct classes: short HTTP and P2P signaling connections and long downloads such as: PDF, MP3, and so on. Thus, according to this observation, we consider a special selection of the parameters of the job size distribution function with two phases and find the approximation of the optimal threshold, when the variance of the job size distribution goes to infinity.

We show, that the approximated value of the threshold tends to the optimal threshold, when the second moment of the distribution function goes to infinity. We found that the gain of the TLPS system compared to the standard PS system could reach 45% when the load of the system increases. Also the system is not too sensitive to the selection of the optimal value of the threshold.

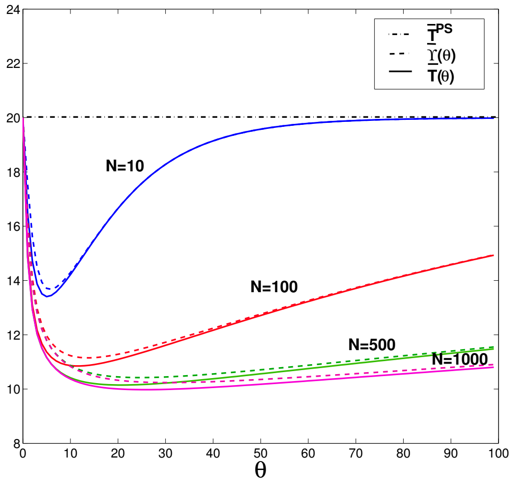

Figure 3: The expected sojourn time $\overline{T}(\theta)$ and its upper bound $\overline{\Upsilon}(\theta))$ for $N = 10$, 100, 500, 1000

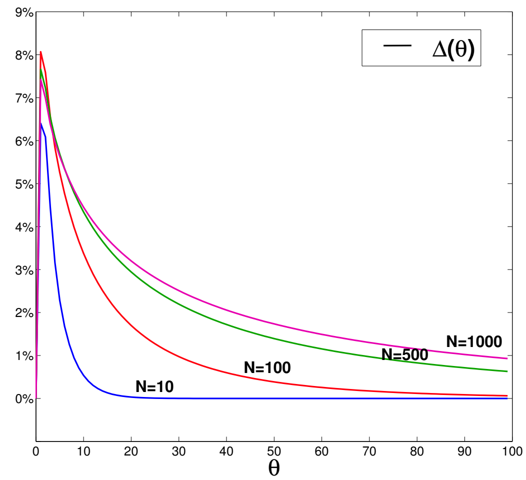

Figure 4: The relative error $\Delta(\theta) = (\overline{T}(\theta) - \overline{\Upsilon}(\theta))/\overline{T}(\theta)$ for $N = 10$, 100, 500, 1000

In Section 4 we have studied the TLPS model when the job size distribution is a hyper-exponential function with many phases. We provide an expression of the expected conditional sojourn time as a solution of the system of linear equations. Also we apply the iteration method to find the expression of the expected conditional sojourn time in the form of operator series and using the obtained expression we provide an upper bound for the expected sojourn time function. With the experimental results we show that the upper bound is very tight and could be used as an approximation of the expected sojourn time function. We show numerically, that the relative error between the upper bound and expected sojourn time function decreases when the variation of the job size distribution function increases. The obtained upper bound could be used to identify an approximation of the optimal value of the optimal threshold for TLPS system when the job size distribution is heavy-tailed.

We study the properties of the expected sojourn time function, when the parameters of the job size distribution function are selected in such a way, that it approximates a heavy-tailed distribution as the number of phases of the job size distribution increases. As the number of phases increases the gain of the TLPS system compared with the standard PS system increases and the sensitivity of the system with respect to the selection of the optimal threshold decreases.

6 Appendix: Proof of Lemma 8

Let us take any function $\beta(x) > 0$ and define $\beta_j = \int_0^\infty \beta(x) e^{-\mu_j x} dx, \quad j = 1, ..., N$. Let us show for $\beta(x) \geq 0$ that if

$$\frac{d(\beta_j \mu_j)}{d\mu_j} \geq 0, \quad j = 1, ..., N, \qquad \text{then it follows that} \quad \Phi_2\left(\Phi_1(\beta(x))\right) \leq q \Phi_2\left(\beta(x)\right).$$

As

$$\int_0^\infty \int_0^x \beta(y)\overline{F}(x-y+\theta)\overline{F}(x+\theta)dydx = \int_0^\infty \int_0^\infty \beta(y)\overline{F}(x_1+\theta)\overline{F}(x_1+y+\theta)dx_1dy$$

and

$$\begin{aligned}\Phi_2(\Phi_1(\beta(x))) &= \gamma(\theta)\int_0^\infty \int_0^\infty \beta(y)\overline{F}(x+y+\theta)\overline{F}(x+\theta)dydx \\ &\quad +\gamma(\theta)\int_0^\infty \int_0^x \beta(y)\overline{F}(x-y+\theta)\overline{F}(x+\theta)dydx,\end{aligned}$$

then

$$\begin{aligned}\Phi_2(\Phi_1(\beta(x))) &= 2\gamma(\theta)\int_0^\infty \int_0^\infty \beta(x)\overline{F}(x+\theta)\overline{F}(x+y+\theta)dydx = \\ &= 2\gamma(\theta)\int_0^\infty \beta(x)\sum_{i,j}\frac{\overline{F^i_\theta}\overline{F^j_\theta}}{\mu_i+\mu_j}e^{-\mu_j x}dx = 2\gamma(\theta)\sum_{i,j}\frac{\overline{F^i_\theta}\overline{F^j_\theta}}{\mu_i+\mu_j}\beta_j.\end{aligned}$$

Also for $\Phi_2(\beta(x))$, taking into account that $q=\gamma(\theta)\sum_i \frac{\overline{F^i_\theta}}{\mu_i}$, we obtain

$$q\Phi_2(\beta(x)) = \gamma(\theta)\sum_i \frac{\overline{F^i_\theta}}{\mu_i}\sum_j \overline{F^j_\theta}\int_0^\infty \beta(x)e^{-\mu_j x}dx = \gamma(\theta)\sum_{i,j}\frac{\overline{F^i_\theta}\overline{F^j_\theta}}{\mu_i}\beta_j.$$

Thus, a sufficient condition for the inequality $\Phi_2(\Phi_1(\beta(x))) \le q\Phi_2(\beta(x))$ to be satisfied is that for every pair i,j:

$$\frac{2}{\mu_i+\mu_j}\beta_j + \frac{2}{\mu_j+\mu_i}\beta_i \le \frac{1}{\mu_i}\beta_j + \frac{1}{\mu_j}\beta_i \quad \Longleftrightarrow \quad -(\beta_j\mu_j-\beta_i\mu_i)(\mu_j-\mu_i)\le 0.$$

The inequality is indeed satisfied when $\beta_j\mu_j$ is an increasing function of μ_j. We conclude that $\Phi_2(\Phi_1(\beta(x))) \le q\Phi_2(\beta(x))$, which proves Lemma 8.

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