Boolean algebras arising from information systems*
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1 Introduction

In recent years a wide variety of intensional modal-like logics with the propositional operators determined by relations play an important role in a number of application areas such as spatial reasoning, cognitive agent technologies, knowledge-based systems, etc. Many of these applications require much more involved logical systems than the ordinary modal logics can offer. In one direction, one considers operations which are intended to express properties not expressible by the possibility operator, such as the inaccessibility of Humberstone [10], or the suffiency operators of Gargov et al. [6]. Such operators lead to various classes of Boolean algebras with operators, and in earlier work we have introduced the classes of *sufficiency algebras* (SUA) and *mixed algebras* (MIA) and have started an investigation of their properties [5].

Other novel features of the logics that have been considered in connection with information systems are – on the level of semantics – frames with a family $\{R_Q : Q \subseteq PAR\}$ of *relative relations.* These are relations which are indexed by the elements of the powerset of a set *PAR* in such a way that the set operations on 2^{PAR} pose restrictions on the relations, for example,

$$R_{P \cup Q} = R_P \cap R_Q. \tag{1.1}$$

The need to consider such systems arises from the fact that in the context of information systems, dependencies among attributes are usually present in some form which have to be modelled.

In this paper we continue the development of algebraic counterparts of logics arising from information systems which we have begun in [5] and extend some results to reasoning about relative relations.

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In Section 2 we recall some classes of frames with relative relations derived from an information system and their abstract characterisations. In Section 3 we present some new results on sufficiency algebras. In Section 4 we extend the notion of canonical extension of an algebra to sufficiency algebras and we investigate properties of these canonical extensions. Section 5 is devoted to mixed algebras which have both modal and sufficiency operators in their signature. In Section 6 we introduce the concept of Boolean algebras with relative operators, and we present several classes of such algebras. These are meant to provide an abstract characterisation of the corresponding algebras derived from information systems.

2 Frames and information systems

A *frame* is a structure $\langle U, \mathcal{R} \rangle$, where U is a set and $\mathcal{R}$ is a family of binary relations on U. If R is a binary relation on U, we let $\mathrm{dom}(R) = \{x \in U : (\exists y \in U) xRy\}$, and for $x \in U$

$$R(x) = \{y \in U : xRy\}$$

is the *R-range of x*. The *converse of R* is the relation

$$R^{\smile} = \{\langle x, y \rangle : yRx\}.$$

For later use we recall several relation properties: R is called

- *weakly reflexive*, if $x \in \mathrm{dom}(R)$ implies xRx,
- *weakly irreflexive*, if $x \in \mathrm{dom}(R)$ implies $x(-R)x$,
- *3–transitive*, if $wRxRyRz$ implies wRz.

If P is a property of relations, we say that R is *co - P*, if $-R$ has the property P.

By an information system we mean a structure $S = \langle OB, AT, \{VAL_a : a \in AT\} \rangle$ such that OB is a nonempty set of objects, AT is a finite nonempty set of attributes, and each VAL_a is a set of values for attribute a. Note that we allow $a(x)$ to be the empty set, since it may be meaningful in various applications. For example, if $a(x)$ consists of the symptoms which a medical expert x assigns to an illness, then one can allow $a(x) = \emptyset$ to express that the expert does not have (or does not want to publicise) an opinion [see 3].

Each attribute a defines various *information relations* [14, 16, 17] on the Universe OB in the following way: If T is a relation on 2^{VAL_a}, we let R_T be the relation on U defined by

$$xR_a^T y \iff a(x)Ta(y).$$

If $A \subseteq OB$, we let

$$xR_A^{T,s} y \iff xR_a^T y \text{ for all } a \in A,$$
$$xR_A^{T,w} y \iff xR_a^T y \text{ for some } a \in A.$$

At times, we will write a more suggestive name for R^T. Of particular interest are those relations which arise from the set theoretic operations and relations on 2^{VAL_a}. We will, in particular consider the following relations:

$$x\mathsf{SIM}_a y \Longleftrightarrow a(x) \subseteq a(y), \tag{2.1}$$

$$x\mathsf{DISJ}_a y \Longleftrightarrow a(x) \cap a(y) = \emptyset, \tag{2.2}$$

$$x\mathsf{COMP}_a y \Longleftrightarrow a(x) = -a(y). \tag{2.3}$$

By a *frame derived from an information system* $S = \langle OB, AT, \{VAL_a : a \in AT\}\rangle$ we mean a relational system $\langle K_S, R\rangle = \langle OB, \{R_A : A \subseteq AT\}\rangle$, where $\{R_A : A \subseteq AT\}$ is a family of information relations. Observe that relations in these frames depend on subsets of AT, and in this sense, they are relative. Such relations provide twofold information, namely, the information which objects are related and the information with respect to which attributes those objects are related. We conclude that in order to represent adequately all the ingredients of information provided in an information system, we need to consider the frames with relative relations. In a general setting, we will deal with families of relations indexed with subsets of a set of parameters, that is, structures of the form $\langle U, \{R_P : P \subseteq PAR\}\rangle$. Relative relations in these frames may satisfy local conditions such as reflexivity, transitivity etc., but also conditions that say how a relation indexed with a compound set (such as $R_{P\cup Q}$) depends on the relations indexed with the component sets (such as R_P and R_Q). These conditions are relevant for the family of relations as a whole, and therefore they are referred to as *global conditions.*Typical examples for such global conditions are

$$R_{P\cup Q} = R_P \cap R_Q, \tag{2.4}$$

$$R_\emptyset = U \times U, \tag{2.5}$$

$$R_{P\cup Q} = R_P \cup R_Q, \tag{2.6}$$

$$R_\emptyset = \emptyset. \tag{2.7}$$

Any family of relative relations satisfying (2.4) and (2.5) (resp. (2.6) and (2.7)) for all $P, Q \subseteq PAR$ is called a family of *strong* (resp. *weak*) relations. With some abuse of notation we will identify singleton sets with the element they contain; in particular, we will write R_a instead of $R_{\{a\}}$ for $a \in PAR$.

In Table 1 we characterise the abstract relative counterparts to the relations (2.1) – (2.3) in terms of local and global properties. Representability theorems which exhibit the connections between the relations of Table 1 can be found in [1] and [4].

3 Modal and sufficiency algebras

Throughout, we let $\langle B, +, \cdot, -, 0, 1\rangle$ be a Boolean algebra, which we will just call B. If A is any Boolean algebra, then $At(A)$ is the set of all atoms of A, and we set $At^0(A) = At(A) \cup \{0\}$. An *operator* on B is just a mapping $f : B \to B$; observe that this is more general than the terminology in [12]. If f is an operator on B, then its *dual operator* f^∂ is defined by

$$f^\partial(x) = -f(-x), \tag{3.1}$$

Table 1: Families of information relations

Strong similarity	Weak similarity
Strong, symmetric, weakly reflexive	Weak, symmetric, each R_a is weakly reflexive
Strong disjointness	**Weak disjointness**
Strong, symmetric, co-weakly reflexive	Weak, symmetric, each R_a is co-weakly reflexive,
Strong complementarity	**Weak complementarity**
Strong, irreflexive, symmetric, 3-transitive	Weak, symmetric, each R_a is irreflexive and 3-transitive.

and its *complementary counterpart* $f^{\star}$ by

$$f^{\star}(x) = -f(x). \tag{3.2}$$

The next Lemma is easily established:

Lemma 3.1. *If f,g are operators on B then*

$$f^{\partial\partial} = f, \tag{3.3}$$

$$f^{\star\star} = f, \tag{3.4}$$

$$f^{\star\partial} = f^{\partial\star}, \tag{3.5}$$

$$(f(x)+g(x))^{\partial} = f^{\partial}(x)\cdot g^{\partial}(x), \tag{3.6}$$

$$(f(x)+g(x))^{\star} = f^{\star}(x)\cdot g^{\star}(x). \tag{3.7}$$

□

An operator $h : B \to B$ is called

1. *completely additive*, if

$$\text{If } \sum_{i\in I} b_i \text{ exists, then } \sum_{i\in I} h(b_i) \text{ exists, and is equal to } h\left(\sum_{i\in I} b_i\right). \tag{3.8}$$

2. *completely co-additive*, if

$$\text{If } \sum_{i\in I} b_i \text{ exists, then } \sum_{i\in I} h(b_i) \text{ exists, and is equal to } h\left(\prod_{i\in I} b_i\right). \tag{3.9}$$

A *modal operator* on B is a mapping $f : B \to B$ for which

$$f(0) = 0, \qquad \text{Normal} \tag{3.10}$$

$$f(a+b) = f(a)+f(b). \qquad \text{Additive} \tag{3.11}$$

If f is a modal operator, then

$$f^{\partial}(1) = 1, \tag{3.12}$$

$$f^{\partial}(a \cdot b) = f^{\partial}(a) \cdot f^{\partial}(b). \tag{3.13}$$

The dual of a modal operator is often called a *necessity operator*. Observe that each modal operator and its dual are isotone. A *modal algebra* (MOA) is a Boolean algebra with additional modal operators. With some abuse of language, we denote the class of these algebras by MOA as well.

Let us briefly recall the connection between frames and modal algebras: If $\langle U, R\rangle$ is a frame, we define two mappings $\langle R\rangle, [R] : 2^U \to 2^U$ by

$$\langle R\rangle(X) = \{x \in U : R(x) \cap X \neq \emptyset\}, \tag{3.14}$$

$$[R](X) = \{x \in U : R(x) \subseteq X\}. \tag{3.15}$$

In other words,

$$\langle R\rangle(X) = \{x \in U : (\exists y \in X) xRy\}, \tag{3.16}$$

$$[R](X) = \{x \in U : (\forall y \in U)[xRy \Rightarrow y \in X]\}, \tag{3.17}$$

Proposition 3.2. *[12] Suppose that $K = \langle U, R\rangle$ is a frame.*

1. *$\langle R\rangle$ is a complete modal operator on 2^U, $[R]$ is a necessity operator, and both are dual to each other.*
2. *If f is a modal operator on 2^U, then there is exactly one binary relation S_f on U such that $\langle S_f\rangle = f$, and $[S_f] = f^{\partial}$.*
3. *$S_{\langle R\rangle} = R$.*

The algebra $Cm_{\mathrm{MOA}}(K) = \langle 2^U, \langle R\rangle\rangle$ is called the *full modal complex algebra of K*.

Correspondence theory investigates the relationship between properties of the relations of a frame and properties of its complex algebra [18]. Examples, which we will need later – and which are easily proved –, are

$$R \text{ is reflexive} \iff [R](X) \subseteq X, \tag{3.18}$$

$$R \text{ is weakly reflexive} \iff \langle R\rangle(U) \cap X \subseteq \langle R\rangle(X), \tag{3.19}$$

$$R \text{ is symmetric} \iff \langle R\rangle[R](X) \subseteq X, \tag{3.20}$$

$$R \text{ is 3-transitive} \iff \langle R\rangle\langle R\rangle\langle R\rangle(X) \subseteq \langle R\rangle(X) \tag{3.21}$$

Several simple properties of binary relations, however, cannot be expressed by modal sentences, for example, irreflexivity. Noting that a relation is irreflexive if and only if its complement is reflexive – and reflexivity is modally expressible – Humberstone [10] introduced an “inaccessibility ” operator, which was determined by the complement of a frame relation; a similar idea was put forward in [6] where a "sufficiency" operator is used. These considerations lead to the following definitions: A *sufficiency operator* on B is a function $g : B \to B$ which satisfies

$$g(0) = 1, \qquad \text{Co-normal} \tag{3.22}$$

$$g(a+b) = g(a) \cdot g(b), \qquad \text{Co-additive} \tag{3.23}$$

for all $a, b \in B$.

A *sufficiency algebra* (SUA) is a Boolean algebra with additional sufficiency operators [5]. With some abuse of language, we denote the class of these algebras by SUA as well. It is easy to see that Boolean complementation is a sufficiency operator, and that a sufficiency operator as well as its dual are antitone. The general connection between modal operators and sufficiency operators is given by the next Proposition, the easy proof of which is left to the reader.

Proposition 3.3. *f is a modal operator if and only if $f^\star$ is a sufficiency operator.*

If $\mathfrak{B} = \langle B, f_1, \dots, f_n \rangle \in$ MOA we let $\mathfrak{B}^\star = \langle B, f_1^\star, \dots f_n^\star \rangle$, which is in SUA by the preceding result.

Suppose that $\mathcal{L}$ is a language containing symbols for the Boolean operations and constants as well as unary operators $h_1, \dots, h_n$. If τ is a term in $\mathcal{L}$, we let $\tau^\star$ be obtained by replacing each occurrence of h_i in τ with $-h_i$.

The fundamental properties relating MOA to SUA are given by

Proposition 3.4. *Suppose that $\mathfrak{A}, \mathfrak{B} \in$ MOA.*

1. *Let $\alpha : \mathfrak{A} \to \mathfrak{B}$ be a MOA homomorphism. Then, the assignment*
   (a) $\mathfrak{A} \mapsto \mathfrak{A}^\star$,
   (b) $\alpha \mapsto \alpha$

   defines a bijective co-variant functor between MOA and SUA.
2. *For all terms τ, σ of $\mathcal{L}$*

$$\mathfrak{B} \models \tau = \sigma \iff \mathfrak{B}^\star \models \tau^\star = \sigma^\star. \tag{3.24}$$

Proof. 1. It is clearly sufficient to show that for each modal operator f on A we have $\alpha(f^\star(x)) = f^\star(\alpha(x))$:

$$\begin{aligned}
\alpha(f^\star(x)) &= \alpha(-f(x)) \\
&= -\alpha(f(x)) \\
&= -f(\alpha(x)) \\
&= f^\star(\alpha(x)),
\end{aligned}$$

which was to be shown.

2. Suppose that v is a valuation of variables of $\mathfrak{B}$ for which $\tau^{\mathfrak{B}}(v) = \sigma^{\mathfrak{B}}(v)$ is true in $\mathfrak{B}$. Since $- - f_i(x)) = f_i(x)$ holds for all $x \in B$, we have $\tau^{\mathfrak{B}}(v) = (\tau^*)^{\mathfrak{B}^*}(v)$ from which "$\Rightarrow$" follows. The other direction is analogous. $\square$

If g is a sufficiency operator, then

$$g^{\partial}(1) = 0, \tag{3.25}$$

$$g^{\partial}(a \cdot b) = g^{\partial}(a) + g^{\partial}(b). \tag{3.26}$$

The relationship of sufficiency operators to frames is as follows: Suppose that $\langle U, R\rangle$ is a frame, and define $[[R]] : 2^U \to 2^U$ by

$$[[R]](X) = \{x \in U : X \subseteq R(x)\}. \tag{3.27}$$

Then,

$$\begin{aligned} x \in [[R]](X) &\iff (\forall y)[y \in X \Rightarrow xRy], \\ &\iff y \in X \text{ is sufficient for } xRy, \end{aligned}$$

which explains the name. We denote the dual operator of $[[R]]$ by $\langle\langle R\rangle\rangle$, and obtain

$$\langle R\rangle(X) = \langle\langle -R\rangle\rangle(-X), \tag{3.28}$$

$$[R](X) = [[-R]](-X). \tag{3.29}$$

Correspondences include

$$R \text{ is irreflexive} \iff X \subseteq -[[R]](X), \tag{3.30}$$

$$R \text{ is co-weakly reflexive} \iff [[R]](X) \cap X \subseteq [[R]](U), \tag{3.31}$$

$$R \text{ is symmetric} \iff X \subseteq [[R]]([[R]](X)). \tag{3.32}$$

In analogy to Proposition 3.2 we have

Proposition 3.5. *Suppose that $K = \langle U, R\rangle$ is a frame.*

1. *$[[R]]$ is a complete sufficiency operator on 2^U.*
2. *If g is a sufficiency operator on 2^U, then there is exactly one binary relation S_g on U such that $[[S_g]] = g$.*
3. *$S_{[[R]]} = R$.*

We invite the reader to consult [5] for details. The algebra $Cm_{\mathrm{SUA}}(K) = \langle 2^U, [[R]]\rangle$ is called the *full sufficiency complex algebra of K.*

B. Jónsson [11] has remarked that MOA is generated by its finite members. A similar result is true for SUA.

Proposition 3.6. SUA *is generated by its finite members.*

Proof. Let $\langle B, g \rangle \in$ SUA. It is our aim to show that $\langle B, g \rangle$ is a subalgebra of a direct product of finite elements of SUA.

Let $\mathcal{C}$ be the collection of finite Boolean subalgebras of B with at least two atoms. Suppose that $C \in \mathcal{C}$ with atoms $\{c_0, \dots, c_n\}$. For each $i \leq n$ choose an ultrafilter U_i of B which contains c_i and no c_j, $j \neq i$. Let $h_C : B \to C$ be defined by

$$h_C(b) = \sum \{c_i : b \in U_i\}.$$

Then, h_C is a retraction of B onto C, see [13], p. 78. For each $x \in B$ let

$$g_C(h_C(x)) = h_C(g(x)).$$

Then, g_C is an operator on C, and $h_C : \langle B, g \rangle \to \langle C, g_C \rangle$ is a homomorphism. Now, $g_C(0) = g_C(h_C(0)) = h_C(g(0)) = h_C(1) = 1$, and, for $x, y \in C$,

$$\begin{aligned}
g_C(x+y) &= g_C(h_C(x+y)) && \text{since } h_C \text{ is a retraction} \\
&= h_C(g(x+y)) && \text{by definition of } g_C \\
&= h_C(g(x) \cdot g(y)) && \text{since } g \text{ is a SUA operator} \\
&= h_C(g(x)) \cdot h_C(g(y)) && \text{since } h_C \text{ is a Boolean homomorphism} \\
&= g_C(h_C(x)) \cdot g_C(h_C(y)) && \text{by definition of } g_C \\
&= g_C(x) \cdot g_C(y) && \text{since } h_C \text{ is a retraction.}
\end{aligned}$$

Thus, $\langle C, g_C \rangle \in$ SUA, and $h_C : B \to C$ is a SUA homomorphism.

Next, let $\langle A, g_A \rangle = \prod_{C \in \mathcal{C}} \langle C, g_C \rangle$, and let π_C be the projection of A to C. The mapping $f : B \to A$ defined by $\pi_C(f(x)) = h_C(x)$ is a SUA homomorphism, since each π_C and each h_C is a SUA homomorphism. All that is left to show is that f is one-one: If $x, y \in B, x \neq y$, we let C be the Boolean subalgebra of B generated by $\{x, y\}$. Now,

$$\pi_C(f(x)) = h_C(x) = x \neq y = h_C(y) = \pi_C(f(y)),$$

which implies $f(x) \neq f(y)$. ☐

4 Canonical extensions

The *canonical extension of a Boolean algebra B* is a complete and atomic Boolean algebra B^σ containing an isomorphic copy of B as a subalgebra with the properties

(4.1) Every atom of B^σ is the meet of elements of B.

(4.2) If $A \subseteq B$ such that $\sum_{B^\sigma} A = 1$, then there is a finite subset of A whose join is 1.

The set $K_B = \{\prod_{B^\sigma} M : M \subseteq B\}$ is called the set of *closed elements of* B^σ, and $(-K)_B = \{-x : x \in K_B\}$ is the *set of open elements*. We will drop the subscript if no confusion can arise.

It is well known, that each Boolean algebra has a canonical extension which is unique up to isomorphism [12]. One such construction is given by Stone's representation theorem for Boolean algebras: Let B^σ be the powerset algebra of the set of ultrafilters $S(B)$ of B, and embed B into B^σ by $h(b) = \{U \in S(B) : b \in U\}$. Observe that h need not preserve infinite joins or meets: If, for example, $U \in S(B)$ is non-principal, then $\prod_B U = 0$, and $\prod_{B^\sigma} h[U] = \{U\}$. Unless stated otherwise, we will suppose in the sequel that B is a subalgebra of B^σ, and that joins and meets are taken in B^σ.

If $f : B \to B$, we let

$$f^\sigma(x) = \sum\left\{\prod\{f(z) : z \in B,\ p \leq z\} : p \in K,\ p \leq x\right\}, \tag{4.3}$$

$$f^\pi(x) = \prod\left\{\sum\{f(z) :\ z \in B,\ p \leq z\} :\ p \in K,\ p \leq x\right\}. \tag{4.4}$$

If f is a modal, respectively, a sufficiency operator, the equations (4.3) and (4.4) have the simpler form

$$f^\sigma(x) = \sum\left\{\prod\{f(z) : z \in B,\ p \leq z\} : p \in At(B^\sigma),\ p \leq x\right\}, \tag{4.5}$$

$$f^\pi(x) = \prod\left\{\sum\{f(z) :\ z \in B,\ p \leq z\} :\ p \in At(B^\sigma),\ p \leq x\right\}. \tag{4.6}$$

In particular, if $p \in At(B^\sigma)$, then we have

$$f^\sigma(p) = \prod\{f(z) : z \in B,\ p \leq z\}, \tag{4.7}$$

$$f^\pi(p) = \sum\{f(z) : z \in B,\ p \leq z\}. \tag{4.8}$$

The proof of the following Lemma is done by simple computation:

Lemma 4.1. *If f is a modal operator, and g a sufficiency operator on B, then*

$$(f^\sigma)^* = (f^*)^\pi, \tag{4.9}$$

$$(g^\pi)^* = (g^*)^\sigma. \tag{4.10}$$

Results, analogous to those for MOA, hold for sufficiency operators:

Proposition 4.2. *(Düntsch & Orłowska [5])*

1. *Extension Theorem: If g is antitone, then g^π is antitone and $g^\pi \upharpoonright B = g$. If g is a sufficiency operator, then g^π is a completely co-additive sufficiency operator.*

2. *Representation Theorem: If $\langle B, g\rangle$ is a sufficiency algebra, then there is (up to isomorphism) a unique frame $K = \langle U, R\rangle$, such that $Cm_{\mathrm{SUA}}(K) \cong \langle B^\sigma, g^\pi\rangle$.*

If $\mathfrak{B} = \langle B, f_1, \ldots, f_n\rangle \in$ MOA, we denote by $\mathfrak{B}^\sigma$ the MOA $\langle B^\sigma, f_1^\sigma, \ldots, f_n^\sigma\rangle$, and call it the *canonical extension of* $\mathfrak{B}$. Similarly, we define the canonical extension of a SUA $\mathfrak{B}$ and denote it by $\mathfrak{B}^\pi = \langle B^\sigma, f_1^\pi, \ldots, f_n^\pi\rangle$. Note that the universes of $\mathfrak{B}^\sigma$ and $\mathfrak{B}^\pi$ are the same.

Proposition 4.3. *If* $\mathfrak{B} \in \text{MOA}$, *then* $\mathfrak{B}^{\sigma\star} = \mathfrak{B}^{\star\pi}$.

Proof. Since * does not change the Boolean part of an algebra and the canonical extensions of the Boolean part are the same both for modal and sufficiency algebras, all we need to show is that for each modal operator f on A we have $f^{\sigma\star} = f^{\star\pi}$. This is just (4.9). □

The relationships between canonical extension and dual operators is as follows:

Proposition 4.4. *[11] Suppose that* $f : B \to B$ *is isotone.*

1. $f^{\partial\sigma} \leq f^{\sigma\partial}$.
2. *If* f *is a modal operator, then* $f^{\partial\sigma} = f^{\sigma\partial}$.

A similar result holds for antitone - resp. sufficiency - operators:

Proposition 4.5. *Suppose that* $g : B \to B$ *is antitone. Then,*

1. $g^{\pi\partial} \leq g^{\partial\pi}$.
2. *If* g *is a sufficiency operator, then* $g^{\partial\pi} = g^{\pi\partial}$.

Proof. 1. Let $g : B \to B$ be antitone. Then, for all $x \in B$,

$$\begin{aligned} g^{\pi\partial}(x) \leq g^{\partial\pi}(x) &\iff -g^{\partial\pi}(x) \leq -g^{\pi\partial}(x) \\ &\iff g^{\partial\pi\star}(x) \leq g^{\pi\partial\star}(x) \\ &\iff g^{\star\partial\sigma}(x) \leq g^{\star\sigma\partial}(x) && \text{by (3.5),} \end{aligned}$$

which is true by Lemma 4.4(1).

2. Let $g : B \to B$ be a sufficiency operator. Then, for all $x \in B$,

$$\begin{aligned} g^{\pi\partial}(x) &= -g^{\pi}(-x) = g^{\pi\star}(-x) = g^{\star\sigma}(-x) \\ &= -g^{\star\sigma\partial}(x) = -g^{\star\partial\sigma}(x) \quad \text{by Lemma 4.4(2)} \\ &= g^{\star\partial\sigma\star}(x) \quad = g^{\star\partial\star\pi}(x) \quad \text{by (4.10)} \\ &= g^{\partial\pi}(x), \end{aligned}$$

which was to be shown. □

Proposition 4.6.

1. *Let* f, g *be modal operators on* B *and* $h : B \to B$ *such that* $h(x) = f(x) + g(x)$. *Then,* h *is a modal operator, and* $h^\sigma(x) = f^\sigma(x) + g^\sigma(x)$. *Furthermore,* $h^{\partial\sigma}(x) = f^{\partial\sigma}(x) \cdot g^{\partial\sigma}(x)$.
2. *Let* f, g *be sufficiency operators on* B *and* $h : B \to B$ *such that* $h(x) = f(x) \cdot g(x)$. *Then,* h *is a sufficiency operator, and* $h^\pi(x) = f^\pi(x) \cdot g^\pi(x)$. *Furthermore,* $h^{\partial\pi}(x) = f^{\partial\pi}(x) + g^{\partial\pi}(x)$.

Proof. 1. This follows from [12], Theorem 2.8

2. It is easily checked that h is a sufficiency operator. For the rest, we need to show that for all $x \in B^\sigma$

$$\begin{aligned}\prod\{\sum\{f(z)\cdot g(z) : z \in B, p \leq z\} : p \in At(B^\sigma), p \leq x\} = \\ \prod\{\sum\{f(z) : z \in B, p \leq z\} : p \in At(B^\sigma), p \leq x\} \cdot \\ \prod\{\sum\{g(z) : z \in B, p \leq z\} : p \in At(B^\sigma), p \leq x\}.\end{aligned}$$

The direction $\leq$ is obvious, and we just show $\geq$: Suppose that $r \in At(B^\sigma)$ such that for all $p \in At(B^\sigma),\ p \leq x$

$$r \leq \sum\{f(z) : z \in B, p \leq z\} \cdot \sum\{g(z) : z \in B, p \leq z\} \tag{4.11}$$

and assume that

$$r \not\leq \prod\{\sum\{f(z)\cdot g(z) : z \in B, p \leq z\} : p \in At(B^\sigma), p \leq x\}.$$

Since r is an atom, there is some $q \in At(B^\sigma)$ such that $r \not\leq \sum\{f(z)\cdot g(z) : z \in B, q \leq z\}$, in other words, we have $r \cdot f(z) \cdot g(z) = 0$ for all $z \in B$ for which $q \leq z$. Since, by our hypothesis, $r \leq \sum\{f(z) : z \in B, q \leq z\} \cdot \sum\{g(z) : z \in B, q \leq z\}$, there are $u, v \in B,\ q \leq u, v$ such that $r \leq f(u) \cdot g(v)$. Observing that f and g are antitone, we obtain that $r \leq f(u\cdot v) \cdot g(u \cdot v)$. The fact, that $q \leq u \cdot v \in B$ now contradicts our assumption $r \cdot f(u\cdot v) \cdot g(u \cdot v) = 0$.

The rest is easily established. $\square$

The preservation of identities of a modal algebra B by its canonical extension B^σ is one of the major questions in the theory of MOA. We will restrict our considerations to algebras with one extra operator.

Let $\mathfrak{C}$ be a class of Boolean algebras with operators. An equation $\tau = \rho$ is said to be $\mathfrak{C}$ – canonical if for every algebra $B \in \mathfrak{C}$, if $\tau = \rho$ is true in B, then it is true in the canonical extension of B. Observe that we have the following fact:

Proposition 4.7. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be classes of Boolean algebras with operators such that $\mathfrak{D} \subseteq \mathfrak{C}$. Then, if an equation $\tau = \rho$ is $\mathfrak{C}$ – canonical, then it is $\mathfrak{D}$ – canonical.*

A class C of Boolean algebras with operators is *canonical* if it is closed under the appropriate canonical extension.

Proposition 4.8. *Let τ, ρ be terms in a language $\mathcal{L}$ defined on p. 6. Then,*

$$\tau = \rho \textit{ is } \text{MOA} - \textit{canonical} \iff \tau^\star = \rho^\star \textit{ is } \text{SUA} - \textit{canonical} \tag{4.12}$$

Proof. Suppose $\tau = \rho$ is MOA - canonical, and $\langle B, f\rangle \models \tau = \rho$. Then,

$$\begin{aligned} &\langle B, f^*\rangle \models \tau^* = \rho^* && \text{by (3.24)} \\ &\langle B^\sigma, f^\sigma\rangle \models \tau = \rho. && \text{since } \tau = \rho \text{ is MOA canonical} \\ &\langle B^\sigma, (f^\sigma)^*\rangle \models \tau^* = \rho^* && \text{by (3.24)} \\ &\langle B, (f^*)^\pi\rangle \models \tau^* = \rho^* && \text{by (4.9)} \end{aligned}$$

Thus, if $\tau = \rho$ is MOA - canonical, then $\tau^* = \rho^*$ is SUA - canonical. Conversely, by an analogous argument, one shows that for every SUA - canonical equation $\tau = \rho$ the equation $\tau^\star = \rho^\star$ is MOA - canonical. $\square$

It was shown in [12] that for modal operators f, g on B,

$$f^\sigma \circ g^\sigma = (f \circ g)^\sigma, \tag{4.13}$$

and the general form of this result was used to show that every positive equation which holds in $\langle B, f\rangle$ also holds in $\langle B^\sigma, f^\sigma\rangle$. Several preservation results for MOA are discussed in[11]. Since the composition of sufficiency operators usually is not a sufficiency operator, a similar result for this class cannot be obtained. What one might hope to show would be a result such as

$$f^\pi \circ g^\pi = (f \circ g)^\sigma.$$

The following example shows that this need not be true in SUA:

Let $\omega = \{0, 1, 2, \dots\}$, $U = \omega \cup \{\omega\}$, and B be the subalgebra of 2^U generated by $\{\{n\} : n \in \omega\}$. Then, B is isomorphic to the finite-cofinite subalgebra of 2^ω, and $2^U \cong B^\sigma$. Define $f : B \to B$ by

$$f(X) = -\{n+1 : n \in X\}.$$

Then, f is a sufficiency operator, and we have

$$\begin{aligned} f^\pi(\{\omega\}) &= \bigcup\{f(X) : X \in B,\ \omega \in X\}, \\ &= \bigcup\{-\{n+1 : n \in X\} : X \in B,\ \omega \in X\}, \\ &= -\{\omega\}, \end{aligned}$$

and

$$\begin{aligned} f^\pi(f^\pi(\{\omega\}) &= f^\pi(-\{\omega\}), \\ &= \bigcap\{\bigcup\{f(X) : n \in X\} : n \in \omega\}, \\ &= \bigcap\{\bigcup\{-\{m+1 : m \in X\} : n \in X\} : n \in \omega\}, \\ &= \bigcap\{-\{n+1 : n \in \omega\}\}, \\ &= \{0, \omega\}. \end{aligned}$$

On the other hand, we have for each $X \in B$

$$\begin{aligned} f(f(X)) &= f(-\{n+1 : n \in X\}), \\ &= -\{m+1 : m \notin \{n+1 : n \in X\}\}, \\ &= -\{m+1 : m-1 \notin X\}, \\ &= \{n+2 : n \in X\}. \end{aligned}$$

Now,

$$\begin{aligned}
(f\circ f)^{\sigma}(\{\omega\}) &= \bigcap\{f(f(X):X\in B,\ \omega\in X\},\\
&= \bigcap\{\{n+2:n\in X\}:X\in B,\ \omega\in X\},\\
&= \{\omega\}.
\end{aligned}$$

Examples of a similar nature can be found in [7].

5 Mixed algebras

Now that we have the concept of a canonical extension, we can look at the algebras arising from algebras of the form $\langle 2^U,\langle R\rangle,[[R]]\rangle$ which we have introduced in [5]. A *mixed algebra* (MIA) is a structure $\langle B,f,g\rangle$, where B is a Boolean algebra, f is a modal operator, g a sufficiency operator, and

$$f^{\sigma}(p)=g^{\pi}(p) \tag{5.1}$$

for all atoms p of B^{σ}. With some abuse of language we will denote the class of mixed algebras by MIA as well. The canonical extension of $\langle B,f,g\rangle\in$ MIA is the algebra $B^{\sigma\pi}=\langle B,f^{\sigma},g^{\pi}\rangle$. If $K=\langle U,R\rangle$ is a frame, the algebra $Cm_{\mathrm{MIA}}(K)=\langle 2^U,\langle R\rangle,[[R]]\rangle$ is called the *full mixed complex algebra of* K. We have shown in [5] that mixed algebras are the appropriate structures for mixed complex algebras:

Proposition 5.1. *For each* MIA $\langle B,f,g\rangle$ *there is (up to isomorphism) a unique frame* $K=\langle U,R\rangle$ *such that* $Cm_{\mathrm{MIA}}(K)\cong\langle B^{\sigma},f^{\sigma},g^{\pi}\rangle$ *and* $xRy\iff x\in\langle R\rangle(\{y\})\cap[[R]](\{y\})$.

A *language* $\mathbb{L}$ *for mixed logic* is a language for classical propositional logic enhanced by unary operator symbols $\langle\ \rangle$ and $[[\ ]]$ which are interpreted as in (3.14) and (3.27), respectively. As with modal or sufficiency algebras, one can easily prove the following result:

Proposition 5.2. *If* φ *is an* $\mathbb{L}$ *- formula, then there is a term* τ_{φ} *in the language of* MIA *such that for any frame* $K=\langle U,R\rangle$*,*

$$K\models\varphi\iff Cm_{\mathrm{MIA}}(K)\models\tau_{\varphi}=U.$$

Before we exhibit some structural (non-) properties of MIA, we prove the following:

Lemma 5.3. *Let* $\langle B,f\rangle\in$ MOA *and* P *be a non-principal ideal of* B *such that* f *is the identity on* P*. Then,* $\langle B,f,g\rangle\notin$ MIA *for any operator* g *on* B*.*

Proof. Suppose that $\langle B,f,g\rangle\in$ MIA for some $g:B\to B$, and set $U=\{-x:x\in P\}$. Let $x\in U$; since P is prime and non-principal, there are $y,z\in P\setminus\{0\}$ such that $y\cdot z=0$ and $y+z\leq x$. Then,

$$\begin{aligned}
&g(y)\leq f(y)=y,\ g(z)\leq f(z)=z,\\
&g(x)\leq g(y+z)=g(y)\cdot g(z)\leq y\cdot z=0.
\end{aligned}$$

Thus,

$$g^\pi(U) = \sum\{g(x) : x \in U\} = 0.$$

On the other hand, $f^\sigma(U) \neq 0$: First, note that $x \in U$ implies $f(x) \in U$; otherwise, $f(x) \in P$, and, since $-x \in P$, we have

$$f(1) = f(x + (-x)) = f(x) + f(-x) = f(x) + (-x) \in P.$$

Since P is non-principal, there is some $z \in P$ such that $f(1) \lneq z$. Then, $z \leq 1$ and $f(1) \lneq z = f(z)$, a contradiction. Thus, $f(x) \in U$ for all $x \in U$, and it follows that

$$f^\sigma(U) = \prod\{f(x) : x \in U\} \neq 0 = g^\pi(U),$$

contradicting (5.1). $\square$

If B is an atomic Boolean algebra, we let $g_{At(B)} : B \to B$ be the mapping

$$g_{At(B)}(x) = \begin{cases} 1, & \text{if } x = 0, \\ x, & \text{if } x \text{ is an atom }, \\ 0, & \text{otherwise.} \end{cases}$$

It is not hard to see that $g_{At(B)}$ is a sufficiency operator, and that $\langle B, f, g_{At(B)}\rangle \in$ MIA if and only if f is the identity on B. We are now ready to prove

Proposition 5.4. *1.* MIA *is not closed under subalgebras.*

2. MIA *is not closed under homomorphic images.*

3. MIA *is not closed under finite products.*

4. MIA*is not closed under ultraproducts.*

Proof. 1. Let B be a Boolean algebra with exactly three atoms a, b, c, and f be the identity on B. Furthermore, let C be the Boolean subalgebra of B with atoms $a + b$ and c. Now, $\langle B, f, g_{At(B)}\rangle \in$ MIA, and $\langle C, f \restriction C, g_{At(B)} \restriction C\rangle$ is a subalgebra of $\langle B, f, g_{At(B)}\rangle$, i.e.closed under all basic operations. However, since $f \restriction C$ is the identity on C, and $g_{At(B)} \restriction C \neq g_{At(C)}$, it cannot be in MIA.

2. Let B, C be as above, and $h : B \to C$ be induced by $h(a) = h(b) = a + b$, $h(c) = c$. Then h is a retraction onto C which preserves all basic operators. Using the same argument as above, the claim follows.

3. Let U be nonempty and finite, R be a nonempty binary relation on U, and B be the mixed complex algebra of $\langle U, R\rangle$. Since U is finite, we have $B = B^\sigma$, $\langle R\rangle = \langle R\rangle^\sigma$, $[[R]] = [[R]]^\pi$ and also $(B \times B)^\sigma = B^\sigma \times B^\sigma$. If $\langle R\rangle^2$ and $[[R]]^2$ are the operators on B^2 arising from $\langle R\rangle$ and $[[R]]$, then, for any atom $\langle \emptyset, \{a\}\rangle$ of $B \times B$ we have

$$\langle R\rangle^2(\langle \emptyset, \{a\}\rangle) = \langle \langle R\rangle(\emptyset), \langle R\rangle(\{a\})\rangle = \langle \emptyset, \langle R\rangle(\{a\})\rangle \neq \langle U, [[R]](\{a\})\rangle = [[R]]^2(\langle \emptyset, \{a\}\rangle),$$

which contradicts (5.1).

4. Let $I = \{n \in \omega : 2 \leq n\}$, and B_n be the finite Boolean algebra with exactly n atoms. For each $n \in I$, we let f_n be the identity on B_n, and $g_n = g_{At(B_n)}$. Then, each $\langle B_n, f_n, g_n\rangle \in$ MIA. Let U be a non-principal ultrafilter on I, and $\langle B, f, g\rangle = \prod_{n\in I}\langle B_n, f_n, g_n\rangle/U$ be the ultraproduct of the algebras $\langle B_n, f_n, g_n\rangle$ over U. Since $f_n(x) = x$ is true for all $n \in I$, f is the identity. B is infinite, and therefore, it contains a non-principal prime ideal. Hence, it cannot be made into a MIA by the preceding Lemma. It follows that MIA is not closed under ultraproducts. □

Since a class which is first order axiomatisable needs to be closed under ultraproducts, we obtain

Corollary 5.5. MIA *is not first order axiomatisable.*

Proposition 5.6. MIA *has a ternary discriminator.*

Proof. We have shown in [5] that the mapping $m : B \to B$ defined by $m(x) = f^{\partial}(x) \cdot g(-x)$ satisfies

$$m(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $x, y, z \in B$, then

$$t(x,y,z) = z \cdot m(-(x \oplus y)) + x \cdot m(-(x \oplus y))$$

is the ternary discriminator, see e.g. [19]. Here, $x \oplus y$ is the symmetric difference. □

We can use this to show a correspondence result. Suppose that $\mathbb{L}$ is a language for mixed logic. Let P_1 be the set of all first order sentences φ in a language with one binary relation symbol and equality, such that there is some formula ψ of $\mathbb{L}$ with

$$\langle U, R\rangle \models \varphi \iff \langle U, R\rangle \models \psi. \tag{5.2}$$

Proposition 5.7. *P_1 is closed under all Boolean connectives.*

Proof. We show by example that P_1 is closed under negation. Suppose that φ is a first order sentence, ψ a formula witnessing (5.2), and $K = \langle U, R\rangle$ a frame. By Proposition 5.2 there is a term τ_ψ in the language of MIA such that for any frame $K = \langle U, R\rangle$

$$K \models \psi \iff Cm_{\mathrm{MIA}}(K) \models \tau_\psi = 1.$$

Now,

$$K \models \neg\varphi \iff K \not\models \varphi \iff Cm_{\mathrm{MIA}}(K) \models \tau_\psi \neq 1 \iff m(\tau_\psi) = 0 \iff -m(\tau_\psi) = 1,$$

where m is the MIA - definable operator of Proposition 5.4. □

A relational property which can be expressed by MIA expression, but not by MOA or SUA alone is antisymmetry [see also 9]:

Proposition 5.8. *Let $K = \langle U, R\rangle$ be a frame. Then,*

$$K \models (\forall x)(\forall y)[xRy \land yRx \Rightarrow x = y] \iff Cm_{\mathrm{MIA}}(K) \models \langle R\rangle([[R]](-X) \cap X) \subseteq X.$$

Proof. "$\Rightarrow$": Suppose that $z \in \langle R\rangle([[R]](-X) \cap X)$ and $z \notin X$. Then, there is some $y \in U$ such that zRy, $-X \subseteq R(y)$, and $y \in X$. Now, $z \notin X$ and $-X \subseteq R(y)$ imply yRx; furthermore, $y \in X$ shows that $z \neq y$.

"$\Leftarrow$": Suppose that zRy, yRz, $z \neq y$, and set $X = U \setminus \{z\}$. Since $y \neq z$, we have $y \in X$; furthermore, yRz implies $-X \subseteq R(y)$, so that $y \in [[R]](-X) \cap X$. Finally, zRy shows that $z \in \langle R\rangle([[R]](-X) \cap X)$, but $z \notin X$. □

This shows that antisymmetry is MIA expressible, and it follows from a construction of [5], Proposition 18, that antisymmetry is not MOA or SUA expressible.

Proposition 5.9. *Let $\langle B, f, g\rangle \in$ MIA. Then, $B \models f(g(-x)\cdot x) \leq x$ implies $B^{\sigma\pi} \models f^{\sigma}(g^{\pi}(-x)\cdot x) \leq x$.*

Proof. Let $K = \langle U, R\rangle$ such that $Cm_{\mathrm{MIA}}(K) \cong \langle B^{\sigma}, f^{\sigma}, g^{\pi}\rangle$ and $xRy \iff x \in \langle R\rangle(\{y\}) \cap [[R]](\{y\})$; these exist by Proposition 5.1. Assume that $B \models f(g(-x)\cdot x) \leq x$, but $B^{\sigma\pi} \not\models f^{\sigma}(g^{\pi}(-x)\cdot x) \leq x$; then, $Cm_{\mathrm{MIA}}(K) \not\models \langle R\rangle([[R]](-X) \cap X) \subseteq X$. By Proposition 5.8, R is not antisymmetric, and thus, there are $a, b \in U$ such that aRb, bRa, and $a \neq b$. In particular, $a \in \langle R\rangle(\{b\})$ and $b \in [[R]](\{a\})$, and it follows that there are $p, q \in At(B^{\sigma\pi})$ such that $p \neq q$ and

$$p \leq f^{\sigma}(q), \tag{5.3}$$

$$q \leq g^{\pi}(p). \tag{5.4}$$

From (4.7) and (5.3) we obtain

$$(\forall z \in B)[q \leq z \Rightarrow p \leq f(z)], \tag{5.5}$$

and (4.8) and (5.4) give us

$$(\exists t \in B)[p \leq -t \text{ and } q \leq g(-t)]. \tag{5.6}$$

Furthermore, $p \neq q$ implies that there is some $s \in B$ such that

$$q \leq s,\ p \leq -s. \tag{5.7}$$

Now, we set $x = s + t$. Then, from (5.6) and (5.7),

$$q \leq x,\ p \leq -x. \tag{5.8}$$

Since $t \leq x$, and g is antitone, we have $g(-t) \leq g(-x)$, and thus, (5.6) and (5.8) imply that $q \leq g(-x)\cdot x$. It follows from (5.5) and our hypothesis $f(g(-x)\cdot x) \leq x$ that $p \leq f(g(-x)\cdot x) \leq x$, contradicting (5.8). □

Table 2: Subclasses of BARO

SMOA Strong modal algebras	**WMOA** Weak modal algebras
1. f_P is a modal operator. 2. If $x \neq 0$, then $f_\emptyset(x) = 1$. 3. $f^\sigma_{P\cup Q}(p) = f^\sigma_P(p) \cdot f^\sigma_Q(p)$ for all $p \in At(B^\sigma)$.	1. f_P is a modal operator. 2. $f_\emptyset(x) = 0$. 3. $f_{P\cup Q}(x) = f_P(x) + f_Q(x)$
SDMOA Strong dual modal algebras	**WDMOA** Weak dual modal algebras
1. f_P is a dual modal operator. 2. If $x \neq 1$, then $f_\emptyset(x) = 0$. 3. $f^\sigma_{P\cup Q}(-p) = f^\sigma_P(-p) + f^\sigma_Q(-p)$ for all $p \in At(B^\sigma)$.	1. f_P is a dual modal operator. 2. $f_\emptyset(x) = 1$. 3. $f_{P\cup Q}(x) = f_P(x) \cdot f_Q(x)$
SSUA Strong sufficiency algebras	**WSUA** Weak sufficiency algebras
1. g_P is a sufficiency operator. 2. $g_\emptyset(x) = 1$. 3. $g_{P\cup Q}(x) = g_P(x) \cdot g_Q(x)$.	1. g_P is a sufficiency operator. 2. If $x \neq 0$, then $g_\emptyset(x) = 0$. 3. $g^\pi_{P\cup Q}(p) = g^\pi_P(p) + g^\pi_Q(p)$ for all $p \in At(B^\sigma)$.
SDSUA Strong dual sufficiency algebras	**WDSUA** Weak dual sufficiency algebras
1. g_P is a dual sufficiency operator. 2. $g_\emptyset(x) = 0$. 3. $g_{P\cup Q}(x) = g_P(x) + g_Q(x)$.	1. g_P is a dual sufficiency operator. 2. If $x \neq 1$, then $g_\emptyset(x) = 1$. 3. $g^\pi_{P\cup Q}(-p) = g^\pi_P(-p) \cdot g^\pi_Q(-p)$ for all $p \in At(B^\sigma)$.
SMIA Strong mixed algebras	**WMIA** Weak mixed algebras
1. $\langle B, \{f_P : P \subseteq A\}\rangle$ is in SMOA. 2. $\langle B, \{g_P : P \subseteq A\}\rangle$ is in SSUA. 3. $f^\sigma_p = g^\pi_p$ for all $p \in At(B^\sigma)$.	1. $\langle B, \{f_P : P \subseteq A\}\rangle$ is in WMOA. 2. $\langle B, \{g_P : P \subseteq A\}\rangle$ is in WSUA. 3. $f^\sigma_p = g^\pi_p$ for all $p \in At(B^\sigma)$.

6 Relative operators

A *Boolean algebra with relative operators* (BARO) is a structure $\mathfrak{B} = \langle B, +, \cdot, -, 0, 1, \{h_P : P \subseteq \mathrm{PAR}\}\rangle$ such that $\langle B, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra and each h_P is a unary operator on B. We will usually identify algebras with their underlying set; for example, we will write $\langle B, \{h_P : P \subseteq \mathrm{PAR}\}\rangle$ for a BARO. With some abuse of language, we denote the class of all Boolean algebras with relative operators by BARO as well. The operators h_P will usually be modal or sufficiency operators; we will assume the convention, that we write f_P for modal operators, and g_P for sufficiency operators. There will be certain connections between the set operations on the set 2^{PAR} and the properties of the corresponding operators. We have already encountered these global conditions with the strong and weak frames on p. 3. Several subclasses of BARO are defined in Table 2.

Proposition 6.1. *If $\langle B, \{h_P : P \subseteq A\}\rangle$ is in one of the classes of Table 2, then so is its appropriate canonical extension.*

Proof. It follows from Proposition 4.2, that for each type above, f^σ or g^π is in the same class of operators as f or g. Therefore, condition 1. of the definition of the classes is fulfilled for the operators in question. We will only prove several exemplary cases; the others are either analogous or immediately clear.

WMOA: Let $x \in B^\sigma$. Then,

$$\begin{aligned} f_\emptyset^\sigma(x) &= \sum\{\prod\{f_\emptyset(z) : z \in B,\ p \leq z\} : p \in At(B^\sigma), p \leq x\}, \\ &= 0, \end{aligned}$$

since by our assumption, $f_\emptyset(z) = 0$ for all $z \in B$.

Condition 3. follows from Lemma 4.6.1 and $f_{P\cup Q}(x) = f_P(x) + f_Q(x)$.

SDMOA Let $1 \neq x \in B^\sigma$. Then,

$$\begin{aligned} f_\emptyset^\sigma(x) &= \prod\left\{\sum\{-f_\emptyset^\partial(z) : z \in B,\ p \leq z\} : p \in At(B^\sigma), p \leq -x\right\} \\ &= \prod\{\sum\{f_\emptyset(-z) : z \in B,\ p \leq z\} : p \in At(B^\sigma), p \leq -x\} \\ &= 0, \end{aligned}$$

since there is some $p \in At(B^\sigma)$ with $p \leq -x$ by $x \neq 1$, and $f_\emptyset(-z) = 0$ for all $0 \neq p \leq z$.

Condition 3. follows from the fact that the operator of taking canonical extensions is idempotent.

SSUA Let $x \in B^\sigma$. Then,

$$\begin{aligned} g_\emptyset^\sigma(x) &= \prod\{\sum\{g_\emptyset(z) :\ z \in B,\ p \leq z\} :\ p \in At(B^\sigma), p \leq x\}, \\ &= 1, \end{aligned}$$

since by our assumption, $g_\emptyset(z) = 1$ for all $z \in B$.

Condition 3. follows from Lemma 4.6.1 and $g_{P\cup Q}(x) = g_P(x) \cdot g_Q(x)$.

SMIA, WMIA: This follows from the considerations above and the fact that the operator of taking canonical extensions is idempotent. □

Algebraic counterparts to information frames are

$$\begin{aligned} \mathfrak{B} &= \langle B, \{f_P : P \subseteq A\}\rangle \in \text{MOA}, \\ \mathfrak{B} &= \langle B, \{g_P : P \subseteq A\}\rangle \in \text{SUA}, \\ \mathfrak{B} &= \langle B, \{f_P : P \subseteq A\}, \{g_P : P \subseteq A\}\rangle \in \text{MIA}, \end{aligned}$$

where the functions f_P (g_P) are modal (sufficiency) operators. The classes corresponding to the frames with the relations of Table 1 are listed in Table 3, and we have

Proposition 6.2. *Each class of algebras listed in Table 3 is canonical.*

Table 3: Examples of BARO classes

Strong similarity algebras	**Weak similarity algebras**
1. $\mathfrak{B} \in$ **SMOA**.	1. $\mathfrak{B} \in$ **WMOA**.
2. $x \leq f_P^\partial f_P(x)$.	2. $x \leq f_P^\partial f_P(x)$.
3. $f_P(1) \cdot x \leq f_P(x)$.	3. $f_P(1) \cdot x \leq f_P(x)$.
Strong disjointness algebras	**Weak disjointness algebras**
1. $\mathfrak{B} \in$ **SSUA**.	1. $\mathfrak{B} \in$ **WSUA**.
2. $x \leq g_P g_P(x)$.	2. $x \leq g_P g_P(x)$.
3. $g_P(x) \cdot x \leq g_P(1)$.	3. $g_a(x) \cdot x \leq g_a(1)$.
Strong complementarity algebras	**Weak complementarity algebras**
1. $\mathfrak{B} \in$ **SMIA**.	1. $\mathfrak{B} \in$ **WMIA**.
2. $x \leq g_P g_P(x)$.	2. $x \leq g_P g_P(x)$.
3. $g_P(x) \leq -x$, if $P \neq \emptyset$.	3. $g_P(x) \leq -x$.
4. $f_P f_P f_P(x) \leq f_P(x)$.	4. $f_a f_a f_a(x) \leq f_a(x)$.

Proof. Observe that each of the axioms for the classes of algebras in Table 3 can be presented as an equation which is MOA – canonical [8, 11], or it leads to a SUA canonical equation after the transformation *, defined on p. 3. For example,

$$x \leq f_P^\partial f_P(x)$$

is canonical, since the inequalities

$$x \leq f(x), \tag{6.1}$$

$$x \leq f^\partial(x) \tag{6.2}$$

are canonical: Observing that $f(x)$ is a positive term, hence expanding by Theorem 5.5 of [11], and x is a contracting term, we conclude from Proposition 1.3 of [11] that (6.1) is canonical. Since f is a modal operator, $f(x)$ is isotone, and thus, Theorem 5.3. of [11] tells us that (6.2) is canonical as well. □

7 Conclusion and outlook

In this paper we have continued the study of extensions of the classical theory of Boolean algebras with normal and additive operators (BAO). The extensions we propose are motivated by a quest for algebraic tools for representation of and reasoning about incomplete information. We have shown that various classes of relations that can be derived from any collection of data in the form of an object - properties assignment leads in a natural way to the corresponding classes of Boolean algebras endowed with relative (i.e. indexed with subsets of a set) modal, sufficiency or mixed operators. We have investigated the underlying classes of algebras along the lines of the methodology of the classical BAO theory. Further work on the problem of closure of various classes of sufficiency and mixed algebras derived from information systems under the appropriate canonical extensions is needed. Proposition 4.8 opens the way to obtain general closure results for classes of sufficiency algebras through reformulation of the corresponding results for modal algebras, e.g. a Sahlqvist result. The closure results for mixed algebras are an open problem. An extended list of Boolean algebras with relative operators as well as the associated logics, can be found in [2].

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