COORDINATE SUBSPACE ARRANGEMENTS and MONOMIAL IDEALS

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Abstract: We relate the (co)homological properties of real coordinate subspace arrangements and of monomial ideals.

1. Introduction

In [PRW] we describe the cohomological properties of a real diagonal subspace arrangement via a minimal free resolution over a certain quotient of a polynomial ring by a monomial ideal. Here we relate the (co)homological properties of two objects: square-free monomial ideals and real coordinate subspace arrangements. The interest in studying such arrangements comes from the facts that they provide examples of arbitrary torsion in the cohomology of the complement of the arrangement [Bj] and the complements provide examples of manifolds with properties similar to toric varieties [DJ], and toric varieties as quotients (see for example [BCo]). A comparison of our formula [GPW, Theorem 2.1] for monomial ideals with the Goresky-MacPherson Formula [GM, III.1.5. Theorem A] for the cohomology of the complement of a subspace arrangement leads to Theorem 3.1. This result states that the i-dimensional cohomology of the complement of a real coordinate subspace arrangement is computed by the Betti numbers in the i-strand in the minimal free resolution of a certain square-free monomial ideal. In Corollaries 3.4 and 3.5 we show how this reveals an equivalence of results, which on

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the one hand were proved for subspace arrangements by Björner [Bj] and on the other hand were recently proved for monomial ideals by Eagon-Reiner and Terai [ER, Te]: Very recently Terai obtained a formula which expresses the regularity of a square-free monomial ideal in terms of the projective dimension of another monomial ideal and which immediately implies that the regularity of a monomial ideal is bounded by its arithmetic degree; Corollary 3.4 shows that Terai's formula is equivalent to Björner's result [Bj, Theorem 11.2.1(ii)].

Motivated by our work Babson-Chan [BCh] proved that the cohomology algebra of the complement of the complexification of a coordinate subspace arrangement is isomorphic to the Tor-algebra of a monomial ideal, see Remark 3.8; later the same result was proved by Buchstaber and Panov.

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2. Multigraded Betti numbers

In this section we recall how to obtain the multigraded Betti numbers of a monomial ideal using the lcm-lattice. Consider the polynomial ring $S = k[x_1, \dots, x_n]$ over a field k as $\mathbf{N}^n$-graded by letting $\deg(x_i)$ be the i^{th} standard basis vector in $\mathbf{R}^n$. Let I be a monomial ideal minimally generated by monomials $m_1, \dots, m_d$. The ideal I and the minimal free resolution of S/I over S are $\mathbf{N}^n$-graded. Therefore we have $\mathbf{N}^n$-graded Betti numbers

$$b_{i,\mathbf{x}^\alpha}(S/I) \,=\, \dim_k \operatorname{Tor}^S_{i,\alpha}(S/I, k)$$

for $i \geq 0$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The *lcm-lattice* L_I of I is the partially ordered set on the set of least common multiples $\operatorname{lcm}(B)$ of all subsets $B \subseteq \{m_1, \dots, m_r\}$ ordered by divisibility. Clearly, L_I is a lattice (i.e., infima and suprema exist) with $1 = \operatorname{lcm}(\emptyset)$ as its minimal element and $\operatorname{lcm}(m_1, \dots, m_r)$ as its maximal element. Taylor's resolution (cf. [Ei, p. 439]) shows that $b_{i,m}(S/I) = 0$ if $m \notin L_I$.

Let L be a lattice with minimal element $\hat{0}$ and $p \in L$. We write $(\hat{0}, p)_L$ for the *open interval* $\{q \in L \mid \hat{0} < q < p\}$ in L. In particular, for $m \in L_I$ we denote by $(\hat{0}, m)_{L_I}$ the open lower interval in L_I below m.

Theorem 2.1. [GPW, Theorem 2.1] *For $i \geq 1$ and $m \in L_I$ we have*

$$b_{i,m}(S/I) \;=\; \dim \widetilde{\mathrm{H}}_{i-2}\big((\hat{0}, m)_{L_I}; k\big)\,.$$

We present a short application of the above result. We apply Theorem 2.1 to a class of specific monomial ideals similar to those studied in [BH, Section 6]. If I is square-free, then L_I can be identified with a join-sublattice of the Boolean lattice on an n-element set generated by the supports of the monomials generating I. Let $M_{n,\ell}$ be the monomial ideal generated by the monomials with support $[i, i+\ell-1]$ for $i = 1, \dots, n-\ell+1$. Then $L_{M_{n,\ell}}$ is isomorphic to the interval generated sublattice of the Boolean lattice generated by $[i, i+\ell-1]$ for $i = 1, \dots, n-\ell+1$. By Björner-Wachs [BW, Corollary 8.4] we get

$$\Delta(L^{\circ}_{M_{n,\ell}}) \simeq \begin{cases} S^{2n/(\ell+1)-2}, & \text{if } n \equiv 0 \pmod{\ell+1} \\ S^{2(n+1)/(\ell+1)-3}, & \text{if } n \equiv -1 \pmod{\ell+1} \\ \mathbf{pt}., & \text{otherwise.} \end{cases}$$

If $m \in L_{M_{n,\ell}}$, then the support set of m is the disjoint union of intervals $A_i = [j_i, l_i]$, $i = 1, \dots, r$ with $l_i+2 \leq j_{i+1}$. For $n_i = l_i - j_i + 1$ we then have $(\hat{0}, m)_{L_{M_{n,\ell}}} \cong \Big(L_{n_1,\ell} \times \cdots \times L_{n_q,\ell}\Big)^{\circ}$. Set

$$p = 2 + \sum_{n_i \equiv 0 \mod \ell+1} (2n/(\ell+1)-2) + \sum_{n_i \equiv -1 \mod \ell+1} (2(n+1)/(\ell+1)-2) + 2(\ell-1).$$

We conclude that $b_{i,m}(S/M_{n,\ell}) = 0$ if and only if there exists a j such that $n_j \not\equiv 0, -1 \pmod{\ell+1}$ or $i \neq p$; otherwise $b_{i,m} = 1$.

Proposition 2.2. *Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in L_I. For $m' = x_1^{\beta_1} \cdots x_n^{\beta_n}$ strictly dividing m we define $s(m') = \{i \mid \alpha_i = \beta_i\} \subseteq [n]$. Let $T(m)$ be the poset of subsets of $[n]$ obtained from $(\hat{0}, m)_{L_I}$ by applying the map s. Then*
(a) $(\hat{0}, m)_{L_I}$ and $T(m)$ are homotopy equivalent.
(b) $\widetilde{H}_i((\hat{0}, m)_{L_I}; k) = 0$ for $i > n-2$.

(c) Suppose that there exists a minimal monomial generator g of I such that for each $1 \leq i \leq n$ we have that x_i^p divides g implies that x_i^{p+1} divides m. Then $\widetilde{H}_i((\hat{0}, m)_{L_I}; k) = 0$ for $i \geq 0$.

In view of Theorem 2.1, we see that Proposition 2.2(b) gives a combinatorial proof of Hilbert's Syzygy Theorem for monomial ideals (cf. [Ei, Corollary 19.7]), and Proposition 2.2(c) is an analogue to [BPS, Theorem 3.2].

Proof. Let $A \in T(m)$ be a set. Then the lower fiber $s^{-1}\big(\{B \in T(m) \mid B \subseteq A\}\big)$ has the lcm of all minimal generators m' of I with $s(m') \subseteq A$ as its maximal element. In particular, the lower fiber is contractible. Applying Quillen's Fiber Lemma [Bj2, Theorem 10.5] we conclude that $(\hat{0}, m)_{L_I}$ and $T(m)$ are homotopy equivalent. This proves (a).

The claim (b) holds since the order complex $\Delta(T(m))$ has dimension $\leq n-2$. Finally, note that under the assumption of (c), $T(m)$ has the empty set $\emptyset = s(g)$ as its least element and therefore is contractible. $\square$

3. (Co)homology of real coordinate subspace arrangements and square-free monomial ideals

In this section we relate the (co)homological properties of real coordinate subspace arrangements and of square-free monomial ideals.

Let Δ be a simplicial complex on the vertex set $[n]$ and $F(\Delta)$ the set of facets (i.e., maximal faces) of Δ. Fix an orthonormal basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$. The *real coordinate subspace arrangement* defined by Δ is

$$\mathcal{K}_\Delta = \big\{ \mathrm{span}(e_j \,|\, j \in \sigma) \,\big|\, \sigma \in F(\Delta) \big\}.$$

The *union* $\mathcal{V}_\Delta = \bigcup_{\sigma \in F(\Delta)} \mathrm{span}(e_j \,|\, j \in \sigma)$ is a real algebraic variety. We denote by $\widehat{\mathcal{V}}_\Delta$ the one-point compactification of $\mathcal{V}_\Delta$ inside the unit n-sphere (which is the one-point compactification of $\mathbf{R}^n$) and by $\mathcal{M}_\Delta = \mathbf{R}^n \backslash \mathcal{K}_\Delta$ the set-theoretic complement of the arrangement in $\mathbf{R}^n$. Furthermore, we denote by $\mathcal{L}_\Delta$ the intersection lattice of the arrangement $\mathcal{K}_\Delta$; it consists of all intersections $\big\{ \bigcap_{V \in \mathcal{B}} V \;\big|\; \mathcal{B} \subseteq \mathcal{K}_\Delta \big\}$ ordered by reversed inclusion. In particular, the intersection corresponding to

$\mathcal{B} = \mathcal{K}_\Delta$ serves as the maximal element and the intersection corresponding to $\mathcal{B} = \emptyset$ is regarded as $\mathbf{R}^n$ and serves as the minimal element in $\mathcal{L}_\Delta$.

On the other hand consider the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k. Let I be a monomial ideal minimally generated by monomials $m_1, \ldots, m_d$. The role of the intersection lattice is played by the lcm-lattice L_I with elements the least common multiples of $m_1, \ldots, m_d$ ordered by divisibility. We define the total degree of x_i by $\operatorname{tdeg}(x_i) = 1$ for $1 \le i \le n$. The $\mathbf{N}$-graded Betti numbers lead to *bigraded Betti numbers* $b_{i,j}(S/I) = \dim_k \operatorname{Tor}^S_{i,j}(S/I, k) = \sum_{\substack{\operatorname{tdeg}(m)=j \\ m \in L_I}} b_{i,m}(S/I)$. The *total Betti numbers* are defined by $b_i(S/I) = \sum_{j \ge 0} b_{i,j}(S/I)$ for $i \ge 0$. The complexity of the resolution is measured by its length $\operatorname{pd}(S/I) = \max\{\, i \mid b_i(S/I) \neq 0 \,\}$ and also by the invariant $\operatorname{reg}(S/I) = \max \left\{ j - i \,|\, b_{i,j}(S/I) \neq 0 \right\}$, called the *regularity*.

Let

$$\Delta^\vee = \{\, [n] \setminus \{j_1, \ldots, j_p\} \mid \{j_1, \ldots, j_p\} \notin \Delta \,\}$$

be the Alexander dual complex of Δ. The Stanley-Reisner ideal of $\Delta^\vee$ is

$$\begin{aligned} I_{\Delta^\vee} &= \left(\left\{ \frac{x_1 \cdots x_n}{x_{j_1} \cdots x_{j_p}} \,\middle|\, \{j_1, \ldots, j_p\} \in \Delta \right\} \right) \\ &= \left(\left\{ \text{monomial } t \,\middle|\, \gcd(t, t') \neq 1 \text{ for each monomial } t' \in I_\Delta \right\} \right). \end{aligned}$$

Note that we have

$$I_{\Delta^\vee} = \left(\left\{ x_{j_1} \cdots x_{j_p} \,\middle|\, x_{j_1} = \cdots = x_{j_p} = 0 \text{ defines a subspace in } \mathcal{K}_\Delta \right\} \right).$$

Theorem 3.1. *Let Δ be a simplicial complex on the vertex set $[n]$, $\mathcal{K}_\Delta$ the real coordinate subspace arrangement defined by Δ, and $I_{\Delta^\vee}$ the Stanley-Reisner monomial ideal associated to $\Delta^\vee$. We have*

$$\dim \widetilde{\mathrm{H}}_{n-1-i}(\widehat{\mathcal{V}}_\Delta; k) = \dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) = \sum_{j \ge 0} b_{j,i+j}(S/I_{\Delta^\vee}) \quad \text{for } i \ge 0,$$

$$\max\{\, j \mid \widetilde{\mathrm{H}}^j(\mathcal{M}_\Delta; k) \neq 0 \,\} = \operatorname{reg}(S/I_{\Delta^\vee}).$$

Thus $\dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k)$ picks up the cohomology of the i-strand in the minimal free resolution of $S/I_{\Delta^\vee}$. Therefore, the dimensions of the cohomology groups can be computed in concrete examples by the computer algebra system Macaulay 2 [GS] by computing the bigraded Betti numbers of $S/I_{\Delta^\vee}$ and then applying Theorem 3.1. Also, Theorem 3.1 yields $\dim \widetilde{\mathrm{H}}^*(\mathcal{M}_\Delta; k)$ for some special types of arrangements when explicit formulas for the Betti numbers of the corresponding monomial ideals are known; for example, a result of Bayer-Peeva-Sturmfels gives the Betti numbers for polarizations of generic monomial ideals and a result of Aramova-Herzog-Hibi provides the Betti numbers for weakly stable arrangements.

Proof of Theorem 3.1: For a lattice L with minimal element $\hat{0}$ we write $L_{>\hat{0}}$ for the poset obtained from L by removing the minimal element $\hat{0}$.

The equalities $\dim \widetilde{\mathrm{H}}_{n-1-i}(\widehat{\mathcal{V}}_\Delta; k) = \dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k)$ are well known and are proved by Alexander duality. Applying a formula of Goresky-MacPherson [GM, III.1.5. Theorem A] to the intersection lattice $\mathcal{L}_\Delta$ of $\mathcal{K}_\Delta$ we get

$$\dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) = \sum_{m \in (\mathcal{L}_\Delta)_{>\hat{0}}} \dim \widetilde{\mathrm{H}}_{\operatorname{codim}(m)-2-i}((\hat{0}, m)_{\mathcal{L}_\Delta}; k)\,.$$

Note that $\mathcal{L}_\Delta$ is the lattice of all non-empty intersections of facets of Δ ordered by reversed inclusion and enlarged by an additional minimal element $\hat{0}$ and maximal element $\hat{1}$. By Proposition 2.3, the facets of Δ correspond bijectively to the minimal monomial generators of $I_{\Delta^\vee}$. Furthermore, if $\sigma_1, \ldots, \sigma_r$ are facets of Δ, then we identify

$$\text{(3.2)} \qquad \bigcap_{1 \le i \le r} \sigma_i \in \mathcal{L}_\Delta \quad \longleftrightarrow \quad \operatorname{lcm}\left(\frac{x_1 \cdots x_n}{\mathbf{x}_{\sigma_i}} \,\Big|\, 1 \le i \le r \right) \in L_{I_{\Delta^\vee}}\,.$$

Thus $\mathcal{L}_\Delta$ coincides with the lcm-lattice $L_{I_{\Delta^\vee}}$ of the monomial ideal $I_{\Delta^\vee}$. Also, (3.2) yields that

$$\operatorname{codim}\Big(\bigcap_{1 \le i \le r} \mathcal{K}_{\sigma_i} \Big) = n - \Big| \bigcap_{1 \le i \le r} \sigma_i \Big| = \left| \operatorname{supp}\left(\operatorname{lcm}\left(\frac{x_1 \cdots x_n}{\mathbf{x}_{\sigma_i}} \,\Big|\, 1 \le i \le r \right)\right) \right|.$$

Therefore, $\dim \widetilde{\mathrm{H}}_{\operatorname{codim}(m)-2-i}((\hat{0}, m)_{\mathcal{L}_\Delta}; k) = \widetilde{\mathrm{H}}_{\operatorname{tdeg}(m)-2-i}((\hat{0}, m)_{L_{I_{\Delta^\vee}}}; k)$, where m is considered as an element in $(\mathcal{L}_\Delta)_{>\hat{0}}$ on the left-hand side of the formula and m is considered as an element in $(L_{I_{\Delta^\vee}})_{>\hat{0}}$ on the right-hand side of the formula.

By Theorem 2.1 there are equalities $\dim \widetilde{\mathrm{H}}_{j-2}\big((\hat{0},m)_{L_{I_{\Delta^\vee}}};k\big) \;=\; b_{j,m}(S/I_{\Delta^\vee})$ for $j \geq 1$; also note that $b_{0,m} = 0$ for $m \in (L_{I_{\Delta^\vee}})_{>\hat{0}}$. Combining this with the Goresky-MacPherson formula above we obtain the equalities

$$\dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta;k) \;=\; \sum_{m\in(L_{I_{\Delta^\vee}})_{>\hat{0}}} b_{\mathrm{tdeg}(m)-i,m}(S/I_{\Delta^\vee})$$

for $i \geq 0$. Taylor's (possibly non-minimal) resolution of $S/I_{\Delta^\vee}$ (cf. [Ei, p. 439] or the proof of Theorem 3.3) implies that $b_{i,m}(S/I_{\Delta^\vee}) = 0$ if $m \notin L_{I_{\Delta^\vee}}$. Therefore,

$$\sum_{m\in(L_{I_{\Delta^\vee}})_{>\hat{0}}} b_{\mathrm{tdeg}(m)-i,m}(S/I_{\Delta^\vee}) \;=\; \sum_{j\geq 0} b_{j,i+j}(S/I_{\Delta^\vee}).$$

Thus $\dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta;k) \;=\sum_{j\geq 0} b_{j,i+j}(S/I_{\Delta^\vee})$ as desired. The statement about the regularity of $S/I_{\Delta^\vee}$ follows immediately. ☐

The proof of Theorem 3.1 is based on an identification of the intersection lattice of an arrangement with the lcm-lattice of a monomial ideal, and then comparison of Theorem 2.1 with the Goresky-MacPherson Formula. The important point is that the codimension of an element in the intersection lattice equals the total degree of this element in the lcm-lattice. Such a construction can be also built for arrangements other than the real coordinate subspace arrangements.

Corollary 3.3. *The following two properties are equivalent:*
(a) $\min\big\{\, i \;\big|\; \widetilde{\mathrm{H}}_i(\widehat{\mathcal{V}}_\Delta;k) \neq 0 \,\big\} \;=\; \mathrm{depth}(S/I_\Delta)$*;*
(b) $\mathrm{pd}(S/I_\Delta) - 1 = \mathrm{reg}(S/I_{\Delta^\vee})$.

Property (a) is proved to hold by Björner [Bj1, Theorem 11.2.1(ii)]. Property (b) is proved by Terai [Te].

Proof: On the one hand we have the following equalities:

$$\begin{aligned}\mathrm{reg}(S/I_{\Delta^\vee}) \;&=\; \max\big\{\, i \mid \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta;k) \neq 0 \,\big\}\\ &=\; \max\big\{\, i \;\big|\; \widetilde{\mathrm{H}}_{n-1-i}(\widehat{\mathcal{V}}_\Delta;k) \neq 0 \,\big\}\\ &=\; n-1-\min\big\{\, j \;\big|\; \widetilde{\mathrm{H}}_j(\widehat{\mathcal{V}}_\Delta;k) \neq 0 \,\big\}.\end{aligned}$$

On the other hand, the Auslander-Buchsbaum equality implies that

$$\operatorname{pd}(S/I_{\Delta^\vee}) - 1 \,=\, n - 1 - \operatorname{depth}(S/I_\Delta)\,.$$

Therefore, (a) and (b) are equivalent. □

A particular case of Corollary 3.3 says that the following two properties are equivalent:

(a) Δ is Cohen-Macaulay if and only if $\dim \widetilde{\mathrm{H}}_i(\widehat{\mathcal{V}}_\Delta; k) = 0$ for all $i \leq \dim(\Delta)$;

(b) Δ is Cohen-Macaulay if and only if the minimal free resolution of $I_{\Delta^\vee}$ is linear.

Property (a) is proved to hold by Björner [Bj1, Theorem 11.2.2]. Property (b) is proved to hold by Eagon and Reiner [ER]; it provides a topological characterization of the linearity of a monomial resolution.

Corollary 3.4. *The following two properties are equivalent:*

(a) $\max\left\{ i \,\middle|\, \widetilde{\mathrm{H}}_i(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \right\} \,=\, \dim(S/I_\Delta)$*;*

(b) $\min\left\{ i \,\middle|\, b_{j,j+i} \neq 0 \text{ for some } j \right\}$ *is the minimal degree of a minimal monomial generator of* $I_{\Delta^\vee}$ *minus one.*

Property (a) holds by Björner [Bj1, Theorem 11.2.1(i)]. It is easy to check that Property (b) holds.

Proof: On the one hand, we have the following equalities:

$$\begin{aligned}\min\left\{ i \,\middle|\, b_{j,j+i} \neq 0 \text{ for some } j \right\} \,&=\, \min\left\{ i \,\middle|\, \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) \neq 0 \right\} \\ &=\, \min\left\{ i \,\middle|\, \widetilde{\mathrm{H}}_{n-1-i}(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \right\} \\ &=\, n - 1 - \max\left\{ j \,\middle|\, \widetilde{\mathrm{H}}_j(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \right\}.\end{aligned}$$

On the other hand we have that

$$\begin{aligned}&\min\left\{ \text{ degree of a minimal monomial generator of } I_{\Delta^\vee} \right\} - 1 \\ &\quad= n - \max\left\{ |\sigma| \,\middle|\, \sigma \text{ is a facet of } \Delta \right\} - 1 \\ &\quad= n - 1 - \dim(S/I_\Delta)\end{aligned}$$

Therefore, (a) and (b) are equivalent. □

Remark 3.5. *(Complexification)*

Fix a standard basis $f_1, \ldots, f_n$ of $\mathbf{C}^n$. The *complexification* of $\mathcal{K}_\Delta$ is the complex coordinate subspace arrangement

$$\mathcal{K}_\Delta \otimes \mathbf{C} \,=\, \big\{ \mathrm{span}(f_j \,|\, j \in \sigma) \,\big|\, \sigma \in F(\Delta) \,\big\}\,.$$

Denote $\mathcal{M}_\Delta \otimes \mathbf{C} = \mathbf{C}^n \setminus \big(\mathcal{K}_\Delta \otimes \mathbf{C}\big)$. The algebraic analogue of this complexification is the ring $S^{\bullet 2} = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and the ideal

$$I^{\bullet 2}_{\Delta^\vee} \,=\, \big(x_{i_1} \cdots x_{i_s} y_{i_1} \cdots y_{i_s} \,|\, x_{i_1} \cdots x_{i_s} \text{ is a minimal monomial generator of } I_{\Delta^\vee} \,\big)\,.$$

Theorem 3.1 shows that

$$\widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta \otimes \mathbf{C}; k) \,\cong \bigoplus_{j \geq 0} \mathrm{Tor}^{S^{\bullet 2}}_{j, i+j}(S^{\bullet 2}/I^{\bullet 2}_{\Delta^\vee}, k)\,.$$

The ideal $I^{\bullet 2}_{\Delta^\vee}$ can be depolarized by setting $x_i = y_i$ for $1 \leq i \leq n$; in this way, one obtains a version of the above formula over the ring S. Motivated by our work, Babson and Chan proved the following result:

Theorem. **[BCh]** *The rings* $\widetilde{H}^*(\mathcal{M}_\Delta \otimes \mathbf{C}; k)$ *and* $\mathrm{Tor}^{S^{\bullet 2}}_{*,*}(S^{\bullet 2}/I^{\bullet 2}_{\Delta^\vee}, k)$ *are isomorphic if the characteristic of* k *is not* 2.

This theorem shows that in the case of a complex coordinate subspace arrangement, the Koszul complex computing $\mathrm{Tor}^S_{*,*}(S/I, k)$ provides a much simpler model for the cohomology ring than the models in [DP] and [Yu].

An analogue of Theorem 3.6 is not valid for the structure of the cohomology algebra of the complement of a real coordinate subspace arrangement. Despite the isomorphism of vector spaces $\widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) \cong \oplus_{j \geq 0} \mathrm{Tor}^S_{j,i+j}(S/I_{\Delta^\vee}, k)$ given by Theorem 3.1, in general the algebras $\mathrm{H}^*(\mathcal{M}_\Delta; k)$ and $\mathrm{Tor}^S_{*,*}(S/I_{\Delta^\vee}, k)$ are not isomorphic. This is easily seen when $I_{\Delta^\vee} = (x_1, \ldots, x_n)$; in this case $\mathrm{Tor}^S_{*,*}(S/I_{\Delta^\vee}, k)$ is an exterior algebra, while $\mathrm{H}^*(\mathcal{M}_\Delta; k)$ is an algebra generated by commuting idempotents.

Corollary 3.7. *Suppose that* Δ *has* d *facets. Then*

$$\sum_{i \geq 0} \dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) \,= \sum_{i \geq 0} b_i(S/I_{\Delta^\vee}) \,\leq\, \sum_{i \geq 0} c_i(n, d)\,,$$

where $c_i(n,d)$ is the maximum number of i-dimensional faces of an n-dimensional polytope having d vertices.

There are explicit formulas for $c_i(n,d)$ (see e.g. [Zi, Corollary 8.28]) and the cyclic n-polytope $C(n,d)$ with d vertices achieves the numbers $c_i(n,d)$.

Proof: Theorem 3.1 implies that the equality in Corollary 3.7 holds. The inequality follows from [BPS, Theorem 6.3]. ☐

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