The Fundamental Properties of Natural Numbers

Grzegorz Bancerek
Warsaw University
Bialystok

Summary. Some fundamental properties of addition, multiplication, order relations, exact division, the remainder, divisibility, the least common multiple, the greatest common divisor are presented. A proof of Euclid algorithm is also given.

MML Identifier: NAT_1

WWW: http://mizar.org/JFM/Voll/nat_1.html

The articles [4], [6], [1], [2], [5], and [3] provide the notation and terminology for this paper.

A natural number is an element of \( \mathbb{N} \).

For simplicity, we use the following convention: \( x \) is a real number, \( k, l, m, n \) are natural numbers, \( h, i, j \) are natural numbers, and \( X \) is a subset of \( \mathbb{R} \).

The following proposition is true (2)

Let \( n, k \) be natural numbers. Then \( n + k \) is a natural number.

Let \( n, k \) be natural numbers. Note that \( n \cdot k \) is natural.

In this article we present several logical schemes. The scheme \textit{Ind} concerns a unary predicate \( P \), and states that:

For every natural number \( k \) holds \( P[k] \)

provided the parameters satisfy the following conditions:

- \( P[0] \), and
- For every natural number \( k \) such that \( P[k] \) holds \( P[k + 1] \).

The scheme \textit{Nat Ind} concerns a unary predicate \( P \), and states that:

For every natural number \( k \) holds \( P[k] \)

provided the following conditions are satisfied:

- \( P[0] \), and
- For every natural number \( k \) such that \( P[k] \) holds \( P[k + 1] \).

Let \( n, k \) be natural numbers. Then \( n \cdot k \) is a natural number.

Let \( n, k \) be natural numbers. Observe that \( n \cdot k \) is natural.

Next we state several propositions:

\( 0 \leq i \).

\( 0 \neq i, \text{ then } 0 < i. \)

\( i \leq j, \text{ then } i \cdot h \leq j \cdot h. \)

---

1 The proposition (1) has been removed.

2 The propositions (3)–(17) have been removed.
(21) $0 \neq i + 1$.

(22) $i = 0$ or there exists $k$ such that $i = k + 1$.

(23) If $i + j = 0$, then $i = 0$ and $j = 0$.

One can check that there exists a natural number which is non zero.

Let $m$ be a natural number and let $n$ be a non zero natural number. Observe that $m + n$ is non zero and $n + m$ is non zero.

The scheme Def by Ind deals with a natural number $A$, a binary functor $F$ yielding a natural number, and a binary predicate $P$, and states that:

For every $k$ there exists $n$ such that $P[k, n]$ and for all $k, n, m$ such that $P[k, n]$ and $P[k, m]$ holds $n = m$ provided the parameters meet the following requirement:

- For all $k, n$ holds $P[k, n]$ iff $k = 0$ and $n = A$ or there exist $m, l$ such that $k = m + 1$ and $P[m, l]$ and $n = F(k, l)$.

We now state four propositions:

(26) For all $i, j$ such that $i \leq j + 1$ holds $i \leq j$ or $i = j + 1$.

(27) If $i \leq j$ and $j \leq i + 1$, then $i = j$ or $j = i + 1$.

(28) For all $i, j$ such that $i \leq j$ there exists $k$ such that $j = i + k$.

(29) $i \leq i + j$.

Now we present three schemes. The scheme Comp Ind concerns a unary predicate $P$, and states that:

For every $k$ holds $P[k]$ provided the parameters have the following property:

- For every $k$ such that for every $n$ such that $n < k$ holds $P[n]$ holds $P[k]$.

The scheme Min concerns a unary predicate $P$, and states that:

There exists $k$ such that $P[k]$ and for every $n$ such that $P[n]$ holds $k \leq n$ provided the following requirement is met:

- There exists $k$ such that $P[k]$.

The scheme Max deals with a natural number $A$ and a unary predicate $P$, and states that:

There exists $k$ such that $P[k]$ and for every $n$ such that $P[n]$ holds $n \leq k$ provided the parameters meet the following requirements:

- For every $k$ such that $P[k]$ holds $k \leq A$, and
- There exists $k$ such that $P[k]$.

We now state three propositions:

(37) If $i \leq j$, then $i \leq j + h$.

(38) $i < j + 1$ iff $i \leq j$.

(40) If $i \cdot j = 1$, then $i = 1$ and $j = 1$.

The scheme Regr concerns a unary predicate $P$, and states that:

$P[0]$ provided the parameters meet the following conditions:

- There exists $k$ such that $P[k]$, and
- For every $k$ such that $k \neq 0$ and $P[k]$ there exists $n$ such that $n < k$ and $P[n]$.

In the sequel $t$ denotes a natural number.

We now state two propositions:

$^3$ The propositions (24) and (25) have been removed.

$^4$ The propositions (30)–(36) have been removed.

$^5$ The proposition (39) has been removed.
For every $m$ such that $0 < m$ and for every $n$ there exist $k$, $t$ such that $n = m \cdot k + t$ and $t < m$.  

For all natural numbers $n, m, k, k_1, t, t_1$ such that $n = m \cdot k + t$ and $t < m$ and $n = m \cdot k_1 + t_1$ and $t_1 < n$ holds $k = k_1$ and $t = t_1$.

Let $k, l$ be natural numbers. The functor $k \div l$ yields a natural number and is defined by:

(Def. 1) There exists $t$ such that $k = l \cdot (k \div l) + t$ and $t < l$ or $k \div l = 0$ and $l = 0$.

The functor $k \bmod l$ yielding a natural number is defined by:

(Def. 2) There exists $t$ such that $k = l \cdot t + (k \bmod l)$ and $k \bmod l < l$ or $k \bmod l = 0$ and $l = 0$.

We now state two propositions:

(46) $\quad$ If $0 < i$, then $j \mod i < i$.

(47) $\quad$ If $0 < i$, then $j = i \cdot (j \div i) + (j \mod i)$.

Let $k, l$ be natural numbers. The predicate $k \mid l$ is defined as follows:

(Def. 3) There exists $t$ such that $l = k \cdot t$.

Let us note that the predicate $k \mid l$ is reflexive.

We now state several propositions:

(49) $\quad$ $j \mid i$ iff $i = j \cdot (i \div j)$.

(50) $\quad$ If $i \mid j$ and $j \mid h$, then $i \mid h$.

(51) $\quad$ If $i \mid j$ and $j \mid i$, then $i = j$.

(52) $\quad$ $i \mid 0$ and $1 \mid i$.

(53) $\quad$ If $0 < j$ and $i \mid j$, then $i \leq j$.

(54) $\quad$ If $i \mid j$ and $i \mid h$, then $i \mid j + h$.

(55) $\quad$ If $i \mid j$, then $i \mid j \cdot h$.

(56) $\quad$ If $i \mid j$ and $i \mid j + h$, then $i \mid h$.

(57) $\quad$ If $i \mid j$ and $i \mid j \mod h$, then $i \mid j \mod h$.

Let $k, n$ be natural numbers. The functor $\text{lcm}(k, n)$ yields a natural number and is defined by:

(Def. 4) $\quad$ $k \mid \text{lcm}(k, n)$ and $n \mid \text{lcm}(k, n)$ and for every $m$ such that $k \mid m$ and $n \mid m$ holds $\text{lcm}(k, n) \mid m$.

Let us observe that the functor $\text{lcm}(k, n)$ is commutative and idempotent.

Let $k, n$ be natural numbers. The functor $\text{gcd}(k, n)$ yielding a natural number is defined as follows:

(Def. 5) $\quad$ $\text{gcd}(k, n) \mid k$ and $\text{gcd}(k, n) \mid n$ and for every $m$ such that $m \mid k$ and $m \mid n$ holds $m \mid \text{gcd}(k, n)$.

Let us observe that the functor $\text{gcd}(k, n)$ is commutative and idempotent.

The scheme Euklides deals with a unary functor $\mathcal{F}$ yielding a natural number and natural numbers $\mathcal{A}$, $\mathcal{B}$, and states that:

There exists $n$ such that $\mathcal{F}(n) = \text{gcd}(\mathcal{A}, \mathcal{B})$ and $\mathcal{F}(n + 1) = 0$ provided the following conditions are satisfied:

- $0 < \mathcal{B}$ and $\mathcal{B} < \mathcal{A}$,
- $\mathcal{F}(0) = \mathcal{A}$ and $\mathcal{F}(1) = \mathcal{B}$, and
- For every $n$ holds $\mathcal{F}(n + 2) = \mathcal{F}(n) \bmod \mathcal{F}(n + 1)$.

One can check that every natural number is ordinal.

Let us observe that there exists a subset of $\mathbb{R}$ which is non empty and ordinal.

---

6 The proposition (41) has been removed.
7 The propositions (44) and (45) have been removed.
8 The proposition (48) has been removed.
9 The proposition (50) has been removed.
REFERENCES


Received January 11, 1989

Published January 2, 2004