Equilibrium States for Partially Hyperbolic Horseshoes

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Abstract

We study ergodic properties of invariant measures for the partially hyperbolic horseshoes, introduced in [9]. These maps have a one dimensional center direction E^c, and are at the boundary of the (uniformly) hyperbolic diffeomorphisms (they are constructed bifurcating hyperbolic horseshoes *via* heterodimensional cycles).

We prove that every ergodic measure is hyperbolic, but the set of Lyapunov exponents in the central direction has gap: all ergodic invariant measures have negative exponent, with the exception of one ergodic measure with positive exponent. As a consequence, we obtain the existence of equilibrium states for any continuous potential. We also prove that there exists a phase transition for the smooth family of potentials given by $\phi_t = t \log |DF|_{E^c}|$.

1 Introduction

Hyperbolic maps were extensively studied from the topological and ergodic viewpoints in the 70´s and 80´s. The contributions of several authors build a rather complete theory for these maps. Among other things, they proved that, given f an expanding map (or Axiom A diffeomorphism) and ϕ a continuous real function, there are *equilibrium states* for ϕ and f. Recall that an equilibrium state is a measure that maximizes, among all invariant probabilities, the sum of the entropy and the integral of the potential. More precisely, if $h_\mu(f)$ denotes the entropy of the map f with respect to the invariant probability μ, then ν is an equilibrium state with respect to the potential ϕ if

$$h_\nu(f) + \int \phi \, d\nu = \sup\{h_\mu(f) + \int \phi \, d\mu \colon \mu \text{ is a } f\text{-invariant probability}\}.$$

If ϕ is Hölder continuous, the equilibrium state is unique on each transitive component of the map. In this case, this measure is the Gibbs measure of the system and has full support. Most of the features of the equilibrium states in the hyperbolic case are now very well understood, as decay of correlations, recurrence properties and limit theorems (see [23, 4, 20, 2] for precise statements and references).

The importance of equilibrium states and/or Gibbs measures in the description of some dynamical properties of the map is considerable. Just to mention few examples, when f is a $C^{1+\alpha}$ expanding map of a compact (connected) manifold, there exists a unique invariant measure absolutely continuous with respect to the Lebesgue measure, and it is the unique equilibrium state associated to the potential $\phi = -\log|\det Df|$. If we consider the zero potential, an equilibrium state is just a maximum for the metric entropy, and gives the statistical distribution of periodic orbits on the manifold. Another application concerns the Hausdorff dimension of a conformal $C^{1+\alpha}$ expanding attractor Λ. If t_0 is the Hausdorff dimension of the attractor Λ, the equilibrium measure associated to the potential $\phi = -t_0 \log |Df|$ is the unique invariant measure absolutely continuous with respect to the t_0-dimensional Hausdorff measure of Λ.

The picture beyond hyperbolic systems is pretty much incomplete. To begin with, to prove the existence of such measures is a difficult problem in general, due to the lack of (any) regularity of the entropy function (see [15, 12, 16]). Moreover, for symbolic systems and some one-dimensional maps, as intermittent maps and some rational maps, there exist *phase transitions*, i.e., some potentials have more than one equilibrium state, even assuming regularity of the potential (see [21, 6, 18]). To find, understand and classify the difficulties and different types of phenomena of the non-uniform setting is a challenging problem.

In this direction, some advances were obtained recently, for non-uniformly hyperbolic maps that have some "hyperbolic flavor". This includes the Hénon-like maps (see [24]), some one dimensional maps (see [5, 17]), horseshoes with homoclinic tangencies (see [12, 13]), rational maps on the Riemannian sphere (see [14, 19]), countable Markov shifts and piecewise expanding maps (see [7, 22, 25]). Results were also obtained when f is a *partially hyperbolic* diffeomorphism (see [3], Chapter 11), which is just the case analyzed here.

One way of having some hyperbolic flavor is to consider systems at the boundary of the hyperbolic ones. To do this, one can consider, for instance, a one parameter family of hyperbolic horseshoes that loose hyperbolicity through a first parameter of bifurcation. The bifurcating system may display a homoclinic tangency, a saddle-node periodic point, a heterodimensional cycle, etc. In [9], Díaz *et al* proposed a model of destruction of higher dimensional horseshoes *via* heterodimensional cycles. They proved that there exists a family of diffeomorphisms F at the boundary of the uniformly hyperbolic ones, defined in a neighborhood of a region R, satisfying

(a) **partial hyperbolicity-** for each map F and each point in R, the tangent space splits into three DF-invariant directions, $E^s \oplus E^c \oplus E^u$, where E^u is uniformly expanded, E^s is uniformly contracted and E^c is *central*. By central we mean that the possible expansion and contraction in this direction are weaker than in the other two.

(b) **heterodimensional cycles-** each diffeomorphism F has a cycle associated to two hyperbolic saddles P and Q with different indices: the one dimensional stable manifold of the point Q has a non-empty intersection with the one dimensional unstable manifold of the point P. Similarly, the two dimensional unstable manifold of the point Q has a non-empty intersection with the two dimensional stable manifold of the point P. See Figure 1.

Recall that the homoclinic class of a hyperbolic fixed point P for the map F, $H(P, F)$, is the closure of the transverse intersections of its stable and unstable manifolds.

(c) **homoclinic classes-** The homoclinic class of Q is $\{Q\}$ and the homoclinic class of P is non-trivial and contains the saddle Q. Moreover, $H(P, F)$ is the both the maximal invariant set and the non-wandering set of F in R.

(d) **central curves-** A remarkable property of the homoclinic class $H(P,F)$ is that it contains infinitely many curves tangent to the central direction E^c. In particular, it contains a curve tangent to E^c joining P and Q.

(e) **semi-conjugacy with the shift-** If Σ_{11} denotes the subshift of finite type in $\{0,1\}^{\mathbb{Z}}$ where only the transition $1 \to 1$ is forbidden, then there is a continuous surjection

$$\Pi\colon H(P,F) \to \Sigma_{11}, \qquad \text{with} \quad \Pi \circ F = \sigma \circ \Pi. \tag{1}$$

All the points in the same central curve have the same image by the projection Π above. Therefore, Π defines a semi-conjugacy of the two dynamics.

The property (d) above imply that the map is a *skew product* defined in $\Sigma_{11} \times I$, where I is the unitary interval $[0,1]$, representing the central direction. We explore this viewpoint in Section 3.

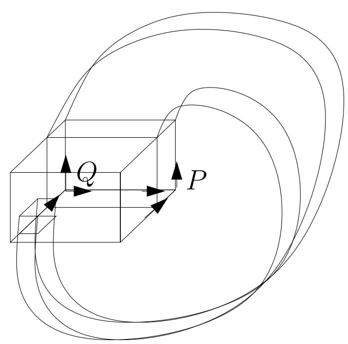

Figure 1: The twisted horseshoe

Here, we give an ergodic counterpart of [9] for these maps. We prove in Theorem 2.1 that *all* ergodic invariant measures are hyperbolic (have non-zero Lyapunov exponents) and that these maps have a gap in the set of central Lyapunov exponents. Indeed, the unique ergodic invariant measure with positive central exponent is the Dirac measure at Q. We remark that, for a C^1-generic system (that is, a system in a countable intersection of C^1 open and dense sets of diffeomorphisms) there is no such a gap (see [1]). The nonexistence of such a central gap seems to be the main reason why in the C^1-generic case there are non-hyperbolic measures with uncountable support (see [8]).

As consequence of the hyperbolicity of the ergodic measures supported in $H(P,F)$, we prove in Theorem 2.2 that *any* continuous potential has equilibrium states, and, for a residual set of potentials in the C^0 topology, the equilibrium state is unique. However, it is unclear if these equilibrium states are Gibbs measures or not. Concerning uniqueness of the equilibrium states, we prove in Theorem 2.3 that the one-parameter family $\phi_t = t \log |DF|_{E^c}|$ of C^∞ potentials has a phase transition: there exists $t_0 > 0$ such that ϕ_{t_0} admits at least two different equilibrium states. As far as we know, this is the first example of diffeomorphism with equilibrium states for all potentials that exhibits phase transition. Finally, for $t > t_0$, the Dirac measure supported in Q is the unique equilibrium state associated to the potential ϕ_t (see Theorem 2.3).

In view of the recent results of [16], it is likely that for each map F, there exists a positive constant C_F depending only on the topological entropy and the expansion/contraction rates of F, such that for any Hölder potential ϕ satisfying $(\sup \phi - \inf \phi) \leq C_F$ there exists a unique equilibrium state associated to ϕ. Moreover this equilibrium state should be a non-lacunary Gibbs measure.

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1.1 Definition of the family of diffeomorphisms

In this section we define he maps F that we consider. For simplicity, we define the $F(x, y, z)$ for $(x, y, z) \in \mathbb{R}^3$. The results remain valid for $(x, y, z) \in \mathbb{R}^i \oplus \mathbb{R} \oplus \mathbb{R}^j$, $i, j \geq 1$, with small changes.

We consider in $\mathbb{R}^3$ a family of horseshoe maps $F = F_{\lambda_0,\lambda_1,\beta_0,\sigma,\beta_1} : R \to \mathbb{R}^3$, on the cube $R = I^3$, where I denote the interval $I = [0, 1]$. Define the sub-cubes $R_0 = I \times I \times [0, 1/6]$, and $R_1 = I \times I \times [5/6, 1]$ of R. The restrictions F_i of F to R_i, $i = 0, 1$, are defined by:

- $F_0(x, y, z) = F_0(x, y, z) = (\lambda_0 x, f(y), \beta_0 z)$, with $0 < \lambda_0 < 1/3$, $\beta_0 > 6$ and f is the time one map of a vector field to be defined later;
- $F_1(x, y, z) = (3/4 - \lambda_1 x, \sigma(1 - y), \beta_1(z - 5/6))$, with $0 < \lambda_1 < 1/3$, $0 < \sigma < 1/3$ and $3 < \beta_1 < 4$.

First define the map $f : I \to I$ as the time one of the vector field

$$y' = y(1 - y),$$

depicted in Figure 2.

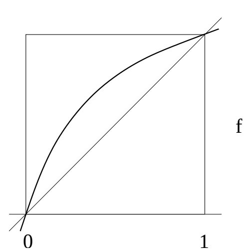

Figure 2: The central map f

Observe that $f(0) = 0$ and $f(1) = 1$, $f'(0) = e$ and $f'(1) = 1/e$ (see Equations (1) and (2) in [9]). Since we have $f(0) = 0$ and $f(1) = 1$, the point $Q = (0, 0, 0)$ is a fixed saddle of index 1 of F (*i.e.* Q has a one-dimensional stable direction), and the point $P = (0, 1, 0)$ is a fixed saddle of index 2 of F.

From the definition, the x-direction is the stable direction E^s, the y-direction is the central direction E^c and the z-direction is the unstable direction E^u. We shall thus also denote a point by (x^s, x^c, x^u). A central curve is a segment of the form $\{x\} \times [a, b] \times \{z\}$.

The projection $\Pi\colon H(P,F) \to \Sigma_{11}$ in (1) is defined as follows. Let Λ be the maximal invariant set in the cube R. Namely

$$\Lambda = \bigcap_{n\in\mathbb{Z}} F^n(R).$$

For $X \in \Lambda$, associate the sequence $\Pi(X) = (\theta_i)_{i\in\mathbb{Z}} \in \Sigma_{11}$ such that $\theta_i = j$ if $F^i(X) \in R_j$, $j = 0, 1$.

We recall that a point X is said to be recurrent if for every neighborhood $X \in U$, there exists some integer $n \neq 0$ such that $F^n(X)$ belongs to U. It is forward (resp. backward) recurrent if we add the constraint $n > 0$ (resp. $n < 0$).

Remark 1. A recurrent point X different from P and Q returns infinitely many times to R_1. Owing to the fact that $R_1 = \Pi^{-1}([1])$ is an open set, the coding $\Pi(X)$ has infinitely many 1's.

This fact has an important consequence (see Proposition 3.3): a point $X \in H(P,F)$ which belongs to a central curve contained in the homoclinic class $H(P,F)$ cannot be recurrent.

2 Statement of the main results

Our first result describes the central Lyapunov exponents for ergodic invariant measures. We prove they are negative, except for the Dirac measure δ_Q supported in Q. Using this information, we prove the existence of equilibrium measures associated to continuous potentials.

Given an ergodic invariant measure μ, its *central Lyapunov exponent* is:

$$\lambda^c_\mu = \int \log |DF|_{E^c}|\, d\mu.$$

Since E^c is one-dimensional, the Birkhoff theorem yields

$$\lambda^c_\mu = \lim_{n\to+\infty} \frac{1}{n} \log |DF^n(X)|_{E^c}|,$$

for μ almost every point $X \in \Lambda$.

Theorem 2.1. *The following properties of F hold true:*

1. *For any recurrent point $X \in \Lambda$, different from Q:*

   $$\liminf_{n\to+\infty} \frac{1}{n} \log |DF^n(X)|_{E^c}| \leq 0.$$

   Moreover, any ergodic F-invariant measure $\mu \neq \delta_Q$ has negative central Lyapunov exponent.

2. *If μ_n is a sequence of ergodic invariant measures such that $\lambda^c_{\mu_n}$ converges to zero, then μ_n converges to $\frac{\delta_Q+\delta_P}{2}$ in the weak* topology.*

Let $\phi : \Lambda \to \mathbb{R}$ be a continuous function. Let η be a F-invariant probability measure supported in Λ and $h_\eta(f)$ its entropy. The the ϕ-pressure of the measure η is defined by

$$\mathcal{P}(\phi) = h_\eta(f) + \int \phi \, d\eta. \tag{2}$$

We recall that η is called an *equilibrium state* for the potential ϕ if its ϕ-pressure maximizes the ϕ-pressures among all F-invariant probabilities. Our second result is:

Theorem 2.2. *Any continuous function $\phi \in C^0(\Lambda)$ admits an equilibrium state. Moreover, there exists a residual set of $C^0(\Lambda)$ such that the equilibrium measure is unique.*

If μ is an equilibrium state for some continuous potential ϕ, the ϕ-pressure of μ is also the *topological pressure* of ϕ. A natural question that arises from the previous theorem is if Hölder regularity of ϕ implies uniqueness of the equilibrium measure. A negative answer to this question for a particular potential is given in Theorem 2.3 below: $\phi_t = t \log |DF|_{E^c}|$ admits a phase transition.

Theorem 2.3. *Consider the one parameter family ϕ_t of C^∞ potentials given by $\phi_t(X) = t \log |DF(X)|_{E^c}|$. Then, there exists a positive real number t_0 such that:*

1. *For $t > t_0$, δ_Q is the unique equilibrium state.*
2. *For $t < t_0$, any equilibrium state for ϕ_t has negative central Lyapunov exponent. In particular, this measure is singular with respect to δ_Q.*
3. *For $t = t_0$, δ_Q is an equilibrium state for ϕ_t, and there exists at least another equilibrium state, singular with respect to δ_Q.*

Remark 2. In fact, the parameter t_0 is the supremum of the expression $\left\{\frac{h_\mu(F)}{1-\lambda^c_\mu}\right\}$ among all F-invariant measures μ different from δ_Q. Observe that it is not *a priori* clear that the supremum above is finite. However, this supremum is well defined, by Theorem 2.1.

3 Central Lyapunov exponents

In this section we study some interesting features of F. For $X = (x^s, x^c, x^u)$ in Λ, we denote by $W^u(X)$ and $W^s(X)$ the strong unstable and strong stable leaves of X. The central leaf $W^c(X)$, will denote the set of points on the form (x^s, y, x^u), with $y \in I$. We prove that, if $X \in \Lambda$ is (forward and backward) recurrent and different from P and Q, then $W^c(X) \cap \Lambda = \{X\}$ (see Proposition 3.3). This means that the central intervals that Λ contains are formed by non-wandering but not recurrent points. We also prove that the central Lyapunov exponent of any ergodic measure different from δ_Q is negative.

3.1 Central Lyapunov exponents for recurrent points

The main tool to prove the results in this section is the reduction of the dynamics to a one-dimensional system of iterated functions. Here we study these systems, and improve some of the results in [9]. Consider the maps $f_0, f_1 \colon I \to \mathbb{R}$ defined by

$$\begin{aligned} f_0(y) &= f(y), \\ f_1(y) &= \sigma\,(1 - y). \end{aligned}$$

Among the properties of the map Π proved in [9], there is the fact that the map F admits a well defined projection along the central direction to I^2. In fact, the image of a central segment of the cube is contained in one central segment, Π is constant on central segments, and the (well defined) projection of F is conjugated to the shift in Σ_{11}. Using this conjugacy, F can be thought as the skew product

$$\begin{array}{rcl} \tilde{F}\colon \Sigma_{11}\times I & \to & \Sigma_{11}\times I \\ (\theta, x) & \mapsto & (\tilde{\sigma}\theta, f_\theta(x)), \end{array}$$

where $f_\theta = f_{\theta_0} \in \{f_0, f_1\}$ and $\tilde{\sigma}$ is the shift map.

For $X = (x_0^s, x_0^c, x_0^u) \in \Lambda$ and $k \geq 0$, let $X_k = F^k(X) = (x_k^s, x_k^c, x_k^u)$. By the definition of F, the central coordinate x_k^c, of X_k is

$$x_k^c = f_{i_{k-1}} \circ f_{i_{k-2}} \circ \cdots \circ f_{i_0}(x_0^c).$$

Note that the numbers $i_0, \ldots, i_{k-1} \in \{0, 1\}$ are determined by the coordinates $x_0^u, \ldots, x_{k-1}^u$.

In what follows, we consider the dynamics associated to the system of iterated functions generated by f_0 and f_1.

Given a sequence $(i_n) \in \Sigma_{11}^+$, for each given $k \geq 0$ we consider the k-block $\varrho_k = \varrho_k(i_n) = (i_0, i_1, \ldots, i_k)$ associated to (i_n). For each k-block ϱ_k, we consider the map Φ_{ϱ_k} defined by

$$\Phi_{\varrho_k}(x) = f_{i_k} \circ f_{i_{k-1}} \circ \cdots \circ f_{i_0}(x).$$

The computation of the contraction in the central direction is based on an explicit computation of the derivative of the functions Φ_{ϱ_k}. First, consider a point $y \in (0, 1]$. Then we have (see Lemma 3.3 in [9])

$$|(f_1 \circ f_0^\alpha)'(y)| = \left(\frac{w}{y\,(1-y)}\right)\left(1 - \frac{w}{\sigma}\right), \quad \text{where } f_0^\alpha(y) = 1 - w/\sigma.$$

Note that $f_1 \circ f_0^\alpha(y) = w$. This implies that, if we chose a sequence $(i_n') \in \Sigma_{11}^+$ such that (i_n') is the concatenation of blocks of type $(0, \ldots, 0, 1)$, with the 1's occurring in the positions k_i, we have (see formula (7) in [9])

$$\Phi'_{\varrho_{k_i}}(y) = \prod_{j=1}^{i} \frac{w_j\,(1 - w_j/\sigma)}{w_{j-1}\,(1 - w_{j-1})} \quad \text{where } w_0 = y \text{ and } w_j = \Phi_{\varrho_{k_j}}(y). \tag{3}$$

Lemma 3.1. *Let $(i_n) \in \Sigma_{11}^+$ be a sequence with infinitely many 1's. Assume that $i_0 = 1$. Let $n_0, n_1, n_2 \ldots$ be the successive positions of the symbol 1 in (i_n). Then, there exist a sequence of positive real numbers $(\delta_j)_{j\geq 0}$ and a positive real number C such that*

(i) C depends only on n_0,

(ii) each δ_j depends only on the n_i's, $i \leq j$ and belongs to the interval $[0, \sigma]$,

(iii) for every $i > 0$ and for every y in $[0, 1]$, $|\Phi'_{\varrho_{n_i}}(y)| \leq C \displaystyle\prod_{j=1}^{i-1} \frac{1 - \delta_j/\sigma}{1 - \delta_j}$.

Proof. Let ϱ' be the block of (i_n) starting at the first symbol and finishing at the second 1. Let $N = n_0$ be its size, and (i'_n) be the sequence obtained from (i_n) by removing ϱ'. Then, for $k > N$ and $y \in (0, 1]$,

$$\Phi'_{\varrho_k}(y) = \Phi'_{\varrho'_{(k-N)}}(\Phi_{\varrho_N}(y)).\Phi'_{\varrho_N}(y). \tag{4}$$

Let $A = \max\{|\Phi'_{\varrho_N}(\xi)|, \xi \in I\}$. Note that A only depends on n_0.

Let $w_0 = \Phi_{\varrho_N}(y)$, and $w_j = \Phi_{\varrho'_{n_j-N}}(w_0)$. Observe that $\Phi_{\varrho_N}(I) \subset (0, \sigma]$; we set

$$\delta_0 = \min \Phi_{\varrho_N}(I) \text{ and } \delta_j = \min \Phi_{\varrho'_{n_j-N}}([\delta_0, \sigma]) \leq \sigma.$$

Then, (3) yields

$$|\Phi'_{\varrho'_{n_i-N}}(w_0)| = \frac{w_i\,(1 - w_i/\sigma)}{w_0\,(1 - w_0)} \prod_{j=1}^{i-1} \frac{1 - w_j/\sigma}{1 - w_j}.$$

Observe that, if $w_j > 0$, the factor of the product corresponding to it is strictly smaller than 1. Moreover, it is a decreasing function of $w_j \in [0, \sigma]$. Therefore we have

$$|\Phi'_{\varrho'_{n_i-N}}(w_0)| = \frac{w_i\,(1 - w_i/\sigma)}{w_0\,(1 - w_0)} \prod_{j=1}^{i-1} \frac{1 - w_j/\sigma}{1 - w_j} \leq \frac{1}{3\delta_0(1 - \delta_0)} \prod_{j=1}^{i-1} \frac{1 - \delta_j/\sigma}{1 - \delta_j}. \tag{5}$$

Therefore, (4) and (5) yield (i), with $C = \dfrac{A}{3\delta_0(1 - \delta_0)}$. Note that C only depends on n_0. Moreover each δ_j only depends on the n_i's, with $i \leq j$. This finishes the proof of the lemma. □

Lemma 3.2. *Let $(i_n) \in \Sigma^+_{11}$ be a recurrent sequence for the shift such that $i_0 = 1$. Then there exist a real number a in $(0, 1)$ and an increasing sequence of stopping times $(m_j)_{j\geq 0}$ such that for every y in $[0, 1]$,*

$$|\Phi'_{\varrho_{m_j}}(y)| \leq C.a^j,$$

where C is obtained from (i_n) as in Lemma 3.1.

Proof. Note that as the sequence (i_n) is recurrent, it has infinitely many symbols 1. We can thus apply Lemma 3.1. In particular, we use the notations of its proof.

Since each factor in the product in (iii)-Lemma 3.1 is strictly less than 1, it remains to show that there are infinitely many factors bounded from above by a number strictly smaller than 1. This is equivalent to show that there are infinitely many values of j such that δ_j is uniformly bounded away from zero.

The first block of ϱ' is composed by $n_1 - 1$ zeros and one 1. This implies that $\Phi_{\varrho'_{n_1-N}}[0, \sigma] \subset [f_1 \circ f_0^{n_1-1}(\sigma), \sigma]$, and so $\delta_1 > f_1 \circ f_0^{n_1-1}(\sigma)$. By the recurrence of the sequence (i'_n), this first block repeats infinitely many times. For each time j that it repeats, using the same argument, we conclude that $\delta_{j+1} \in [f_1 \circ f_0^{n_1-1}(\sigma), \sigma]$. This concludes the proof. □

Remark 3. A direct consequence of Lemma 3.2 is that any periodic point is hyperbolic, and if it is different from Q, it admits a negative Lyapunov exponent in the central direction.

Remark 4. The hypothesis "(i_n) recurrent" in Lemma 3.2 is not necessary, and it can be replaced by the weaker assumption: "One block of the form $(1, \underbrace{0 \ldots, 0}_{k}, 1)$, with a fixed k, appears infinitely many times in (i_n)".

3.2 Proof of Theorem 2.1

Let X be a recurrent point for F for forward and backward iterations. Assume that X is different from Q and P. Let us consider the one-sided sequence $\Pi(X)^+$. By Remark 1 $\Pi(X)^+$ contains infinitely many 1's and it is also recurrent in Σ_{11}. Hence, we can apply Lemma 3.2 to obtain

$$\liminf_{n\to+\infty} \frac{1}{n} \log |DF^n(X)|_{E^c}| \leq 0.$$

This gives estimates for the forward iteration, but we can also get estimates for the backward iterations:

Proposition 3.3. *Let X be a recurrent point for F (for forward and backward iterations) different from Q and P. Then*

$$\cap_{n\in\mathbb{Z}} F^n(R) \cap W^c(X) = \{X\}.$$

Proof. Let $\Pi(X) = (i_n) \in \Sigma_{11}$. By Remark 1, $\Pi(X)$ has infinitely many 1's, thus replacing X by some forward iterate we can assume that $i_0 = 1$. Let ϱ_k denote any block $(i_0, \ldots, i_k)$ of the sequence (i_n). We denote by (i_n^+) the one-sided sequence associated to (i_n). Again, we use vocabulary and notations from the proofs of Lemmas 3.1 and 3.2.

The infinite block $[(i_0, i_1, \ldots)]$ begins with the concatenation of the blocks ϱ_{n_0} and $\varrho'_{n_1-n_0}$. By recurrence of (i_n), we know that the block $\varrho_{n_0}\varrho'_{n_1-n_0}$ appears infinitely many times in the sequence $(\ldots, i_{-2}, i_{-1}, i_0)$. We thus consider a decreasing sequence of integers $k_j \to -\infty$ such that $\sigma^{-k_j}((i_n))$ coincides with (i_n) at the positions $0, 1, \ldots, n_1$. We also ask that $k_j - k_{j+1} > n_1$.

Now, we use Lemmas 3.1 and 3.2. The constant C is as in Lemma 3.1 and only depends on n_0. The Lemma 3.2 is used with the sequence $(i_{k_j}, i_{k_j+1}, \ldots)$ and the sequence of m_j's is the sequence of appearances of the entire block $\varrho'_{n_1-n_0}$ ("shifted" to the end of the whole block). Hence, for every j and for every y in $[0,1]$, we have

$$|\Phi'_{[(i_{k_j},\ldots,i_{-1})]}(y)| \leq C.a^j. \tag{6}$$

Let $L_j \subset I$ be the image of the interval I by the map $\Phi_{[(i_{k_j},\ldots,i_{-1})]}$. Points in $\cap_{n\in\mathbb{Z}} F^n(R) \cap W^c(X)$ have their central coordinates belonging to the intersection of the sets L_j, $j > 0$. Now, (6) implies that the diameter of L_j converges to zero. We also have that each L_j is non-empty, compact and $L_{j+1} \subset L_j$. Thus, their intersection is a single point. This completes the proof of the proposition. □

We define the *cylinder* associated with the block $\varrho = (i_0, \ldots, i_k)$ as follows:

$$[\varrho] = [i_0, \ldots, i_k] = \{x \in \Lambda; F^j(x) \in R_{i_j}, \text{ for } j = 0, \ldots, k\} = \bigcap_{j=0}^{k} F^{-j}(R_{i_j}) \cap \Lambda.$$

The last expression in the definition above tells us that these sets are always closed sets, since they are finite intersection of closed sets $F^{-j}(R_{i_j})$.

We say that a point X has *positive frequency* for a set $A \subset \Lambda$ if

$$\gamma(X, A) := \liminf \frac{\#\{0 \leq j < n; f^j(X) \in A\}}{n} > 0.$$

Definition 3.4. *Let b be a negative number. We say that a point X is of b-contractive type if*

$$\liminf_{n\to+\infty} \frac{1}{n} \log |DF^n(X)_{|E^c}| \le b < 0.$$

Next proposition is a tool to finish the proof of Item *1* in Theorem 2.1.

Proposition 3.5. *Let l be a positive integer. There exists a real number $a \in (0,1)$ which depends only on l, such that every $X \in \Lambda$ with positive frequency $\gamma > 0$ for the l-block $\theta = (1, 0, \dots, 0, 1)$ is of $\gamma \log a$-contractive type.*

Proof. Let us pick some positive integer l and consider X with frequency γ for $\theta = (1, 0, \dots, 0, 1)$. We simply use Lemma 3.2 and Remark 4. The constant $C = C(X)$ depends on the first block in $\Pi(X)^+$ which ends after the second 1. The sequence of "stopping times" considered is the sequence of appearance of the block θ. Hence, there exists $a \in (0,1)$, depending only on the length of the cylinder θ, such that, for every n satisfying $F^n(X) \in \theta$,

$$|DF^{n+l}(X)|_{E^c}| \le C(X) a^{\#\{0 \le j \le n,\ F^j(X) \in A\}}. \tag{7}$$

Then, (7) yields

$$\liminf_{n\to+\infty} \frac{1}{n} \log |DF^{n+l}(X)|_{E^c}| \le \gamma(X, \theta) \log a < 0.$$

By definition $\gamma(X, \theta) = \theta$. This finishes the proof of Proposition 3.5. □

Corollary 3.6. *Every ergodic and F-invariant probability μ which is not δ_Q has a negative Lyapunov exponent in the central direction.*

Proof. If $\mu(R_1) = 0$, then μ has its support in $R_0 = [0]$, and hence in $[0, 0, \dots]$, by invariance. We recall that the cylinder $[0, 0, \dots]$ is the set $\{0\} \times [0, 1] \times \{0\}$. Since every point in $[0, 0, \dots] \setminus \{Q\}$ is attracted to P, the cylinder $[0, 0, \dots]$ supports only two ergodic F-invariant measures, namely, δ_Q and δ_P. Thus, if μ is an ergodic measure different from δ_Q such that $\mu([0, 0, \dots]) = 1$, μ must be δ_P. For this measure, the Lyapunov exponent in the central direction is -1.

Let us now assume that $\mu(R_1) > 0$. We claim that, if we define the k-block $\theta_k = [1, \underbrace{0, \dots, 0}_{k \text{ terms}}, 1]$, then there exist $\varepsilon > 0$ and $l \in \mathbb{N}^*$ such that $\mu([\theta_l]) > \varepsilon$. Set $\theta_\infty = [1, 0, 0, \dots]$, and observe that

$$[1] = \theta_\infty \cup \bigcup_{k=1}^{\infty} [\theta_k].$$

Clearly points in θ_∞ are non-recurrent, hence Poincaré recurrence's theorem yields that $\mu(\theta_\infty) = 0$ (μ is singular with δ_P and δ_Q). Thus, there exist some positive ε and some l-block $\theta_l = [1, 0, \dots, 0, 1]$ such that $\mu([\theta_l]) > \varepsilon$. By ergodicity, there exists a set of full μ-measure, $B_1 \subset \Lambda$, such that every $X \in B_1$ has frequency of θ_l equal to $\mu([\theta_l]) > \varepsilon > 0$. On the other hand, since μ is ergodic, there exists a set $B_2 \subset \Lambda$ with full μ-measure such that for every $X \in B_2$, the central Lyapunov exponent is well-defined and coincides with λ^c_μ. Taking any $X \in B_1 \cap B_2$ and observing Proposition 3.5, we have that $\lambda^c_\mu \le \mu([\theta_l]) \log a < 0$. □

Let us now prove Item *2* in Theorem 2.1.

Proposition 3.7. *Let (μ_k) be a sequence of ergodic measures such that the sequence of central Lyapunov exponents $(\lambda^c_{\mu_k})$ converges to zero. Then, the sequence of measures (μ_k) converges to $\Delta = (1/2)\delta_Q + (1/2)\delta_P$.*

Proof. Given $\varepsilon > 0$ and $\theta_l = [1, \underbrace{0, \dots, 0}_{l \text{ terms}}, 1]$ an $(l+2)$-block, we define $E_{\varepsilon,l}$ by:

$$E_{\varepsilon,l} = \{\mu \text{ ergodic and } F\text{-invariant}; \mu(\theta_l) > \varepsilon\}.$$

From Proposition 3.5, there exists a constant $a = a(l) \in (0,1)$, such that

$$\lambda^c_{\mu_k} = \mu_k([\theta_l]) \log a.$$

Therefore $\lim_{k\to+\infty} \mu_k([\theta_l]) = 0$. Since $[\theta_l]$ is open and closed in Λ, if μ is any accumulation point for the weak* topology, we get $\mu([\theta_l]) = 0$; this holds for every l, which means that $\mu([\theta_l]) = 0$ for any $l \in \mathbb{N}$. Hence, $\mu([0, 0, \dots]) = 1$, thus $\mu = \alpha\delta_Q + (1-\alpha)\delta_P$, for some $\alpha \in I$.

Finally, we observe that $\log |DF|_{E^c}|$ is continuous; since μ is a weak* accumulation point for the sequence (μ_k), and $\lim_{k\to+\infty} \lambda^c_{\mu_k} = 0$, we get

$$0 = \int \log |DF|_{E^c}| \, d\mu = \alpha\lambda^c_{\delta_Q} + (1-\alpha)\lambda^c_{\delta_P}.$$

Since $\lambda^c_{\delta_Q} = 1$ and $\lambda^c_{\delta_P} = -1$, we must have $\alpha = 1/2$. In particular, the sequence (μ_k) admits a unique accumulation point for the weak* topology. It thus converges to Δ and the proof is finished. □

Remark 5. Using the structure provided by the heteroclinic cycle and the explicit expression of F, we can prove that there exists a sequence of periodic points p_n such that the Lyapunov exponents of the sequence of measures $\mu_n = (1/n)\sum_{i=0}^{n-1} \delta_{f^i(p_n)}$ converges to zero.

4 Proofs of Theorems 2.2 and 2.3

4.1 Existence of equilibrium states

In this section we prove that the entropy function $\mu \to h_\mu(F)$ is upper-semi continuous. As a consequence, we are able to prove the existence of equilibrium states for any continuous potential.

We recall that expansiveness is a sufficient condition to get the upper semi-continuity for the metric entropy. However, here, F is not a expansive map. It can be easily deduced observing that points in the central segment connecting Q and P have same α and ω limits, and F (respectively, F^{-1}) is a contraction when it is restricted to a neighborhood of P (respectively, Q). This segment is contained in the homoclinic class of P. Nevertheless, we have the following:

Lemma 4.1. *Let μ be any F-invariant probability in Λ. Then every partition $\mathcal{P}$ of Λ with diameter smaller than $1/2$ is generating for μ.*

Proof. Let $\mathcal{P}$ be any partition with diameter smaller than 1/2. For any X in Λ we denote by $\mathcal{P}(X)$ the unique element of the partition which contains X. If n and m are two positive integers, we set

$$\mathcal{P}_{-m+1}^{n-1}(X) := \bigcap_{k=-m+1}^{n-1} F^{-k}(\mathcal{P}(F^k(X))),$$

and $\mathcal{P}_{-\infty}^{+\infty}(X)$ is the intersection of all $\mathcal{P}_{-m}^{n}(X)$. We have just to prove that for μ almost every point X, $\mathcal{P}_{-\infty}^{+\infty}(x) = \{X\}$.

Consider the set of recurrent points in Λ for F. This set has full μ-measure. Moreover, if X is recurrent, then its projection $\Pi(X)$ in Σ_{11} is also recurrent. If the bi-infinite sequence $\rho(X)$ contains at least one 1, it contains infinitely many 1's (forward and backward) and Proposition 3.3 proves that $\mathcal{P}_{-\infty}^{+\infty}(X) \cap W^c(X) = \{X\}$. Hence, the uniform hyperbolicity in the two other directions yields $\mathcal{P}_{-\infty}^{+\infty}(X) = \{X\}$.

If the bi-infinite sequence $\Pi(X)$ does not contain any 1, then X must be in the segment $[Q, P]$. Therefore, $X = P$ or $X = Q$. Let us first assume that $X = Q$; then for any $Y \in (Q, P]$, $\lim_{n\to+\infty} F^n(Y) = P$. Hence,

$$\bigcap_{n\geq 0} F^{-n}(\mathcal{P}(Q)) \cap [Q, P] = \{Q\}.$$

Again, the uniform hyperbolicity in the two other directions yields $\mathcal{P}_{-\infty}^{+\infty}(Q) = \{Q\}$. If $X = P$, then for any $Y \in [Q, P)$, $\lim_{n\to+\infty} F^{-n}(Y) = Q$. The same argument yields $\mathcal{P}_{-\infty}^{+\infty}(P) = \{P\}$. □

Remark 6. If fact any partition with diameter strictly smaller than 1 is generating with respect to any invariant probability.

We now recall a classical result about entropy:

Proposition 4.2 (Bowen, Proposition 2.19 in [4])**.** *Let (Ω, T) be a dynamical system with Ω compact and T continuous. Suppose that for some $\varepsilon > 0$ one has $h_\mu(T, \mathcal{P}) = h_\mu(T)$ whenever μ is a T-invariant probability and diam$(\mathcal{P}) < \varepsilon$. Then every $\varphi \in C^0(\Omega)$ has an equilibrium state.*

In fact, Bowen proves that the hypothesis yield the upper semi-continuity of the metric entropy. Now, we recall that for every generating partition $\mathcal{P}$, Kolmogorov-Sinai's Theorem gives

$$h_\mu(F, \mathcal{P}) = h_\mu(F).$$

Therefore Proposition 4.2 and Lemma 4.1 prove that the metric entropy is a upper-semi continuous function defined on the compact set of all invariant probabilities. Thus, it attains its maximum. This implies the existence of equilibrium states for any continuous potential. Now, Theorem V.9.8 in [10] also yields uniqueness for any potential in a residual set of $C^0(M)$, since $(\phi, \mu) \to h_\mu(f) + \int \phi\, d\mu$ is upper semi-continuous on the set of invariant measures, and is a convex function for $\phi \in C^0(M)$.

4.2 Phase transition: proof of Theorem 2.3

We denote by $\mathcal{P}(t)$ the topological pressure of $\phi_t = t\log|DF|_{E^c}|$. For convenience it is also referred as the topological t-pressure.

The function $t \mapsto \mathcal{P}(t)$ is convex, thus continuous on $\mathbb{R}$. Hence we can define $t_0 \leq +\infty$ as the supremum of the set

$$\mathcal{T} = \{\xi > 0,\ \forall\, t \in [0,\xi),\ \mathcal{P}(t) > t\}.$$

By continuity the set $\mathcal{T}$ is not empty because $\mathcal{P}(0) = h_{top}(F) > 0$.

Lemma 4.3. *For t in $[0,t_0)$, any equilibrium state μ_t for ϕ_t is singular with respect to δ_Q.*

Proof. Let us assume, by contradiction, that μ_t is an equilibrium state for ϕ_t with $\mu_t(\{Q\}) > 0$, for some $t \in [0,t_0)$. By the theorem of decomposition of measures, there exists a F-invariant measure ν, singular with respect to δ_Q such that $\mu_t = \mu_t(\{Q\})\delta_Q + (1-\mu_t(\{Q\}))\nu$.

Note that $DF_{|E^c}(Q) = f'(0) = e$, hence $\int \phi_t\, d\delta_Q = t$. Since the metric entropy is affine, we have

$$\begin{aligned} t < \mathcal{P}(t) &= \mu_t(\{Q\})t + \big(1-\mu_t(\{Q\})\big)\left(h_\nu(F) + \int \phi_t\, d\nu\right) \\ &< \mu_t(\{Q\})\mathcal{P}(t) + (1-\mu_t(\{Q\}))\left(h_\nu(F) + \int \phi_t\, d\nu\right). \end{aligned}$$

In particular we get $\mathcal{P}(t) < h_\nu(F) + \int \phi_t\, d\nu$, which is absurd.

One could also note that any ergodic component of an equilibrium state is also an equilibrium state, and, in this case, $t = h_{\delta_Q}(F) + \int \phi_t d\delta_Q < P(t)$, which is a contradiction. □

Corollary 4.4. *Given t in $[0,t_0)$ and μ_t any equilibrium state for ϕ_t,*

$$\lambda^c_{\mu_t} = \int \log|DF|_{E^c}|d\mu_t < 0.$$

Proof. Let $\left(\nu_{t,\xi}\right)_{\xi\in\mathbb{T}^1}$, be the ergodic decomposition of μ_t (see [11] page 139, with $\mathbb{T}^1$ the unit circle). Since $\mu_t(\{Q\}) = 0$, we have that for Lebesgue-almost every $\xi \in \mathbb{T}^1$, $\nu_{t,\xi}(\{Q\}) = 0$. Corollary 3.6 says that for each of such ξ, we have $\displaystyle\int \log|DF|_{E^c}|d\nu_{t,\xi} < 0$. Therefore

$$\lambda^c_{\mu_t} := \int \log|DF|_{E^c}|d\mu_t = \int_{\mathbb{T}^1}\left(\int \log|DF|_{E^c}|\, d\nu_{t,\xi}\right) d\xi < 0.$$

□

Lemma 4.5. *The function $\mathcal{P}$ is decreasing on $[0,t_0)$.*

Proof. Let $t < t'$ be in $[0, t_0)$. Let us consider two equilibrium states for ϕ_t and $\phi_{t'}$, μ_t and $\mu_{t'}$. Then we have

$$\begin{aligned}
\mathcal{P}(t') &= h_{\mu_{t'}}(F) + t'\lambda^c_{\mu_{t'}} \\
&= h_{\mu_{t'}}(F) + t\lambda^c_{\mu_{t'}} + (t'-t)\lambda^c_{\mu_{t'}} \\
&\leq \mathcal{P}(t) + (t'-t)\lambda^c_{\mu_{t'}} \\
&< \mathcal{P}(t),
\end{aligned}$$

where the last inequality yields from Corollary 4.4. □

Corollary 4.6. *The term t_0 is a positive real number.*

Proof. Lemma 4.5 implies that $\mathcal{P}(t)$ is less than $h_{top}(F)$ on $[0, t_0)$. On the other hand, observe that $h_{\delta_Q}(F) + \int \phi_t \, d\delta_Q = t$, which means that $\mathcal{P}(t)$ is greater or equal to t. Therefore $t_0 \leq h_{top}(F) < +\infty$ (see figure 3 for $t \leq t_0$). □

We can now finish the proof of Theorem 2.3. Note that the existence of the real number t_0 and Item *(2)* are already proved (see Lemma 4.3 and Corollary 4.6).

Now, we prove Item *(3)* that is that δ_Q is an equilibrium state for $t = t_0$ and that there exists at least one other equilibrium state.

By definition of t_0 and by continuity of $t \mapsto \mathcal{P}(t)$, we must have $\mathcal{P}(t_0) = t_0$, thus δ_Q is an equilibrium state for t_0. Moreover, we claim that any weak accumulation point for μ_t, as t *increases* to t_0, is an equilibrium state for t_0. Indeed, let us pick any acumulation point μ for μ_t as t goes to t_0. We have

$$\mathcal{P}(t) = h_{\mu_t}(F) + t \int \log |DF_{|E^c}| \, d\mu_t,$$

and the left hand side term goes to $\mathcal{P}(t_0)$ by continuity of the pressure (in fact by convexity) when t goes to t_0. The second term in the right hand side term goes to $t_0 \int \log |DF_{|E^c}| \, d\mu$ if we pick the limit along the chosen subsequence of t. Thus, the entropy $h_{\mu_t}(F)$ must converge and the upper semi-continuity yields $h_\mu(F) \geq \lim_{t \to t_0} h_{\mu_t}(F)$.

Hence we get

$$\mathcal{P}(t_0) \leq h_\mu(F) + t_0 \int \log |DF_{|E^c}| \, d\mu \leq \mathcal{P}(t_0).$$

Note that the continuity of $\log \left|DF_{|E^c}\right|$ also yields $\int \log |DF_{|E^c}| \, d\mu \leq 0$, thus the measure μ is different from δ_Q. This finishes the proof of Item *(3)*.

Now, we prove Item *(1)*.

Let us pick $t > t_0$. Let μ_t be any equilibrium state for t. We have

$$\begin{aligned}
t \leq \mathcal{P}(t) &= h_{\mu_t}(F) + t\lambda^c_{\mu_t} \\
&= h_{\mu_t}(F) + t_0\lambda^c_{\mu_t} + (t-t_0)\lambda^c_{\mu_t} \\
&\leq \mathcal{P}(t_0) + (t-t_0)\lambda^c_{\mu_t}.
\end{aligned}$$

Hence we get $t_0 + (t - t_0) = t \leq t_0 + (t - t_0)\lambda^c_{\mu_t}$. This yields $\lambda^c_{\mu_t} \geq 1$. Again, considering the ergodic decomposition of μ_t, $(\nu_{t,\xi})$, we prove like in the proof of Corollary 4.4 that for almost every ξ, $\nu_{t,\xi} = \delta_Q$. In particular, this means that δ_Q is the unique equilibrium state for $t > t_0$ (see figure 3 for $t \geq t_0$) and that $\mathcal{P}(t) = t$ for every $t > t_0$. This complete the proof of Theorem 2.3.

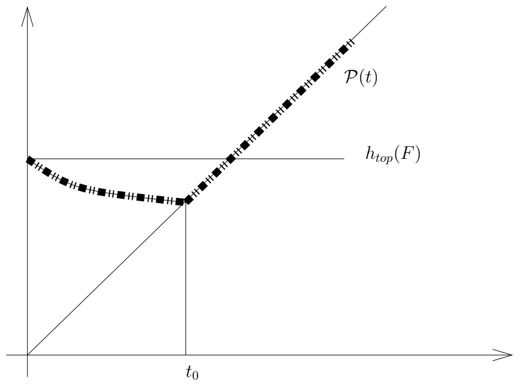

Figure 3: $t \mapsto \mathcal{P}(t)$

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