On the variance of the Gaussian quadrature rule

H. Brass

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Institut für Angewandte Mathematik der TU Braunschweig, Abteilung Numerik.
D 38106 Braunschweig, Pockelsstr. 14, Germany.

Abstract - Denote by $\sum_{\nu=1}^{n} a_{\nu} f\left(x_{\nu}\right)$ the Gaussian quadrature rule for the integral $\int_{-1}^{1} f(x)\, dx$. We give a simple explicit expression for the "variance" $\sum_{\nu=1}^{n} a_{\nu}^{2}$. The method can be used to obtain similar results for the Lobatto rule.

1. The result

A quadrature rule (on $[-1,1]$) is a functional Q_n on $C[-1,1]$ of the form

$$(1) \qquad Q_n[f] = \sum_{\nu=1}^{n} a_\nu f(x_\nu) , \qquad a_\nu \in \mathbb{R}$$

$$-1 \leq x_1 < x_2 < \ldots < x_n \leq 1 .$$

The "variance of Q_n"

$$\operatorname{Var} Q_n := \sum_{\nu=1}^{n} a_\nu^2$$

is of interest in the statistical theory of error propagation, for a survey cf. Förster [3].

Among all quadrature rules the Gaussian rule Q_n^{G} (to be defined below) is the most interesting and the most useful rule. Förster and Petras [4] proved

$$(2) \qquad \frac{\pi^2}{2n+1}\left[1 - \frac{1}{(2n+1)^2}\right] < \operatorname{Var} Q_n^{\mathrm{G}} < \frac{\pi^2}{2n+1} ,$$

$$n = 3, 4, \ldots .$$

Our aim is the proof of

Theorem 1

$$(3) \qquad \operatorname{Var} Q_n^{\mathrm{G}} = \frac{6}{2n+1}\left(1 - \frac{1}{(2n-1)(2n+3)}\right)\sum_{\nu=1}^{n}\frac{1}{\nu^2} + \frac{12}{(2n-1)(2n+3)} .$$

The relation

$$\lim_{n\to\infty} (2n+1) \operatorname{Var} Q_n^{\mathrm{G}} = \pi^2$$

is an immediate consequence, with a little computation we obtain more precisely: The sequence $(2n+1)\operatorname{Var} Q_n^{\mathrm{G}}$ ($n = 4, 5, \ldots$) tends increasingly to π^2, and the sequence $\left[(2n+1) + (2n+1)^{-1}\right] \operatorname{Var} Q_n^{\mathrm{G}}$ ($n = 1, 2, \ldots$) tends decreasingly to π^2, then (2) (and a little more) follows.

Our proof of (3) uses only quite elementary properties of the Legendre polynomials, whereas the proof of (2) in [4] depends on the bounds for the coefficients a_ν of Q_n^{G}, whose proofs are long and difficult.

2. The proof

We start with the Legendre polynomial

$$P_n(x) := \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \left[(x^2-1)^n\right] .$$

It is well known (for this and further results on P_n and Q_n^{G} see Szegö [7] or Brass [1]) that P_n has n simple zeros located in the interior of $[-1,1]$. Q_n^{G} is defined by using the zeros $x_1,\ldots,x_n$ as evaluation points in (1) and by defining

$$a_\nu = a_\nu^{\mathrm{G}} := \int_{-1}^{1} \frac{P_n(x)}{(x-x_\nu)\,P_n'(x_\nu)}\,dx\,. \tag{4}$$

Applying some identities for the P_n, one obtains as a further expression

$$a_\nu^{\mathrm{G}} = \frac{2}{(1-x_\nu^2)\,[P_n'(x_\nu)]^2} \tag{5}$$

(Szegö [7] p. 352 or Brass [1] p. 151). The main property of the Gaussian quadrature rule Q_n^{G} is

$$\sum_{\nu=1}^{n} a_\nu^{\mathrm{G}} p(x_\nu) = \int_{-1}^{1} p(x)\,dx\,, \tag{6}$$

holding for any polynomial p of degree $2n-1$. The idea of the proof consists in constructing a polynomial $\hat{p}$ of degree $2n-1$ with

$$\hat{p}(x_\nu) = a_\nu \qquad \nu = 1,\ldots,n\,,$$

then we have

$$\operatorname{Var} Q_n^{\mathrm{G}} = \sum_{\nu=1}^{n} \left(a_\nu^{\mathrm{G}}\right)^2 = \sum_{\nu=1}^{n} a_\nu^{\mathrm{G}}\hat{p}(x_\nu) = \int_{-1}^{1} \hat{p}(x)\,dx\,. \tag{7}$$

To this end we introduce the "associated Legendre polynomials" $\hat{P}_n$ by

$$\hat{P}_n(x) := \int_{-1}^{1} \frac{P_n(y)-P_n(x)}{y-x}\,dy\,.$$

We obtain from (4)

$$a_\nu^{\mathrm{G}} = \frac{\hat{P}_n(x_\nu)}{P_n'(x_\nu)}$$

and by applying (5)

$$\left(a_\nu^{\mathrm{G}}\right)^2 = \left[\frac{\hat{P}_n(x_\nu)}{P_n'(x_\nu)}\right]^2 = a_\nu^{\mathrm{G}} \frac{(1-x_\nu^2)\left[\hat{P}_n(x_\nu)\right]^2}{2}$$

hence we have

$$a_\nu^{\mathrm{G}} = \frac{(1-x_\nu^2)\left[\hat{P}_n(x_\nu)\right]^2}{2} \equiv \tilde{p}(x_\nu)\,.$$

The polynomial $\tilde{p}$ has the degree $2n$ and its main coefficient is (-2) times the main coefficient of P_n^2, so we may choose

$$\hat{p}(x) := \tilde{p}(x) + 2P_n^2(x)$$

and we obtain according to (7)

$$\operatorname{Var} Q_n^{\mathrm{G}} = \int_{-1}^{1} \tilde{p}(x)\,dx + 2\int_{-1}^{1} P_n^2(x)\,dx\,.$$

The relation

$$\int_{-1}^{1} P_i(x)\,P_j(x)\,dx = \begin{cases} 0 & i \neq j \\ \frac{2}{2j+1} & i = j \end{cases} \tag{8}$$

is fundamental in the theory of Legendre polynomials. So we arrive at

$$\operatorname{Var} Q_n^{\mathrm{G}} = \frac{1}{2}\int_{-1}^{1} \left(1-x^2\right)\left[\hat{P}_n(x)\right]^2 dx + \frac{4}{2n+1}\,. \tag{9}$$

The crucial tool in our proof is the following identity of Christoffel

$$\hat{P}_n(x) = \sum_{\nu=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} c_\nu P_{n-2\nu-1}(x) \qquad \text{with } c_\nu := 2\frac{2n-4\nu-1}{(2\nu+1)(n-\nu)}\,. \tag{10}$$

A proof can be found in Hobson ([5] p. 53), the simplest proof is by induction using the recurrence relations

$$(n+1)\,P_{n+1}(x) = (2n+1)\,xP_n(x) - nP_{n-1}(x) \tag{11}$$

$$P_0(x) = 1\,,\quad P_1(x) = x$$

and (easy consequence of (11))

$$(n+1)\,\hat{P}_{n+1}(x) = (2n+1)\,x\hat{P}_n(x) - n\hat{P}_{n-1}(x)$$

$$\hat{P}_0(x) = 0\,,\quad \hat{P}_1(x) = 2\,.$$

If we introduce (10) in (9) and apply (8), we get

$$\begin{aligned} \operatorname{Var} Q_n^{\mathrm{G}} &= \frac{1}{2}\sum_{\nu=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} c_\nu^2 \int_{-1}^{1}\left(1-x^2\right)P_{n-2\nu-1}^2(x)\,dx \\ &\quad + \sum_{\nu=1}^{\left\lfloor\frac{n-1}{2}\right\rfloor} c_{\nu-1}c_\nu \int_{-1}^{1}\left(1-x^2\right)P_{n-2\nu-1}(x)\,P_{n-2\nu+1}(x)\,dx + \frac{4}{2n+1}\,. \end{aligned}$$

The integrals can be determined using (11) and (8). We obtain

$$\int_{-1}^{1}\left(1-x^2\right)P_j^2(x)\,dx = \frac{1}{2j+1}\left[1 - \frac{1}{(2j-1)(2j+3)}\right],$$

$$\int_{-1}^{1}\left(1-x^2\right)P_{j-1}(x)\,P_{j+1}(x)\,dx = \frac{-2j(j+1)}{(2j-1)(2j+1)(2j+3)}\,.$$

Hence we have

$$
\begin{aligned}
\operatorname{Var} Q_n^{\mathrm{G}} &= \frac{1}{2} \sum_{\nu=0}^{\lfloor \frac{n-1}{2} \rfloor} 4 \frac{(2n-4\nu-1)^2}{(2\nu+1)^2 (n-\nu)^2} \frac{1}{2n-4\nu-1} \left[1 - \frac{1}{(2n-4\nu-3)(2n-4\nu+1)} \right] \\
&- 8 \sum_{\nu=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2n-4\nu-1}{(2\nu+1)(n-\nu)} \frac{2n-4\nu+3}{(2\nu-1)(n-\nu+1)} \frac{(n-2\nu)(n-2\nu+1)}{(2n-4\nu-1)(2n-4\nu+1)(2n-4\nu+3)} \\
&+ \frac{4}{2n+1} .
\end{aligned}
$$

In the last step we have to simplify this expression by partial fraction expansion of the summands. After some work we obtain the theorem.

3. Remarks

(i) Chawla and Ramakrishnan [2] were the first to give an explicit expression for the variance of a quadrature rule in a nontrivial case. Their result concerns the Polya quadrature rule and their method can be applied to the rules of Filippi and Clenshaw/Curtis (for the definitions cf. Brass [1]), but it does not seem to be possible to obtain theorem 1 with their method.

(ii) Our method can be extended to the Gaussian rules with ultraspherical weight functions. The generalization of the crucial identity (10) is known, see e.g. Lewanowicz [6] (3.2). But I had no success in the simplification of the obtained expression, and so it seems to be of minor interest.

(iii) Our method can be applied to the Lobatto quadrature rule Q_{n+1}^{Lo} (mainly by replacing P_n by $P_{n+1} - P_{n-1}$). We give the result as

Theorem 2

$$
\begin{aligned}
\operatorname{Var} Q_{n+1}^{\mathrm{Lo}} &= \frac{6}{2n+1} \left[1 + \frac{3}{(2n-1)(2n+3)} \right] \sum_{\nu=1}^{n} \frac{1}{\nu^2} \\
&+ \frac{12}{(2n-1)(2n+3)} - \frac{36}{(2n-1)(2n+3)\, n\,(n+1)} .
\end{aligned}
$$

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