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PROBLEMS AND RESULTS ON 3-CHROMATIC HYPERGRAPHS AND SOME RELATED QUESTIONS

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A *hypergraph* is a collection of sets. This paper deals with finite hypergraphs only. The sets in the hypergraph are called *edges*, the elements of these edges are *points*. The *degree* of a point is the number of edges containing it. The hypergraph is *r-uniform* if every edge has r points.

A hypergraph is *simple* if any two edges have at most one common point, and it is called a *clique* if any two edges have at least one common point.

The *chromatic number* of a hypergraph is the least number k such that the points can be k-colored so that no edge is monochromatic. As far as we know families of sets with chromatic number 2 were first investigated systematically by Miller (who used the term property B) in the case of infinite edges. There now is a large literature of this subject both for finite and infinite sets.

The main idea behind our investigations is that being simple or being a clique imposes surprisingly strict properties on 3-chromatic hypergraphs.

The reason why we relate these two properties with chromatic number is the following trivial observation:

If a hypergraph has chromatic number $\geqslant 3$ *then it has two edges with exactly one common point.*

Let $m_k(r)$ be the minimum number of edges of a $(k+1)$-chromatic r-uniform hypergraph. It is known [5], [9]

$$\frac{r}{r+2}\, 2^{r-1} \leqslant m_2(r) \leqslant r^2 2^r .$$

Perhaps $r2^r$ is the correct order of magnitude of $m_2(r)$; it seems likely that

$$\frac{m(r)}{2^r} \to \infty .$$

A stronger conjecture would be: Let $\{E_k\}_{k=1}^m$ be a 3-chromatic (not necessarily uniform) hypergraph. Let

$$f(r) = \min \sum_{k=1}^{m} \frac{1}{2^{|E_k|}} ,$$

where the minimum is extended over all hypergraphs with $\min |E_k| = r$. We conjecture that $f(r) \to \infty$ as $r \to \infty$.

Let $n_k^*(r)$, $m_k^*(r)$ denote the minimum number of points and edges in a $(k+1)$-chromatic r-uniform *simple* hypergraph. We shall prove

Theorem 1.

$$\lim_{r \to \infty} \sqrt[r]{n_k^*(r)} = k ,$$

$$\lim_{r \to \infty} \sqrt[r]{m_k^*(r)} = k^2 .$$

Thus in particular,

$$c_1 \frac{4^r}{r^3} < m_2^*(r) < c_2 r^4 4^r ,$$

i.e. $m_2^*(r)$ is much larger then $m_2(r)$.

In fact, we will prove a more general theorem (Theorem 1′) which constructs small hypergraphs with large chromatic number and girth; see [4].

L o v á s z [6] and W o o d a l l [7] proved that in every 3-chromatic r-uniform hypergraph there is a vertex of degree $\geqslant r$. We improve this result showing

Theorem 2. *A* $(k+1)$*-chromatic* r*-uniform hypergraph contains an edge which is intersected by at least* $k^{r-1}/4$ *other edges. Thus, the valency of at least one vertex is* $> k^{r-1}/4r$.

S t r a u s formulated the following problem: Is there a function $f(k)$ such that if S in any set of integers with $|S| = f(k)$ then the integers can be k-colored so that each color meets every translated copy of S (i.e. every set of form $S + a = \{x + a\colon x \in S\}$).

A stronger form of this problem asserts that if $f(k)$ is large then each color occurs at least $(1 - \epsilon)\dfrac{f(k)}{k}$ times in each and similar statement hold for the lattice points of the n-dimensional space. This problem will follow from the method of the proof of Theorem 2. In fact, a general theorem on hypergraph coloration can be obtained:

Theorem 3. *If each edge of an* r*-uniform hypergraph* H *meets at most* $k^{r-1}/4(k-1)^r$ *other edges then the vertices of* H *can be* k*-colored in such a way that each color meets each edge. We also prove the stronger version of* S t r a u s s' *conjecture* (Theorem 4.)

For simple hypergraphs, we will prove the following sharpening of Theorem 2:

Theorem 5. *If* H *is a simple* $(k+1)$*-chromatic* r*-uniform hypergraph then it contains at least* $k^{r-2}/4(r-1)$ *points with degree* $\geqslant k^{r-2}/4(r-1)$.

This theorem will be needed to prove Theorem 1. E r d ő s and S h e l a h [3] observed that in every simple 3-chromatic r-uniform hypergraph there are two disjoint edges if r is large enpugh. Theorem 5 will imply

Corollary 2 to Theorem 5. *A simple* $(k+1)$*-chromatic* r*-uniform hypergraph contains* $\frac{k^{r-2}}{4r(r-1)}$ *independent edges.*

The previously mentioned result of Lovász and Woodall states that, if H is a hypergraph such that, for each $H' \subseteq H$,

$$(1) \qquad \Big| \bigcup_{E \in H'} E \Big| \geq |H'| + 1$$

then H is 2-chromatic. Woodall made the surprising observation that (1) is best possible in the sense that there is an r-uniform 3 chromatic hypergraph H such that (1) holds for each $H' \subset H$ (but, of course, not for $H' = H$). In Woodall's example $|H| \approx r!$ and we suspect that $|H|$ cannot be much smaller. We also conjecture that for simple hypergraphs (1) can be replaced by a much weeker a assumption. Perhaps

$$\Big| \bigcup_{E \in H'} E \Big| \geq |H'|/2^{r(1-\epsilon)}, \qquad (\forall H' \subseteq H)$$

will imply that H. is 2-chromatic, provided H is simple.

Consider now r-uniform cliques. Obviously, a clique can have chromatic number 2 or 3 only; we are interested in those with chromatic number 3. Let $m^{**}(r)$ denote the minimum number of edges in such a hypergraph; we prove

Theorem 6. $m^{**}(r) \leq 7^{\frac{r-1}{2}}$ *for infinitely many* r.

We do not know if $\sqrt[r]{m^{**}(r)}$ is greater than 2; we cannot even show $m^{**}(r) > m(r)$.

Somewhat surprisingly, there are only finitely many 3-chromatic r-uniform cliques for a given r, so we may ask for the maximum number $M(r)$ of edges in them. We have the inequalities

Theorem 7. $r!(e-1) \leq M(r) \leq r^r$.

To obtain the upper bound we only use the fact that the edges of a 3-chromatic r-uniform hypergraph cannot be represented by $r-1$ points.

Theorem 8. *Let* $N(r)$ *denote the maximum number of points in a 3-chromatic* r*-uniform clique. Then*

$$\frac{1}{2}\binom{2r-2}{r-1} + 2r - 2 \leqslant N(r) \leqslant \frac{r}{2}\binom{2r-1}{r-1}.$$

Shelah and the authors observed that if H is a 3-chromatic r-uniform clique then there are two edges E, F with

$$|E \cap F| \geqslant \frac{r}{\log r}.$$

Perhaps the right hand side can be replaced by $c \cdot r$ or even $r - c$, since the worst example we have is an r-uniform 3-chromatic clique with

$$|E \cap F| \leqslant r - 2$$

(for infinitely many values of r), and we have no single example with

$$|E \cap F| \leqslant r - 3.$$

Theorem 9. *If* r *is large enough and* H *is an* r*-uniform 3-chromatic clique then the cardinalities* $|E \cap F|$, $E, F \in H$ *take at least 3 distinct values.*

We make some further remarks on the distribution of $|E \cap F|$, where E, F are edges in 3-chromatic cliques, but we know here very little.

Finally, we consider the following problem. Denote by $q(r)$ the smallest integer for which there is an r-uniform clique which cannot be covered by less than r points (r points, obviously, always cover an r-uniform clique; e.g. the r points of an edge). We prove

Theorem 10. $\frac{8}{3}r - 3 \leqslant q(r) \leqslant c \cdot r^{3/2}\log r.$

It is a challenging problem to prove or disprove $q(r) < c \cdot r$. We feel sure that $q(r) < c \cdot r \cdot \log r$ holds.

1.

We prove the following statement, which yields the upper bounds (i.e. $\overline{\lim}\sqrt[r]{n_k^* r} \leqslant k$, $\overline{\lim}\sqrt[r]{m_k^* r} \leqslant k^2$) in Theorem 1. The lower bounds will be proved later (Corollary 2 to Theorem 2 and Corollary 3 to Theorem 5).

Theorem 1′. *Let* $s \geqslant 2$, $r \geqslant 2$, $k \geqslant 2$; $n = 4 \cdot 20^{s-1} r^{3s-2} \cdot$ $\cdot k^{(s-1)(r+1)}$, $m = 4 \cdot 20^s \cdot r^{3s-2} \cdot k^{s(r+1)}$, $d = 20r^2 \cdot k^{r-1}$.

Then there exists an r-uniform hypergraph H on $k \cdot n$ points with at most m edges and with degrees $\leqslant d$ which does not contain any circuits of length $\leqslant s$ and in which each set of n points contains an edge.

This hypergraph is, obviously, at least $(k+1)$-chromatic.

Proof. S be any set of $k \cdot n$ points. We construct our hypergraph $H = \{E_i : i = 1, \dots, t\}$ inductively. Suppose $E_1, \dots, E_p$ have already been chosen so that

(α) $E_1, \dots, E_p$ from no circuit of length $\leqslant s$;

(β) no point is contained in more than d of them.

Let $S_1, \dots, S_{x_p}$ be those n-element sets containing no one of $E_1, \dots, E_p$. If there is no such S_1 we are finished. Suppose $x_p \geqslant 1$. Choose now E_{p+1} in such a way that $E_1, \dots, E_{p+1}$ satisfy (α) and (β) and E_{p+1} is contained in as many S_i, $(1 \leqslant i \leqslant x_p)$ as possible. We will show that this is possible and that E_{p+1} will be contained in at least $\frac{1}{20} x_p / k^r$ sets as long as $p < m$. This will imply

$$x_{p+1} \leqslant x_p \left(1 - \frac{1}{20k^r}\right). \tag{2}$$

Suppose we know that if $p < m$ then (2) holds. Then

$$x_m \leqslant x_0 \left(1 - \frac{1}{20k^r}\right)^m < 2^{kn} \cdot e^{\frac{-m}{20k^r}} < e^{kn - \frac{m}{20k^r}} = 1$$

thus our procedure stops before the m-th step, i.e. we get a hypergraph satisfying the requirements with $< m$ edges.

We still have to show (2). Suppose $s = 2s'$ is even; the odd case can be treated similarly. Let $1 \leqslant j \leqslant x_p$; we estimate how many r-tuples of S_j could be chosen for E_{p+1} without violating (α) and (β).

Let N be the number of those points of S_j with degree d. Then

$$d \cdot N \leqslant r \cdot p \leqslant r \cdot m, \qquad N \leqslant \frac{r \cdot m}{d} = \frac{n}{r}.$$

Therefore, the number of those points in S_j with degree $< d$ is

$$n - N \geqslant n\left(1 - \frac{1}{r}\right).$$

Any r-tuple chosen from these points will satisfy (β). Let us see, how many r-tuples are excluded by (α). We can describe these r-tuples as those not containing any pair of points which is at distance $\leqslant 2s' - 1$ in $\{E_1, \dots, E_p\}$; or which are both at distance $\leqslant s' - 1$ from a certain edge E_i, $1 \leqslant i \leqslant p$. Now there are at most $r^{s'} \cdot d^{s'-1}$ points at distance $\leqslant s' - 1$ from E_i; therefore, E_i excludes at most

$$\binom{r^{s'} \cdot d^{s'-1}}{2} < r^{2s'} \cdot d^{2s'-2}$$

pairs and so, there are at most

$$p \cdot r^{2s'} \cdot d^{2s'-2} \leqslant m \cdot r^{2s'} \cdot d^{2s'-2}$$

excluded pairs. One excluded pair forbids at most $\binom{n-2}{r-2}$ r-tuples of S_j; thus, the total number of r-tuples of S_j forbidden by (β) is

$$< \binom{n-2}{r-2} \cdot m \cdot r^{2s'} d^{2s'-2}$$

and so, the number of r-tuples of S_j which are candidates for E_{k+1} is

$$> \binom{n\left(1 - \frac{1}{r}\right)}{r} - \binom{n-2}{r-2} \cdot m \cdot r^{2s'} \cdot d^{2s'-2} \sim$$

$$\sim \frac{1}{e}\binom{n}{r} - \frac{m \cdot r^{2s'+2} \cdot d^{2s'-2}}{n^2}\binom{n}{r} = \left(\frac{1}{e} - \frac{1}{4}\right)\binom{n}{r} > \frac{1}{20}\binom{n}{r}.$$

Thus, there are altogether

$$\geq \frac{x_p}{20}\binom{n}{r}$$

r-tuples of $S_1, \ldots, S_{x_p}$ which can be chosen.

Since the total number of r-tuples is $\binom{kn}{r}$ there must be an r-tuple which is counted in at least

$$\frac{x_p \cdot \binom{n}{r}}{20\binom{kn}{r}} \sim \frac{x_p}{20k^r}$$

n-tuples. This proves (2).

2.

Lemma. *Let G be a (finite) graph with maximum degree d and vertices $v_1, \ldots, v_n$. Let us associate an event A_i with v_i $(i = 1, \ldots, n)$ and suppose that A_i is independent of the set*

$$\{A_j \colon (v_i, v_j) \in E(G)\}.$$

Also suppose

$$P(A_i) \leq \frac{1}{4d}. \tag{3}$$

Then

$$P(\bar{A}_1 \ldots \bar{A}_n) > 0. \tag{4}$$

Proof. We prove more, namely that

$$P(A_1 \mid \bar{A}_2 \ldots \bar{A}_n) \leq \frac{1}{2d}. \tag{5}$$

This formula makes sense because we may assume by induction

$$P(\bar{A}_2 \ldots \bar{A}_n) > 0 .$$

Then (5) obviously implies (4).

We prove (5) by induction on n. For $n = 1$ it is trivial. Let $v_2, \ldots, v_q$ be the points adjacent to v_1, $(q \leqslant d + 1)$. Then we have

$$P(A_1 \mid \bar{A}_2 \ldots \bar{A}_n) = \frac{P(A_1 \bar{A}_2 \ldots \bar{A}_q \mid \bar{A}_{q+1} \ldots \bar{A}_n)}{P(\bar{A}_2 \ldots \bar{A}_q \mid \bar{A}_{q+1} \ldots \bar{A}_n)} .$$

Here, by (3)

$$P(A_1 \bar{A}_2 \ldots \bar{A}_q \mid \bar{A}_{q+1} \ldots \bar{A}_n) \leqslant$$
$$\leqslant P(A_1 \mid \bar{A}_{q+1} \ldots \bar{A}_n) = P(A_1) \leqslant \frac{1}{4d} ,$$

and on the other hand

$$P(\bar{A}_2 \ldots \bar{A}_q \mid \bar{A}_{q+1} \ldots \bar{A}_n) =$$
$$= 1 - P(A_2 + \ldots + A_q \mid \bar{A}_{q+1} \ldots \bar{A}_n) \geqslant$$
$$\geqslant 1 - \sum_{i=2}^{q} P(A_i \mid \bar{A}_{q+1} \ldots \bar{A}_n) \geqslant 1 - (q-1)\frac{1}{2d} \geqslant \frac{1}{2}$$

by the induction hypothesis. Thus

$$P(A_1 \mid \bar{A}_2 \ldots \bar{A}_n) \geqslant \frac{1}{4d} \Big/ \frac{1}{2} = \frac{1}{2d} .$$

This proves the lemma.

Proof of Theorem 2. Let us color each point of H with colors $1, \ldots, k$ at random, independently of each other and with equal probability. Let $E(H) = \{E_1, \ldots, E_m\}$ and let A_i denote the event that E_i is monochromatic. Then

$$P(A_i) = \frac{1}{k^{r-1}} .$$

Let G be the line-graph of H i.e. a graph with points $v_1, \ldots, v_m$ where v_i is adjacent to v_j iff $E_i \cap E_j \neq \phi$. Then the events A_i are associated with the points of G and obviously, A_i is independent of the set of all A_j's such that $E_i \cap E_j = \phi$, i.e. $(v_i, v_j) \notin E(G)$. Moreover, the maximum degree of G is, obviously $d \leqslant k^{r-1}/4$ and thus

$$P(A_i) = \frac{1}{k^{r-1}} \leqslant \frac{1}{4d} ,$$

i.e. (3) is satisfied. Thus, the lemma gives

$$P(\bar{A}_1 \dots \bar{A}_m) > 0 .$$

But in any case when $\bar{A}_1 \dots \bar{A}_m$ occurs we get a k-coloration of H. Thus H is k-colorable.

Corollary 1. *If each point of an r-uniform hypergraph H has degree $\leqslant k^{r-1}/4r$ then the chromatic number of H is $\leqslant k$.*

Corollary 2. *If H is a simple $(k+1)$-chromatic r-uniform hypergraph then $|V(H)| > c \cdot k^{r-1}$.*

Proof of Theorem 3. Let $H = \{E_1 \dots E_m\}$. Color the points of H with colors $1, \dots, k$ at random, independently of each other. Let A_i denote the event that E_i does not get all colors. Obviously,

$$P(A_i) \leqslant k\left(1 - \frac{1}{k}\right)^r .$$

Considering the line-graph of H again, we get that the maximum degree is

$$d \leqslant k^{r-1}/4(k-1)^r$$

by the assumption, thus

$$P(A_i) \leqslant \frac{1}{4d}$$

holds, and the lemma implies that $P(\bar{A}_1 \dots \bar{A}_m) > 0$; this means there exists a desired coloration.

Theorem 3 immediately implies the weak form of Strauss' conjecture; in fact, $f(k) = c \cdot k \cdot \log k$ will be appropriate. The stronger version would have a similar generalization to hypergraphs, but it would be lengthy to formulate it, so we leave it to the reader and only prove

Theorem 4. *Let* $\epsilon > 0$, $k \geqslant 2$, $n \geqslant 1$. *Then there is an* $r_0 = r_0(k, \epsilon)$ *such that if* S *is any set of lattice points in the* n*-dimensional space with* $|S| = r \geqslant r_0$ *then the lattice points can be* k*-colored so that each set* $S + a$ *obtained by translating* S *with an integer vector* a *contains at least* $(1 - \epsilon)\frac{r}{k}$ *points of any given color.*

Proof. By compactness, it suffices to show this for a finite collection H of translated copies of S. Let us color the vertices of H with one of k given colors at random, independently of each other. The probability of the event A_i that the i-th translated copy contains $< (1 - \epsilon)\frac{r}{k}$ of a given color is

$$P(A_i) < (1 - \delta)^r \qquad (\delta > 0)$$

where δ depends on k and ϵ but not on r (this follows from the central limit theorem). On the other hand, each copy of S meets less than r^2 other copies (since if $S + a$ meets $S + b$ then $b - a$ must be one of the vectors joining two points of S). Thus if

$$(1 - \delta)^r < \frac{1}{4r^2}$$

then we can conclude as in the two previous cases.

Proof of Theorem 5. Let, for each edge $E \in H$, $\xi(E)$ be a point of E with maximum degree and set $E' = E - \{\xi(E)\}$, $H' = \{E' : E \in H\}$. Obviously, H' cannot be k-colorable (any k-coloration of H' would yield one of H) thus by Theorem 2, H' contains a vertex of degree $\geqslant k^{r-2}/4(r-1)$. Let $E_1', \ldots, E_t'$ be those edges of H' containing x, $t \geqslant k^{r-2}/4(r-1)$. Then $\xi(E_1), \ldots, \xi(E_t)$ must have degree $\geqslant t$ in H by definition, which proves the assertion.

Corollary 1. *A* $(k + 1)$*-chromatic* r*-uniform simple hypergraph cannot be covered by less than* $k^{r-2}/4(r-1)$ *points.*

Proof. Suppose T covers all edges where $|T| < k^{r-2}/4(r-1)$. By

Theorem 5, there is a point x with degree $> |T|$ not belonging to T. But then T cannot cover all edges adjacent to x as H is simple, a contradiction.

This assertion immediately implies

Corollary 2. *A $(k+1)$-chromatic r-uniform simple hypergraph contains at least $k^{r-2}/4r(r-1)$ disjoint edges.*

Corollary 3. *A $(k+1)$-chromatic r-uniform simple hypergraphs has at least $k^{2(r-2)}/16r(r-1)^2$ edges.*

3.

This paragraph contains constructions of 3-chromatic r-uniform cliques, and proves some simple properties in general.

(a) All r-tuples from $2r-1$ points form a 3-chromatic r-uniform clique.

(b) Let S be a set, $|S| = 2r-2$. For each partition $P = \{S_1, S_2\}$ of S with $|S_1| = |S_2| = r-1$ take a new point x_P. Define H to consist of all r-tuples from S plus all r-tuples of the form $S_1 \cup \{x_P\}$ where $P = \{S_1, S_2\}$ is a partition as above. Then H is a 3-chromatic r-uniform clique.

(c) Let H be a 3-chromatic r-uniform clique. Let $T \cap V(H) = \phi$, $|T| = r+1$ and define H' to consist of T and all $(r+1)$-tuples of the form $E \cup \{t\}$, $E \in H$, $t \in T$. Then H' is an $(r+1)$-uniform 3-chromatic clique.

(d) Let H be a 3-chromatic r-uniform clique, $V(H) = \{1, \ldots, n\}$. Let $H_1, \ldots, H_n$ be 3-chromatic ρ-uniform cliques, $V(H_i) \cap V(H_j) = \phi$. Define

$$H^* = \{E_{i_1} \cup \ldots \cup E_{i_r} : E_i \in H_i,\ \{i_1, \ldots, i_r\} \in H\}.$$

Then H^* is a (ρr)-uniform 3-chromatic clique.

The proof is straightforward in all cases. Obviously, (c) and (d) yield several families of 3-chromatic cliques when applied with different initial 3-chromatic cliques. We will use two initial hypergraphs, the triangle and the Fano plane on seven points. Let us collect the consequences of the above constructions.

Proof of Theorem 6. Apply (d) inductively, with $H^{(1)}$ the Fano plane, $H = H^{(k)}$ and $H_1, \dots, H_{|V(H)|}$ Fano planes to get $H^{(k+1)}$. Then $H^{(k)}$ is 3^k-uniform and

$$|H^{(k+1)}| = 7^{3^k} \cdot |H^{(k)}|,$$

whence $|H^{(k)}| = 7^{1+3+\ldots+3^{k-1}} = 7^{\frac{3^k-1}{2}}$.

Proof of Theorem 7.

I. Starting with the triangle, apply (c) repeatedly. It is easy to see that the obtained r-uniform 3-chromatic cliques have $(e-1)r!$ edges.

II. Suppose there is a 3-chromatic r-uniform clique with more than r^r edges. In fact, we will not use that H is 3-chromatic only that H is an r-uniform clique which cannot be covered by less than r points.

Let $E \in H$. Since there are $\geqslant r^r$ other edges, there will be a point $x_1 \in E$ with degree $> r^{r-1}$.

Let us define $x_1, \dots, x_r$ inductively as follows. Suppose $x_1, \dots, x_i$ are defined in such a way that more than r^{r-i} edges contain all of them. Since $x_1, \dots, x_i$ do not cover all the edges, there is an edge E_i not containing any of them. All the edges containing $x_1, \dots, x_i$ meet E_i; therefore, E_i contains a point x_{i+1} such that more than r^{r-i-1} edges contain $x_1, \dots, x_i$ and x_{i+1}.

Now more than one edge contains $x_1, \dots, x_r$, a contradiction.

Proof of Theorem 8.

I. The lower bound immediately follows from construction (b).

II. For every $x \in V(H)$, there are two edges $E, F \in H$ with $E \cap F = \{x\}$; for let E be any edge adjacent to x, and consider $E - x$. This set does not cover all edges, therefore there is an edge F avoiding $E - x$. Since $E \cap F \neq \phi$ we must have $E \cap F = \{x\}$.

Thus the assertion will be implied by the following

Lemma. *If H is an r-uniform clique such that, for each point x, there are two edges with $E \cap F = \{x\}$ then*

$$|V(H)| \leq \frac{r}{2} \cdot \binom{2r-1}{r-1}.$$

This is a sharpening of a theorem of Calczynska-Karlowicz [1]. The proof uses a method due to Lubell [2].

Proof. Let $(x_1, \ldots, x_n)$ be a permutation of $V(H)$. There is at most one point x_i such that both $\{x_1, \ldots, x_i\}$ and $\{x_i, \ldots, x_n\}$ contain an edge, since H is a clique. If we count this for each permutation of the points each point x is counted; in fact, if $E, F \in H$ with $E \cap F = x$ then order $E \cup F$ so that the points of E be on the first r plances or on the last r places. This can be done in $2(r-1)!^2$ ways; then choose the $2r-1$ places of $E \cup F$, this can be done in $\binom{n}{2r-1}$ ways; finally, place the $n - 2r + 1$ remaining points on the remaining places, this can be done in $(n - 2r + 1)!$ ways. Thus the number of times x is counted is

$$2[(r-1)!]^2 \binom{n}{2r-1} (n - 2r + 1)! = \frac{2n!}{r\binom{2r-1}{r}}$$

Thus we count at most $n!$ points, each point at least $\dfrac{2n!}{r\binom{2r-1}{r}}$ times. Hence $|V(G)| \leq \frac{r}{2}\binom{2r-1}{r}$.

Set, for a hypergraph H,

$$A(H) = \{|E \cap F| : E, F \in H, E \neq F\}.$$

Let H be a 3-chromatic r-uniform clique. The same proof as that of Theorem 7 yields

$$\max A(H) \geqslant \frac{r}{\log r}\,.$$

We don't know how sharp this estimation is; the construction in the proof of Theorem 6 yields a 3-chromatic 3^k-uniform clique with

$$A(H) = \{1, 3, \ldots, 3^k - 2\}\,.$$

But we do not know any example with

$$\max A(H) \leqslant r - 3\,.$$

Also note the interesting property of the preceding example that $A(H)$ consists of odd numbers only. How "lacunary" can $A(H)$ be? We cannot even prove

$$|A(H)| \to \infty \quad \text{as} \quad r \to \infty$$

for r-uniform 3-chromatic cliques. The best we can show is $|A(H)| \geqslant 3$ for large enough r.

Proof of Theorem 9. Suppose $|A(H)| \leqslant 2$. We know $1 \in A(H)$. As known, $|H| \geqslant 2^{r-1}$. Similarly as in the proof of Theorem 4, we find two points x, y contained in at least $2^{r-1}/r^2$ edges $F_1, \ldots, F_t$. Any two of these have at least two points in common. Hence $|A(H)| = 2$, say $A(H) = \{1, k\}$. Any two of the edges containing x and y must have exactly k points in common. As

$$t \geqslant \frac{2^{r-1}}{r^2} > r^2 - r + 1$$

if r is large enough, a theorem of Deza [8] implies $F_i \cap F_j = M = \text{const.}$ for any i, j. Now any edge not covered by M has at least $t > r$ points, a contradiction.

4.

Modifying slightly the definition of 3-chromatic r-uniform cliques, let us consider now r-uniform cliques which cannot be covered by less then r points. As pointed out, the proof of Theorem 4 works, so for the maximum number of edges in such a clique we have the same bounds as for $M(r)$.

The question of the minimum number $q(r)$ of edges is more confusing (or more interesting), as Theorem 10 shows.

Proof of Theorem 10. I. Suppose there are $< \frac{8}{3} r - 3$ edges in an r-uniform clique H, we show it can be covered by $r-1$ points. Let x_1 be a point of H with maximum degree, let x_2 be a point of $H - x_1$ with maximum degree, etc., let x_{i+1} be a point of $H - x_1 - \ldots - x_i$ with maximum degree ($H - x_1 - \ldots - x_i$ denotes the hypergraph obtained by removing all edges which meet $\{x_1, \ldots, x_i\}$). Observe that the degree of x_{i+1} in $H - x_1 - \ldots - x_i$ is $\geqslant 4$ if $|H - x_1 - \ldots - x_i| > 2r + 1$; it is $\geqslant 3$ if $|H - x_1 - \ldots - x_i| > r + 1$, and it is $\geqslant 2$ if $|H - x_1 - \ldots - x_i| > 1$. (This immediately follows from the assumption that H is a clique). Hence, if there are $\sim \frac{8}{3} r$ edges to begin with, in $\sim \frac{r}{6}$ step we get down to $\leqslant 2r + 1$ edges, in another $\sim \frac{r}{3}$ steps we will only have $\leqslant r + 1$ edges, which can be covered by $\sim \frac{r}{2}$ points. These are $\sim r$ points altogether. The accurate calculation with integral parts yields that if $|H| < \frac{8}{3} r - 3$ then, in fact, we use only $r - 1$ points to cover all edges.

II. For sake of simplicity let $r = p^\alpha + 1$ and our edges will be lines of a finite plane. Set $t = 4r^{3/2} \log r$. We can choose t lines $\binom{r^2 - r + 1}{t}$ ways; we will show that all but $o\binom{r^2 - r + 1}{t}$ choices of the lines cannot be represented by fewer than r points.

To prove this we make a few simple known remarks about lines in a finite geometry. Let $v_1, \ldots, v_{r-1}$ be vertices and $l_1, \ldots, l_k$ be the lines determined by them. Let e_i be the number of v_j's on l_i. Clearly

$$(12) \qquad \sum_{i=1}^{k} \binom{e_i}{2} = \binom{r-1}{2} .$$

Let $e_1 \geqslant e_2 \geqslant \ldots \geqslant e_k$ and let B be the number of lines disjoint from $\{v_1, \ldots, v_k\}$. Simple computation shows

$$B = \sum_{i=1}^{k} (e_i - 1) + 1$$

and so from (12)

$$(13)1 \qquad (r-1)(r-2) = \sum_{i=1}^{k} e_i(e_i - 1) \leqslant e_1 \sum_{i=1}^{k} (e_i - 1) = e_1(B-1)\,.$$

Another simple argument shows

$$(14) \qquad B \geqslant e_1(r - e_1)\,.$$

What we need is the following

Lemma.

$$B \geqslant \begin{cases} \sqrt{r}(r - \sqrt{r}) & \text{if} \quad e_1 \leqslant r - \sqrt{r}\,, \\ r - 1 & \text{otherwise}\,. \end{cases}$$

This immediately follows from (13) if $e_1 \leqslant \sqrt{r}$ or $e_1 \geqslant r - \sqrt{r}$, and from (14) if $\sqrt{r} \leqslant e_1 \leqslant r - \sqrt{r}$.

Now we are ready to prove out theorem. Assume first $e_1 > r - \sqrt{r}$. The number of ways of choosing such a system of points is

$$< (r^2 - r + 1)\binom{r}{\sqrt{r}}\binom{r^2 - r + 1}{\sqrt{r}} < r^{3\sqrt{r}}\,.$$

Thus the number of choices of t lines which can be represented by a system of $r-1$ points with more than $r - \sqrt{r}$ on a line is

$$< r^{3\sqrt{r}} \cdot \binom{r^2 - 2r + 2}{t}$$

and so the percentage of such choices of t lines among all choices is

$$< r^{3\sqrt{r}} \frac{\binom{r^2 - 2r + 2}{t}}{\binom{r^2 - r + 1}{t}} < r^{3\sqrt{r}} \cdot \left(1 - \frac{1}{r-1}\right)^t = o(1)\,.$$

Suppose now $e_1 < r - \sqrt{r}$. We can only say that the number of ways of choosing $r-1$ such points is

$$< \binom{r^2 - r + 1}{r - 1} < (er)^r .$$

The number of choices of t lines covered by such systems of $r - 1$ points is

$$< \binom{r^2 - r + 1 - \sqrt{r}\,(r - \sqrt{r}\,)}{t} .$$

Hence the precentage of such choices among all choices is

$$< \frac{(er)^r \binom{r^2 - r + 1 - \sqrt{r}(r - \sqrt{r})}{t}}{\binom{r^2 - r + 1}{t}} < (er)^r \left(1 - \frac{\sqrt{r}(r - \sqrt{r})}{r^2 - r + 1} \right)^t = o(1) .$$

We remark that we feel the natural boundary of the method is $r \log r$. The first part of the proof, i.e. the case $e_1 > r - \sqrt{r}$ could be improved easily to yield this. However, if e_1 is small then sometimes – very rarely – B can be close to $r^{3/2}$ (let, say, $v_1, \ldots, v_{r-1}$ be points of a subplane of order $\sqrt{r - 1}$). We have no good estimation how often can this happen.

Added in proof. Recently J. Beck (Budapest) proved that $m(r)/2^r \to \to \infty$ (oral communication).

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