Bispectral quantum Knizhnik-Zamolodchikov equations

Michel van Meer

Bispectral quantum Knizhnik-Zamolodchikov equations

Michel van Meer

Bispectral quantum Knizhnik-Zamolodchikov equations

Bispectral quantum Knizhnik-Zamolodchikov equations
Michel van Meer
Proefschrift Universiteit van Amsterdam
Druk: GVO drukkers & vormgevers B.V. | Ponsen & Looijen, Ede
ISBN 978-90-6464-395-8

Dit proefschrift werd mede mogelijk gemaakt door:

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

NWO
Nederlandse Organisatie voor Wetenschappelijk Onderzoek

Bispectral quantum Knizhnik-Zamolodchikov equations

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. D.C. van den Boom
ten overstaan van een door het college voor promoties ingestelde
commissie, in het openbaar te verdedigen in de Agnietenkapel
op dinsdag 6 juli 2010, te 14:00 uur

door

Michel van Meer

geboren te Roosendaal en Nispen

Promotiecommissie

Promotor:	prof. dr. E.M. Opdam
Copromotor:	dr. J.V. Stokman
Overige leden:	prof. dr. G.J. Heckman
	prof. dr. H.T. Koelink
	prof. dr. T.H.K. Koornwinder
	prof. dr. B. Nienhuis
	prof. dr. N.Y. Reshetikhin
	prof. dr. S.N.M. Ruijsenaars
	dr. P. Zinn-Justin

FACULTEIT DER NATUURWETENSCHAPPEN, WISKUNDE EN INFORMATICA

Ja hoor!

Maxim Hartman

Contents

1 Introduction **1**
1.1 Linear q-difference equations . . . 1
1.2 Holonomy . . . 2
1.3 Compatible systems in auxiliary parameters . . . 6
1.4 Trigonometric Knizhnik-Zamolodchikov equations . . . 10
1.5 Difference KZ equations . . . 13
1.6 Compatibility of the trigonometric KZ and the difference KZ equations 15
1.7 Bispectral quantum KZ equations . . . 16
1.8 BqKZ and bispectral problems . . . 19
1.9 Overview . . . 20
1.10 Global conventions . . . 21

2 Bispectral quantum Knizhnik-Zamolodchikov equations **23**
2.1 Introduction . . . 23
2.2 The double affine Hecke algebra . . . 25
2.3 The bispectral quantum KZ equations . . . 31
2.4 The explicit form of the bispectral quantum KZ equations . . . 35
2.5 Solutions of the bispectral quantum KZ equations . . . 43

3 Correspondence of BqKZ with bispectral problems **55**
3.1 Introduction . . . 55
3.2 The monodromy cocycle . . . 56
3.3 The correspondence . . . 57
3.4 Bispectral Harish-Chandra series . . . 65
3.5 Specialized central character and Harish-Chandra series . . . 66

4 Polynomial solutions of qKZ equations and Macdonald polynomials **69**
4.1 Introduction . . . 69
4.2 Constructing polynomial solutions of qKZ . . . 70
4.3 Duality . . . 72
4.4 Relation to the basic asymptotically free solution . . . 72
4.5 Relation to symmetric self-dual Macdonald polynomials . . . 76

5 Bispectral quantum KZ equations for arbitrary root systems **79**
5.1 Introduction 79
5.2 The double affine Hecke algebra 80
5.3 Bispectral quantum KZ equations 87
5.4 Generic principal series representation and the cocycle values 95
5.5 Solutions of the bispectral quantum KZ equations 97
5.6 Correspondence with bispectral problems 106
5.7 Harish-Chandra series 110

Appendix: Holonomic systems of q-difference equations **115**

Bibliography **123**

Samenvatting **129**

Dankwoord **133**

Curriculum vitae **135**

Chapter 1

Introduction

In this thesis we define and analyze an explicit holonomic system of linear first-order q-difference equations for vector-valued functions, which we call the *bispectral quantum Knizhnik-Zamolodchikov equations*. The first goal of this introduction is to explain the concepts involved on a non-expert level. Secondly, we like to demonstrate how the bispectral quantum Knizhnik-Zamolodchikov equations can be exploited to gain insight in the well-known (trigonometric) Knizhnik-Zamolodchikov equations and quantum Knizhnik-Zamolodchikov equations, systems that have naturally emerged from mathematical physics and representation theory.

1.1 Linear q-difference equations

As a first step, we explain in this section the notion of a scalar linear q-difference equation. Let q be some fixed real number satisfying $0 < q < 1$ and let $\gamma \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Consider the equation

$$f(qt) = \gamma f(t) \tag{1.1.1}$$

for a function $f\colon \mathbb{C}^\times \to \mathbb{C}$. By means of the (renormalized) Jacobi theta function $\theta_q(t) := \prod_{m\geq 0}(1 - q^m t)(1 - q^{m+1}/t)$, which is a holomorphic function in $t \in \mathbb{C}^\times$ satisfying

$$\theta_q(qt) = -t^{-1}\theta_q(t),$$

we readily find a nonzero meromorphic solution f_γ of (1.1.1), given by

$$f_\gamma(t) = \frac{\theta_q(t)}{\theta_q(\gamma t)}.$$

In other words, f_γ solves the eigenvalue problem

$$\mathcal{T}_q f = \gamma f_\gamma,$$

where $\mathcal{T}_q$ is the so-called q-dilation operator $(\mathcal{T}_q f)(t) := f(qt)$. Note the reminiscence with the fact that the exponential map $u \mapsto e^{\lambda u}$ for $u \in \mathbb{C}$ is an eigenfunction of the differential operator $\frac{d}{du}$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$.

Definition 1.1.1. A (homogeneous) *linear q-difference equation* for a meromorphic function f on $\mathbb{C}^\times$ is an equation of the form

$$a_n(t)(\mathcal{T}_q^n f)(t) + \cdots + a_1(t)(\mathcal{T}_q f)(t) + a_0(t)f(t) = 0, \tag{1.1.2}$$

in which $a_0, \ldots, a_n$ are fixed meromorphic functions on $\mathbb{C}^\times$. If $a_n, a_0 \neq 0$, then we call n the *order* of the q-difference equation (see, e.g., [1]).

Note that (1.1.1) is a first-order linear q-difference equation. Another elementary example of a first-order linear q-difference equation is

$$\frac{f(qt) - f(t)}{qt - t} = a(t)f(t), \tag{1.1.3}$$

where $a(t)$ is a given meromorphic function. This is actually a rather special example, for, if we let $q \to 1$, then the left-hand side of (1.1.3) tends to the derivative $\frac{df}{dt}(t)$ of f at t (if it exists). Thus, in some formal sense, q-difference equations generalize differential equations. We call a q-difference equation of which a given differential equation is the result when formally taking the limit $q \to 1$, a *q-deformation* of this differential equation.

If we change the variable $t = e^u$ and write $q = e^h$ for some $h > 0$, then the q-dilation operator becomes the translation operator $\mathcal{T}$ acting on $2\pi i$-periodic meromorphic functions f on $\mathbb{C}$, given by $(\mathcal{T}f)(u) := f(u + h)$. Accordingly, a linear q-difference equation turns into a linear *difference* equation.

Definition 1.1.2. We call an equation for meromorphic functions f on $\mathbb{C}$ of the form

$$a_n(u)(\mathcal{T}^n f)(u) + \cdots + a_1(u)(\mathcal{T}f)(u) + a_0(u) = 0, \tag{1.1.4}$$

where the a_i are given meromorphic functions on $\mathbb{C}$ with $a_n, a_0 \neq 0$, a (homogeneous) n^{th}-order *linear difference equation*. Often, h is taken to be 1, so that $(\mathcal{T}f)(u) = f(u + 1)$ (see [48]).

The difference analog of the differential equation $\frac{df}{du} = af$ reads

$$\frac{f(u + h) - f(u)}{h} = a(u)f(u). \tag{1.1.5}$$

Indeed, considering the limit $h \to 0$ in the left-hand side of (1.1.5) yields the familiar formula for the derivative of f at u.

1.2 Holonomy

The bispectral quantum KZ equations, which are the main characters of the thesis, form a *holonomic system* of linear first-order q-difference equations for meromorphic vector-valued functions on some complex torus. The goal of this section is to explain, in various settings, the notion of holonomy.

1.2.1 Covariant derivatives

Let V be a finite-dimensional complex vector space. We write $\mathrm{End}(V)$ for the algebra of complex linear endomorphisms of V. Fix an integer $N \geq 1$. Let A_i $(1 \leq i \leq N)$ be holomorphic functions on an open subset $U \subset \mathbb{C}^N$ with values in $\mathrm{End}(V)$ and consider the system of linear first-order differential equations

$$\frac{\partial f}{\partial u_i} = A_i(u)f(u), \qquad 1 \leq i \leq N \tag{1.2.1}$$

for V-valued holomorphic functions f on U. We call such a system of differential equations *holonomic*, if

$$\frac{\partial A_j}{\partial u_i} - \frac{\partial A_i}{\partial u_j} = [A_i, A_j], \tag{1.2.2}$$

where the bracket $[,]$ denotes the usual commutator $[x,y] := xy - yx$ for ring elements x and y.

Note that this is the natural condition in order to expect nontrivial solutions of (1.2.1) to exist, since for a solution f of (1.2.1), both application of $\frac{\partial A_j}{\partial u_i} + A_j A_i$ and $\frac{\partial A_i}{\partial u_j} + A_i A_j$ to f yield $\frac{\partial^2 f}{\partial u_i \partial u_j}$.

In the language of differential geometry, the system of differential equations (1.2.1) gives rise to *covariant derivatives* in the standard i^{th} direction

$$\nabla_i := \frac{\partial}{\partial u_i} - A_i(u) \tag{1.2.3}$$

acting on holomorphic V-valued functions on U. These covariant derivatives may occur as local coordinate expressions $\nabla_i = \nabla_{\partial/\partial u_i}$ for a connection ∇ on some holomorphic vector bundle, locally trivialized as $U \times V$. The system (1.2.1) then becomes

$$\nabla_i f = 0, \qquad 1 \leq i \leq N, \tag{1.2.4}$$

while the holonomy condition translates to the familiar *flatness* condition

$$[\nabla_i, \nabla_j] = 0, \qquad 1 \leq i, j \leq N. \tag{1.2.5}$$

We call a holomorphic function $f \colon U \to V$ satisfying (1.2.4) *flat* with respect to ∇.

1.2.2 Covariant q-derivatives

It is tempting to try to develop similar notions for systems of linear first-order q-difference equations. The situation is substantially different though, since we are then dealing with *nonlocal* operators: whereas for the differential operator $\frac{d}{du}$, the value of $\frac{df}{du}$ at a point u_0 (for some differentiable function f), only depends on the values of f in an arbitrary small neighborhood of u_0, this obviously is not the case for the q-dilation operator $\mathcal{T}_q$. This makes it rather unclear what the appropriate notion of a connection should be in the q-difference setting, but the local description of a covariant derivative with respect to a chosen coordinate patch quite naturally allows a q-difference analog. This is the path we will now follow. In the q-difference setting it is natural to work with meromorphic functions.

Write $q^{\epsilon_i}t = (t_1, \ldots, qt_i, \ldots, t_N)$ for $1 \leq i \leq N$ and $t \in (\mathbb{C}^\times)^N$. For a meromorphic function $f\colon (\mathbb{C}^\times)^N \to V$ we consider the system of q-difference equations

$$A_i(t)(\mathcal{T}_{q,i}f)(t) = f(t), \qquad 1 \leq i \leq N, \tag{1.2.6}$$

where $(\mathcal{T}_{q,i}f)(t) := f(q^{\epsilon_i}t)$ and the A_i are $\mathrm{End}(V)$-valued meromorphic functions on $(\mathbb{C}^\times)^N$. Fully written out, it reads

$$A_i(t_1, \ldots, t_N)f(t_1, \ldots qt_i, \ldots, t_N) = f(t_1, \ldots, t_N), \qquad 1 \leq i \leq N.$$

The system (1.2.6) is called *holonomic* if $A_1, \ldots, A_N$ satisfy

$$A_i(t)(\mathcal{T}_{q,i}A_j)(t) = A_j(t)(\mathcal{T}_{q,j}A_i)(t) \tag{1.2.7}$$

for all $1 \leq i, j \leq N$. Again, holonomy is the natural condition in order to expect nontrivial solutions of (1.2.6). Indeed, for a solution f of (1.2.6), application of either side of (1.2.7) to $f(t_1, \ldots, qt_i, \ldots, qt_j, \ldots, t_N)$ yields $f(t)$.

The system of q-difference equations (1.2.6) gives rise to what we will be calling *covariant q-derivatives* ∇_i^q, defined by

$$\nabla_i^q := A_i\mathcal{T}_{q,i} - \mathrm{id}_V, \qquad 1 \leq i \leq N,$$

acting on meromorphic V-valued functions on $(\mathbb{C}^\times)^N$. The holonomy of (1.2.6) is then equivalent to the *flatness* condition

$$[\nabla_i^q, \nabla_j^q] = 0, \qquad 1 \leq i, j \leq N \tag{1.2.8}$$

of $\nabla^q := \{\nabla_i^q\}_{i=1}^N$, to which we sometimes will refer as the associated *q-connection*. We call a meromorphic function $f\colon (\mathbb{C}^\times)^N \to V$ *flat* with respect to ∇^q if $\nabla_i^q f = 0$ for $1 \leq i \leq N$. Hence, f is flat with respect to ∇^q if and only if it satisfies (1.2.6).

Remark 1.2.1. We need to mention that we use terms like 'q-connection' and 'covariant q-derivative' just because we like to think of them as q-analogs of locally trivialized connections and covariant derivatives, but that it is by no means standard terminology. The same goes for the terminology introduced in the following section where we consider the difference setting.

1.2.3 Covariant difference derivatives

Now that we discussed analogs of holonomy and covariant derivatives for q-difference equations, it is a small step to construct corresponding notions for difference equations.

For a function f depending on $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$, let $\mathcal{T}_i$ $(1 \leq i \leq N)$ be the difference operator $(\mathcal{T}_i f)(\lambda) := f(\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_N)$. For holomorphic functions $f\colon \mathbb{C}^N \to V$ we consider the system of linear first-order difference equations

$$A_i(\lambda)(\mathcal{T}_i f)(\lambda) = f(\lambda), \qquad 1 \leq i \leq N, \tag{1.2.9}$$

where the A_i are End(V)-valued holomorphic functions on $\mathbb{C}^N$. We call (1.2.9) *holonomic* if

$$A_i(\lambda)(\mathcal{T}_i A_j)(\lambda) = A_j(\lambda)(\mathcal{T}_j A_i)(\lambda) \tag{1.2.10}$$

for all $1 \leq i, j \leq N$. In a similar way as in the previous subsection, one observes that holonomy is the natural condition to expect the existence of nontrivial solutions of (1.2.9).

We say that the system (1.2.9) gives rise to *covariant difference derivatives* ∇_i $(1 \leq i \leq N)$, defined by

$$\nabla_i := A_i(\lambda)\mathcal{T}_i - \mathrm{id}_V$$

for $1 \leq i \leq N$. If the system (1.2.9) is holonomic, then the ∇_i satisfy the *flatness* conditions

$$[\nabla_i, \nabla_j] = 0, \qquad 1 \leq i, j \leq N. \tag{1.2.11}$$

We will call $\nabla := \{\nabla_i\}_{i=1}^N$ the corresponding *difference connection*. Let us complete the analogy with Subsections 1.2.1 and 1.2.2 by calling a smooth function $f\colon \mathbb{C}^N \to V$ *flat* with respect to ∇, if $\nabla_i f = 0$ for $1 \leq i \leq N$, hence, equivalently, if f is a solution of (1.2.9).

1.2.4 Holonomy for mixed systems

In this subsection we will discuss a concept of holonomy in case we combine different types of systems of equations. Concretely, we explain what we will mean by a holonomic system consisting of both linear first-order differential and difference equations.

Let U be an open subset of $\mathbb{C}^M$ and let A_i, B_k be holomorphic End(V)-valued functions on $U \times \mathbb{C}^N$ for $1 \leq i \leq M$, $1 \leq k \leq N$. Now define

$$\begin{aligned} \nabla_i &:= \frac{\partial}{\partial u_i} - A_i, \qquad 1 \leq i \leq M \\ \widehat{\nabla}_k &:= B_k \mathcal{T}_k^\lambda - \mathrm{id}_V, \quad 1 \leq k \leq N, \end{aligned} \tag{1.2.12}$$

viewed as linear operators acting on holomorphic V-valued functions in $(u, \lambda) \in U \times \mathbb{C}^N$ (here $\mathcal{T}_k^\lambda$ is the operator $\mathcal{T}_k$ defined in the previous subsection acting in the variable $\lambda \in \mathbb{C}^N$). Now consider the linear system of first-order differential/difference equations corresponding to (1.2.12),

$$(\nabla_i f)(u,\lambda) = 0, \qquad 1 \le i \le M, \tag{1.2.13}$$

$$(\widehat{\nabla}_k f)(u,\lambda) = 0, \qquad 1 \le k \le N \tag{1.2.14}$$

for holomorphic functions $f\colon U \times \mathbb{C}^N \to V$. We call the total system (1.2.13)–(1.2.14) *holonomic* if

$$[\nabla_i, \nabla_j] = 0, \quad [\widehat{\nabla}_k, \widehat{\nabla}_l] = 0, \quad [\nabla_i, \widehat{\nabla}_k] = 0$$

for all $1 \le i, j \le M$ and $1 \le k, l \le N$. Note that the first two conditions are equivalent to $\frac{\partial A_j}{\partial u_i} - \frac{\partial A_i}{\partial u_j} = [A_i, A_j]$ and $B_k \mathcal{T}_k^\lambda(B_l) = B_l \mathcal{T}_l^\lambda(A_k)$, respectively (cf. (1.2.2) and (1.2.10)), and a small computation shows that the third condition is equivalent to

$$\frac{\partial B_k}{\partial u_i} - A_i B_k = -B_k \mathcal{T}_k^\lambda(A_i). \tag{1.2.15}$$

These are the natural conditions to expect nontrivial solutions. For the first two conditions this is clear from the previous subsections, for the third we remark that application of either side of (1.2.15) to $\mathcal{T}_k^\lambda(f)$ for a solution f of (1.2.13)–(1.2.14), yields $-B_k \frac{\partial}{\partial u_i}(\mathcal{T}_k^\lambda(f))$.

Analogously to the other cases considered in this section, we might say that $\nabla := \{\nabla_i\}_{i=1}^M$ and $\widehat{\nabla} := \{\widehat{\nabla}_k\}_{k=1}^N$ together define a *mixed connection* and call a solution f of the system (1.2.13)–(1.2.14) *flat* with respect to $\nabla, \widehat{\nabla}$.

1.3 Compatible systems in auxiliary parameters

In this section we will give a reinterpretation of the holonomic systems discussed in the previous section as holonomic systems depending on auxiliary parameters, together with a compatible holonomic system of equations operating on the auxiliary parameters. First we consider the differential case.

Suppose we have covariant derivatives $\nabla_i = \frac{\partial}{\partial u_i} - A_i$ ($1 \le i \le L$) on $U \subset \mathbb{C}^L$ (see Subsection 1.2.1), satisfying the flatness conditions $[\nabla_i, \nabla_j] = 0$ ($1 \le i, j \le L$). Now let $L = M + N$ and write $u = (v, w) \in U_1 \times U_2 \subset U$, with $U_1 \subset \mathbb{C}^M$ and $U_2 \subset \mathbb{C}^N$ open. For fixed $w \in U_2$, the ∇_i ($1 \le i \le M$) give rise to flat covariant derivatives

$$\nabla_i(w) := \frac{\partial}{\partial v_i} - A_i(\cdot, w) \tag{1.3.1}$$

acting on holomorphic V-valued functions on U_1. Note that $\nabla_i(w)$ depends holomorphically on $w \in U_2$. In other words, the second set of variables $w \in U_2 \subset \mathbb{C}^N$ are interpreted here as auxiliary parameters. We thus have a family of flat connections $\nabla(w) = \{\nabla_i(w)\}_{i=1}^M$ ($w \in U_2$) depending holomorphically on $w \in U_2$ and operating on V-valued functions on U_1.

Similarly, for fixed $v \in U_1$, the ∇_{k+M} $(1 \le k \le N)$ give rise to flat covariant derivatives

$$\widehat{\nabla}_k(v) := \frac{\partial}{\partial w_k} - A_{k+M}(v,\cdot) \tag{1.3.2}$$

acting on holomorphic V-valued functions on U_2, depending holomorphically on $v \in U_1$.

Corresponding to (1.3.1), for all $w \in U_2$, we have a holonomic system of differential equations for holomorphic functions $f\colon U_1 \to V$

$$(\nabla_i(w)f)(v) = 0, \qquad 1 \le i \le M. \tag{1.3.3}$$

Similarly, corresponding to (1.3.2), for all $v \in U_1$ we have a holonomic system of linear first-order differential equations acting on the auxiliary parameter w of $\nabla_i(w)$,

$$(\widehat{\nabla}_k(v)f)(w) = 0, \qquad 1 \le k \le N, \tag{1.3.4}$$

for holomorphic functions $f\colon U_2 \to V$. Now the fact that the holonomic systems (1.3.3) and (1.3.4) come from a flat connection $\nabla = \{\nabla_i\}_{i=1}^{M+N}$ implies that these two systems are mutually compatible in the sense that $\frac{\partial A_{k+M}}{\partial v_i} - \frac{\partial A_i}{\partial w_k} = [A_i, A_{k+M}]$ for $1 \le i \le M$ and $1 \le k \le N$ (which is equivalent to $[\nabla_i, \nabla_{k+M}] = 0$; see (1.2.2)).

Definition 1.3.1. Let $U_1 \subset \mathbb{C}^M$ and $U_2 \subset \mathbb{C}^N$ be open and let $A_i, B_k\colon U_1 \times U_2 \to \mathrm{End}(V)$ be holomorphic. Suppose that for each $w \in U_2$, the covariant derivatives

$$\nabla_i(w) := \frac{\partial}{\partial v_i} - A_i(\cdot, w), \qquad 1 \le i \le M \tag{1.3.5}$$

acting on holomorphic V-valued functions on U_1 are holonomic. Furthermore, suppose that for each $v \in U_1$ the covariant derivatives

$$\widehat{\nabla}_k(v) := \frac{\partial}{\partial w_j} - B_k(v,\cdot), \qquad 1 \le k \le N \tag{1.3.6}$$

acting on holomorphic V-valued functions on U_2 are holonomic. Then the families of flat connections $\{\nabla(w)\}_{w\in U_2}$ and $\{\widehat{\nabla}(v)\}_{v\in U_1}$ are called *compatible*, if

$$\frac{\partial B_k}{\partial v_i} - \frac{\partial A_i}{\partial w_k} = [A_i, B_k] \tag{1.3.7}$$

for $1 \le i \le M$ and $1 \le k \le N$ as $\mathrm{End}(V)$-valued functions on $U_1 \times U_2$. We will also say that the corresponding (two families of) holonomic systems of differential equations are compatible.

Now if we actually have such a compatible family of connections as in the definition, then we can construct a flat connection $\nabla = \{\nabla_i\}_{i=1}^{M+N}$ acting on holomorphic V-valued functions f on $U_1 \times U_2$, by setting

$$(\nabla_i f)(v,w) := \big(\nabla_i(w) f(\cdot, w)\big)(v), \qquad 1 \le i \le M,$$
$$(\nabla_{k+M} f)(v,w) := \big(\widehat{\nabla}_k(v) f(v,\cdot)\big)(w), \qquad 1 \le k \le N$$

for all $(v, w) \in U_1 \times U_2$. In particular, given holonomic covariant derivatives (1.3.5) depending holomorphically on auxiliary parameters $w \in U_2$, it is natural to ask whether it extends to a compatible system of equations in the sense of the definition.

We just discussed compatible systems of equations for the differential case (which corresponds to Subsection 1.2.1), but the story for difference equations (Subsection 1.2.3) and q-difference (Subsection 1.2.2) is completely analogous.

Moreover, in the same spirit, we may reinterpret the mixed system we discussed in Subsection 1.2.4 as a compatible system of holonomic covariant derivatives $\nabla_i(\lambda)$ $(1 \leq i \leq M)$ determined by $\nabla_i(\lambda) := \frac{\partial}{\partial u_i} - A_i(\cdot, \lambda)$ acting on holomorphic V-valued functions on U, and of holonomic covariant difference derivatives $\widehat{\nabla}_k(u)$ $(1 \leq k \leq N)$ determined by $\widehat{\nabla}_k(u) := B_k(u, \cdot)\mathcal{T}_k^\lambda - \mathrm{id}_V$ and acting on holomorphic V-valued functions g on $\mathbb{C}^N$. The compatibility condition in this case, that is, the analog of (1.3.7), is given by (1.2.15). As we will see, the main nontrivial example we examine in this introduction is of this kind (see Section 1.6).

One way that holonomic systems of equations in auxiliary parameters, compatible to a given system, naturally arise, is in isomonodromy theory (see [28]). Another, and closely related, feature of such compatible systems in the case of difference or q-difference equations, is that they give rise to nontrivial isomorphisms between spaces of solutions of the original system with respect to different values of the auxiliary parameters (see Proposition 2.4.9 for an example related to the bispectral quantum KZ equations).

Remark 1.3.2. In this thesis, compatible systems of q-difference equations in the auxiliary parameters will also give rise to nontrivial *bispectral problems* for higher-order scalar q-difference operators (bispectrality in the sense of Duistermaat and Grünbaum [13]). We come back to this in Section 1.8 (see also Chapter 3).

The rest of this section is devoted to a couple of very elementary examples of compatible systems. In subsequent sections we will deal with the nontrivial case involving (various instances of) the Knizhnik-Zamolodchikov equations, which is the main subject of this thesis.

1.3.1 Differential / differential

For $\lambda \in \mathbb{C}$, consider the covariant derivative $\nabla(\lambda) := \frac{d}{du} - \lambda$ operating on holomorphic complex-valued functions on $\mathbb{C}$. Note that the function $f_\lambda(u) := e^{\lambda u}$ satisfies $\nabla(\lambda) f_\lambda = 0$. Now for $u \in \mathbb{C}$, define $\widehat{\nabla}(u)$ by

$$\widehat{\nabla}(u) := \frac{\partial}{\partial \lambda} - u,$$

operating on holomorphic complex-valued functions on $\mathbb{C}$. Then, clearly $A(\cdot, \lambda) := \lambda$ and $B(u, \cdot) := u$ satisfy (1.3.7), so $\{\nabla(\lambda)\}_{\lambda \in \mathbb{C}}$ and $\{\widehat{\nabla}(u)\}_{u \in \mathbb{C}}$ are compatible. Accordingly, if we define ∇ and $\widehat{\nabla}$ by

$$(\nabla g)(\cdot,\lambda) := \nabla(\lambda)g(\cdot,\lambda),$$
$$(\widehat{\nabla} g)(u,\cdot) := \widehat{\nabla}(u)g(u,\cdot),$$

for $g\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ holomorphic, then ∇ and $\widehat{\nabla}$ define flat covariant derivatives acting on holomorphic functions on $\mathbb{C}\times\mathbb{C}$. Consequently, we have a holonomic system of first-order linear differential equations

$$\begin{aligned}(\nabla f)(u,\lambda) &= 0,\\ (\widehat{\nabla} f)(u,\lambda) &= 0\end{aligned} \tag{1.3.8}$$

for holomorphic complex-valued functions on $\mathbb{C}\times\mathbb{C}$ in the sense of Subsection 1.2.1.

If we regard $f_\lambda(u)$ as a function of both $u,\lambda\in\mathbb{C}$, then the resulting function $f(u,\lambda) := e^{\lambda u}$ is a nontrivial solution of (1.3.8). Moreover, $f(u,\lambda)$ is a *self-dual* solution of (1.3.8), that is,

$$f(u,\lambda) = f(\lambda,u)$$

for $u,\lambda\in\mathbb{C}$. The existence of self-dual solutions of (1.3.8) is related to the duality symmetry

$$\nabla = \phi\circ\widehat{\nabla}\circ\phi$$

of the system (1.3.8), where ϕ is the involution of the space of complex-valued functions on $\mathbb{C}\times\mathbb{C}$ defined by $(\phi g)(u,\lambda) = g(\lambda,u)$. It follows, in particular, that if we start with a holomorphic function $g\colon\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ such that $\nabla(\lambda)g(\cdot,\lambda)=0$ for all $\lambda\in\mathbb{C}$ and satisfying $g(u,\lambda)=g(\lambda,u)$ ($u\in\mathbb{C}$, $\lambda\in\mathbb{C}$), then g automatically also satisfies $\widehat{\nabla}(u)g(u,\cdot)=0$ for all $u\in\mathbb{C}$. Hence, g is a self-dual solution of (1.3.8). In this thesis we exploit such duality symmetry extensively (see Section 2.5).

1.3.2 q-Difference / q-difference

To equation (1.1.1) we can associate a covariant q-derivative $\nabla^q(\gamma) = \gamma^{-1}\mathcal{T}_q - \mathrm{id}$ acting on meromorphic functions on $\mathbb{C}^\times$ for all $\gamma\in\mathbb{C}^\times$. We extend it as follows. For a meromorphic function f on $\mathbb{C}^\times\times\mathbb{C}^\times$ write $(\mathcal{T}_q^t f)(t,\gamma) = f(qt,\gamma)$ and $(\mathcal{T}_q^\gamma f)(t,\gamma) = f(t,q\gamma)$. We put

$$\nabla^q := \gamma^{-1}\mathcal{T}_q^t - \mathrm{id},$$
$$\widehat{\nabla}^q := t^{-1}\mathcal{T}_q^\gamma - \mathrm{id}.$$

Then one easily checks that $[\nabla^q,\widehat{\nabla}^q]=0$, and so the set of equations

$$\begin{aligned}(\nabla^q f)(t,\gamma) &= 0,\\ (\widehat{\nabla}^q f)(t,\gamma) &= 0,\end{aligned} \tag{1.3.9}$$

define a holonomic system of q-difference equations for meromorphic functions on $\mathbb{C}^\times \times \mathbb{C}^\times$ in the sense of Subsection 1.2.2. Note that we can reinterpret (1.3.9) as a compatible system of a q-difference equation determined by the covariant q-derivative $\nabla^q(\gamma) = \gamma^{-1}\mathcal{T}_q - \mathrm{id}$ depending meromorphically on an auxiliary parameter $\gamma \in \mathbb{C}^\times$ and a q-difference equation determined by $\widehat{\nabla}^q(t) := t^{-1}\mathcal{T}_q^\gamma - \mathrm{id}$ operating on that auxiliary parameter. If we multiply the solution $\frac{\theta_q(t)}{\theta_q(\gamma t)}$ of (1.1.1) by $\theta_q(\gamma)$, then by a similar analysis as in Subsection 1.3.1, we obtain a nontrivial meromorphic solution

$$g(t,\gamma) := \frac{\theta_q(t)\theta_q(\gamma)}{\theta_q(\gamma t)}$$

of (1.3.9), which is self-dual, i.e. invariant under $t \leftrightarrow \gamma$. As in the previous subsection, this relates to a duality symmetry of (1.3.9).

1.3.3 Differential / difference

For the final basic example, first recall the covariant derivative $\nabla(\lambda) := \frac{\partial}{\partial u} - \lambda$ on $\mathbb{C}$ from Subsection 1.3.1. For a function f depending on $\lambda \in \mathbb{C}$, put $(\mathcal{T}^\lambda f)(\lambda) := f(\lambda+1)$. Define

$$\widehat{\nabla}(u) := e^{-u}\mathcal{T}^\lambda - \mathrm{id}.$$

We readily see that $A(\cdot,\lambda) := \lambda$ and $B(u,\cdot) := e^{-u}$ satisfy (1.2.15), hence $\{\nabla(\lambda)\}_{\lambda\in\mathbb{C}}$ and $\{\widehat{\nabla}(u)\}_{u\in\mathbb{C}}$ are compatible.

In other words, we have a flat mixed connection given by

$$\begin{aligned}(\nabla f)(\cdot,\lambda) &:= \nabla(\lambda)f(\cdot,\lambda),\\ (\widehat{\nabla} f)(u,\cdot) &:= \widehat{\nabla}(u)f(u,\cdot)\end{aligned} \tag{1.3.10}$$

acting on holomorphic functions $\mathbb{C}\times\mathbb{C}\to\mathbb{C}$. Accordingly, the system of linear first-order differential/difference equations

$$\begin{aligned}(\nabla f)(u,\lambda) &= 0,\\ (\widehat{\nabla} f)(u,\lambda) &= 0\end{aligned} \tag{1.3.11}$$

for holomorphic functions f on $\mathbb{C}\times\mathbb{C}$, is holonomic (in the sense of Subsection 1.2.4).

Note that we have a nontrivial solution of (1.3.11) given by

$$f(u,\lambda) := e^{u\lambda}.$$

This is the same function as we obtained in Subsection 1.3.1 as a solution of (1.3.8).

1.4 Trigonometric Knizhnik-Zamolodchikov equations

We now come to a nontrivial example of a holonomic system of linear first-order differential equations, which can be regarded as forming the classical background of the theory developed in this thesis: the so-called *trigonometric Knizhnik-Zamolodchikov equations*. They form a system of linear first-order differential equations for holomorphic functions on

$$U_{\mathrm{KZ}} := \{u \in \mathbb{C}^N \mid u_i - u_j \notin 2\pi\sqrt{-1}\mathbb{Z} \text{ for all } i \neq j\},$$

with values in a vector space V of dimension $N!$ (here and in the rest of this introduction we take $N \geq 2$). The Knizhnik-Zamolodchikov (KZ) equations were originally obtained as a holonomic system of differential equations satisfied by the correlation functions in a Wess-Zumino-Witten conformal field theory (cf. [33]). A mathematical rigorous treatment of these equations was provided by Tsuchiya and Kanie [62]. The corresponding covariant derivatives in this case are defined in terms of a classical r-matrix (by definition a solution of the classical Yang-Baxter equation; see e.g. [15, §3.9]), which is constructed by means of representation theory of affine Lie algebras.

In general, one can associate flat covariant derivatives to any r-matrix (again, see [15, §3.9]). Cherednik developed an alternative way to produce r-matrices, namely via representations of so-called degenerate affine Hecke algebras (see [10, §1.1]), and the corresponding holonomic system of differential equations are Cherednik's *affine trigonometric KZ equations* (cf. [10, §1.1]). In this thesis we employ Cherednik's approach to (q-analogs of) the KZ equations, but in the introduction we merely present the various KZ equations rather than actually construct them. The natural construction of the trigonometric KZ equations can be found in [10]; for the construction in the quantum case see Chapter 2.

In order to write down the KZ equations, we need some facts about the symmetric group. Let S_N denote the group of permutations of the set $\{1, \ldots, N\}$. For $i \neq j$, write $s_{ij} \in S_N$ for the transposition $i \leftrightarrow j$. We will identify S_N with the group presented by generators $s_1, \ldots, s_{N-1}$, subject to the relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i, \quad (|i-j| > 1). \end{aligned} \tag{1.4.1}$$

The identification is made by relating the generator s_i ($1 \leq i \leq N-1$) to the transposition $s_{i,i+1}$. For $w \in S_N$, we call an expression for w of the form

$$w = s_{i_1} \cdots s_{i_r}$$

reduced, if $r \in \mathbb{Z}_{\geq 0}$ is as small as possible. In this case, the integer r is referred to as the length of w and is denoted by $\ell(w)$.

Let V be a complex vector space of dimension $\#S_N = N!$ and let $\{v_w\}_{w \in S_N}$ denote a distinguished basis of V over $\mathbb{C}$. Define a $\mathbb{C}$-linear left action of S_N on V by

$$w v_{w'} := v_{ww'}, \tag{1.4.2}$$

for $w, w' \in S_N$. Later on, we will also need the $\mathbb{C}$-linear *right* action of S_N on V, which is defined by

$$\overline{w} v_{w'} := v_{w'w} \tag{1.4.3}$$

for $w, w' \in S_N$.

Remark 1.4.1. The map $w \mapsto v_w$ defines an isomorphism of vector spaces $\mathbb{C}[S_N] \simeq V$, where $\mathbb{C}[S_N]$ is the group algebra of S_N. The left (respectively right) action of S_N on V we just defined corresponds to the left (respectively right) regular action of S_N on $\mathbb{C}[S_N]$.

Now fix $\kappa \in \mathbb{C}$ and let h be a formal parameter. We write $\mathbb{C}[[h]]$ for the ring of formal power series in h and put $V[[h]] := \mathbb{C}[[h]] \otimes V$. The actions (1.4.2) and (1.4.3) of S_N on V give rise to S_N-actions on $V[[h]]$ by $\mathbb{C}[[h]]$-linear extension. We define operators T_i ($1 \leq i < N$) and $\pi(\lambda)$ (depending on $\lambda \in \mathbb{C}^N$) in $\mathrm{End}_{\mathbb{C}[[h]]}(V[[h]]) \simeq \mathbb{C}[[h]] \otimes \mathrm{End}(V)$ by setting

$$T_i(v_w) = \begin{cases} v_{s_i w} & \text{if } \ell(s_i w) = \ell(w) + 1, \\ v_{s_i w} + (e^{h\kappa} - e^{-h\kappa}) v_w & \text{if } \ell(s_i w) = \ell(w) - 1, \end{cases} \tag{1.4.4}$$

$$\pi(\lambda) v_w = e^{h\lambda_{w^{-1}(N)}} v_{\sigma w} \tag{1.4.5}$$

for $w \in S_N$, where $\sigma \in S_N$ is given by $\sigma(i) = i + 1$ ($1 \leq i < N$) and $\sigma(N) = 1$. Note that the T_i are invertible with inverse $T_i^{-1} = T_i - e^{h\kappa} + e^{-h\kappa}$. In fact, the operators $T_1, \ldots, T_{N-1}, \pi(\lambda)$ represent the so-called *extended affine Hecke algebra* $H = H(e^{h\kappa})$ (see Subsection 2.2.2). The Hecke algebra H contains a large commutative subalgebra ($\mathbb{C}_Y[T]$, see Theorem 2.2.3). Its image in $\mathbb{C}[[h]] \otimes \mathrm{End}(V)$ is generated (as an algebra) by $Y_i(\lambda)^{\pm 1}$, where

$$Y_i(\lambda) = T_{i-1}^{-1} \cdots T_1^{-1} \pi(\lambda) T_{N-1} \cdots T_i,$$

for $1 \leq i \leq N$.

For $F \in \mathbb{C}[[h]] \otimes \mathrm{End}(V)$, let $F_n \in \mathrm{End}(V)$ be the coefficient of h^n, that is,

$$F = \sum_{n \geq 0} F_n h^n.$$

Note that with this notation, $T_{i,0} = s_i$ as operators on V. Furthermore, we have $Y_i(\lambda)_0 = \mathrm{id}_V$, and hence, since $[Y_i(\lambda), Y_j(\lambda)] = 0$, also $[Y_i(\lambda)_1, Y_j(\lambda)_1] = 0$.

Remark 1.4.2. The operators $T_{i,0}$ ($1 \leq i < N$) and $Y_j(\lambda)_1$ ($1 \leq j \leq N$) define a representation of the so-called degenerate affine Hecke algebra (cf. [10, §1.1]).

Define $\mathrm{End}(V)$-valued holomorphic functions A_i ($1 \leq i \leq N$) on $U_{\mathrm{KZ}} \times \mathbb{C}^N$ by

$$A_i(u, \lambda) := Y_i(\lambda)_1 + 2\kappa \left(\sum_{j=i+1}^{N} \frac{s_{ij} - 1}{e^{u_j - u_i} - 1} - \sum_{j=1}^{i-1} \frac{s_{ij} - 1}{e^{u_i - u_j} - 1} + i - \frac{N+1}{2} \right) \tag{1.4.6}$$

and set

$$\nabla_i(\lambda) := \frac{\partial}{\partial u_i} - A_i(\cdot, \lambda), \qquad 1 \leq i \leq N.$$

Theorem 1.4.3. *The system of linear differential equations*

$$(\nabla_i(\lambda)f)(u) = 0, \qquad 1 \leq i \leq N \tag{1.4.7}$$

for V-valued holomorphic functions f in $u \in U_{\mathrm{KZ}}$ is holonomic (see (1.2.5)*).*

Definition 1.4.4. The holonomic system of equations (1.4.7) is called the system of *trigonometric Knizhnik-Zamolodchikov equations* (with respect to the degenerate affine Hecke algebra module V).

Theorem 1.4.3 is due to Cherednik (see [10]). The trigonometric form of the KZ equations was also found by Frenkel and Reshetikhin in [21], then in the setting of affine Lie algebras. In this introduction, we will see that Theorem 1.4.3 follows by degenerating a specific holonomic system of q-difference equations, called the *quantum KZ equations* (see Subsection 1.7.2).

Note that (1.4.7) is defined in terms of a flat covariant derivative depending holomorphically on an auxiliary parameter $\lambda \in \mathbb{C}^N$, so it is natural to look for a compatible system in the auxiliary parameter (see Section 1.3). This will be constructed in the following section.

1.5 Difference KZ equations

In this section we construct a difference analog of the KZ equations, which will turn out to be compatible with the trigonometric KZ equations (1.4.7). In order to do so, we need to introduce operators $\mathbb{C}[[h]] \otimes \mathrm{End}(V)$ dual to the T_i and $\pi(\lambda)$ defined in previous section. First, let ι be the involution of V determined by $\iota(v_w) = v_{w^{-1}}$. We view ι as element of $\mathrm{End}_{\mathbb{C}[[h]]}(V[[h]]) = \mathbb{C}[[h]] \otimes \mathrm{End}(V)$ by $\mathbb{C}[[h]]$-linear extension. Then for fixed $u \in \mathbb{C}^N$, we define

$$\widehat{T}_i := \iota T_i \iota, \quad \widehat{\pi}(u) := \iota\pi(-u/h)\iota$$

for $1 \leq i < N$, where the second formula (involving "h^{-1}") should be understood to mean that $\widehat{\pi}(u)v_w = e^{-u_{w(N)}}v_{w\sigma^{-1}}$ for all $w \in S_N$ (compare with (1.4.5)). Similarly as the T_i and $\pi(\lambda)$ did before, these operators define a representation of the extended affine Hecke algebra. Accordingly, we obtain invertible commuting operators

$$\widetilde{Y}_i(u) := \widehat{T}_{i-1}^{-1} \cdots \widehat{T}_1^{-1}\widehat{\pi}(u)\widehat{T}_{N-1} \cdots \widehat{T}_i$$

in $\mathbb{C}[[h]] \otimes \mathrm{End}(V)$, for $1 \leq i \leq N$.

Remark 1.5.1. We have $\widehat{T}_{i,0} = \overline{s}_i$ (see (1.4.3)) as elements of $\mathrm{End}(V)$, and one can easily check that

$$\widetilde{Y}_i(u)_0 v_w = e^{-u_{w(i)}}v_w$$

for $1 \leq i \leq N$ and $w \in S_N$. In particular, the $\widetilde{s}_1, \ldots, \overline{s}_{N-1}$ and $\widetilde{Y}_i(u)_0$ ($1 \leq i \leq N$) define a representation of the affine Weyl group $W = S_N \ltimes \mathbb{Z}^N$ on V, where the action of the standard basis element $\epsilon_i \in \mathbb{Z}^N$ $(1 \leq i \leq N)$ is by $\widetilde{Y}_i(u)_0$ (see Subsection 2.2.1).

Define $\mathrm{End}(V)$-valued rational functions $\widehat{\mathcal{R}}_i$ on $\mathbb{C}$ by

$$\widehat{\mathcal{R}}_i(z) := \frac{z}{z+2\kappa}(\overline{s}_i - 1) + \mathrm{id}_V$$

for $1 \leq i < N$. The $\widehat{\mathcal{R}}_i$ satisfy

$$\begin{aligned} \widehat{\mathcal{R}}_i(z)\widehat{\mathcal{R}}_{i+1}(z+w)\widehat{\mathcal{R}}_i(w) &= \widehat{\mathcal{R}}_{i+1}(w)\widehat{\mathcal{R}}_i(z+w)\widehat{\mathcal{R}}_{i+1}(z), \quad 1 \leq i < N-1, \\ \widehat{\mathcal{R}}_j(z)\widehat{\mathcal{R}}_j(-z) &= \mathrm{id}_V, \quad 1 \leq j < N. \end{aligned} \tag{1.5.1}$$

An R-matrix, compared to a classical r-matrix, is by definition a solution of the *quantum* Yang-Baxter equation (with spectral parameter). It gives rise to a flat q-connection, just like an r-matrix gives rise to a flat connection. The first identity in (1.5.1) is reminiscent of the quantum Yang-Baxter equations (with spectral parameter), and though the $\widehat{\mathcal{R}}_i$ are no genuine R-matrices (but rather generalized R-matrices in the sense of Cherednik), they do give rise to a flat q-connection, as we will see shortly.

We define $\mathrm{End}(V)$-valued rational functions A_j on $U_{\mathrm{KZ}} \times \mathbb{C}^N$ by

$$\begin{aligned} \widehat{A}_j(u,\lambda) :=& \widehat{\mathcal{R}}_{j-1}(\lambda_j - \lambda_{j-1})\widehat{\mathcal{R}}_{j-2}(\lambda_i - \lambda_{j-2}) \cdots \widehat{\mathcal{R}}_1(\lambda_j - \lambda_1)\widehat{\pi}(u)_0 \\ & \times \widehat{\mathcal{R}}_{N-1}(\lambda_j - \lambda_N + 1)\widehat{\mathcal{R}}_{N-2}(\lambda_j - \lambda_{N-1} + 1) \cdots \widehat{\mathcal{R}}_j(\lambda_j - \lambda_{j+1} + 1) \end{aligned} \tag{1.5.2}$$

for $1 \leq j \leq N$. Furthermore, we define the covariant difference derivatives

$$\widehat{\nabla}_j(u) := \widehat{A}_j(u,\cdot)\mathcal{T}_j^\lambda - \mathrm{id}_V,$$

for $1 \leq j \leq N$.

Theorem 1.5.2. *For fixed $u \in \mathbb{C}^N$, the system of linear first order difference equations*

$$(\widehat{\nabla}_j(u)f)(\lambda) = 0, \qquad 1 \leq j \leq N \tag{1.5.3}$$

for V-valued holomorphic functions f in $\lambda \in \mathbb{C}^N$ is holonomic (in the sense of Subsection 1.2.3).

Definition 1.5.3. We call the holonomic system of difference equations (1.5.3) the *difference KZ equations*.

Again, just as we announced for the trigonometric KZ equations, Theorem 1.5.2 will follow by degenerating a specific holonomic system of q-difference equations, which are in some sense *dual* quantum KZ equations. The assertion of Theorem 1.5.2 is actually a partial result of the fact that the system (1.5.3) is one 'portion' of a certain compatible mixed system, as we will see in the next section.

1.6 Compatibility of the trigonometric KZ and the difference KZ equations

The following theorem asserts that the trigonometric KZ equations (1.4.7) and the difference KZ equations (1.5.3) we defined in the previous two subsections are compatible in the sense of Section 1.3.

Theorem 1.6.1. *Let A_i and $\widehat{A}_j$, as* $\mathrm{End}(V)$*-valued functions in $(u,\lambda) \in U_{\mathrm{KZ}} \times \mathbb{C}^N$, be given by* (1.4.6) *and* (1.5.2)*, respectively. For $1 \leq i, j \leq N$ define covariant derivatives*

$$\nabla_i := \frac{\partial}{\partial u_i} - A_i$$

and covariant difference derivatives

$$\widehat{\nabla}_j := \widehat{A}_j \mathcal{T}_j^{\lambda} - \mathrm{id}_V,$$

acting on holomorphic V-valued functions on $U_{\mathrm{KZ}} \times \mathbb{C}^N$. Then the system

$$\begin{aligned} (\nabla_i f)(u,\lambda) &= 0, \qquad 1 \leq i \leq N, \\ (\widehat{\nabla}_j f)(u,\lambda) &= 0, \qquad 1 \leq j \leq N, \end{aligned} \tag{1.6.1}$$

is a holonomic system of differential/difference equations for V-valued holomorphic functions f on $U_{\mathrm{KZ}} \times \mathbb{C}^N$ in the sense of Subsection 1.2.4.

Note that Theorem 1.6.1 implies Theorem 1.4.3 as well as Theorem 1.5.2. The proof of Theorem 1.6.1 follows from degenerating the result that the *bispectral quantum Knizhnik-Zamolodchikov equations*, to be defined in the following section, form a holonomic system of q-difference equations. This approach was used before by Takeyama [56] in a different setting.

The search for systems of equations compatible with the Knizhnik-Zamolodchikov equations was initiated by Felder, Markov, Tarasov and Varchenko. In their paper [20] they found a system of differential equations compatible with the rational KZ equations and they called this compatible system the (rational) *dynamical differential equations*. Subsequently, the trigonometric KZ equations were considered by Tarasov and Varchenko in [58]. Here, the resulting compatible system is a system of difference equations, so-called *dynamical difference equations*. Under the limiting process which turns the trigonometric KZ equations into the standard (rational) KZ equations, the dynamical difference equations are turned into the rational dynamical differential equations from [20].

Accordingly, we now know two different cases of trigonometric KZ equations forming a compatible system together with a certain holonomic system of difference equations: trigonometric KZ equations associated with affine Lie algebras and a compatible system of difference equations constructed via quantum groups (Tarasov

[57], Tarasov-Varchenko [58]), and, secondly, (affine) trigonometric KZ equations associated with degenerate affine Hecke algebras (Cherednik) and a compatible system of difference equations by degenerating the bispectral quantum KZ equations. Comparing Theorem 1.6.1 with Theorem 4.1 of [57], it is likely that both compatible systems are (gauge) equivalent.

1.7 Bispectral quantum KZ equations

The *bispectral quantum Knizhnik-Zamolodchikov equations* constitute a holonomic system of q-difference equations on a complex torus, which in some sense contains two families of the *quantum Knizhnik-Zamolodchikov equations*. These quantum KZ equations are a q-deformation of the Knizhnik-Zamolodchikov equations and are due to Frenkel and Reshetikhin [21] (from the quantum group perspective) and Cherednik [6] (from the Hecke algebra perspective).

In this section we provide an ad hoc definition of the bispectral quantum KZ equations by explicitly writing out the covariant q-derivatives. The conceptually more justified definition of the bispectral quantum KZ equations as well as the proof of the holonomy (which heavily depends on the conceptual definition) are given in the main text (see Section 2.3).

1.7.1 The bispectral quantum KZ equations

Fix a complex parameter $k \in \mathbb{C}^\times$, and let $h > 0$ such that $q = e^h$. Accordingly, we view the operators T_i, $\widehat{T}_i$, $\pi(\lambda)$ and $\widehat{\pi}(u)$ from Sections 1.4 and 1.5, as linear operators on V. We write $T := (\mathbb{C}^\times)^N$. The generalized R-matrices $R_i^q(z)$ and $\widehat{R}_i^q(z)$ on which the construction relies are the $\mathrm{End}(V)$-valued rational functions in $z \in \mathbb{C}$ defined by

$$\begin{aligned} R_i^q(z) &:= \frac{1-z}{k^{-1}-kz}(T_i - k) + \mathrm{id}_V, \\ \widehat{R}_i^q(z) &:= \frac{1-z}{k^{-1}-kz}(\widehat{T}_i - k) + \mathrm{id}_V \end{aligned}$$

for $1 \leq i < N$. Using representation theory of the affine Hecke algebra, it can be shown that

$$\begin{aligned} R_i^q(z)R_{i+1}^q(zw)R_i^q(w) &= R_{i+1}^q(w)R_i^q(zw)R_{i+1}^q(z), \quad && 1 \leq i < N-1, \\ R_j^q(z)R_j^q(z^{-1}) &= \mathrm{id}_V, && 1 \leq j < N, \end{aligned}$$

(cf. (1.5.1)) and similarly for the $\widehat{R}_i^q(z)$ ($1 \leq i < N$). Define $C_i^q(t,\gamma)$ and $\widehat{C}_j^q(t,\gamma)$ ($1 \leq i, j \leq N$) as $\mathrm{End}(V)$-valued meromorphic functions in $(t,\gamma) \in T \times T$ by

$$\begin{aligned} C_i^q(t,\gamma) =& R_{i-1}^q(t_{i-1}/t_i)R_{i-2}^q(t_{i-2}/t_i)\cdots R_1^q(t_1/t_i)\pi(\log(\gamma)/h) \\ & \times R_{N-1}^q(qt_N/t_i)\cdots R_{i+1}^q(qt_{i+2}/t_i)R_i^q(qt_{i+1}/t_i) \end{aligned}$$

and

$$\begin{aligned}\widehat{C}_j^q(t,\gamma) =& \widehat{R}_{j-1}^q(\gamma_j/\gamma_{j-1})\widehat{R}_{j-2}^q(\gamma_j/\gamma_{j-2})\cdots\widehat{R}_1^q(\gamma_j/\gamma_1)\widehat{\pi}(\log(t))\\ &\times \widehat{R}_{N-1}^q(q\gamma_j/\gamma_N)\cdots\widehat{R}_{j+1}^q(q\gamma_j/\gamma_{j+2})\widehat{R}_j^q(q\gamma_j/\gamma_{j+1}),\end{aligned}$$

where $\log(y)$, for $y=(y_1,\ldots,y_N)\in T$, is allowed to be any $x=(x_1,\ldots,x_N)\in\mathbb{C}^N$ such that $e^{x_i}=y_i$. In other words, $\pi(\log(\gamma)/h)v_w=\gamma_{w^{-1}}(N)v_{\sigma w}$ and $\widehat{\pi}(\log(t))v_w = t_{w(N)}^{-1}v_{w\sigma^{-1}}$ for $w\in S_N$. We have the following result.

Theorem 1.7.1. *The system of linear first-order q-difference equations*

$$\begin{aligned}C_i^q(t,\gamma)f(q^{-\epsilon_i}t,\gamma)&=f(t,\gamma), \qquad 1\le i\le N,\\ \widehat{C}_j^q(t,\gamma)f(t,q^{\epsilon_j}\gamma)&=f(t,\gamma), \qquad 1\le j\le N\end{aligned} \tag{1.7.1}$$

for V-valued meromorphic functions f in $(t,\gamma)\in T\times T$ is holonomic.

Remark 1.7.2. The system (1.7.1) can be written in the familiar form (1.2.6), so that it is clear what holonomy means here. Alternatively, we can reformulate it as a compatible system of flat q-connections, which is what we will do below.

Definition 1.7.3. We call (1.7.1) the system of *bispectral quantum KZ equations*.

Theorem 1.7.1 is one the main results in this thesis (see Section 2.3). The construction of the bispectral quantum KZ equations depends on a sophisticated algebraic structure called the *double affine Hecke algebra*, which was introduced by Cherednik (cf. [7]). Its fundamental properties, in particular a certain nontrivial duality symmetry (see [10, Theorem 1.4.8]), turn out to force the holonomy of the total system (1.7.1).

In the language of Section 1.3, (1.7.1) gives rise to a compatible system of flat q-connections on the complex torus. Concretely, the two systems are given by

$$\nabla_i^q(\gamma) := \mathcal{T}_{q,i}^t(C_i^q(\cdot,\gamma)^{-1})\mathcal{T}_{q,i}^t - \mathrm{id}_V, \qquad 1\le i\le N, \tag{1.7.2}$$

operating on meromorphic V-valued functions in $t\in T$, and

$$\widehat{\nabla}_j^q(t) := \widehat{C}_j^q(t,\cdot)\mathcal{T}_{q,j}^\gamma - \mathrm{id}_V, \qquad 1\le j\le N, \tag{1.7.3}$$

operating on meromorphic V-valued functions in $\gamma\in T$.

Both the linear first-order system of q-difference equations determined by (1.7.2) and the system determined by (1.7.3) are examples of Cherednik's quantum KZ equations [10, (1.3.12)]. For (1.7.2), the auxiliary parameters γ then relate to the central character of the affine Hecke algebra module naturally associated to the quantum KZ equation. Our key result, Theorem 1.7.1, thus states that the quantum KZ equations (associated to (1.7.2)) and the dual quantum KZ equations (associated to (1.7.3)) form a *compatible* system in the sense of Definition 1.3.1.

Let us return to Theorem 1.6.1. We do not have a direct and conceptual proof (as we do have for the holonomy of (1.7.1)), which is due to the absence of a duality symmetry of the associated degenerate double affine Hecke algebra in this case. Instead, Theorem 1.6.1 can be shown by degenerating the bispectral quantum KZ equations, as we will see in the next subsection.

1.7.2 Compatibility of the trigonometric and difference KZ equations revisited

In order to employ the result of Theorem 1.7.1 to come to a proof of Theorem 1.6.1, it is convenient to rewrite the bispectral quantum KZ equations as a system of difference equations for meromorphic functions on $\mathbb{C}^N \times \mathbb{C}^N$. For that purpose, we apply a change of variables by setting $t_i = e^{u_i}$, $\gamma_j = e^{h\lambda_j}$, and we let $e^{h\kappa} = k$. Define

$$R_i(z) := R_i^q(e^z), \qquad \widehat{R}_i(z) := \widehat{R}_i^q(e^z),$$

for $1 \leq i < N$ as End(V)-valued meromorphic functions in $z \in \mathbb{C}$. Note that we have

$$\begin{aligned} R_i(z)R_{i+1}(z+w)R_i(w) &= R_{i+1}(w)R_i(z+w)R_{i+1}(z), \quad && 1 \leq i < N-1, \\ R_j(z)R_j(-z) &= \mathrm{id}_V, && 1 \leq j < N, \end{aligned}$$

and similarly for the $\widehat{R}_i(z)$ ($1 \leq i < N$). Correspondingly, $C_i^q(t,\gamma)$ and $\widehat{C}_j^q(t,\gamma)$ become the operators

$$\begin{aligned} C_i(u,\lambda) :=& R_{i-1}(u_{i-1}-u_i)R_{i-2}(u_{i-2}-u_i)\cdots R_1(u_1-u_i)\pi(\lambda) \\ &\times R_{N-1}(u_N-u_i+h)\cdots R_{i+1}(u_{i+2}-u_i+h)R_i(u_{i+1}-u_i+h) \end{aligned}$$

and

$$\begin{aligned} \widehat{C}_j(u,\lambda) :=& \widehat{R}_{j-1}(h(\lambda_j-\lambda_{j-1}))\widehat{R}_{j-2}(h(\lambda_j-\lambda_{j-2}))\cdots\widehat{R}_1(h(\lambda_j-\lambda_1))\widehat{\pi}(u) \\ &\times \widehat{R}_{N-1}(h(\lambda_j-\lambda_N+1))\cdots\widehat{R}_{j+1}(h(\lambda_j-\lambda_{j+2}+1))\widehat{R}_j(h(\lambda_j-\lambda_{j+1}+1)), \end{aligned}$$

as End(V)-valued meromorphic functions in $(u,\lambda) \in \mathbb{C}^N \times \mathbb{C}^N$. For $u = (u_1,\ldots,u_N) \in \mathbb{C}^N$ and $c \in \mathbb{C}$, write $u + c\epsilon_i := (u_1,\ldots,u_i+c,\ldots,u_N)$. Theorem 1.7.1 can then be reformulated as follows.

Theorem 1.7.4. *The system of linear difference equations*

$$\begin{aligned} C_i(u,\lambda)f(u-h\epsilon_i,\lambda) &= f(u,\lambda), \qquad && 1 \leq i \leq N, \\ \widehat{C}_j(u,\lambda)f(u,\lambda+\epsilon_j) &= f(u,\lambda), && 1 \leq j \leq N \end{aligned} \tag{1.7.4}$$

for V-valued meromorphic functions f in $(u,\lambda) \in \mathbb{C}^N \times \mathbb{C}^N$ is holonomic.

First note that Theorem 1.7.4 is equivalent to conditions

$$\begin{aligned} C_i(u,\lambda)C_j(u-h\epsilon_i,\lambda) &= C_j(u,\lambda)C_i(u-h\epsilon_j,\lambda)\\ \widehat{C}_i(u,\lambda)\widehat{C}_j(u,\lambda+\epsilon_i) &= \widehat{C}_j(u,\lambda)\widehat{C}_i(u,\lambda+\epsilon_j)\\ C_i(u,\lambda)\widehat{C}_j(u-h\epsilon_i,\lambda) &= \widehat{C}_j(u,\lambda)C_i(u,\lambda+\epsilon_j) \end{aligned} \tag{1.7.5}$$

for $1 \le i,j \le N$ (cf. (1.2.10)). In order to establish the link with (1.6.1), it is convenient to put

$$D_i(u,\lambda) := C_i(u+h\epsilon_i,\lambda) - \mathrm{id}_V$$

for $1 \le i \le N$, and also write $\widehat{D}_j(u,\lambda) := \widehat{C}_j(u,\lambda)$ for $1 \le j \le N$. Then (1.7.4) reads

$$\begin{aligned} D_i(u,\lambda)f(u,\lambda) &= f(u+h\epsilon_i,\lambda) - f(u,\lambda),\\ \widehat{D}_j(u,\lambda)f(u,\lambda+\epsilon_j) &= f(u,\lambda) \end{aligned} \tag{1.7.6}$$

for $1 \le i,j \le N$. Now to derive Theorem 1.6.1 as a consequence of Theorem 1.7.4, we will view h as a formal parameter, and accordingly we will view $C_i(u,\lambda)$, $\widehat{C}_j(u,\lambda)$, etc. as elements of $\mathbb{C}[[h]] \otimes \mathrm{End}(V)$, and the equations (1.7.5) as identities in $\mathbb{C}[[h]] \otimes \mathrm{End}(V)$. By a direct computation one can now show that

$$\begin{aligned} A_i(u,\lambda) &= D_i(u,\lambda)_1\\ \widehat{A}_j(u,\lambda) &= \widehat{D}_j(u,\lambda)_0, \end{aligned}$$

for $1 \le i,j \le N$. Considering the h^1-term in the first and the h^0-term in the second set of equations of (1.7.6), we formally obtain the mixed system (1.6.1). Picking out the appropriate h^1-terms and h^0 terms in the holonomy conditions (1.7.5) then leads to the holonomy of (1.6.1), and hence to a proof of Theorem 1.6.1, as a consequence of the holonomy of the bispectral quantum KZ equations (1.7.1).

1.8 BqKZ and bispectral problems

From the results in this thesis it follows that solutions of a compatible system of differential, difference or q-difference equations can lead to solutions of *bispectral problems* in the sense of Duistermaat and Grünbaum [13]. In this section we explain the concept of bispectrality and its relevance to the compatible system of q-difference equations (1.7.1).

In general, the bispectral problem is concerned with a commutative algebra $\mathcal{A}$ of scalar differential (difference/q-difference) operators L, which is said to be *bispectral* if there exists a family of common eigenfunctions $f(u,\lambda)$ (depending on a spectral parameter λ)

$$(Lf)(u,\lambda) = \phi_L(\lambda)f(u,\lambda), \qquad L \in \mathcal{A}, \tag{1.8.1}$$

which is also a family of common eigenfunctions of a commutative algebra $\mathcal{B}$ of scalar linear differential (difference/q-difference) operators Λ with respect to λ

$$(\Lambda f)(u,\lambda) = \theta_\Lambda(u)f(u,\lambda), \qquad \Lambda \in \mathcal{B}. \tag{1.8.2}$$

The bispectral problem has its origins in the work of Duistermaat and Grünbaum, who encountered the case where $\mathcal{A}$ is the algebra of differential operators that commute with a Schrödinger operator $L = -\frac{d^2}{du^2} + V(u)$ for some specific potential V. Since then, many other systems of differential equations have been considered, as well as generalizations to difference and q-difference equations. Bispectrality has shown up in various areas of mathematical physics (see [24]).

It is sometimes possible to find an explicit correspondence between solutions of a *holonomic system* of linear first-order differential (difference or q-difference) equations and solutions of a certain *spectral problem* of the form (1.8.1). For example, Cherednik derived a correspondence [7, Theorem 4.4] between the quantum KZ equations and the spectral problem involving Ruijsenaars' commuting trigonometric q-difference operators [50] (also known as Macdonald-Ruijsenaars operators). In this thesis we consider a refinement of this map, which leads to a correspondence between solutions of the *bispectral* quantum KZ equations (1.7.1), and solutions of a *bispectral* problem of the form (1.8.1)–(1.8.2), where now $\mathcal{A}$ and $\mathcal{B}$ are commutative algebras of scalar linear q-difference operators, generated by the Ruijsenaars operators.

In particular, asymptotic solutions of the bispectral quantum KZ equations that we construct in this thesis are transferred to bispectral analogs of Harish-Chandra series solutions ([16], [17], [31] and [36]) of the spectral problem of the Ruijsenaars operators. These *bispectral Harish-Chandra series solutions* can be exploited to obtain new results on the singularities and the convergence of the Harish-Chandra series (see Chapter 3).

Furthermore, fixing one of the variables of the asymptotic solutions in an appropriate way, we obtain Laurent polynomial solutions of the bispectral quantum KZ equations. Pushing these polynomials forward through the correspondence, we end up with the well-known *symmetric self-dual Macdonald polynomials*. By exploiting the results concerning the solutions of the bispectral quantum KZ equations, we obtain new proofs of the duality and evaluation formulas for the symmetric Macdonals polynomials [42] (see Chapter 4).

1.9 Overview

We conclude the introduction with a brief outline of the contents of the remaining chapters in this thesis. For a more detailed description per chapter, including references to the literature, the reader is referred to the introduction with which each chapter starts off.

In Section 1.7 we already introduced the system of bispectral quantum Knizhnik-Zamolodchikov equations, which, at a first encounter, probably seemed to come from nowhere. Its 'natural' construction and also the proof of its holonomy heavily relies on the theory of Cherednik's double affine Hecke algebra. In Chapter 2 we recall this theory and give a detailed construction of the bispectral quantum KZ equations as a holonomic system of linear first-order q-difference equations. Moreover, we construct asymptotic solutions which constitute a basis of the solution space of the bispectral KZ equations, regarded as a vector space over the field of q-dilation invariant meromorphic functions (which serves as field of constants).

The correspondence between the solutions of the bispectral quantum KZ equations and the solutions of a bispectral problem for the Ruijsenaars operators (see Section 1.8) will be established in full detail in the first part of Chapter 3. We already mentioned in 1.8 that it is worthwhile to see what, under this correspondence, happens to the asymptotic solutions of the bispectral quantum KZ equations. The resulting so-called bispectral Harish-Chandra series solutions of the bispectral problem, as well as its applications to the theory of (ordinary) Harish-Chandra series, are the object of study in the remaining part of Chapter 3.

In the final paragraph of Section 1.8 we mentioned that the correspondence between the bispectral quantum KZ equations and the bispectral problem on the level of polynomial solutions has several interesting applications in the theory of Macdonald polynomials. These are addressed in Chapter 4.

To any root system one can associate (quantum) KZ equations. Up till Chapter 5 (including this introduction), we consider only one particular root system, namely the root system of type A_{N-1}. In Chapter 5 we extend the theory we developed in Chapters 2 and 3 to arbitrary root systems. In particular, we construct the bispectral quantum KZ equations by means of the double affine Hecke algebra, now corresponding to an arbitrary irreducible reduced root system. We will not discuss the polynomial theory for arbitrary root systems in this thesis, but we expect that many of the results obtained in Chapter 4 can be transferred to the arbitrary root system case as well.

The appendix contains a detailed account on the general theory of power series solutions of holonomic systems of q-difference equations, which plays a crucial role in Chapter 2 (and Chapter 5), when we construct the asymptotic solutions of the bispectral quantum KZ equations.

The material presented in Chapters 2–4 and the appendix can be found in [45]. Chapter 5 is a slightly adapted version of [44].

1.10 Global conventions

Throughout the text we adopt the following notations and conventions.

- For a module M over a commutative ring R and a ring extension $R \subset S$, we write $M^S = S \otimes_R M$.
- Without subscript, $\otimes$ always stands for tensor product over $\mathbb{C}$.
- For a module M over $\mathbb{C}$, End(M) stands for $\mathbb{C}$-linear endomorphisms.
- $\mathbb{N} = \{1, 2, \ldots\}$.
- For $a, r \in \mathbb{R}$ with $a > 0$, we choose a^r to be the positive real branch of the power function.

Chapter 2

Bispectral quantum Knizhnik-Zamolodchikov equations

2.1 Introduction

Let V be an $N!$-dimensional complex vector space and let T denote the complex torus $T := (\mathbb{C} \setminus \{0\})^N$. In this chapter we derive an explicit holonomic system of q-difference equations on V-valued meromorphic functions on $T \times T$, which we call the bispectral quantum Knizhnik-Zamolodchikov (BqKZ) equations. It contains as a subsystem, quantum affine KZ equations of a particular type [10, 21]. From this perspective, the additional compatible equations may be thought of as the associated quantum isomonodromy transformations. Let us briefly explain the ideas involved and the relation of the BqKZ equations to the quantum affine KZ equations introduced by Cherednik.

Let $W = S_N \ltimes \mathbb{Z}^N$ be the (extended) affine Weyl group of type GL_N, that is, the semidirect product of the symmetric group S_N and the lattice $\mathbb{Z}^N$. The group S_N acts on T by permuting the coordinates. Choose $0 < q < 1$. The action of S_N on T extends to an action of W on T by letting $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ act via

$$t = (t_1, \ldots, t_N) \mapsto q^\lambda t := (q^{\lambda_1} t_1, \ldots, q^{\lambda_N} t_N).$$

The BqKZ system which we will introduce, is a system of q-difference equations of the form

$$C_{(\lambda,\mu)}(t, \gamma) f(q^{-\lambda} t, q^\mu \gamma) = f(t, \gamma), \qquad \lambda, \mu \in P^\vee,$$

for meromorphic functions f on $T \times T$ with values in V. Here $C_{(\lambda,\mu)}$ ($\lambda, \mu \in \mathbb{Z}^N$) are explicit $\mathrm{End}(V)$-valued meromorphic functions on $T \times T$, satisfying the following cocycle property

$$C_{(\lambda+\nu,\mu+\xi)}(t,\gamma) = C_{(\lambda,\mu)}(t,\gamma)C_{(\nu,\xi)}(q^{-\lambda}t, q^{\mu}\gamma), \qquad \lambda,\mu,\nu,\xi \in \mathbb{Z}^N, \tag{2.1.1}$$

which reflects the fact that BqKZ is holonomic.

BqKZ contains, in some sense, two families of Cherednik's quantum affine KZ equations associated with the principal series representation of the affine Hecke algebra H of type GL_N. We recall [10] (for arbitrary root systems, see also Subsection 5.3.2) that the quantum affine KZ equations associated with a finite dimensional H-module M is a consistent system of q-difference equations of the form

$$F_\lambda^M(t)f(q^{-\lambda}t) = f(t), \qquad \lambda \in P^\vee,$$

for meromorphic functions f on T with values in M, and where F_λ^M ($\lambda \in \mathbb{Z}^N$) are $\mathrm{End}(M)$-valued meromorphic functions on T satisfying cocycle conditions similar to (2.1.1). Now the first family of quantum affine KZ equations inside BqKZ is parameterized by $\gamma \in T \simeq \{1\} \times T \subset T \times T$. More precisely, if we fix $\gamma = \zeta \in T$, we have

$$C_{(\lambda,e)}(t,\zeta) = F_\lambda^{M_\zeta}(t),$$

where M_ζ is the principal series representation of H with central character ζ, which as a vector space can be identified with V via a ζ-dependent isomorphism. Similarly, BqKZ contains a second family of quantum affine KZ equations, parameterized by $t \in T$, related to the affine Hecke algebra module $M_{t^{-1}}$. It is obtained from the first by interchanging the roles of the torus variables t and γ^{-1} and by conjugating the cocycle matrices by an explicit complex linear automorphism C_ι of V. Hence, it acts only on the second T-component of $T \times T$ and as such realizes $\mathrm{qKZ}_{t^{-1}}$ for fixed $t \in T$. In particular, this provides a quantum isomonodromic interpretation of qKZ. This should be compared with the interpretation of rational KZ equations as quantizations of Schlesinger equations, see [49] and [23].

Etingof and Varchenko [19] used quantum group methods to construct systems of q-difference equations (so-called dynamical q-difference equations) that are compatible with Frenkel and Reshetikhin's quantum KZ equations associated with evaluation representations of quantum affine algebras. It is likely that the system of dynamical q-difference equations associated with qKZ_ζ is equivalent to the dual qKZ subsystem in BqKZ.

Preceding the above mentioned work [19] of Etingof and Varchenko, dynamical equations for various degenerations of quantum KZ equations have been analyzed in detail; see, e.g., [20], [60], [57], [59], [58] and [35]. An interesting aspect in, e.g., [60] and [59], is the observation that various degenerations of quantum KZ equations are the duals of their associated dynamical equations with respect to $(\mathfrak{gl}_r, \mathfrak{gl}_s)$ duality. In the present set-up (which corresponds to $r = s = N$), this duality is incorporated by the automorphism C_ι, which reflects Cherednik's duality anti-involution of the double affine Hecke algebra $\mathbb{H}$ on the level of BqKZ.

Related to this observation, an important virtue of the present approach is worth mentioning. The double affine Hecke algebra and its symmetry (embodied by the duality involution) give rise to the bispectral quantum KZ equations, and thus a way to construct a system of q-difference equations compatible with the quantum KZ equations, which is relatively easy compared to cases considered in the above mentioned works of Etingof, Felder, Markov, Tarasov and Varchenko.

We conclude this introduction with a detailed outline of this chapter. In Section 2.2 we introduce Cherednik's [10] double affine Hecke algebra $\mathbb{H}$ of type GL_N, on which the construction of BqKZ relies in the following way. As a vector space $\mathbb{H}$ is isomorphic to $\mathbb{C}[T]\otimes H \simeq \mathbb{C}[T]\otimes H_0\otimes\mathbb{C}[T] \simeq \mathbb{C}[T\times T]\otimes H_0$ with H_0 the finite Hecke algebra of type A_{N-1}. Cherednik's anti-algebra involution $*\colon \mathbb{H}\to\mathbb{H}$ essentially interchanges, under the above vector space identification, the role of the two copies of $\mathbb{C}[T]$. For $w, w' \in W = S_N \ltimes \mathbb{Z}^N$ we consider the map $h \mapsto \widetilde{S}_w h \widetilde{S}^*_{w'}$ ($h\in\mathbb{H}$), where the $\widetilde{S}_w \in \mathbb{H}$ are Cherednik's nonnormalized (X-)intertwiners. Restricted to $w, w' \in \mathbb{Z}^N$, suitable renormalizations of these maps become the cocycle matrices of BqKZ, with H_0 playing the role of V. The anti-involution $*$ of $\mathbb{H}$ gives rise to the automorphism C_ι interchanging the qKZ subsystem of BqKZ with its dual subsystem in BqKZ. This construction of BqKZ is described in Section 2.3.

In Section 2.4 we make the BqKZ equations explicit by calculating the cocycle matrices $C_{(\lambda,\mu)}$ for suitable $\lambda,\mu\in\mathbb{Z}^N$. This enables us to relate qKZ$_\zeta$ to Frenkel and Reshetikhin's [21] quantum KZ equations associated with the N-fold tensor product of the vector representation of quantum $\mathfrak{sl}_N$ (see [10, §1.3.2]). A special case of qKZ$_\zeta$ was considered earlier by Smirnov [54].

In Section 2.5 we investigate the space SOL of M-valued meromorphic solutions of BqKZ in detail. We first analyze BqKZ in a suitable asymptotic region. It leads to a solution Φ of BqKZ which is self-dual, in the sense that $\Phi(t,\gamma) = C_\iota\Phi(\gamma^{-1}, t^{-1})$ as M-valued meromorphic functions in $(t,\gamma)\in T\times T$. We construct a basis of SOL in terms of Φ, and we give an explicit formula for the leading term of $\Phi(t,\gamma)$ as function of t.

The contents of this chapter agree with Sections 2–5 of [45].

2.2 The double affine Hecke algebra

2.2.1 The extended affine Weyl group

Let $N\geq 2$ and let $D = D_N$ be the affine Dynkin diagram of affine type $\widehat{A}_{N-1}$ (the cyclic graph with N vertices if $N\geq 3$). The N vertices are labeled by the numbers $0,1,\ldots,N-1$ (anticlockwise if $N\geq 3$). Occasionally we identify the set of labels with the group $\mathbb{Z}_N$ of integers modulo N.

Write W_Q for the affine Weyl group of affine type $\widehat{A}_{N-1}$. In terms of its Coxeter generators s_i ($i\in\mathbb{Z}_N$), the characterizing group relations are the quadratic relations $s_i^2 = 1$ and, if $N\geq 3$, the braid relations

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i, \qquad i-j\neq 0,\pm 1. \end{aligned} \tag{2.2.1}$$

The subgroup generated by $s_1, \ldots, s_{N-1}$ is isomorphic to the symmetric group S_N in N letters, where s_i is identified with the simple transposition $i \leftrightarrow i+1$.

Let Aut(D) be the group of automorphisms of the affine Dynkin diagram of type $\widehat{A}_{N-1}$. Let $c \in \mathrm{Aut}(D)$ be the element of order N, acting on the label set $\mathbb{Z}_N$ of the vertices of D by $c(i) = i+1$. We view c as automorphism of W_Q by $c(s_i) = s_{i+1}$.

Let $\Omega = \langle \pi \rangle$ be the infinite cyclic group with cyclic generator π. It acts by group automorphisms on W_Q by $\pi \mapsto c$. Accordingly, we can define the semidirect product group $W = W_Q \rtimes \Omega$, which is called the extended affine Weyl group (associated with GL_N). We write e for the identity element of W.

Since $s_0 = \pi s_{N-1} \pi^{-1}$, the subgroups S_N and Ω already generate W as a group. Furthermore, we have $W \simeq S_N \ltimes \mathbb{Z}^N$. The cyclic generator π of Ω corresponds to $\pi = \sigma \epsilon_N$, where $\{\epsilon_i\}_{i=1}^N$ denotes the standard $\mathbb{Z}$-basis of $\mathbb{Z}^N$ and $\sigma = s_1 s_2 \cdots s_{N-1} \in S_N$ is the "clockwise rotation" which maps N to 1 and all other i to $i+1$. Conversely,

$$\epsilon_j = s_{j-1} \cdots s_2 s_1 \pi s_{N-1} s_{N-2} \cdots s_j \tag{2.2.2}$$

for $j = 1, \ldots, N$.

Remark 2.2.1. Under the identification $W \simeq S_N \ltimes \mathbb{Z}^N$, we have $W_Q = S_N \ltimes Q$ with $Q \subset \mathbb{Z}^N$ the sublattice of rank $N-1$ consisting of N-tuples of integers that sum up to zero (this is the (co)root lattice of the root system $R = \{\epsilon_i - \epsilon_j\}_{1 \leq i \neq j \leq N}$ of type A_{N-1}).

For $w \in W$, let $w' \in W_Q$ and $\omega \in \Omega$ be the unique group elements such that $w = w'\omega$. Then we define the length $\ell(w)$ of w to be the length of $w' \in W_Q$, i.e., it is the minimal number r such that w' can be expressed as

$$w' = s_{i_1} \cdots s_{i_r}$$

for some $i_k \in \mathbb{Z}_N$ (such an expression of w', as well as the resulting expression for $w = w'\omega$, is called a reduced expression). Thus Ω consists of the elements of W of length zero.

A central role in this thesis is played by an action of the extended affine Weyl group W by q-difference reflection operators on suitable function spaces on $T := (\mathbb{C}^\times)^N$, where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Here q is taken to be real and strictly between zero and one (with minor technical adjustments the condition on q may be relaxed to $0 < |q| < 1$, and a parallel theory can be developed for $|q| > 1$). Since q is fixed once and for all, we will in general suppress the dependence on q in notations. We start with an action of W on T by

$$\begin{aligned} wt &= (t_{w^{-1}(1)}, \ldots, t_{w^{-1}(N)}), \qquad w \in S_N, \\ \lambda t &= (q^{\lambda_1} t_1, \ldots, q^{\lambda_N} t_N), \qquad \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N, \end{aligned} \tag{2.2.3}$$

for $t = (t_1, \ldots, t_N) \in T$. It is convenient to introduce $\kappa^\lambda := (\kappa^{\lambda_1}, \ldots, \kappa^{\lambda_N})$ for $\kappa \in \mathbb{C}^\times$ and $\lambda \in \mathbb{Z}^N$, so that the action of $\lambda \in \mathbb{Z}^N$ on $t \in T$ can simply be written as

$$\lambda t = q^\lambda t$$

in standard vector notation. Note that the action of $\pi \in \Omega$ is given by

$$\pi(t_1, \ldots, t_N) = (qt_N, t_1, \ldots, t_{N-1}).$$

Consider the algebra $\mathbb{C}[T] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ of complex-valued regular functions on T, where $x_i(t) := t_i$ for $t = (t_1, \ldots, t_N) \in T$ are the standard coordinate functions. We write

$$x^\lambda := x_1^{\lambda_1} \cdots x_N^{\lambda_N} \in \mathbb{C}[T]$$

for $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$, which form the monomial basis of $\mathbb{C}[T]$.

Let $\mathbb{C}(T)$ be the field of rational functions on T, $\mathcal{O}(T)$ be the ring of analytic functions on T, and $\mathcal{M}(T)$ be the field of meromorphic functions on T. Note that $\mathcal{M}(T)$ is the quotient field of $\mathcal{O}(T)$ (cf. [25, Theorem 7.4.6]). The W-action on T gives rise to a left W-action by algebra automorphisms on $\mathbb{C}[T]$, $\mathbb{C}(T)$, $\mathcal{O}(T)$ and $\mathcal{M}(T)$, via

$$(wf)(t) = f(w^{-1}t)$$

for $w \in W$, $t \in T$. We can, in particular, form the smash product algebra $\mathbb{C}(T)\#W$. Recall that if G is a group and A is a G-algebra over $\mathbb{C}$ (that is, a unital associative algebra over $\mathbb{C}$ endowed with a left G-action by algebra automorphisms), then the smash product algebra $A\#G$ is the unique complex unital associative algebra such that

(i) $A\#G = A \otimes \mathbb{C}[G]$ as a complex vector space;

(ii) the canonical linear embeddings $A \hookrightarrow A\#G$, $\mathbb{C}[G] \hookrightarrow A\#G$ are algebra homomorphisms; and

(iii) the cross relations

$$(a \otimes g) \cdot (b \otimes h) = ag(b) \otimes gh,$$

are satisfied for $a, b \in A$ and $g, h \in G$.

We will always write $ag := a \otimes g \in A\#G$ ($a \in A$, $g \in G$). Observe that $A\#G$ canonically acts on any G-algebra B containing A as a G-subalgebra.

Note that the smash product algebra $\mathbb{C}(T)\#W$ depends on q, since the W-action on $\mathbb{C}(T)$ depends on q (see (2.2.3)). Sometimes it is convenient to emphasize its q-dependence, in which case we write $\mathbb{C}(T)\#_q W$ instead of $\mathbb{C}(T)\#W$.

The canonical left $\mathbb{C}(T)\#W$-action on $\mathbb{C}(T)$ (and $\mathcal{M}(T)$) is faithful and realizes $\mathbb{C}(T)\#W$ as the algebra of q-difference S_N-reflection operators with coefficients in $\mathbb{C}(T)$. If $f \in \mathbb{C}(T)$, then we write $f(X)$ for the associated element in $\mathbb{C}(T)\#W$ (it is the operator defined as multiplication by f). In particular, X_i is multiplication by the coordinate function x_i.

2.2.2 The extended affine Hecke algebra and Cherednik's basic representation

In this subsection we recall some constructions and results due to Cherednik (see, e.g., [10, Chapter 1] and references therein).

Fix a nonzero complex number k.

Definition 2.2.2. The affine Hecke algebra $H_Q = H_Q(k)$ is the complex unital associative algebra generated by T_i ($i \in \mathbb{Z}_N$) and satisfying

(i) if $N \geq 3$, the braid relations

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, \\ T_iT_j &= T_jT_i, \qquad i-j \neq 0, \pm 1; \end{aligned}$$

(ii) the quadratic relations $(T_i - k)(T_i + k^{-1}) = 0$.

The Dynkin diagram automorphism $c \in \mathrm{Aut}(D)$ can also be viewed as automorphism of H_Q by $c(T_i) = T_{i+1}$. Accordingly, Ω acts by algebra automorphisms on H_Q by $\pi \mapsto c$. The extended affine Hecke algebra $H = H(k)$ is the associated smash product algebra $H_Q \# \Omega$.

For a reduced expression $w = s_{i_1} \cdots s_{i_r}\omega \in W$ ($i_k \in \mathbb{Z}_N$, $\omega \in \Omega$), the element

$$T_w := T_{i_1} \cdots T_{i_r}\omega \in H$$

is well-defined. The T_w ($w \in W$) form a linear basis of H. For $k = 1$, the extended affine Hecke algebra H is isomorphic to the group algebra $\mathbb{C}[W]$ of W via the identification $T_w \leftrightarrow w$ ($w \in W$).

The finite Hecke algebra is the subalgebra H_0 of H generated by $T_1, \ldots, T_{N-1}$. The elements T_w ($w \in S_N$) form a linear basis of H_0. Note that H is already generated as algebra by H_0 and $\pi^{\pm 1}$, since $T_0 = \pi T_{N-1}\pi^{-1}$.

Put

$$Y_i := T_{i-1}^{-1} \cdots T_2^{-1}T_1^{-1}\pi T_{N-1}T_{N-2} \cdots T_i \in H \tag{2.2.4}$$

for $i = 1, \ldots, N$. Note that Y_i becomes the translation element ϵ_i in W if $k = 1$. We furthermore write $Y^\lambda := Y_1^{\lambda_1} \cdots Y_N^{\lambda_N}$ for $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$. We have the following characterization of H, due to Bernstein. For details we refer to Lusztig [38] or Macdonald [42, §4.2].

Theorem 2.2.3. *H is the unique unital complex associative algebra, such that*

(i) *$H_0 \otimes \mathbb{C}[T] \simeq H$ as complex vector spaces, via $h \otimes f \mapsto hf(Y)$ for $h \in H_0$, $f \in \mathbb{C}[T]$, where $f(Y) = \sum_\lambda c_\lambda Y^\lambda$ if $f = \sum_\lambda c_\lambda x^\lambda \in \mathbb{C}[T]$;*

(ii) *the canonical maps $H_0, \mathbb{C}[T] \hookrightarrow H$ are algebra embeddings; we write $\mathbb{C}_Y[T] = \mathrm{span}_{\mathbb{C}}\{Y^\lambda\}_{\lambda \in \mathbb{Z}^N}$ for the image of $\mathbb{C}[T]$ in H; and*

(iii) *the following cross relations*

$$T_i^{-1} Y_i T_i^{-1} = Y_{i+1},$$
$$Y_j T_i = T_i Y_j, \qquad i \neq j-1, j$$

are satisfied for $1 \leq i < N$ *and* $1 \leq j \leq N$.

Cherednik realized the affine Hecke algebra H inside the algebra $\mathbb{C}(T)\#W$ of q-difference reflection operators as follows.

Theorem 2.2.4. *There is a unique injective algebra homomorphism* $\rho = \rho_{k,q} \colon H(k) \to \mathbb{C}(T)\#_q W$ *satisfying*

$$\rho(T_i) = k + c_k(X_i/X_{i+1})(s_i - 1),$$
$$\rho(\pi) = \pi,$$

for $i = 1, \ldots, N-1$, *where*

$$c_k(z) := \frac{k^{-1} - kz}{1 - z}. \tag{2.2.5}$$

Note that for the affine Hecke algebra $H = H(k)$ with fixed parameter k, Theorem 2.2.4 yields a one-parameter family of realizations of H (the additional parameter being q).

Remark 2.2.5. The image $\rho(H)$ preserves $\mathbb{C}[T]$, viewed as a subspace of the canonical $\mathbb{C}(T)\#W$-module $\mathbb{C}(T)$. The resulting representation of H on $\mathbb{C}[T]$ is faithful and is called the basic representation of H.

We frequently identify H with its image under ρ in $\mathbb{C}(T)\#W$.

We now come to the definition of Cherednik's double affine Hecke algebra which depends, besides on k, on the additional parameter q.

Definition 2.2.6. The double affine Hecke algebra $\mathbb{H} = \mathbb{H}(k,q)$ is the subalgebra of $\mathbb{C}(T)\#_q W$ generated by $\rho_{k,q}(H)$ and by the multiplication operators $f(X)$ ($f \in \mathbb{C}[T]$).

Let $\mathbb{L} = \mathbb{C}[T] \otimes \mathbb{C}[T] \simeq \mathbb{C}[T \times T]$ denote the complex-valued regular functions on $T \times T$. We view $\mathbb{H}$ as $\mathbb{L}$-module by

$$(f \otimes g) \cdot h := f(X) h g(Y) \tag{2.2.6}$$

for $f, g \in \mathbb{C}[T]$ and $h \in \mathbb{H}$. The following theorem is the so-called Poincaré-Birkhoff-Witt (PBW) property of the double affine Hecke algebra.

Theorem 2.2.7. *We have* $\mathbb{H} \simeq H_0^{\mathbb{L}} = \mathbb{L} \otimes H_0$ *as* $\mathbb{L}$-*modules.*

The PBW property is an essential ingredient in deriving the characterizing relations for the double affine Hecke algebra $\mathbb{H}$ in terms of its algebraic generators T_i

$(1 \leq i < N)$, $\pi^{\pm 1}$ and $X_j^{\pm 1}$ $(1 \leq j \leq N)$. Since we are not going to use this presentation explicitly here, we refer the reader to [10] for further details. We use though one of its direct consequences, namely the existence of the duality anti-isomorphism (see Cherednik [10, Theorem 1.4.8]):

Theorem 2.2.8. *There exists a unique $\mathbb{C}$-linear anti-algebra involution $*\colon \mathbb{H} \to \mathbb{H}$ determined by*

$$\begin{aligned} T_w^* &= T_{w^{-1}}, \qquad && w \in S_N, \\ (Y^\lambda)^* &= X^{-\lambda}, && \lambda \in \mathbb{Z}^N, \\ (X^\lambda)^* &= Y^{-\lambda}, && \lambda \in \mathbb{Z}^N. \end{aligned}$$

2.2.3 Intertwiners

In this subsection we recall the construction of the (nonnormalized) affine intertwiners associated to the double affine Hecke algebra $\mathbb{H}$. The intertwiners play an important role in the construction of a nontrivial $W \times W$-cocycle in the next section. Consider the elements

$$\begin{aligned} \widetilde{S}_i &= (k - k^{-1} X_{i+1}/X_i) s_i, \qquad 1 \leq i < N, \\ \widetilde{S}_0 &= (k - k^{-1} q^{-1} X_1/X_N) s_0, \\ \widetilde{S}_\pi &= \pi \end{aligned}$$

in $\mathbb{C}(T)\#_q W$. The following facts are well known (cf., e.g., [10, §1.3]). For the convenience of the reader, we give a short sketch of the proof.

Proposition 2.2.9. *Let $w \in W$ and let $w = s_{j_1} \cdots s_{j_r} \pi^m$ be a reduced expression ($j_l \in \mathbb{Z}_N$, $m \in \mathbb{Z}$).*

- **(i)** *$\widetilde{S}_w := \widetilde{S}_{j_1} \cdots \widetilde{S}_{j_r} \widetilde{S}_\pi^m$ is a well-defined element of $\mathbb{C}(T)\#W$;*
- **(ii)** *$\widetilde{S}_w \in \mathbb{H}$;*
- **(iii)** *the $\widetilde{S}_i$ ($i \in \mathbb{Z}_N$) satisfy the $\widehat{A}_{N-1}$-type braid relations;*
- **(iv)** *$\widetilde{S}_w f(X) = (wf)(X) \widetilde{S}_w$ in $\mathbb{C}(T)\#W$ for all $f \in \mathbb{C}(T)$; and*
- **(v)** *$\widetilde{S}_i \widetilde{S}_i = (k - k^{-1} X_{i+1}/X_i)(k - k^{-1} X_i/X_{i+1})$ for $1 \leq i < N$.*

Proof. (i) Set $d_i := (k - k^{-1} X_{i+1}/X_i)$ $(1 \leq i < N)$ and $d_0 := (k - k^{-1} q^{-1} X_1/X_N)$. We have

$$\widetilde{S}_w = d_{j_1} (s_{j_1} d_{j_2}) \cdots (s_{j_1} \cdots s_{j_{r-1}} d_{j_r}) w$$

in $\mathbb{C}(T)\#W$. By, e.g., Macdonald [42, (2.2.9)], we know that

$$d_w := d_{j_1} (s_{j_1} d_{j_2}) \cdots (s_{j_1} \cdots s_{j_{r-1}} d_{j_r}) \tag{2.2.7}$$

is independent of the reduced expression of w. Hence $\widetilde{S}_w \in \mathbb{C}(T)\#W$ is well defined.
(ii) Note that $\widetilde{S}_i$ can be written as

$$\widetilde{S}_i = (1 - X_{i+1}/X_i)(T_i - k) + k - k^{-1}X_{i+1}/X_i \qquad (2.2.8)$$

for $1 \leq i < N$, which shows that it lies in $\mathbb{H}$. Furthermore, $\pi^{\pm 1} \in \mathbb{H}$, hence $\widetilde{S}_\pi^{\pm 1} \in \mathbb{H}$ and $\widetilde{S}_0 = \pi \widetilde{S}_{N-1}\pi^{-1} \in \mathbb{H}$. Consequently, $\widetilde{S}_w \in \mathbb{H} \subset \mathbb{C}(T)\#W$.
(iii) is immediate from (i), while (iv) and (v) are clear from the definition of the $\widetilde{S}_i$ and $\widetilde{S}_\pi$. □

Definition 2.2.10. The elements $\widetilde{S}_w$ ($w \in W$) are called the affine intertwiners of $\mathbb{H}$.

2.3 The bispectral quantum KZ equations

2.3.1 Construction of the cocycle

Let ι denote the nontrivial element of the two group $\mathbb{Z}_2$. We define the group $\mathbb{W}$ as the semidirect product

$$\mathbb{W} := \mathbb{Z}_2 \ltimes (W \times W),$$

where $\iota \in \mathbb{Z}_2$ acts on $W \times W$ by switching the components: $\iota(w, w') = (w', w)\iota$ for $w, w' \in W$. We first use the double affine Hecke algebra, its affine intertwiners, and its duality anti-isomorphism to construct a group homomorphism $\tau = \tau_k \colon \mathbb{W} \to \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$ depending on the Hecke algebra parameter k, where $\mathbb{K} := \mathcal{M}(T \times T)$ is the field of meromorphic functions on $T \times T$.

The representation τ will be constructed from the complex linear endomorphisms $\widetilde{\sigma}_{(w,w')}$ ($w, w' \in W$) and $\widetilde{\sigma}_\iota$ of the double affine Hecke algebra $\mathbb{H}$, defined by

$$\widetilde{\sigma}_{(w,w')}(h) = \widetilde{S}_w h \widetilde{S}_{w'}^*,$$
$$\widetilde{\sigma}_\iota(h) = h^*$$

for $h \in \mathbb{H}$. In the following lemma we collect some elementary properties of the maps $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$. First we introduce some auxiliary notations.

For a regular function $g \in \mathbb{C}[T]$, we write $g(x) \in \mathbb{C}[T \times T]$ (respectively $g(y) \in \mathbb{C}[T \times T]$) for the corresponding regular function on $T \times T$ constant with respect to the second (respectively first) T-component. In particular, the x_i (respectively y_i) are the standard coordinate functions of the first (respectively second) copy of T in $T \times T$. Recall the regular function $d_w \in \mathbb{C}[T]$ ($w \in W$) such that

$$\widetilde{S}_w = d_w(X)w$$

in $\mathbb{C}(T)\#W$; see (2.2.7).

Lemma 2.3.1. *The complex linear endomorphisms $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$ of $\mathbb{H}$ satisfy the following properties:*

(i) *the* $\widetilde{\sigma}_{(s_i,e)}$ *(*$i \in \mathbb{Z}_N$*) satisfy the* $\widehat{A}_{N-1}$ *braid relations;*

(ii) $\widetilde{\sigma}^2_{(s_i,e)} = d_{s_i}(x)(s_i d_{s_i})(x) \cdot \mathrm{id}$ *for* $i \in \mathbb{Z}_N$*;*

(iii) $\widetilde{\sigma}_{(\pi,e)}\widetilde{\sigma}_{(s_i,e)}\widetilde{\sigma}_{(\pi^{-1},e)} = \widetilde{\sigma}_{(s_{i+1},e)}$ *for* $i \in \mathbb{Z}_N$*;*

(iv) $\widetilde{\sigma}^2_\iota = \mathrm{id}$ *and* $\widetilde{\sigma}_{(e,w)} = \widetilde{\sigma}_\iota \widetilde{\sigma}_{(w,e)} \widetilde{\sigma}_\iota$ *for* $w \in W$*; and*

(v) $\widetilde{\sigma}_{(w,e)}\widetilde{\sigma}_{(e,w')} = \widetilde{\sigma}_{(w,w')} = \widetilde{\sigma}_{(e,w')}\widetilde{\sigma}_{(w,e)}$ *for* $w, w' \in W$.

Proof. These are direct consequences of Proposition 2.2.9 and Theorem 2.2.8. □

To construct a $\mathbb{W}$-action from the maps $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$, we need to renormalize the maps appropriately. To do so, we describe as a first step the behavior of the maps $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$ with respect to the $\mathbb{L}$-module structure (2.2.6) on $\mathbb{H}$. This will allow us to extend the maps $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$ to endomorphisms of $H_0^{\mathbb{K}} \simeq \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H}$, which is a suitably flexible surrounding for the normalizations of the maps to take place in.

Consider the group involution ${}^\diamond \colon W \to W$ defined by $w^\diamond = w$ for $w \in S_N$ and $\lambda^\diamond = -\lambda$ for $\lambda \in \mathbb{Z}^N$. Then $\mathbb{W}$ acts on $T \times T$ by

$$(w,w')(t,\gamma) = (wt, w'^\diamond\gamma),$$
$$\iota(t,\gamma) = (\gamma^{-1}, t^{-1})$$

for $w, w' \in W$, where $t^{-1} := (t_1^{-1}, \ldots, t_N^{-1}) \in T$ and the action of W on T is by q-dilations and permutations; see (2.2.3). By transposition, this defines an action of $\mathbb{W}$ on $\mathbb{K} = \mathcal{M}(T \times T)$ by field automorphisms,

$$(\mathrm{w}f)(t,\gamma) = f(\mathrm{w}^{-1}(t,\gamma)), \qquad \mathrm{w} \in \mathbb{W}. \tag{2.3.1}$$

Note that $\mathbb{L} = \mathbb{C}[T \times T]$ is a $\mathbb{W}$-subalgebra of $\mathbb{K}$.

Lemma 2.3.2. *For* $h \in \mathbb{H}$ *and* $f \in \mathbb{L}$ *we have*

$$\begin{aligned} \widetilde{\sigma}_{(w,w')}(f \cdot h) &= ((w,w')f) \cdot \widetilde{\sigma}_{(w,w')}(h), \\ \widetilde{\sigma}_\iota(f \cdot h) &= (\iota f) \cdot \widetilde{\sigma}_\iota(h) \end{aligned} \tag{2.3.2}$$

for $w, w' \in W$.

Proof. From Proposition 2.2.9 we know that $\widetilde{S}_w p(X) = (wp)(X)\widetilde{S}_w$ in $\mathbb{H}$ for $p \in \mathbb{C}[T]$ and $w \in W$. For $p \in \mathbb{C}[T]$, let $p^\diamond \in \mathbb{C}[T]$ be defined by $p^\diamond(t) = p(t^{-1})$. Then we also have

$$p(Y)\widetilde{S}^*_{w'} = (\widetilde{S}_{w'}p^\diamond(X))^* = ((w'p^\diamond)(X)\widetilde{S}_{w'})^* = \widetilde{S}^*_{w'}(w'p^\diamond)^\diamond(Y)$$

in $\mathbb{H}$. Hence for $p, r \in \mathbb{C}[T]$,

$$\widetilde{\sigma}_{(w,w')}(p(X)hr(Y)) = (wp)(X)\widetilde{S}_w h \widetilde{S}^*_{w'}(w'r^\diamond)^\diamond(Y).$$

The first formula of (2.3.2) now follows since $(w'r^\diamond)^\diamond = w'^\diamond r$. The second is immediate from the definition of the duality anti-involution. □

As a direct consequence of Lemma 2.3.2 the maps $\widetilde{\sigma}_{(w,w')}$ $(w, w' \in W)$ and $\widetilde{\sigma}_\iota$ uniquely extend to complex linear endomorphisms of $H_0^{\mathbb{K}} \simeq \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H}$ such that (2.3.2) is valid for all $f \in \mathbb{K}$ and $h \in H_0^{\mathbb{K}}$. We keep the same notations $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$ for these maps. Note that the properties of $\widetilde{\sigma}_{(w,w')}$ and $\widetilde{\sigma}_\iota$ as described in Lemma 2.3.1 also hold true as identities between endomorphisms of $H_0^{\mathbb{K}}$.

Theorem 2.3.3. *There is a unique group homomorphism*

$$\tau\colon \mathbb{W} \to \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$$

satisfying

$$\begin{aligned} \tau(w, w')(f) &= d_w(x)^{-1} d^\diamond_{w'}(y)^{-1} \cdot \widetilde{\sigma}_{(w,w')}(f), \\ \tau(\iota)(f) &= \widetilde{\sigma}_\iota(f) \end{aligned} \tag{2.3.3}$$

for $w, w' \in W$ and $f \in H_0^{\mathbb{K}}$. It satisfies $\tau(\mathrm{w})(g \cdot f) = \mathrm{w}g \cdot \tau(\mathrm{w})(f)$ for $g \in \mathbb{K}$, $f \in H_0^{\mathbb{K}}$ and $\mathrm{w} \in \mathbb{W}$.

Proof. The last statement is clear.

The action τ of $W \times \{e\}$ arises naturally from left multiplication by normalized affine intertwiners on a suitable localization of the double affine Hecke algebra (see Cherednik [10, §1.3]). In the present set-up, one observes that Lemma 2.3.2 and Lemma 2.3.1(i)-(ii) imply that the $\tau(s_i, e)$ $(i \in \mathbb{Z}_N)$ satisfy the $\widehat{A}_{N-1}$ braid relations and the quadratic relations $\tau(s_i, e)^2 = \mathrm{id}_{H_0^{\mathbb{K}}}$. Since furthermore $\tau(\pi, e)$ is a complex linear automorphism of $H_0^{\mathbb{K}}$ with inverse $\tau(\pi^{-1}, e)$, and $\tau(\pi, e)\tau(s_i, e)\tau(\pi^{-1}, e) = \tau(s_{i+1}, e)$ for $i \in \mathbb{Z}_N$ by Lemma 2.3.2 and Lemma 2.3.1(iii), we conclude that the formulas (2.3.3) for the maps $\tau(s_i, e)$ $(i \in \mathbb{Z}_N)$ and $\tau(\pi, e)$ uniquely extend to a group homomorphism $\tau\colon W \times \{e\} \to \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$. It follows from Proposition 2.2.9 and its proof that the resulting group homomorphism satisfies

$$\tau(w, e)f = d_w(x)^{-1} \cdot \widetilde{\sigma}_{(w,e)} f$$

for $w \in W$. This is in accordance with formula (2.3.3).

Combining Lemma 2.3.1(iv) with Lemma 2.3.2 we can relate the complex endomorphism $\tau(e, w)$ (see (2.3.3)) of $H_0^{\mathbb{K}}$ to $\tau(w, e)$ by the formula

$$\tau(e, w) = \tau(\iota)\tau(w, e)\tau(\iota), \qquad w \in W, \tag{2.3.4}$$

where $\tau(\iota)$ is given by the second formula of (2.3.3). Since $\tau(\iota)^2 = \widetilde{\sigma}_\iota^2 = \mathrm{id}_{H_0^{\mathbb{K}}}$ we conclude that $W \ni w \mapsto \tau(e, w)$ (see (2.3.3)) defines a left W-action on $H_0^{\mathbb{K}}$.

By Lemma 2.3.1 and Lemma 2.3.2(v) we have

$$\tau(w, e)\tau(e, w') = \tau(w, w') = \tau(e, w')\tau(w, e)$$

for all $w, w' \in W$. Thus $\tau\colon W \times W \to \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$, defined by the first formula of (2.3.3), is a group homomorphism. Combined with (2.3.4) and $\tau(\iota)^2 = \mathrm{id}_{H_0^{\mathbb{K}}}$ we conclude that τ (2.3.3) indeed defines a complex linear action of $\mathbb{W}$ on $H_0^{\mathbb{K}}$. □

For $\mathrm{w} \in \mathbb{W}$ and $f \in H_0^{\mathbb{K}} = \mathbb{K} \otimes H_0$, we write $\mathrm{w}f$ for the action of w on the $\mathbb{K}$-coefficients of f in its expansion along a basis of H_0. In other words, viewing $f(t,\gamma)$ as H_0-valued meromorphic function in $(t,\gamma) \in T \times T$, the action is given by $(\mathrm{w}f)(t,\gamma) = f(\mathrm{w}^{-1}(t,\gamma))$. Consider $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ as a $\mathbb{W}$-group by the corresponding conjugation action

$$(\mathrm{w}, A) \mapsto \mathrm{w}A\mathrm{w}^{-1}, \qquad \mathrm{w} \in \mathbb{W}, \;\; A \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}}) \tag{2.3.5}$$

by group automorphisms. We have the following direct consequence of the previous theorem.

Corollary 2.3.4. *The map* $\mathrm{w} \mapsto C_{\mathrm{w}} := \tau(\mathrm{w})\mathrm{w}^{-1}$ *is a cocycle of* $\mathbb{W}$ *with values in the* $\mathbb{W}$*-group* $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$*. In other words,* $C_{\mathrm{w}} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ *and*

$$C_{\mathrm{w}\mathrm{w}'} = C_{\mathrm{w}}\mathrm{w}C_{\mathrm{w}'}\mathrm{w}^{-1}$$

for all $\mathrm{w}, \mathrm{w}' \in \mathbb{W}$.

For more details on non-abelian group cohomology, see, e.g., the appendix in [53].

Remark 2.3.5. Interpreting $A \in \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ as $\mathrm{End}(H_0)$-valued meromorphic function $A(t,\gamma)$ in $(t,\gamma) \in T \times T$, the action (2.3.5) becomes $(\mathrm{w}A)(t,\gamma) = A(\mathrm{w}^{-1}(t,\gamma))$. In particular, $\mathrm{w}A\mathrm{w}^{-1} = A$ for all $\mathrm{w} \in \mathbb{W}$ if $A \in \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ is the $\mathbb{K}$-linear extension of a complex linear endomorphism of H_0. This is, for instance, the case for the cocycle value C_ι (see Subsection 2.4.2).

Remark 2.3.6. One may replace in this subsection $\mathbb{K}$ by the field $\mathbb{C}(T \times T)$ of rational functions on $T \times T$. Consequently, the cocycle value $C_{\mathrm{w}}(t,\gamma)$ for $\mathrm{w} \in \mathbb{W}$ is a rational $\mathrm{End}(H_0)$-valued function in $(t,\gamma) \in T \times T$. We presented the results with respect to $\mathbb{K} = \mathcal{M}(T \times T)$ since this is the natural setting for the applications of the cocycle C in the analytic theory of the quantum KZ equations (to which we come at a later stage).

2.3.2 Bispectral quantum KZ equations

In this subsection, we use the cocycle $C_{\mathrm{w}} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ ($\mathrm{w} \in \mathbb{W}$) to define a holonomic system of q-difference equations on the space $H_0^{\mathbb{K}}$ of H_0-valued meromorphic functions on $T \times T$.

The constructions thus far have led to a $\mathbb{C}$-linear action τ of $\mathbb{W}$ on $H_0^{\mathbb{K}}$. In terms of the cocycle C_{w} ($\mathrm{w} \in \mathbb{W}$), it is given by

$$(\tau(\mathrm{w})f)(t,\gamma) = C_{\mathrm{w}}(t,\gamma)f(\mathrm{w}^{-1}(t,\gamma)) \tag{2.3.6}$$

for $\mathrm{w} \in \mathbb{W}$ and $f \in H_0^{\mathbb{K}}$, where (2.3.6) should be read as identities between H_0-valued meromorphic functions in $(t,\gamma) \in T \times T$. It follows that $f \in H_0^{\mathbb{K}}$ is $\tau(\mathbb{Z}^N \times \mathbb{Z}^N)$-invariant if and only if

$$C_{(\lambda,\mu)}(t,\gamma)f(q^{-\lambda}t, q^{\mu}\gamma) = f(t,\gamma) \qquad \forall\, \lambda, \mu \in \mathbb{Z}^N, \tag{2.3.7}$$

viewed as identities between H_0-valued meromorphic functions on $T \times T$.

Definition 2.3.7. We call the q-difference equations (2.3.7) the bispectral quantum KZ (BqKZ) equations. We write SOL for the set of functions $f \in H_0^{\mathbb{K}}$ satisfying the BqKZ equations (2.3.7).

Let $\mathbb{F} \subset \mathbb{K}$ denote the subfield consisting of $f \in \mathbb{K}$ satisfying $(\lambda, \mu)f = f$ for all $\lambda, \mu \in \mathbb{Z}^N$. Let furthermore $\mathbb{S}_N$ denote the subgroup $\mathbb{Z}_2 \ltimes (S_N \times S_N)$ of $\mathbb{W}$.

Corollary 2.3.8. (i) *The BqKZ equations* (2.3.7) *form a holonomic system of q-difference equations. In other words, the cocycle matrices $C_{(\lambda,\mu)}$ ($\lambda, \mu \in \mathbb{Z}^N$) satisfy the compatibility conditions*

$$C_{(\lambda,\mu)}(t,\gamma)C_{(\nu,\xi)}(q^{-\lambda}t, q^{\mu}\gamma) = C_{(\nu,\xi)}(t,\gamma)C_{(\lambda,\mu)}(q^{-\nu}t, q^{\xi}\gamma) \tag{2.3.8}$$

for $\lambda, \mu, \nu, \xi \in \mathbb{Z}^N$, as $\mathrm{End}(H_0)$*-valued meromorphic functions in $(t,\gamma) \in T \times T$.*
(ii) *The solution space* SOL *of BqKZ is a $\tau(\mathbb{S}_N)$-invariant $\mathbb{F}$-subspace of $H_0^{\mathbb{K}}$.*

Proof. (i) By means of the cocycle condition, both sides of (2.3.8) can be seen to be equal to $C_{(\lambda+\nu,\mu+\xi)}(t,\gamma)$.
(ii) Clearly, SOL is an $\mathbb{F}$-subspace of $H_0^{\mathbb{K}}$. Note, furthermore, that $\mathbb{Z}^N \times \mathbb{Z}^N$ is a normal subgroup of $\mathbb{W}$ with quotient group isomorphic to $\mathbb{S}_N$. Hence the $\mathbb{F}$-subspace SOL of $\tau(\mathbb{Z}^N \times \mathbb{Z}^N)$-invariant elements in the $\tau(\mathbb{W})$-module $H_0^{\mathbb{K}}$ is $\tau(\mathbb{S}_N)$-invariant. □

2.4 The explicit form of the bispectral quantum KZ equations

In this section we derive explicit expressions for the cocycle values C_{w} ($\mathrm{w} \in \mathbb{W}$) and, in particular, for the cocycle matrices $C_{(\lambda,\mu)}$ ($\lambda, \mu \in \mathbb{Z}^N$) of the BqKZ equations. It will become apparent that the $C_{(\lambda,e)}(\cdot,\zeta)$ ($\lambda \in \mathbb{Z}^N$) with $\zeta \in T$ fixed coincide with the cocycle matrices of Cherednik's quantum affine KZ equation associated to the principal series module of $H(k)$ with central character ζ. They also turn up as gauged cocycle matrices for a Frenkel-Reshetikhin [21] type quantum KZ equation associated to the quantum affine algebra $\mathcal{U}_k(\widehat{sl}_N)$.

2.4.1 Generic principal series

View the commutative subalgebra $\mathbb{C}_Y[T]$ of H as left $\mathbb{C}_Y[T]$-module by left multiplication. Let $M = \mathrm{Ind}_{\mathbb{C}_Y[T]}^{H}\big(\mathbb{C}_Y[T]\big)$ be the corresponding induced left H-module. With respect to the $\mathbb{C}[T] \simeq \mathbb{C}[\{1\} \times T]$-module structure

$$f \cdot (h \otimes_{\mathbb{C}_Y[T]} g(Y)) = h \otimes_{\mathbb{C}_Y[T]} (fg)(Y) \qquad f, g \in \mathbb{C}[T],\ h \in H$$

on M we have $M \simeq H_0^{\mathbb{C}[\{1\}\times T]} = \mathbb{C}[\{1\} \times T] \otimes H_0$ as $\mathbb{C}[\{1\} \times T]$-modules. The left H-action on M is $\mathbb{C}[\{1\} \times T]$-linear, hence we obtain an algebra homomorphism

$$\eta \colon H \to \mathrm{End}_{\mathbb{C}[\{1\}\times T]}\big(H_0^{\mathbb{C}[\{1\}\times T]}\big).$$

We occasionally view $\eta(h)$ as $\mathrm{End}(H_0)$-valued regular function in $\gamma \in T$, in which case we write it as $T \ni \gamma \mapsto \eta(h)(\gamma)$. Extending the ground ring $\mathbb{C}[\{1\} \times T]$ to $\mathbb{K} = \mathcal{M}(T \times T)$ we obtain an algebra homomorphism

$$H \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}}),$$

which we shall also denote by η. From this viewpoint, $\eta(h)(\gamma)$ is the regular $\mathrm{End}(H_0)$-valued function in $(t, \gamma) \in T \times T$ which is constant in t. Note that $\eta(h)$ for $h \in H_0$ is constant as $\mathrm{End}(H_0)$-valued function on $T \times T$ (see Remark 2.3.5).

Lemma 2.4.1. *For $w \in S_N$ and $1 \leq i < N$ we have*

$$\eta(T_i)T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) = \ell(w) + 1, \\ (k - k^{-1})T_w + T_{s_i w} & \text{if } \ell(s_i w) = \ell(w) - 1, \end{cases} \tag{2.4.1}$$

and

$$\eta(\pi)(\gamma)T_w = \gamma_{w^{-1}(N)} T_{\sigma w} \tag{2.4.2}$$

as regular H_0-valued functions in $\gamma \in T$.

Proof. The first formula follows directly from the definitions. For the second formula it suffices to verify that $\pi T_w = T_{\sigma w} Y_{w^{-1}(N)}$ in H.

If $w = e$ then $\pi = T_\sigma Y_N$ since $\sigma = s_1 s_2 \cdots s_{N-1}$ is a reduced expression. If $w = s_{N-1}$ then

$$\pi T_w = \pi T_{N-1} = T_1 \cdots T_{N-2} Y_{N-1} = T_{s_1 \cdots s_{N-2}} Y_{w^{-1}(N)} = T_{\sigma w} Y_{w^{-1}(N)}.$$

Next, we prove that $\pi T_w = T_{\sigma w} Y_{w^{-1}(N)}$ in H if $w \neq e$ and $\ell(s_i w) = \ell(w) + 1$ for all $1 \leq i \leq N-2$. Then $w = s_{N-1} s_{N-2} \cdots s_j$ for some $1 \leq j < N$ (and this is a reduced expression of w). We find, making repetitive use of the cross relation $T_r Y_{r+1} T_r = Y_r$ $(1 \leq r < N)$ in H,

$$\begin{aligned} \pi T_w &= \pi T_{N-1} T_{N-2} \cdots T_j = T_1 \cdots T_{N-2} Y_{N-1} T_{N-2} \cdots T_j \\ &= T_1 \cdots T_{N-3} Y_{N-2} T_{N-3} \cdots T_j \\ &\vdots \\ &= T_1 \cdots T_{j-1} Y_j = T_{s_1 \cdots s_{j-1}} Y_j \\ &= T_{\sigma w} Y_j = T_{\sigma w} Y_{w^{-1}(N)}, \end{aligned}$$

which is the desired relation in H.

The general case is now proved by induction on $\ell(w)$. Let $w \neq e$ and decompose it as $w = s_i u$ with $1 \leq i < N$ and $u \in S_N$ such that $\ell(s_i u) = \ell(u) + 1$. Suppose that $\pi T_u = T_{\sigma u} Y_{u^{-1}(N)}$ in H. In order to prove that $\pi T_w = T_{\sigma w} Y_{w^{-1}(N)}$ we may, in view of the previous paragraph, assume without loss of generality that $1 \leq i \leq N-2$. Then $s_i(N) = N$ and $\ell(s_{i+1} \sigma u) = \ell(\sigma u) + 1$. (The latter equality is equivalent to

$(\sigma u)^{-1}(i+1) < (\sigma u)^{-1}(i+2)$, which follows from $u^{-1}(i) < u^{-1}(i+1)$, which again is equivalent to the assumption $\ell(s_i u) = \ell(u) + 1$.) Then

$$\begin{aligned}\pi T_w &= \pi T_i T_u = T_{i+1}\pi T_u = T_{i+1} T_{\sigma u} Y_{u^{-1}(N)} \\ &= T_{s_{i+1}\sigma u} Y_{u^{-1}s_i(N)} = T_{\sigma s_i u} Y_{w^{-1}(N)} \\ &= T_{\sigma w} Y_{w^{-1}(N)}\end{aligned}$$

in H, which completes the proof. □

In view of the explicit expression (2.2.8) for the intertwiner $\widetilde{S}_i \in \mathbb{H}$ $(1 \leq i < N)$ and the definition of the duality anti-involution, we have $\widetilde{S}_w^* \in H$ for all $w \in S_N$. We now set

$$\xi_w := \eta(\widetilde{S}^*_{w^{-1}})T_e \in H_0^{\mathbb{K}}, \qquad w \in S_N.$$

Note that $\xi_w \in H_0^{\mathbb{C}[\{1\}\times T]} \subset H_0^{\mathbb{K}}$ for $w \in S_N$. We view ξ_w as regular H_0-valued function in $\gamma \in T$, as well as meromorphic H_0-valued function in $(t,\gamma) \in T \times T$ constant in $t \in T$.

Lemma 2.4.2. *$\{\xi_w\}_{w\in S_N}$ is a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ consisting of common eigenfunctions for the η-action of $\mathbb{C}_Y[T]$ on $H_0^{\mathbb{K}}$. For $p \in \mathbb{C}[T]$ and $w \in S_N$ we have*

$$\eta(p(Y))(\gamma)\xi_w(\gamma) = (w^{-1}p)(\gamma)\xi_w(\gamma) \tag{2.4.3}$$

as H_0-valued regular functions in $\gamma \in T$.

Proof. For $p \in \mathbb{C}[T]$ we have $\eta(p(Y))(\gamma)T_e = p(\gamma)T_e$. Furthermore, observe that $p(Y)\widetilde{S}^*_{w^{-1}} = \widetilde{S}^*_{w^{-1}}(w^{-1}p)(Y)$ in H for $p \in \mathbb{C}[T]$ and $w \in S_N$; see the proof of Lemma 2.3.2. Combining the two observations gives (2.4.3). It follows from Proposition 2.2.9(iv)-(v) that the ξ_w $(w \in S_N)$ are nonzero in $H_0^{\mathbb{K}}$. The eigenvalue equations (2.4.3) then show that the ξ_w $(w \in S_N)$ are $\mathbb{K}$-linearly independent in $H_0^{\mathbb{K}}$. □

2.4.2 The cocycle values

We define

$$R_i(z) = c_k(z)^{-1}(\eta(T_i) - k) + \mathrm{id}, \qquad 1 \leq i < N,$$

viewed as a rational $\mathrm{End}(H_0)$-valued function in z. The results of the previous subsection imply that the $R_i(z)$ satisfy the following Yang-Baxter type equations (see Cherednik [10, §1.3.2]).

Lemma 2.4.3. *We have*

$$C_{(s_i,e)}(t,\gamma) = R_i(t_i/t_{i+1}), \qquad 1 \leq i < N,$$

as rational $\mathrm{End}(H_0)$-valued functions in $(t,\gamma) \in T \times T$. In particular, the $R_i(z)$ satisfy

$$\begin{aligned} R_i(z)R_i(z^{-1}) &= \mathrm{id}, \\ R_j(z)R_{j+1}(zz')R_j(z') &= R_{j+1}(z')R_j(zz')R_{j+1}(z), \end{aligned} \tag{2.4.4}$$

for $1 \leq i < N$ and $1 \leq j < N-1$ as $\mathrm{End}(H_0)$-valued rational functions.

Proof. For $1 \leq i < N$ and $h \in H_0$ we have, as H_0-valued meromorphic functions in $(t, \gamma) \in T \times T$,

$$\begin{aligned} C_{(s_i,e)}(t,\gamma)h &= (\tau(s_i,e)h)(t,\gamma) \\ &= d_{s_i}(t)^{-1}(\widetilde{S}_i h)(t,\gamma) \\ &= c_k(t_i/t_{i+1})^{-1}(\eta(T_i)-k)h + h, \end{aligned}$$

in view of the explicit expression (2.2.8) for $\widetilde{S}_i$, c_k (2.2.5) and d_{s_i} (see the proof of Lemma 2.2.9). For the second statement of the lemma, note that the cocycle property of C implies for $1 \leq i < N$ and $1 \leq j < N-1$ that

$$\begin{aligned} C_{(s_i,e)}(t,\gamma)C_{(s_i,e)}(s_i t,\gamma) &= \mathrm{id}, \\ C_{(s_j,e)}(t,\gamma)C_{(s_{j+1},e)}(s_j t,\gamma)C_{(s_j,e)}(s_{j+1}s_j t,\gamma) & \\ = C_{(s_{j+1},e)}(t,\gamma)&C_{(s_j,e)}(s_{j+1}t,\gamma)C_{(s_{j+1},e)}(s_j s_{j+1}t,\gamma), \end{aligned}$$

as rational $\mathrm{End}(H_0)$-valued functions in $(t,\gamma) \in T \times T$. Then using that $C_{(s_i,e)}(t,\gamma) = R_i(t_i/t_{i+1})$, these formulas imply (2.4.4). □

Observe that C_ι is the $\mathbb{K}$-linear extension of the anti-algebra involution of H_0 mapping T_w to $T_{w^{-1}}$ for all $w \in S_N$. Note furthermore that

$$C_{(\pi,e)} = \eta(\pi). \tag{2.4.5}$$

Together with the explicit description of $C_{(s_i,e)}$ ($1 \leq i < N$) from the previous lemma, these formulas determine the values C_{w} ($\mathrm{w} \in \mathbb{W}$) uniquely (cf. Corollary 2.3.4). In particular, the cocycle property of C implies that

$$C_{(e,w)}(t,\gamma) = C_\iota C_{(w,e)}(\gamma^{-1},t^{-1})C_\iota, \qquad w \in W,$$

as $\mathrm{End}(H_0)$-valued rational functions in $(t,\gamma) \in T \times T$.

Lemma 2.4.4. *Let $w \in W$.*
(i) $C_{(w,e)} \in (\mathbb{C}(T) \otimes \mathbb{C}[T]) \otimes \mathrm{End}(H_0)$.
(ii) *The $\mathbb{C}[T] \otimes \mathrm{End}(H_0)$-valued rational function $t \mapsto C_{(w,e)}(t,\cdot)$ in $t \in T$ is regular at $t \in T \setminus \mathcal{S}$, where*

$$\mathcal{S} = \{t \in T \mid t^\alpha \in k^{-2}q^{\mathbb{Z}} \quad \text{for some } \alpha \in R\}. \tag{2.4.6}$$

Proof. By the cocycle condition, $C_{(w,e)}(t,\gamma)$ can be written as a product of factors $C_{(s_i,e)}(ut,\gamma)$ ($1 \leq i < N$, $u \in W$) and $\eta(\pi^{\pm 1})(\gamma)$. By Lemma 2.4.1 and the fact that $R_i(z)$ has a single pole at $z = k^{-2}$, we conclude (i) and (ii). □

2.4.3 The cocycle matrices

Besides the standard $\mathbb{Z}$-basis $\{\epsilon_i\}_{i=1}^N$ of $\mathbb{Z}^N$, we also have the $\mathbb{Z}$-basis $\{\varpi_i\}_{i=1}^N$ consisting of the fundamental weights $\varpi_i := \sum_{j=1}^i \epsilon_j \in \mathbb{Z}^N$. Note that

$$\varpi_i = \pi^i \sigma^{-i} \tag{2.4.7}$$

in W for all $1 \leq i \leq N$. In the following lemma, we compute the cocycle matrices $C_{(\lambda,e)}$ for $\lambda \in \mathbb{Z}^N$ of BqKZ explicitly in case λ is one of these two types of basis elements of $\mathbb{Z}^N$.

Lemma 2.4.5. **(i)** *For* $1 \leq j \leq N$ *we have*

$$\begin{aligned} C_{(\epsilon_j,e)}(t,\gamma) = R_{j-1}(t_{j-1}/t_j)R_{j-2}(t_{j-2}/t_j)\cdots R_1(t_1/t_j) \\ \times \eta(\pi)(\gamma)R_{N-1}(qt_N/t_j)\cdots R_{j+1}(qt_{j+2}/t_j)R_j(qt_{j+1}/t_j) \end{aligned}$$

as rational $\mathrm{End}(H_0)$*-valued functions in* $(t,\gamma) \in T \times T$.
(ii) *For* $1 \leq i < N$ *we have*

$$\begin{aligned} C_{(\varpi_i,e)}(t,\gamma) = \big(\eta(\pi)(\gamma)\big)^i (R_{N-i}(qt_N/t_1)\cdots R_2(qt_{i+2}/t_1)R_1(qt_{i+1}/t_1)) \\ \times\cdots\times (R_{N-2}(qt_N/t_{i-1})\cdots R_i(qt_{i+2}/t_{i-1})R_{i-1}(qt_{i+1}/t_{i-1})) \\ \times (R_{N-1}(qt_N/t_i)\cdots R_{i+1}(qt_{i+2}/t_i)R_i(qt_{i+1}/t_i)) \end{aligned}$$

as rational $\mathrm{End}(H_0)$*-valued functions in* $(t,\gamma) \in T \times T$.
(iii) *We have*

$$C_{(\varpi_N,e)}(t,\gamma) = \gamma^{\varpi_N}\mathrm{id}$$

as rational $\mathrm{End}(H_0)$*-valued functions in* $(t,\gamma) \in T \times T$.

Proof. (i) By the cocycle property of C and by the expression (2.2.2) for $\epsilon_j \in W$, we obtain an explicit expression for $C_{(\epsilon_j,e)}$ in terms of the $C_{(s_i,e)}$ $(1 \leq i < N)$ and $C_{(\pi,e)}$. Combining (2.4.5) and the previous lemma then gives the desired expression for $C_{(\epsilon_j,e)}(t,\gamma)$.
(ii) σ^i is the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & N-i & N-i+1 & N-i+2 & \cdots & N \\ i+1 & i+2 & \cdots & N & 1 & 2 & \cdots & i \end{pmatrix},$$

so we find a reduced expression

$$\sigma^i = (s_i\cdots s_{N-1})(s_{i-1}\cdots s_{N-2})\cdots(s_2\cdots s_{N-i+1})(s_1\cdots s_{N-i}). \tag{2.4.8}$$

Combined with (2.4.7) we get a reduced expression for ϖ_i. Using the cocycle condition for C_{w} repeatedly, we obtain the desired result.
(iii) Since $\sigma^N = 1$, we get $C_{(\varpi_N,e)}(t,\gamma) = \big(\eta(\pi)(\gamma)\big)^N$, which maps T_w to $\gamma^{\varpi_N}T_w$ for all $w \in S_N$ in view of Lemma 2.4.1. □

We end this subsection by computing the asymptotic leading terms of the cocycle matrices $C_{(\lambda,e)}(t,\gamma)$ $(\lambda \in \mathbb{Z}^N)$ as $|t^{-\alpha_i}| \to 0$ $(1 \leq i < N)$, where we take $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(1 \leq i < N)$ as a base of the root system $R = \{\epsilon_i - \epsilon_j\}_{1\leq i\neq j\leq N}$ of type A_{N-1}. Let $R_+ = \{\epsilon_i - \epsilon_j\}_{1\leq i<j\leq N}$ denote the associated set of positive roots and $Q_+ = \bigoplus_{i=1}^{N-1}\mathbb{Z}_{\geq 0}\alpha_i$ the corresponding cone in the root lattice Q of R. Let furthermore $\delta := (N-1, N-3, \ldots, 1-N) \in \mathbb{Z}^N$ and write $w_0 \in S_N$ for the longest Weyl group element (mapping i to $N-i+1$ for $1 \leq i \leq N$).

Consider the subring $\mathcal{A} := \mathbb{C}[x^{-\alpha_1}, \ldots, x^{-\alpha_{N-1}}]$ of $\mathbb{C}[T\times\{1\}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}] \subset \mathbb{C}[T \times T]$. We write $Q(\mathcal{A})$ for its quotient field and $Q_0(\mathcal{A})$ for the subring of $Q(\mathcal{A})$ consisting of rational functions which are analytic at the point $x^{-\alpha_i} = 0$ $(1 \leq i < N)$. We consider $Q_0(\mathcal{A}) \otimes \mathbb{C}[T]$ as a subring of $\mathbb{C}(T \times T)$ in the natural way. The first part of the following corollary is a refinement of Lemma 2.4.4(i) in case $w \in \mathbb{Z}^N$.

Corollary 2.4.6. *Let $\lambda \in \mathbb{Z}^N$. We have*

$$C_{(\lambda,e)} \in (Q_0(\mathcal{A}) \otimes \mathbb{C}[T]) \otimes \mathrm{End}(H_0). \tag{2.4.9}$$

Writing

$$C^{(0)}_{(\lambda,e)} = C_{(\lambda,e)}|_{x^{-\alpha_1}=0,\ldots,x^{-\alpha_{N-1}}=0} \in \mathbb{C}[T] \otimes \mathrm{End}(H_0),$$

we have $C^{(0)}_{(\lambda,e)} = k^{\langle \delta,\lambda\rangle}\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1})$, where $\langle \cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^N$.

Proof. To prove (2.4.9) it suffices, in view of the cocyle property of C, to verify (2.4.9) for $\lambda = \epsilon_i$. The statement then follows from Lemma 2.4.5(i), Lemma 2.4.1 and the explicit expression of $R_i(z)$.

Observe that $\lim_{z\to 0} R_i(z) = k\eta(T_i^{-1})$ for $1 \leq i < N$. Combined with Lemma 2.4.5(i) and the explicit expression for Y_j (see (2.2.4)) we obtain for $1 \leq j \leq N$,

$$C_{(\epsilon_j,e)}(t,\gamma) \to k^{2j-N-1}\eta(Y_j)(\gamma)$$

as $|t^{\alpha_i}| \to 0$ for all $1 \leq i < N$, hence

$$C_{(\lambda,e)}(t,\gamma) \to k^{-\langle\delta,\lambda\rangle}\eta(Y^{\lambda})(\gamma)$$

as $|t^{\alpha_i}| \to 0$ for all $1 \leq i < N$. To derive the asymptotics of $C_{(\lambda,e)}(t,\gamma)$ as $|t^{-\alpha_i}| \to 0$ for $1 \leq i < N$ we use the cocycle property to write

$$C_{(\lambda,e)}(t,\gamma) = C_{(w_0,e)}(t,\gamma)C_{(w_0(\lambda),e)}(w_0t,\gamma)C_{(w_0,e)}(q^{-w_0(\lambda)}w_0t,\gamma).$$

Note that $C_{(w_0,e)}(t,\gamma) \to k^{\ell(w_0)}\eta(T_{w_0}^{-1})$ if $|t^{\alpha_i}| \to 0$ for all $1 \leq i < N$. Hence

$$C^{(0)}_{(\lambda,e)} = k^{\langle\delta,\lambda\rangle}\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1}),$$

as desired. □

2.4.4 Relation to quantum KZ equations

Fix $\zeta \in T$ and let $\chi_\zeta : \mathbb{C}[\{1\} \times T] \to \mathbb{C}$ denote the corresponding evaluation character $\chi_\zeta(f) = f(\zeta)$. Recall the generic principal series $\eta : H \to \mathrm{End}_{\mathbb{C}[\{1\}\times T]}(M)$. The corresponding complex H-representation $M_\zeta = \mathbb{C} \otimes_{\chi_\zeta} M$ of dimension $N!$ is the principal series module of H with central character ζ. We identify M_ζ with H_0 as a complex vector space, and push the H-action on M_ζ through the linear isomorphism to H_0. We denote the corresponding representation map by

$$\eta_\zeta : H \to \mathrm{End}(H_0).$$

As in Subsection 2.4.1 we have as identities in H_0,

$$\eta_\zeta(T_i)T_w = \begin{cases} T_{s_iw} & \text{if } \ell(s_iw) = \ell(w)+1, \\ (k-k^{-1})T_w + T_{s_iw} & \text{if } \ell(s_iw) = \ell(w)-1, \end{cases}$$

for $1 \leq i < N$ and $w \in S_N$,

$$\eta_\zeta(\pi)T_w = \zeta_{w^{-1}(N)}T_{\sigma w},$$

for $w \in S_N$, as well as

$$\eta_\zeta(f(Y))\xi_w(\zeta) = (w^{-1}f)(\zeta)\xi_w(\zeta),$$

for $w \in S_N$, where $\xi_w(\zeta) \in H_0$ is the regular H_0-valued function $\xi_w(\gamma)$ in $\gamma \in T$ specialized at $\gamma = \zeta$. Extending the base field to $\mathcal{M}(T)$ we get an algebra homomorphism $H \to \mathrm{End}_{\mathcal{M}(T)}(H_0^{\mathcal{M}(T)})$, which is also denoted by η_ζ.

In this subsection we consider the BqKZ for specialized values of γ. In view of Lemma 2.4.4(i) we may specialize $C_{(w,e)}(t,\gamma)$ ($w \in W$) at $\gamma = \zeta$. We write

$$C_w^\zeta(t) := C_{(w,e)}(t,\zeta), \qquad w \in W,$$

for the resulting specialized cocycle values, viewed as $\mathrm{End}(H_0)$-valued rational functions in $t \in T$. For $\zeta \in T$ the map $W \ni w \mapsto C_w^\zeta$ defines a cocycle of W with values in the W-group $\mathrm{GL}_{\mathbb{C}(T)}(H_0^{\mathbb{C}(T)})$. In other words,

$$C_{ww'}^\zeta(t) = C_w^\zeta(t)C_{w'}^\zeta(w^{-1}t), \qquad w, w' \in W,$$

as rational H_0-valued functions in $t \in T$. Comparing the cocycle values C_λ^ζ ($\lambda \in \mathbb{Z}^N$) to the ones in [10, §1.3] we obtain the following result.

Corollary 2.4.7. *Fix $\zeta \in T$. The holonomic system of q-difference equations*

$$C_\lambda^\zeta(t)f(q^{-\lambda}t) = f(t), \qquad \forall \lambda \in \mathbb{Z}^N \tag{2.4.10}$$

for $f \in H_0^{\mathcal{M}(T)}$ is Cherednik's quantum affine KZ equation associated to the principal H-module M_ζ with central character ζ.

We occasionally write qKZ_ζ for the quantum KZ equations (2.4.10). Let $\mathrm{SOL}_\zeta \subset H_0^{\mathcal{M}(T)}$ denote the set of solutions of (2.4.10). Write $\mathcal{E}(T) \subset \mathcal{M}(T)$ for the subfield of meromorphic functions f satisfying $f(q^\lambda t) = f(t)$ for all $\lambda \in \mathbb{Z}^N$ as meromorphic functions in $t \in T$. The set SOL_ζ of solutions is a $\mathcal{E}(T)$-subspace of $H_0^{\mathcal{M}(T)}$. Furthermore, SOL_ζ is invariant for the S_N-action

$$(\varsigma(w)f)(t) := C_w^\zeta(t)f(w^{-1}t), \qquad w \in S_N \tag{2.4.11}$$

on $H_0^{\mathcal{M}(T)}$ (note that ς does not depend on ζ since $C_w^\zeta(t) = C_{(w,e)}(t,\zeta)$ is independent of ζ for $w \in S_N$).

Remark 2.4.8. The quantum KZ equations (2.4.10) are gauge equivalent to Frenkel and Reshetikhin's [21] quantum KZ equations associated with the N-fold tensor product representation $\mathbb{C}^N(t_1)\otimes\cdots\otimes\mathbb{C}^N(t_N)$ of the quantum affine algebra $\mathcal{U}_k(\widehat{sl}_N)$, where $\mathbb{C}^N(t_i)$ is the evaluation representation of the vector representation $\mathbb{C}^N$ of $\mathcal{U}_k(sl_N)$ (see [10, §1.3.2] and [15] for the details).

In view of Corollary 2.4.7 the BqKZ equations (2.3.7) are a holonomic extension of the quantum KZ equations (2.4.10) by q-difference equations in the central character ζ of M_ζ. These may be thought of as analogs of isomonodromy transformations; in fact, in view of Lemma 2.4.2 and Corollary 2.4.6 the q-difference equations in ζ (which are essentially the quantum KZ equations again!) are reminiscent of Schlesinger transformations. This should be compared with the quantum isomonodromic interpretation of (rational) KZ equations as quantizations of Schlesinger equations, see [49] and [23].

From a different perspective we may think of the cocycle values $C_{(e,w)}$ ($w \in W$) as shift operators, in the sense that they map solutions of quantum KZ equations to solutions of quantum KZ equations with respect to shifted central characters. To formulate the precise result, we view in the following proposition $\gamma \mapsto C_{(e,w)}(\cdot,\gamma)$ as $\mathbb{C}[T]\otimes\mathrm{End}(H_0)$-valued rational function in $\gamma \in T$.

Proposition 2.4.9. *Let $w \in W$ and $\zeta \in T$ such that $\gamma \mapsto C_{(e,w)}(\cdot,\gamma)$ is regular at $\gamma = \zeta$. Then $f \mapsto C_{(e,w)}(\cdot,\zeta)f$ defines an S_N-equivariant linear map $\mathrm{SOL}_{w^{\diamond-1}\zeta} \to \mathrm{SOL}_\zeta$.*

Proof. By the cocycle property we have for $f \in \mathrm{SOL}_{w^{\diamond-1}\zeta}$ and $\lambda \in \mathbb{Z}^N$,

$$\begin{aligned} C_\lambda^\zeta(t)\big(C_{(e,w)}(q^{-\lambda}t,\zeta)f(q^{-\lambda}t)\big) &= C_{(\lambda,w)}(t,\zeta)f(q^{-\lambda}t) \\ &= C_{(e,w)}(t,\zeta)C_\lambda^{w^{\diamond-1}\zeta}(t)f(q^{-\lambda}t) \\ &= C_{(e,w)}(t,\zeta)f(t). \end{aligned}$$

Hence $C_{(e,w)}(\cdot,\zeta)f \in \mathrm{SOL}_\zeta$. The S_N-equivariance of the map is again a consequence of the cocycle property of C_{w} ($\mathrm{w} \in \mathbb{W}$); indeed, for $v \in S_N$ and $f \in \mathrm{SOL}_{w^{\diamond-1}\zeta}$ we have

$$\begin{aligned} C_v^\zeta(t)\big(C_{(e,w)}(v^{-1}t,\zeta)f(v^{-1}t)\big) &= C_{(v,w)}(t,\zeta)f(v^{-1}t) \\ &= C_{(e,w)}(t,\zeta)\big(C_v^{w^{\diamond-1}\zeta}(t)f(v^{-1}t)\big), \end{aligned}$$

which is the desired result. □

From the quantum group perspective (see Remark 2.4.8), Proposition 2.4.9 resembles the action of the dynamical Weyl group on solutions of quantum KZ equations from [19]. We expect that the second half of the BqKZ is closely related to the Varchenko-Etingof dynamical difference equations [19, §9]; see also [20], [60], [57], [59], [58] and [35] for detailed studies of various degenerate cases. An interesting aspect, e.g., in [60] and [59], is the observation that KZ equations are dual to the associated dynamical equations using $(\mathfrak{gl}_r, \mathfrak{gl}_s)$ duality (our set-up relates to $r = s = N$). In the present theory this duality is incorporated by the cocycle value C_ι, which relates the cocycle matrices $C_{(\lambda,e)}$ ($\lambda \in \mathbb{Z}^N$) of the quantum KZ equation to the dual cocycle matrices $C_{(e,\lambda)}$ by conjugation,

$$C_{(e,\lambda)}(t,\gamma) = C_\iota C_{(\lambda,e)}(\gamma^{-1}, t^{-1})C_\iota$$

as End(H_0)-valued meromorphic functions in $(t,\gamma) \in T \times T$. In turn, C_ι is a direct reflection of (the existence of) Cherednik's duality anti-isomorphism of the double affine Hecke algebra (see Theorem 2.2.8).

2.5 Solutions of the bispectral quantum KZ equations

In this section we use asymptotic analysis to construct a ι-invariant solution Φ_κ of BqKZ, which we call the basic asymptotically free solution. It depends in a mild way on an auxiliary parameter $\kappa \in \mathbb{C}^\times$ (in fact, $\mathbb{F}\Phi_\kappa$ is independent of κ). The orbit of Φ_κ under the action of $\{e\} \times S_N \subset \mathbb{S}_N$ turns out to be an $\mathbb{F}$-basis of SOL consisting of asymptotically free solutions. Along the way we derive various additional properties of Φ_κ.

2.5.1 The leading term

Let $\theta \in \mathcal{M}(T)$ denote the renormalized Jacobi theta function

$$\theta(z) := \prod_{m \geq 0} (1 - q^m z)(1 - q^{m+1}/z) \tag{2.5.1}$$

for $z \in \mathbb{C}^\times$. It satisfies

$$\theta(q^m z) = (-z)^{-m} q^{-\frac{1}{2}m(m-1)}\theta(z), \qquad m \in \mathbb{Z}. \tag{2.5.2}$$

For $\kappa \in \mathbb{C}^\times$ we define $W_\kappa \in \mathbb{K}$ by

$$W_\kappa(t,\gamma) := \prod_{i=1}^{N} \frac{\theta(\kappa t_i \gamma_{N-i+1}^{-1})}{\theta(\kappa k^{\langle \delta, \epsilon_i \rangle} t_i)\theta(\kappa k^{-\langle \delta, \epsilon_i \rangle} \gamma_{N-i+1}^{-1})}. \tag{2.5.3}$$

By Corollary 2.4.6, the formal asymptotic form of the quantum KZ equations

$$C_{(\lambda,e)}(t,\gamma) f(q^{-\lambda} t, \gamma) = f(t,\gamma), \qquad \lambda \in \mathbb{Z}^N$$

in the asymptotic region $|t^{\alpha_i}| \gg 0$ $(1 \leq i < N)$ is

$$k^{\langle \delta,\lambda\rangle}\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1})(\gamma)f(q^{-\lambda}t,\gamma)=f(t,\gamma), \qquad \lambda\in\mathbb{Z}^N. \tag{2.5.4}$$

Lemma 2.5.1. *$W_\kappa \in \mathbb{K}$ enjoys the following properties.*

(i) $f_\kappa^{(0)}(t,\gamma) := W_\kappa(t,\gamma)T_{w_0}$ *is a solution of* (2.5.4).

(ii) $\iota(W_\kappa) = W_\kappa$ *and* $\tau(\iota)f_\kappa^{(0)} = f_\kappa^{(0)}$.

Proof. (i) Since $\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1})(\gamma)T_{w_0} = \gamma^{w_0(\lambda)}T_{w_0}$ for all $\lambda \in \mathbb{Z}^N$, it suffices to show that

$$W_\kappa(q^{-\lambda}t,\gamma)=k^{-\langle\delta,\lambda\rangle}\gamma^{-w_0(\lambda)}W_\kappa(t,\gamma), \qquad \lambda\in\mathbb{Z}^N,$$

which follows from (2.5.2).

(ii) Clearly $\iota(W_\kappa) = W_\kappa$, i.e. $W_\kappa(\gamma^{-1},t^{-1}) = W_\kappa(t,\gamma)$. Since $C_\iota(T_{w_0}) = T_{w_0}$, it follows that $\tau(\iota)f_\kappa^{(0)} = f_\kappa^{(0)}$. □

Observe that, more generally, $W_\kappa \in \mathbb{K}$ satisfies the q-difference equations

$$W_\kappa(q^{-\lambda}t,q^{\mu}\gamma)=k^{-\langle\delta,\lambda+\mu\rangle}t^{w_0(\mu)}\gamma^{-w_0(\lambda)}q^{-\langle w_0(\lambda),\mu\rangle}W_\kappa(t,\gamma), \qquad \lambda,\mu\in\mathbb{Z}^N. \tag{2.5.5}$$

2.5.2 The basic asymptotically free solution Φ_κ

We now gauge BqKZ by $W_\kappa \in \mathbb{K}$. Concretely, for $\lambda,\mu \in \mathbb{Z}^N$ we write

$$D_{(\lambda,\mu)}(t,\gamma)=W_\kappa(t,\gamma)^{-1}C_{(\lambda,\mu)}(t,\gamma)W_\kappa(q^{-\lambda}t,q^{\mu}\gamma)$$

as $\mathrm{End}(H_0)$-valued meromorphic functions in $(t,\gamma) \in T \times T$. It is independent of κ in view of (2.5.5). For $f \in H_0^{\mathbb{K}}$ we have $f \in \mathrm{SOL}$ if and only if $g := W_\kappa^{-1}f \in H_0^{\mathbb{K}}$ satisfies the holonomic system of q-difference equations

$$D_{(\lambda,\mu)}(t,\gamma)g(q^{-\lambda}t,q^{\mu}\gamma)=g(t,\gamma), \qquad \lambda,\mu\in\mathbb{Z}^N \tag{2.5.6}$$

as H_0-valued rational functions in $(t,\gamma) \in T \times T$.

The existence of a solution $\Psi \in H_0^{\mathbb{K}}$ of (2.5.6) admitting a convergent H_0-valued power series expansion

$$\Psi(t,\gamma)=\sum_{\alpha,\beta\in Q_+}K_{\alpha,\beta}t^{-\alpha}\gamma^{\beta}, \qquad K_{0,0}=T_{w_0} \tag{2.5.7}$$

in the asymptotic region $|t^{\alpha_i}| \gg 0$ and $|\gamma^{-\alpha_i}| \gg 0$ $(1 \leq i < N)$ is guaranteed by the following properties of the gauged cocycle matrices $D_{(\lambda,\mu)}$.

Consider the subring $\mathcal{B} := \mathbb{C}[y^{\alpha_1},\ldots,y^{\alpha_{N-1}}]$ of $\mathbb{C}[\{1\}\times T] = \mathbb{C}[y_1^{\pm1},\ldots,y_N^{\pm1}]$. Write $Q(\mathcal{B})$ for its quotient field and $Q_0(\mathcal{B})$ for the subring of $Q(\mathcal{B})$ consisting of rational functions which are analytic at the point $y^{\alpha_j} = 0$ $(1 \leq j < N)$. We consider $Q_0(\mathcal{A}) \otimes \mathcal{B}$ and $\mathcal{A} \otimes Q_0(\mathcal{B})$ as subrings of $\mathbb{C}(T \times T)$ in the natural way.

Lemma 2.5.2. *Set* $A_i = D_{(\varpi_i,e)}$ *and* $B_i = D_{(e,\varpi_i)}$ *for* $1 \leq i \leq N$.

(i) $A_N = B_N = \mathrm{id}$ *on* $H_0^{\mathbb{K}}$.

(ii) $A_i \in (Q_0(\mathcal{A}) \otimes \mathcal{B}) \otimes \mathrm{End}(H_0)$ *and* $B_j \in (\mathcal{A} \otimes Q_0(\mathcal{B})) \otimes \mathrm{End}(H_0)$.

(iii) *Set* $A_i^{(0,0)} \in \mathrm{End}(H_0)$ *and* $B_j^{(0,0)} \in \mathrm{End}(H_0)$ *for the value of* A_i *and* B_j *at* $x^{-\alpha_r} = 0 = y^{\alpha_s}$ $(1 \leq r, s < N)$. *For* $w \in S_N$ *we have*

$$A_i^{(0,0)}(T_{w_0}T_w) = \begin{cases} 0 & \text{if } w^{-1}w_0(\varpi_i) \neq w_0(\varpi_i), \\ T_{w_0}T_w & \text{if } w^{-1}w_0(\varpi_i) = w_0(\varpi_i) \end{cases} \tag{2.5.8}$$

and

$$B_i^{(0,0)}(T_{w_0}T_w) = \begin{cases} 0 & \text{if } w(\varpi_i) \neq \varpi_i, \\ T_{w_0}T_w & \text{if } w(\varpi_i) = \varpi_i. \end{cases} \tag{2.5.9}$$

Proof. (i) We only give the proof of $A_N = \mathrm{id}$. Since $\varpi_N = \pi^N$ in W, we have

$$A_N(t,\gamma) = W_\kappa(t,\gamma)^{-1} C_{(\varpi_N,e)}(t,\gamma) W_\kappa(q^{-\varpi_N}t,\gamma) = \gamma^{-\varpi_N}(\eta(\pi)(\gamma))^N = \mathrm{id},$$

where we use (2.5.5) and (2.4.5) for the second equality, and Lemma 2.4.1 and $\sigma^N = e$ for the third equality.

(ii) Note that
$$A_i(t,\gamma) = k^{-\langle \delta, \varpi_i\rangle} \gamma^{-w_0(\varpi_i)} C_{(\varpi_i,e)}(t,\gamma)$$
by (2.5.5). Since $\varpi_i = \pi^i \sigma^{-i}$ the cocycle property of C gives

$$A_i(t,\gamma) = k^{-\langle \delta, \varpi_i\rangle} \gamma^{-w_0(\varpi_i)} \big(\eta(\pi)(\gamma)\big)^i C_{(\sigma^{-i},e)}(\pi^{-i}t,\gamma).$$

It follows from the explicit expressions for the cocycle values $C_{(s_i,e)}$ $(1 \leq i < N)$ that the $\mathrm{End}(H_0)$-valued rational function $C_{(\sigma^{-i},e)}(\pi^{-i}t,\gamma)$ in $(t,\gamma) \in T \times T$ lies in $Q_0(\mathcal{A}) \otimes \mathrm{End}(H_0)$ (in particular, it is independent of γ). Furthermore, for $w \in S_N$

$$\gamma^{-w_0(\varpi_i)} \big(\eta(\pi)(\gamma)\big)^i \big(T_w\big) = \gamma^{w^{-1}w_0(\varpi_i) - w_0(\varpi_i)} T_{\sigma^i w}$$

by Lemma 2.4.1, hence the $\mathrm{End}(H_0)$-valued regular function $\gamma^{-w_0(\varpi_i)}\big(\eta(\pi)(\gamma)\big)^i$ in $\gamma \in T$ lies in $\mathcal{B} \otimes \mathrm{End}(H_0)$. Consequently, $A_i \in (Q_0(\mathcal{A}) \otimes \mathcal{B}) \otimes \mathrm{End}(H_0)$. The statement for B_j follows from this result using the cocycle property $C_{(e,\varpi_j)}(t,\gamma) = C_\iota C_{(\varpi_j,e)}(\gamma^{-1},t^{-1})C_\iota$.

(iii) Recall that $\xi_w = \eta(\widetilde{S}^*_{w^{-1}})T_e$ with $\widetilde{S}_w$ the intertwiners of $\mathbb{H}$ (see Proposition 2.2.9). By induction on $\ell(w)$, using the explicit expression (2.2.8) of the intertwiners $\widetilde{S}_i$, it follows that $\xi_w \in \mathcal{B} \otimes H_0$ and that the value of ξ_w at $y^{\alpha_i} = 0$ $(1 \leq i < N)$ is $T_w \in H_0$. Set
$$A_i^{(0)} = A_i|_{x^{-\alpha_1}=0,\ldots,x^{-\alpha_{N-1}}=0} \in \mathcal{B} \otimes \mathrm{End}(H_0).$$

By Corollary 2.4.6 and (2.5.5),

$$A_i^{(0)} = y^{-w_0(\varpi_i)}\eta(T_{w_0}Y^{w_0(\varpi_i)}T_{w_0}^{-1}).$$

Lemma 2.4.2 then gives

$$A_i^{(0)}\big(\eta(T_{w_0})\xi_w\big) = y^{w^{-1}w_0(\varpi_i)-w_0(\varpi_i)}\eta(T_{w_0})\xi_w, \qquad \forall\, w \in S_N \tag{2.5.10}$$

as identities in $\mathcal{B}\otimes H_0$. Specializing (2.5.10) at $y^{\alpha_j}=0$ $(1\le j<N)$ yields (2.5.8).

To prove (2.5.9) we consider

$$\widetilde{B}_j^{(0)} = B_j|_{y^{\alpha_1}=0,\ldots,y^{\alpha_{N-1}}=0} \in \mathcal{A}\otimes \mathrm{End}(H_0).$$

It is the rational $\mathrm{End}(H_0)$-valued function

$$\widetilde{B}_j^{(0)}(t) = t^{w_0(\varpi_j)}C_\iota\big(\eta(T_{w_0}Y^{w_0(\varpi_j)}T_{w_0}^{-1})(t^{-1})\big)C_\iota$$

in $t\in T$. Denoting $\widetilde{\xi}_w\in\mathcal{A}\otimes H_0$ for the rational H_0-valued function $\xi_w(t^{-1})$ in $t\in T$, it follows that

$$\widetilde{B}_j^{(0)}\big(C_\iota\eta(T_{w_0})\widetilde{\xi}_w\big) = x^{-w^{-1}w_0(\varpi_j)+w_0(\varpi_j)}C_\iota\eta(T_{w_0})\widetilde{\xi}_w \tag{2.5.11}$$

for all $w\in S_N$. The value of $C_\iota\eta(T_{w_0})\widetilde{\xi}_w$ at $x^{-\alpha_i}=0$ $(1\le i<N)$ is $C_\iota(T_{w_0}T_w)$. In addition, C_ι restricts to the anti-algebra involution on H_0 mapping T_w to $T_{w^{-1}}$ for $w\in S_N$, hence

$$C_\iota(T_{w_0}T_w) = T_{w^{-1}}T_{w_0} = T_{w_0}T_{w_0w^{-1}w_0}.$$

Formula (2.5.9) then follows from specializing (2.5.11) at $x^{-\alpha_i}=0$ $(1\le i<N)$ and replacing w by $w_0w^{-1}w_0$ in the resulting formula. □

For $\epsilon>0$, put $B_\epsilon := \{t\in T \mid |t^{\alpha_i}|<\epsilon,\ \forall i\}$ and $B_\epsilon^{-1} := \{t\in T\mid t^{-1}\in B_\epsilon\}$.

Theorem 2.5.3. *There exists a unique solution $\Psi\in H_0^{\mathbb{K}}$ of the gauged equations* (2.5.6) *satisfying, for some $\epsilon>0$,*
(i) *$\Psi(t,\gamma)$ admits an H_0-valued power series expansion*

$$\Psi(t,\gamma) = \sum_{\alpha,\beta\in Q_+} K_{\alpha,\beta}t^{-\alpha}\gamma^{\beta}, \qquad (K_{\alpha,\beta}\in H_0) \tag{2.5.12}$$

for $(t,\gamma)\in B_\epsilon^{-1}\times B_\epsilon$ which is normally convergent on compacta of $B_\epsilon^{-1}\times B_\epsilon$. In particular, $\Psi(t,\gamma)$ is analytic at $(t,\gamma)\in B_\epsilon^{-1}\times B_\epsilon$;
(ii) *$K_{0,0}=T_{w_0}$.*

Proof. It follows from the previous lemma that the commuting endomorphisms $A_i^{(0,0)}$, $B_j^{(0,0)}\in\mathrm{End}(H_0)$ $(1\le i,j<N)$ are semisimple. For $a,b\in\mathbb{C}^{N-1}$ set

$$H_0[(a,b)] = \{v\in H_0 \mid A_i^{(0,0)}v=a_iv \ \text{ and }\ B_j^{(0,0)}v=b_jv\ (1\le i,j<N)\},$$

so that $H_0 = \bigoplus_{(a,b)\in S} H_0[(a,b)]$ with S the finite set of $(a,b) \in \mathbb{C}^{N-1} \times \mathbb{C}^{N-1}$ for which $H_0[(a,b)] \neq 0$. By the previous lemma, $(1^{N-1}, 1^{N-1}) \in S$ and we have $H_0[(1^{N-1}, 1^{N-1})] = \mathrm{span}_{\mathbb{C}}\{T_{w_0}\}$. Furthermore, $a_i, b_i \notin q^{-\mathbb{N}}$ for all $(a,b) \in S$ and i. Under these conditions, the holonomic system of q-difference equations (2.5.6) admits a unique solution Ψ satisfying the desired properties; see Theorem A.6 in the appendix (to show that the gauged BqKZ falls in the class of holonomic systems of q-difference equations to which Theorem A.6 applies, one should take $M = 2(N-1)$, $q_i = q$ for $1 \leq i < N$ and variables $z_i = x^{-\alpha_i}$ and $z_{N-1+j} = y^{\alpha_j}$ for $1 \leq i, j < N$ in the appendix). □

Remark 2.5.4. In a small neighborhood of a fixed $(t', \gamma') \in T \times T$, the meromorphic solution Ψ of (2.5.6) can be expressed in terms of the power series expansion (2.5.12) by the formula

$$\begin{aligned}\Psi(t,\gamma) &= D_{(\lambda,\mu)}(t,\gamma)\Psi(q^{-\lambda}t, q^{\mu}\gamma)\\ &= D_{(\lambda,\mu)}(t,\gamma)\sum_{\alpha,\beta\in Q_+} K_{\alpha,\beta}(q^{-\lambda}t)^{-\alpha}(q^{\mu}\gamma)^{\beta},\end{aligned}$$

where $\lambda, \mu \in \mathbb{Z}^N$ are such that $(q^{-\lambda}t', q^{\mu}\gamma') \in B_\epsilon^{-1} \times B_\epsilon$.

Definition 2.5.5. We call $\Phi_\kappa := W_\kappa \Psi \in \mathrm{SOL}$ the basic asymptotically free solution of BqKZ.

Note that $\Phi_\kappa \in \mathbb{F}^\times \Phi_{\kappa'}$ for $\kappa, \kappa' \in \mathbb{C}^\times$. The κ-flexibility will come in handy when we consider specializations of Φ_κ. In the following subsections, we derive various properties of the basic asymptotically free solution Φ_κ.

2.5.3 Duality

Theorem 2.5.6. *The basic asymptotically free solution Φ_κ of BqKZ is self-dual, in the sense that*

$$\tau(\iota)\Phi_\kappa = \Phi_\kappa.$$

Proof. SOL is $\mathbb{S}_N$-invariant, hence $\tau(\iota)\Phi_\kappa \in \mathrm{SOL}$. In addition,

$$\tau(\iota)\Phi_\kappa = W_\kappa(\tau(\iota)\Psi),$$

because $\iota(W_\kappa) = W_\kappa$. Hence $\tau(\iota)\Psi$ is a solution of the gauged equations (2.5.6) having a convergent H_0-valued power series expansion

$$(\tau(\iota)\Psi)(t,\gamma) = C_\iota\Psi(\gamma^{-1}, t^{-1}) = \sum_{\alpha,\beta\in Q_+} C_\iota(K_{\alpha,\beta})\gamma^{\alpha}t^{-\beta}$$

for $(t,\gamma) \in B_\epsilon^{-1} \times B_\epsilon$. Since $C_\iota(K_{0,0}) = C_\iota(T_{w_0}) = T_{w_0}$, we conclude from Theorem 2.5.3 that $\tau(\iota)\Psi = \Psi$, hence $\tau(\iota)\Phi_\kappa = \Phi_\kappa$. □

2.5.4 Singularities

Define

$$\Lambda := \{\lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N\} = \bigoplus_{i=1}^{N-1} \mathbb{Z}_{\geq 0}\varpi_i \oplus \mathbb{Z}\varpi_N, \tag{2.5.13}$$

i.e., Λ consists of the $\lambda \in \mathbb{Z}^N$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in R_+$. Set

$$\mathcal{S}_+ := \{t \in T \mid t^\alpha \in k^{-2}q^{-\mathbb{N}} \text{ for some } \alpha \in R_+\}.$$

Write $\Psi(t,\gamma) = \sum_{\alpha \in Q_+} \Gamma_\alpha(\gamma) t^{-\alpha}$ for $(t,\gamma) \in B_\epsilon^{-1} \times B_\epsilon$, where Γ_α is the H_0-valued analytic function on B_ϵ defined by the H_0-valued power series

$$\Gamma_\alpha(\gamma) := \sum_{\beta \in Q_+} K_{\alpha,\beta}\gamma^\beta.$$

Lemma 2.5.7. *The Γ_α ($\alpha \in Q_+$) extend uniquely to a meromorphic H_0-valued function on T, analytic on $T \setminus \mathcal{S}_+$, such that $\Psi(t,\gamma)$ admits an H_0-valued power series expansion*

$$\Psi(t,\gamma) = \sum_{\alpha \in Q_+} \Gamma_\alpha(\gamma) t^{-\alpha}$$

for $(t,\gamma) \in B_\epsilon^{-1} \times T \setminus \mathcal{S}_+$, converging normally on compacta of $B_\epsilon^{-1} \times T \setminus \mathcal{S}_+$.

Proof. Using Lemma 2.5.2 we write for $\mu \in \Lambda$,

$$D_{(e,\mu)}(t,\gamma) = \sum_{\beta \in Q_+} F_\beta^\mu(\gamma) t^{-\beta},$$

with $F_\beta^\mu \in Q_0(\mathcal{B}) \otimes \mathrm{End}(H_0)$ for all $\beta \in Q_+^\vee$. Note that $F_\beta^\mu \equiv 0$ for all but finitely many $\beta \in Q_+$.

We first show that $F_\beta^\mu(\gamma)$ is regular at $\gamma \in T \setminus \mathcal{S}_+$. By (2.5.5) and by the cocycle property, $F_\beta^\mu(\gamma)$ is regular at $\gamma = \zeta$ if $C_{(e,\varpi_j)}(\cdot, q^\nu\gamma) \in \mathbb{C}[T] \otimes \mathrm{End}(H_0)$ is regular at $\gamma = \zeta$ for all $1 \leq j \leq N$ and $\nu \in \Lambda$. The latter statement follows from the fact that $R_i(z)$ has only a (simple) pole at $z = k^{-2}$ and from the explicit expression

$$\begin{aligned} C_{(e,\varpi_j)}(t,\gamma) = C_\iota\big(\eta(\pi)(t^{-1})\big)^j &(R_{N-j}(q\gamma_1/\gamma_N)\cdots R_2(q\gamma_1/\gamma_{j+2})R_1(q\gamma_1/\gamma_{j+1}))\\ &\times \cdots \times (R_{N-2}(q\gamma_{j-1}/\gamma_N)\cdots R_j(q\gamma_{j-1}/\gamma_{j+2})R_{j-1}(q\gamma_{j-1}/\gamma_{j+1}))\\ &\times (R_{N-1}(q\gamma_j/\gamma_N)\cdots R_{j+1}(q\gamma_j/\gamma_{j+2})R_j(q\gamma_j/\gamma_{j+1}))C_\iota, \end{aligned} \tag{2.5.14}$$

which follows from Lemma 2.4.5(ii) and the cocycle property of C.

Let $U \subset T \setminus \mathcal{S}_+$ be a relatively compact open subset. Choose $\mu \in \Lambda$ such that the closure of $q^\mu U$ is contained in B_ϵ, where $q^\mu U := \{q^\mu \gamma \,|\, \gamma \in U\}$. As meromorphic H_0-valued function in $(t, \gamma) \in B_\epsilon^{-1} \times U$, we have

$$\begin{aligned}\Psi(t,\gamma) &= D_{(e,\mu)}(t,\gamma)\Psi(t,q^\mu\gamma)\\ &= \sum_{\alpha,\beta\in Q_+} F^\mu_\beta(\gamma)\big(\Gamma_\alpha(q^\mu\gamma)\big)t^{-\alpha-\beta}\\ &= \sum_{\alpha\in Q_+}\Big(\sum_{\beta\in Q_+:\alpha-\beta\in Q_+} F^\mu_\beta(\gamma)\big(\Gamma_{\alpha-\beta}(q^\mu\gamma)\big)\Big)t^{-\alpha},\end{aligned}$$

with the sums converging normally on compacta of $B_\epsilon^{-1} \times U$ (note that the sums over β are finite). It follows that Γ_α ($\alpha \in Q_+$) has a unique H_0-valued meromorphic extension to T which, on U, is given by

$$\Gamma_\alpha(\gamma) = \sum_{\beta\in Q_+:\alpha-\beta\in Q_+} F^\mu_\beta(\gamma)\big(\Gamma_{\alpha-\beta}(q^\mu\gamma)\big), \tag{2.5.15}$$

such that Ψ on $B_\epsilon^{-1} \times U$ admits the power series expansion

$$\Psi(t,\gamma) = \sum_{\alpha\in Q_+} \Gamma_\alpha(\gamma)t^{-\alpha},$$

which converges normally on compacta of $B_\epsilon^{-1} \times U$. It follows from (2.5.15) and the previous paragraph that Γ_α is analytic on $T \setminus \mathcal{S}_+$. □

The arguments from the proof of Lemma 2.5.7, applied to both torus variables of $\Psi(t,\gamma)$ at the same time, directly lead to the following result.

Proposition 2.5.8. *The H_0-valued meromorphic function $\Psi(t,\gamma)$ is analytic at $(t,\gamma) \in T \setminus \mathcal{S}_+^{-1} \times T \setminus \mathcal{S}_+$.*

For specialized spectral parameter, we obtain the following result.

Proposition 2.5.9. *Let $\zeta \in T \setminus \mathcal{S}_+$.*
(i) *The H_0-valued meromorphic function $\Psi(t,\gamma)$ in $(t,\gamma) \in T \times T$ can be specialized at $\gamma = \zeta$, giving rise to a meromorphic H_0-valued function $\Psi(t,\zeta)$ in $t \in T$. It has the power series expansion*

$$\Psi(t,\zeta) = \sum_{\alpha\in Q_+} \Gamma_\alpha(\zeta)t^{-\alpha}$$

for $t \in B_\epsilon^{-1}$, normally converging on compacta of B_ϵ^{-1}.
(ii) *$\Psi(t,\zeta)$ satisfies the gauged q-difference equations*

$$D_{(\lambda,e)}(t,\zeta)\Psi(q^{-\lambda}t,\zeta) = \Psi(t,\zeta), \qquad \forall \lambda \in \mathbb{Z}^N. \tag{2.5.16}$$

Proof. (i) Restricting to $t \in B_\epsilon^{-1}$ for $\epsilon > 0$ small enough, the statement is correct by Lemma 2.5.7. If $t' \in T$ is arbitrary then there exists a $\lambda \in \Lambda$ such that $q^{-\lambda}t' \in B_\epsilon^{-1}$. For $t \in T$ in a small neighborhood of t' we then have

$$\Psi(t,\gamma) = D_{(\lambda,e)}(t,\gamma)\Psi(q^{-\lambda}t,\gamma).$$

Since $D_{(\lambda,e)} \in (Q_0(\mathcal{A}) \otimes \mathcal{B}) \otimes \mathrm{End}(H_0)$ by Lemma 2.5.2(ii) the statement now follows in a small open neighborhood of t'.
(ii) Specializing the gauged q-difference equations $D_{(\lambda,e)}(t,\gamma)\Psi(q^{-\lambda}t,\gamma) = \Psi(t,\gamma)$ ($\lambda \in \mathbb{Z}^N$) to $\gamma = \zeta$ yields the desired result. □

2.5.5 Evaluation formula

We write $\left(z;q\right)_\infty = \prod_{m=0}^\infty (1-q^m z)$ for the q-shifted factorial. Recall the power series expansion $\Psi(t,\gamma) = \sum_{\alpha\in Q_+} \Gamma_\alpha(\gamma)t^{-\alpha}$ for $|t^{\alpha_i}| \gg 0$ ($1 \leq i < N$) from Subsection 2.5.4. We call the following result the evaluation formula for the basic asymptotically free solution $\Phi_\kappa = W_\kappa\Psi$ of BqKZ, since it implies the celebrated evaluation formula for the Macdonald polynomials (see Subsection 4.5).

Theorem 2.5.10. *We have*

$$\Gamma_0(\gamma) = K(\gamma)T_{w_0}$$

with $K \in \mathcal{M}(T)$ explicitly given by

$$K(\gamma) := \prod_{1\leq i<j\leq N} \frac{\left(q\gamma_i/\gamma_j;q\right)_\infty}{\left(qk^2\gamma_i/\gamma_j;q\right)_\infty}. \tag{2.5.17}$$

Proof. We use the notations of Lemma 2.5.2. Recall that Ψ satisfies the gauged q-difference equations

$$A_i(t,\gamma)\Psi(q^{-\varpi_i}t,\gamma) = \Psi(t,\gamma)$$

for $1 \leq i \leq N$. In view of the proof of Lemma 2.5.2 and Lemma 2.5.7, it reduces in the limit $|t^{-\alpha_i}| \to 0$ ($1 \leq i < N$) to

$$\gamma^{-w_0(\varpi_i)}\eta(T_{w_0}Y^{w_0(\varpi_i)}T_{w_0}^{-1})(\gamma)\Gamma_0(\gamma) = \Gamma_0(\gamma)$$

for $1 \leq i \leq N$, as H_0-valued meromorphic functions in $\gamma \in T$. This forces $\Gamma_0(\gamma) = K(\gamma)\eta(T_{w_0})\xi_e(\gamma) = K(\gamma)T_{w_0}$ for some $K \in \mathcal{M}(T)$; see Lemma 2.4.2.

It remains to show that K is explicitly given by (2.5.17). Write $L(\gamma)$ for the right hand side of (2.5.17). Then $L \in \mathcal{M}(T)$ is characterized by the following three properties:

(i) for some $\epsilon > 0$ we have a power series expansion

$$L(\gamma) = \sum_{\alpha\in Q_+} l_\alpha\gamma^\alpha$$

for $\gamma \in B_\epsilon$, converging normally on compacta of B_ϵ;

(ii) $l_0 = 1$; and

(iii) $L(\gamma)$ satisfies the q-difference equations

$$\prod_{\substack{1\le r\le j\\ j+1\le s\le N}} \frac{1-q\gamma_r/\gamma_s}{1-qk^2\gamma_r/\gamma_s} L(q^{\varpi_j}\gamma) = L(\gamma), \qquad 1\le j\le N.$$

It thus suffices to show that $K(\gamma)$ satisfies the three properties (i)–(iii). It is clear that $K\in\mathcal{M}(T)$ satisfies (i); see Subsection 2.5.4. Theorem 2.5.3(ii) implies (ii) for K. What remains is the verification of the q-difference equations (iii) for K. Using the notations of Lemma 2.5.2, we write

$$B_j^{(0)} := B_j|_{x^{-\alpha_1}=0,\ldots,x^{-\alpha_{N-1}}=0} \in Q_0(\mathcal{B})\otimes \mathrm{End}(H_0).$$

We view $B_j^{(0)}(\gamma)$ as an $\mathrm{End}(H_0)$-valued meromorphic function in $\gamma\in T$. Taking the limit $|t^{-\alpha_i}|\to 0$ ($1\le i<N$) in the gauged q-difference equations

$$B_j(t,\gamma)\Psi(t,q^{\varpi_j}\gamma) = \Psi(t,\gamma), \qquad 1\le j\le N$$

and using $\Gamma_0(\gamma) = K(\gamma)T_{w_0}$ we obtain

$$K(q^{\varpi_j}\gamma)B_j^{(0)}(\gamma)T_{w_0} = K(\gamma)T_{w_0}$$

for $1\le j\le N$, as meromorphic H_0-valued functions in $\gamma\in T$. Writing $B_j^{(0)}(\gamma)T_{w_0} = \sum_{w\in S_N} a_w^j(\gamma)T_w$ with $a_w^j\in\mathcal{M}(T)$ it thus suffices to show that

$$a_{w_0}^j(\gamma) = \prod_{\substack{1\le r\le j\\ j+1\le s\le N}} \frac{1-q\gamma_r/\gamma_s}{1-qk^2\gamma_r/\gamma_s} = k^{-\langle\delta,\varpi_j\rangle} \prod_{\substack{1\le r\le j\\ j+1\le s\le N}} c_k(q\gamma_r/\gamma_s)^{-1} \tag{2.5.18}$$

for $1\le j\le N$, where the second equality follows from a direct computation using the explicit expression (2.2.5) of c_k.

By (2.5.5) we have

$$B_j(t,\gamma) = k^{-\langle\delta,\varpi_j\rangle} t^{w_0(\varpi_j)} C_{(e,\varpi_j)}(t,\gamma)$$

and $C_{(e,\varpi_j)}(t,\gamma)$ is given explicitly by (2.5.14). Since $R_i(z) = c_k(z)^{-1}(\eta(T_i)-k)+1$, Lemma 2.4.1 and the reduced expression (2.4.8) for σ^i imply that

$$B_j(t,\gamma)T_{w_0} = k^{-\langle\delta,\varpi_j\rangle} t^{w_0(\varpi_j)} C_\iota(\eta(\pi)(t^{-1}))^j \big(\sum_{w\le\sigma^{-j}} b_w^j(\gamma)T_{ww_0}\big)$$

with $\le$ the Bruhat order on S_N and with

$$b_{\sigma^{-j}}^j(\gamma) = \prod_{\substack{1\le r\le j\\ j+1\le s\le N}} c_k(q\gamma_r/\gamma_s)^{-1}.$$

By Lemma 2.4.1 we have

$$t^{w_0(\varpi_j)} C_\iota(\eta(\pi)(t^{-1}))^j T_{ww_0} = t^{w_0(\varpi_j) - w_0 w^{-1} w_0(\varpi_j)} T_{w_0 w^{-1} \sigma^{-j}}.$$

Hence

$$B_j^{(0)}(\gamma) T_{w_0} = k^{-\langle \delta, \varpi_j \rangle} \sum_w b_w^j(\gamma) T_{w_0 w^{-1} \sigma^{-j}},$$

with the sum running over $w \in S_N$ satisfying $w \leq \sigma^{-j}$ and $w(\varpi_j) = w_0(\varpi_j)$. In particular, $a_{w_0}^j(\gamma) = k^{-\langle \delta, \varpi_j \rangle} b_{\sigma^{-j}}^j(\gamma)$. This completes the proof of (2.5.18). □

2.5.6 Consistency of the bispectral quantum KZ equations

In this subsection, we show that BqKZ is a consistent system of q-difference equations, i.e., $\dim_{\mathbb{F}}(\mathrm{SOL}) = \dim_{\mathbb{C}}(H_0)$, by explicitly constructing an $\mathbb{F}$-basis of SOL. Since the cocycle matrices $C_{(\lambda,\mu)}(t,\gamma)$ ($\lambda, \mu \in \mathbb{Z}^N$) depend rationally on $(t,\gamma) \in T \times T$, the consistency of BqKZ follows also from the abstract arguments in [14, §5].

We start with a preliminary lemma on the cocycle values $C_{(e,w)}$ for $w \in S_N$.

Lemma 2.5.11. *Let $w \in S_N$. We have $C_{(e,w)} \in Q_0(\mathcal{B}) \otimes \mathrm{End}(H_0)$ and*

$$C_{(e,w)}^{(0)}(h) = k^{-\ell(w)} h T_{w^{-1}}, \qquad h \in H_0,$$

where

$$C_{(e,w)}^{(0)} = C_{(e,w)}|_{y^{\alpha_1}=0,\ldots,y^{\alpha_{N-1}}=0} \in \mathrm{End}(H_0).$$

Proof. Let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression for $w \in S_N$ ($1 \leq i_j < N$) and write $\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in R_+$ for $1 \leq j \leq r$, where β_1 should be read as α_{i_1}. By Subsection 2.4.2 and the cocycle property, we have

$$C_{(e,w)}(t,\gamma) = C_\iota C_{(w,e)}(\gamma^{-1}, t^{-1}) C_\iota = C_\iota \big(R_{i_r}(\gamma^{\beta_r}) \cdots R_{i_2}(\gamma^{\beta_2}) R_{i_1}(\gamma^{\beta_1}) \big)^{-1} C_\iota.$$

From the expression for $R_i(z)$ it now follows that $C_{(e,w)} \in Q_0(\mathcal{B}) \otimes \mathrm{End}(H_0)$. Since $\lim_{z \to 0} R_i(z) = k\eta(T_i^{-1})$ we furthermore have

$$C_{(e,w)}^{(0)} = k^{-\ell(w)} C_\iota \eta(T_w) C_\iota.$$

The map C_ι is the $\mathbb{K}$-linear extension of the anti-algebra involution of H_0 mapping T_w to $T_{w^{-1}}$. Hence $C_{(e,w)}^{(0)}(h) = k^{-\ell(w)} h T_{w^{-1}}$ for $h \in H_0$. □

Define $U \in \mathrm{End}(H_0)^{\mathbb{K}} := \mathbb{K} \otimes \mathrm{End}(H_0)$ by

$$U\big(k^{-\ell(w)} T_{w_0} T_{w^{-1}}\big) = \tau(e,w)\Phi_\kappa, \qquad w \in S_N. \tag{2.5.19}$$

Since SOL is $\mathbb{S}_N$-invariant, U is an $\mathrm{End}(H_0)$-valued solution of BqKZ, i.e.

$$C_{(\lambda,\mu)}(t,\gamma) U(q^{-\lambda} t, q^\mu \gamma) = U(t,\gamma), \qquad \lambda, \mu \in \mathbb{Z}^N$$

as $\mathrm{End}(H_0)$-valued meromorphic functions in $(t,\gamma) \in T \times T$.

Lemma 2.5.12. $U \in \mathrm{End}(H_0)^{\mathbb{K}}$ *is invertible.*

Proof. Using the natural identification $\mathrm{End}(H_0)^{\mathbb{K}} \simeq \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ as $\mathbb{K}$-algebras, we need to verify that $U \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$.

Set $\Phi_w := \tau(e,w)\Phi_\kappa$ and $\Psi_w := \tau(e,w)\Psi$ for $w \in S_N$, so that

$$\Phi_w(t,\gamma) = W_\kappa(t, w^{-1}\gamma)\Psi_w(t,\gamma)$$

Since $C_{(e,w)}(t,\gamma)$ is independent of $t \in T$, we simply write it as $C_{(e,w)}(\gamma)$. Recall the W-invariant subset $\mathcal{S} \subset T$ (see (2.4.6)), which contains $\mathcal{S}_+$. By Lemma 2.4.4 and Lemma 2.5.7, we have for some $\epsilon > 0$ the power series expansion

$$\Psi_w(t,\gamma) = \sum_{\alpha \in Q_+} C_{(e,w)}(\gamma)\big(\Gamma_\alpha(w^{-1}\gamma)\big)t^{-\alpha}$$

for $(t,\gamma) \in B_\epsilon^{-1} \times T \setminus \mathcal{S}$, converging normally on compacta of $B_\epsilon^{-1} \times T \setminus \mathcal{S}$. We write $\Gamma_\alpha^w(\gamma) := C_{(e,w)}(\gamma)\big(\Gamma_\alpha(w^{-1}\gamma)\big)$ in the remainder of the proof. It is a meromorphic function in $\gamma \in T$, analytic on $T \setminus \mathcal{S}$, and the power series expansion of Ψ_w becomes

$$\Psi_w(t,\gamma) = \sum_{\alpha \in Q_+} \Gamma_\alpha^w(\gamma)t^{-\alpha}. \tag{2.5.20}$$

Observe that

$$\Gamma_0^w(\gamma) \to C_{(e,w)}^{(0)}(T_{w_0}) = k^{-\ell(w)}T_{w_0}T_{w^{-1}},$$

in the limit $\gamma^{\alpha_i} \to 0$ $(1 \le i < N)$, in view of the previous lemma.

Write $U = V\Xi$ with V, Ξ the $\mathbb{K}$-linear endomorphisms of $H_0^{\mathbb{K}}$ given by

$$\begin{aligned} \Xi\big(k^{-\ell(w)}T_{w_0}T_{w^{-1}}\big)(t,\gamma) &= W_\kappa(t,w^{-1}\gamma)k^{-\ell(w)}T_{w_0}T_{w^{-1}}, \\ V\big(k^{-\ell(w)}T_{w_0}T_{w^{-1}}\big) &= \Psi_w, \end{aligned}$$

for $w \in S_N$. Since $\Xi \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ it suffices to show that $V \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$. Let M be the matrix of V with respect to the $\mathbb{K}$-basis $k^{-\ell(w)}T_{w_0}T_{w^{-1}}$ $(w \in S_N)$ of $H_0^{\mathbb{K}}$. Fix $\zeta \in T \setminus \mathcal{S}$ such that $\zeta^\alpha \notin R$ for all $\alpha \in R$. The matrix $M(t,\gamma)$ may be specialized at $\gamma = \zeta$ and the limit of $M(t,\zeta)$ as $t^{-\alpha_i} \to 0$ $(1 \le i < N)$ exists. We write $M^{(0)}(\zeta)$ for the limit and $V^{(0)}(\zeta)$ for the corresponding linear endomorphism of H_0. We then have

$$V^{(0)}(\zeta)\big(k^{-\ell(w)}T_{w_0}T_{w^{-1}}\big) = \Gamma_0^w(\zeta) = K(w^{-1}\zeta)C_{(e,w)}(\zeta)T_{w_0},$$

with $K(\gamma)$ given by (2.5.17). Note that $K(w^{-1}\zeta) \neq 0$ since $\zeta^\alpha \notin q^{\mathbb{Z}}$ for all $\alpha \in R$. By the explicit expression for the cocycle value $C_{(e,w)}(\zeta) \in \mathrm{End}(H_0)$ (see the proof of the previous lemma) we have

$$C_{(e,w)}(\zeta)(T_{w_0}) = \sum_{v \le w} a_v^w(\zeta)T_{w_0}T_{v^{-1}},$$

with $a_w^w(\zeta) \neq 0$ and with $\le$ the Bruhat order on S_N. This implies that $V^{(0)}(\zeta)$ is a linear automorphism of H_0, hence $\det(M^{(0)}(\zeta)) \neq 0$. Consequently, $\det(M) \in \mathbb{K}^\times$ and $V \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$. □

Proposition 2.5.13. **(i)** *$U' \in \mathrm{End}(H_0)^{\mathbb{K}}$ is an $\mathrm{End}(H_0)$-valued meromorphic solution of BqKZ if and only if $U' = UF$ for some $F \in \mathrm{End}(H_0)^{\mathbb{F}}$.*
(ii) *U, viewed as $\mathbb{K}$-linear endomorphism of $H_0^{\mathbb{K}}$, restricts to an $\mathbb{F}$-linear isomorphism*

$$U\colon H_0^{\mathbb{F}} \to \mathrm{SOL}.$$

(iii) *$\{\tau(e,w)\Phi_\kappa\}_{w\in S_N}$ is an $\mathbb{F}$-basis of* SOL.

Proof. (i) If U' is an $\mathrm{End}(H_0)$-valued meromorphic solution of BqKZ then, since U is invertible, we have for all $\lambda, \mu \in \mathbb{Z}^N$,

$$U(q^{-\lambda}t, q^{\mu}\gamma)^{-1}U'(q^{-\lambda}t, q^{\mu}\gamma) = U(t,\gamma)^{-1}U'(t,\gamma).$$

Hence $U' = UF$ with $F \in \mathrm{End}(H_0)^{\mathbb{F}}$. The converse implication is clear.
(ii) By the previous lemma we have $U : H_0^{\mathbb{F}} \hookrightarrow \mathrm{SOL}$. It is surjective, since for $g \in \mathrm{SOL}$, $f := U^{-1}g \in H_0^{\mathbb{K}}$ satisfies $f(q^{-\lambda}t, q^{\mu}\gamma) = f(t,\gamma)$ for all $\lambda, \mu \in \mathbb{Z}^N$ (cf. the proof of (i)), hence $f \in H_0^{\mathbb{F}}$.
(iii) This is clear from (ii) and from the definition of U. □

By Proposition 2.5.9 and by the proofs of Lemma 2.5.12 and Proposition 2.5.13 we obtain the following consistency statement for the quantum KZ equation (2.4.10) with specialized central character (see [8] and [10]). Recall the W-invariant subset $\mathcal{S} \subset T$ given by (2.4.6). Recall furthermore that $C_{(e,w)}(t,\gamma)$ for $w \in S_N$ only depends on γ, so we simply write it as $C_{(e,w)}(\gamma)$.

Corollary 2.5.14. *Fix $\zeta \in T \setminus \mathcal{S}$ such that $\zeta^\alpha \notin q^{\mathbb{Z}}$ for all $\alpha \in R$. For generic $\kappa \in \mathbb{C}^\times$, the H_0-valued meromorphic functions $\big(\tau(e,w)\Phi_\kappa\big)(t,\gamma)$ in $(t,\gamma) \in T \times T$ ($w \in S_N$) can be specialized at $\gamma = \zeta$, giving rise to*

- **(i)** *a basis $\{C_{(e,w)}(\zeta)\Phi_\kappa(\cdot, w^{-1}\zeta)\}_{w\in S_N}$ of SOL_ζ over $\mathcal{E}(T)$;*
- **(ii)** *an invertible $\mathrm{End}(H_0)$-valued meromorphic solution U_ζ of the quantum KZ equations* (2.4.10)*, where $U_\zeta \in \mathrm{End}(H_0)^{\mathcal{M}(T)}$ is explicitly defined by*

$$U_\zeta(k^{-\ell(w)}T_{w_0}T_{w^{-1}}) := C_{(e,w)}(\zeta)\Phi_\kappa(\cdot, w^{-1}\zeta), \qquad w \in S_N.$$

Chapter 3

Correspondence of BqKZ with bispectral problems

3.1 Introduction

Cherednik [7, Theorem 3.4] constructed for arbitrary root systems a correspondence between solutions of quantum KZ equations and solutions of a system of q-difference equations. In [7, Theorem 4.4], Cherednik made the correspondence precise for GL_N. It yields an explicit map χ_+ from solutions of the quantum KZ equations qKZ_ζ (with fixed central character $\zeta \in T$; see (2.4.10)) to solutions of the spectral problem of Ruijsenaars' [50] commuting trigonometric q-difference operators with spectral parameter ζ^{-1} (the Ruijsenaars operators are also frequently referred to as Macdonald-Ruijsenaars operators). The latter result has been generalized to arbitrary root systems in [30, Theorem 4.6] and [8]. In the classical setting ($q = 1$) it goes back to Matsuo [43]. In this chapter we analyze the map χ_+ in the bispectral setting of Chapter 2. It leads to the interpretation of χ_+ as an embedding of the solution space SOL of BqKZ (see Definition 2.3.7) into the space of meromorphic solutions of a bispectral problem involving the above Ruijsenaars operators as well as Ruijsenaars operators acting on the spectral parameter.

In Section 3.2 we introduce the so-called monodromy cocycle which plays a role in the construction of the correspondence, which will be established in Section 3.3. The techniques employed there are analogous to the ones for the usual correspondence (see [10, Chapter 1]). For the convenience of the reader we provide full details of the arguments involved.

In Section 3.4 we apply the correspondence to the self-dual solution Φ_κ (Definition 2.5.5) of BqZK to obtain a self-dual solution Φ_κ^+ of the bispectral problem of the Ruijsenaars q-difference operators. For reasons which become apparent below, we call this solution the basic Harish-Chandra series solution of the bispectral problem of the Ruijsenaars q-difference operators.

Finally, in Section 3.5, we consider BqKZ for specialized $\gamma = \zeta \in T$ as in Subsection 2.4.4, and in this case reobtain from the bispectral correspondence the (classical) correspondence between solutions of qKZ_ζ and the spectral problem of the Ruijsenaars operators with spectral parameter ζ^{-1}. Specialization of Φ_κ^+ leads to a Harish-Chandra series solution of the Ruijsenaars operators with fixed spectral parameter. These Harish-Chandra solutions to the spectral problem were investigated before in, e.g., [16], [17], [31] and [36]. The present approach to Harish-Chandra series, which uses quantum KZ equations in an essential way, has the advantage that it leads to new results on the convergence and singularities of the Harish-Chandra series. These results, together with Cherednik's recent work [11], form important building blocks in deriving the c-function expansion of Cherednik's global (q,t)-spherical function (see [55]).

The material presented in this chapter coincides with Section 6 of the paper [45].

Convention. Throughout Sections 3.2-3.4, we fix $\kappa \in \mathbb{C}^\times$. Furthermore, we use the notations and conventions of Chapter 2.

3.2 The monodromy cocycle

Recall the $\mathrm{End}(H_0)$-valued meromorphic solution U of BqKZ that we constructed in Subsection 2.5.6. In this section, we will define an auxiliary cocycle $\mathcal{T}_{\mathrm{w}}$ ($\mathrm{w} \in \mathbb{W}$) of $\mathbb{W}$, which can be thought of as a family of monodromy matrices with respect to the fundamental solution U, as explained below.

Observe that $F \in \mathrm{End}(H_0)^{\mathbb{K}}$ is an $\mathrm{End}(H_0)$-valued meromorphic solution of BqKZ if and only if

$$\tau(\mathrm{w})F = F, \qquad \mathrm{w} \in \mathbb{Z}^N \times \mathbb{Z}^N,$$

where the $\mathbb{W}$-action τ on $\mathrm{End}(H_0)^{\mathbb{K}}$ is defined by

$$(\tau(\mathrm{w})F)(t,\gamma) := C_{\mathrm{w}}(t,\gamma)(\mathrm{w}F)(t,\gamma) = C_{\mathrm{w}}(t,\gamma)F(\mathrm{w}^{-1}(t,\gamma))$$

for $\mathrm{w} \in \mathbb{W}$ and $F \in \mathrm{End}(H_0)^{\mathbb{K}}$, viewed as identities between $\mathrm{End}(H_0)$-valued meromorphic functions in $(t,\gamma) \in T \times T$.

By Proposition 2.5.13(i), given an $\mathrm{End}(H_0)$-valued meromorphic solution F of BqKZ, there exists a unique $G \in \mathrm{End}(H_0)^{\mathbb{F}}$ such that $F = UG$. Accordingly, G describes the deviation of F from the fundamental solution U of BqKZ, and therefore can be thought of as a connection matrix (cf. [15, §12.1]). We will consider the special cases when the solutions F are the $\mathrm{End}(H_0)$-valued meromorphic solutions $\tau(\mathrm{w})U$ ($\mathrm{w} \in \mathbb{W}$) of BqKZ.

For $\mathrm{w} \in \mathbb{W}$ we set

$$\mathcal{T}_{\mathrm{w}} := U^{-1}(\tau(\mathrm{w})U) \in \mathrm{End}(H_0)^{\mathbb{F}},$$

that is, $\mathcal{T}_{\mathrm{w}}$ ($\mathrm{w} \in \mathbb{W}$) is the unique element of $\mathrm{End}(H_0)^{\mathbb{F}}$ such that

$$\tau(\mathrm{w})U = U\mathcal{T}_{\mathrm{w}}.$$

Note that $\mathrm{End}(H_0)^{\mathbb{F}}$ is a $\mathbb{W}$-stable subalgebra of $\mathrm{End}(H_0)^{\mathbb{K}}$ with respect to the action $(\mathrm{w}F)(t,\gamma) = F(\mathrm{w}^{-1}(t,\gamma))$. The following lemma now shows that the $\mathcal{T}_{\mathrm{w}}$ ($\mathrm{w} \in \mathbb{W}$) define a cocycle of $\mathbb{W}$ with values in the group of units of $\mathrm{End}(H_0)^{\mathbb{F}}$.

Lemma 3.2.1. **(i)** $\mathcal{T}_{\mathrm{w}} = \mathrm{id}$ *for* $\mathrm{w} \in \mathbb{Z}^N \times \mathbb{Z}^N$.
(ii) *For* $\mathrm{w}, \mathrm{w}' \in \mathbb{W}$ *we have the cocycle relation*

$$\mathcal{T}_{\mathrm{ww}'} = \mathcal{T}_{\mathrm{w}}\mathrm{w}(\mathcal{T}_{\mathrm{w}'})$$

in $\mathrm{End}(H_0)^{\mathbb{F}}$.

Proof. (i) This follows immediately from the fact that U is an $\mathrm{End}(H_0)$-valued meromorphic solution of BqKZ.
(ii) Note that $\mathcal{T}_{\mathrm{w}} = U^{-1}C_{\mathrm{w}}\mathrm{w}(U)$ for $\mathrm{w} \in \mathbb{W}$. By the cocycle condition for $C_{\mathrm{w}} \in \mathrm{End}(H_0)^{\mathbb{K}}$, which reads in the present notations as $C_{\mathrm{ww}'} = C_{\mathrm{w}}\mathrm{w}(C_{\mathrm{w}'})$ for $\mathrm{w}, \mathrm{w}' \in \mathbb{W}$, we have

$$\begin{aligned}\mathcal{T}_{\mathrm{ww}'} &= U^{-1}C_{\mathrm{ww}'}\mathrm{ww}'(U) = U^{-1}C_{\mathrm{w}}\mathrm{w}(C_{\mathrm{w}'}\mathrm{w}'(U)) \\ &= U^{-1}C_{\mathrm{w}}\mathrm{w}(U)\mathrm{w}(U^{-1}C_{\mathrm{w}'}\mathrm{w}'(U)) = \mathcal{T}_{\mathrm{w}}\mathrm{w}(\mathcal{T}_{\mathrm{w}'})\end{aligned}$$

for all $\mathrm{w}, \mathrm{w}' \in \mathbb{W}$. □

Definition 3.2.2. In analogy with the terminology in [10, §1.3.3] for the quantum KZ equation, we call $\{\mathcal{T}_{\mathrm{w}}\}_{\mathrm{w}\in\mathbb{W}}$ the monodromy cocycle of the BqKZ.

Remark 3.2.3. Connection matrices and Riemann-Hilbert problems for ordinary linear q-difference equations have been studied extensively; see, e.g., [3] and [52]. For quantum KZ equations, connection matrices have been computed explicitly in, e.g., [21], [15, §12], and [31].

3.3 The correspondence

Consider the algebra $\mathbb{C}(T\times T)\#\mathbb{W}$, where $\mathbb{W}$ acts as field automorphisms on $\mathbb{C}(T\times T)$ by the formula (2.3.1). Recall that $\mathbb{C}(T \times T)\#\mathbb{W}$ naturally acts on $\mathbb{K}$. We write Df for the action of $D \in \mathbb{C}(T \times T)\#\mathbb{W}$ on $f \in \mathbb{K}$.

We have a representation $\vartheta\colon \mathbb{C}(T \times T)\#\mathbb{W} \to \mathrm{End}(\mathrm{End}(H_0)^{\mathbb{K}})$ given by

$$\begin{aligned}\vartheta(f)F &= fF, \qquad f \in \mathbb{C}(T \times T), \\ \vartheta(\mathrm{w})F &= \mathrm{w}(F), \qquad \mathrm{w} \in \mathbb{W}\end{aligned}$$

for $F \in \mathrm{End}(H_0)^{\mathbb{K}}$. Let $\mathbb{D}$ be the subalgebra $\mathbb{C}(T \times T)\#(\mathbb{Z}^N \times \mathbb{Z}^N)$ of $\mathbb{C}(T \times T)\#\mathbb{W}$. Under the natural action of $\mathbb{C}(T \times T)\#\mathbb{W}$ on $\mathbb{C}(T \times T)$, the subalgebra $\mathbb{D}$ identifies with the algebra of q-difference operators on $T \times T$ with rational coefficients.

Set $H_0^* := \mathrm{Hom}(H_0, \mathbb{C})$. We will regard a linear functional $\chi \in H_0^*$ also as an element of $\mathrm{Hom}_{\mathbb{K}}(H_0^{\mathbb{K}}, \mathbb{K})$ by $\mathbb{K}$-linear extension. For $F \in \mathrm{End}(H_0)^{\mathbb{K}}$, write

$$\phi^F_{\chi,v} := \chi(Fv) \in \mathbb{K}, \qquad \chi \in H_0^*, v \in H_0$$

for its matrix coefficients. Note that for any $D \in \mathbb{C}(T \times T)\#\mathbb{W}$, $\chi \in H_0^*$ and $v \in H_0$,

$$D\phi_{\chi,v}^{F} = \phi_{\chi,v}^{\vartheta(D)F} \tag{3.3.1}$$

for all $F \in \mathrm{End}(H_0)^{\mathbb{K}}$.

Lemma 3.3.1. *For* $\mathrm{w} \in \mathbb{W}$ *we have*

$$\vartheta(\mathrm{w})U = C_{\mathrm{w}}^{-1}U\mathcal{T}_{\mathrm{w}}.$$

In particular, $\vartheta(\mathrm{w})U = C_{\mathrm{w}}^{-1}U$ *for* $\mathrm{w} \in \mathbb{Z}^N \times \mathbb{Z}^N$.

Proof. For $\mathrm{w} \in \mathbb{W}$

$$\vartheta(\mathrm{w})U = \mathrm{w}(U) = C_{\mathrm{w}}^{-1}(\tau(\mathrm{w})U) = C_{\mathrm{w}}^{-1}U\mathcal{T}_{\mathrm{w}}.$$

The second claim follows from the fact that $\mathcal{T}_{\mathrm{w}} = \mathrm{id}$ for $\mathrm{w} \in \mathbb{Z}^N \times \mathbb{Z}^N$. □

We are now going to look for a particular linear functional χ such that the matrix coefficients $\phi_{\chi,v}^{U}$ ($v \in H_0$) of U solve a bispectral problem with respect to two commuting families of Ruijsenaars' trigonometric q-difference operators (one family acting on the first torus component, the second on the second torus component). In view of (3.3.1) and the previous lemma, to obtain q-difference equations for $\phi_{\chi,v}$ we have to deal with the cocycle value C_{w} and the monodromy matrix $\mathcal{T}_{\mathrm{w}}$ in the equations $\mathrm{w}\phi_{\chi,v}^{U} = \phi_{\chi,v}^{C_{\mathrm{w}}^{-1}U\mathcal{T}_{\mathrm{w}}}$. It is convenient to postpone the analysis of the monodromy cocycle by initially absorbing it into the action ϑ of $\mathbb{C}(T \times T)\#\mathbb{W}$ via the twisted algebra homomorphism

$$\vartheta_{\mathcal{T}} \colon \mathbb{C}(T \times T)\#\mathbb{W} \to \mathrm{End}(\mathrm{End}(H_0)^{\mathbb{K}}),$$

defined by

$$\begin{aligned} \vartheta_{\mathcal{T}}(f)F &= fF, & f &\in \mathbb{C}(T \times T), \\ \vartheta_{\mathcal{T}}(\mathrm{w})F &= \mathrm{w}(F)\mathcal{T}_{\mathrm{w}}^{-1}, & \mathrm{w} &\in \mathbb{W} \end{aligned}$$

for $F \in \mathrm{End}(H_0)^{\mathbb{K}}$. Note that $\vartheta_{\mathcal{T}}$ is indeed an algebra homomorphism, thanks to the cocycle condition for $\mathcal{T}$. Moreover, $\vartheta_{\mathcal{T}}|_{\mathbb{D}} = \vartheta|_{\mathbb{D}}$.

For $D \in \mathbb{C}(T \times T)\#\mathbb{W}$ we will occasionally use the notations

$$D = \sum_{\mathrm{w} \in \mathbb{W}} d_{\mathrm{w}}\mathrm{w} = \sum_{\mathrm{v} \in \mathbb{S}_N} D_{\mathrm{v}}\mathrm{v}, \tag{3.3.2}$$

where $d_{\mathrm{w}} \in \mathbb{C}(T \times T)$ ($\mathrm{w} \in \mathbb{W}$) and $D_{\mathrm{v}} = \sum_{\mathrm{u} \in \mathbb{Z}^N \times \mathbb{Z}^N} d_{\mathrm{uv}}\mathrm{u} \in \mathbb{D}$ ($\mathrm{v} \in \mathbb{S}_N$). Reformulating (3.3.1) and Lemma 3.3.1 in terms of the twisted action $\vartheta_{\mathcal{T}}$ yields the following result.

Lemma 3.3.2. **(i)** *For* $\mathrm{w} \in \mathbb{W}$ *we have*

$$\vartheta_{\mathcal{T}}(\mathrm{w})U = C_{\mathrm{w}}^{-1}U.$$

(ii) *For* $D \in \mathbb{C}(T \times T)\#\mathbb{W}$ *we have*

$$\phi_{\chi,v}^{\vartheta_{\mathcal{T}}(D)U} = \sum_{\mathrm{v}\in\mathbb{S}_N} D_{\mathrm{v}}(\phi_{\chi,v}^{C_{\mathrm{v}}^{-1}U})$$

for all $\chi \in H_0^*$ *and* $v \in H_0$.

Proof. (i) This is clear from Lemma 3.3.1 and the definition of $\vartheta_{\mathcal{T}}$.
(ii) By Lemma 3.3.1 and (3.3.1), we obtain

$$\phi_{\chi,v}^{\vartheta_{\mathcal{T}}(D)U} = \sum_{\mathrm{v}\in\mathbb{S}_N} \phi_{\chi,v}^{\vartheta(D_{\mathrm{v}})\vartheta_{\mathcal{T}}(\mathrm{v})U} = \sum_{\mathrm{v}\in\mathbb{S}_N} D_{\mathrm{v}}(\phi_{\chi,v}^{\vartheta_{\mathcal{T}}(\mathrm{v})U}).$$

The result now follows from (i). □

We define the restriction map Res: $\mathbb{C}(T \times T)\#\mathbb{W} \to \mathbb{D}$ to be the $\mathbb{C}(T \times T)$-linear map

$$\mathrm{Res}(D) := \sum_{\mathrm{v}\in\mathbb{S}_N} D_{\mathrm{v}}, \qquad D \in \mathbb{D}.$$

Lemma 3.3.1(ii) implies that if we have a linear functional $\chi_+ \in H_0^*$ that satisfies $\chi_+(C_{\mathrm{v}}^{-1}U) = \chi_+(U)$ for all $\mathrm{v} \in \mathbb{S}_N$, then the corresponding matrix coefficients $\phi_{\chi_+,v}^{U}$ ($v \in H_0$) satisfy

$$\mathrm{Res}(D)(\phi_{\chi_+,v}^{U}) = \phi_{\chi_+,v}^{\vartheta_{\mathcal{T}}(D)U} \tag{3.3.3}$$

for all $D \in \mathbb{C}(T \times T)\#\mathbb{W}$.

Lemma 3.3.3. *Define* $\chi_+ \in H_0^*$ *by* $\chi_+(T_w) = k^{\ell(w)}$ *for all* $w \in S_N$. *Then*

$$\chi_+(C_{\mathrm{v}}^{-1}F) = \chi_+(F)$$

for $F \in \mathrm{End}(H_0)^{\mathbb{K}}$ *and* $\mathrm{v} \in \mathbb{S}_N$.

Proof. Since $C_\iota(T_w) = T_{w^{-1}}$ for $w \in S_N$, we have $\chi_+ \circ C_\iota = \chi_+$. By the cocycle condition for C_{w} ($\mathrm{w} \in \mathbb{S}_N$) it remains to prove that $\chi_+ \circ C_{(s_i,e)} = \chi_+$ for $1 \le i < N$. But this follows from the expression $C_{(s_i,e)}(t,\gamma) = c_k(t_i/t_{i+1})^{-1}(\eta(T_i) - k) + 1$ (see Lemma 2.4.3), since

$$\chi_+((T_i - k)h) = 0 \tag{3.3.4}$$

for $1 \le i < N$ and $h \in H_0$. □

If $D \in \mathbb{C}(T \times T)\#\mathbb{W}$ satisfies $\vartheta_{\mathcal{T}}(D)U = \lambda U$ for some $\lambda \in \mathbb{K}$, then it follows from (3.3.3) that the matrix coefficients $\phi_{\chi_+,v}^{U}$ ($v \in H_0$) are eigenfunctions of $\mathrm{Res}(D)$ with eigenvalue λ. We will now construct such a commuting family of D's. It leads to the interpretation of the $\phi_{\chi_+,v}$ ($v \in H_0$) as solutions of a bispectral problem.

The appropriate elements $D \in \mathbb{C}(T \times T)\#\mathbb{W}$ are obtained as images of elements from the center $Z(H)$ of the affine Hecke algebra H under the faithful algebra homomorphism ρ from Theorem 2.2.4. Since we aim at a bispectral version, we will interpret ρ as algebra map $\rho : H \to \mathbb{C}(T \times T)\#\mathbb{W}$ in two different ways. We have, on the one hand, the algebra homomorphism

$$\rho^x_{k^{-1},q} \colon H(k^{-1}) \to \mathbb{C}(T \times T)\#\mathbb{W},$$

which is the map $\rho_{k^{-1},q}$ from Theorem 2.2.4, interpreted as algebra homomorphism from $H(k^{-1})$ to the subalgebra $\mathbb{C}(T \times \{1\})\#(W \times \{e\})$ of $\mathbb{C}(T \times T)\#\mathbb{W}$. On the other hand, we have an algebra homomorphism

$$\rho^y_{k,q^{-1}} \colon H(k) \to \mathbb{C}(T \times T)\#\mathbb{W},$$

defined as the map $\rho_{k,q^{-1}}$ from Theorem 2.2.4, interpreted as algebra homomorphism from $H(k)$ to the subalgebra $\mathbb{C}(\{1\} \times T)\#(\{e\} \times W)$ of $\mathbb{C}(T \times T)\#\mathbb{W}$. Note that they can be combined into an algebra homomorphism

$$\rho^x_{k^{-1},q} \times \rho^y_{k,q^{-1}} \colon H(k^{-1}) \otimes H(k) \to \mathbb{C}(T \times T)\#\mathbb{W}.$$

Definition 3.3.4. **(i)** For $h \in H(k^{-1})$, define

$$D^x_h := \rho^x_{k^{-1},q}(h) \in \mathbb{C}(T \times T)\#\mathbb{W}.$$

(ii) For $h \in H(k)$, define

$$D^y_h := \rho^y_{k,q^{-1}}(h) \in \mathbb{C}(T \times T)\#\mathbb{W}.$$

Remark 3.3.5. Let $^\circ \colon H(k^{-1}) \to H(k)$ be the algebra isomorphism defined by $\pi^\circ = \pi$ and $T_i^\circ = T_i^{-1}$ for $1 \leq i < N$. Then

$$D^y_{h^\circ} = \iota D^x_h \iota, \qquad \forall h \in H(k^{-1}). \tag{3.3.5}$$

This follows by verifying the identity

$$\rho^y_{k,q^{-1}}(h^\circ) = \iota \rho^x_{k^{-1},q}(h)\iota$$

for the algebraic generators π and T_i ($1 \leq i < N$) of $H(k^{-1})$ using Theorem 2.2.4.

Recall the generic principal series representation, encoded by the algebra homomorphism $\eta \colon H(k) \to \mathrm{End}(H_0)^{\mathbb{K}}$ (see Subsection 2.4.1).

Proposition 3.3.6. **(i)** *For $h \in H(k^{-1})$ we have*

$$\vartheta_{\mathcal{T}}(D^x_h)U = \eta(h^\dagger)U, \tag{3.3.6}$$

where $\dagger \colon H(k^{-1}) \to H(k)$ is the unique anti-algebra isomorphism satisfying

$$T_i^\dagger = T_i^{-1}, \qquad \pi^\dagger = \pi^{-1}$$

for $1 \le i < N$.
(ii) *For* $h \in H(k)$ *we have*

$$\vartheta_{\mathcal{T}}(D^y_h)U = C_\iota \iota\big(\eta(h^\ddagger)\big)C_\iota U, \tag{3.3.7}$$

where $\ddagger\colon H(k) \to H(k)$ *is the unique anti-algebra involution satisfying*

$$T_i^\ddagger = T_i, \qquad \pi^\ddagger = \pi^{-1}$$

for $1 \le i < N$ *(note that* $\dagger = \ddagger \circ {}^\circ$*).*

Proof. (i) We first show that it suffices to prove (3.3.6) for algebraic generators of $H(k^{-1})$. Indeed, if (3.3.6) is valid for $h, h' \in H(k^{-1})$, then we have

$$\begin{aligned}\vartheta_{\mathcal{T}}(D^x_{hh'})U &= \vartheta_{\mathcal{T}}(D^t_h)\vartheta_{\mathcal{T}}(D^x_{h'})U = \vartheta_{\mathcal{T}}(D^x_h)\eta(h'^\dagger)U\\ &= \eta(h'^\dagger)\vartheta_{\mathcal{T}}(D^x_h)U = \eta(h'^\dagger)\eta(h^\dagger)U\\ &= \eta((hh')^\dagger)U,\end{aligned}$$

where the third equality follows since $[\vartheta_{\mathcal{T}}(D^x_h), \eta(h'^\dagger)] = 0$ as endomorphisms of $\mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ (here $\eta(h'^\dagger)$ should be viewed as element in $\mathrm{End}\big(\mathrm{End}(H_0)^{\mathbb{K}}\big)$ by left multiplication). Indeed, since $\eta(h'^\dagger)(t,\gamma)$ does not depend on the torus parameter $t \in T$, it commutes with $\vartheta_{\mathcal{T}}(D^x_h) \in \vartheta_{\mathcal{T}}(\mathbb{C}(T \times \{1\})\#(W \times \{e\}))$ (which involves, besides the action of $W \times \{e\}$, only right multiplication by the monodromy cocycle).

So it remains to verify (3.3.6) for $h = \pi \in H(k^{-1})$ and for $h = T_i \in H(k^{-1})$ ($1 \le i < N$). For $h = \pi \in H(k^{-1})$ we have

$$\vartheta_{\mathcal{T}}(D^x_\pi)U = \vartheta_{\mathcal{T}}((\pi, e))U = C^{-1}_{(\pi,e)}U = \eta(\pi^\dagger)U,$$

where the last equality follows from (2.4.5). For $h = T_i \in H(k^{-1})$ ($1 \le i < N$), we have

$$\begin{aligned}\vartheta_{\mathcal{T}}(D^x_{T_i})U &= (k^{-1} - c_k(x_{i+1}/x_i))U + c_k(x_{i+1}/x_i)\vartheta_{\mathcal{T}}((s_i, e))U\\ &= (k^{-1} - c_k(x_{i+1}/x_i)U + c_k(x_{i+1}/x_i)C^{-1}_{(s_i,e)}U\\ &= \eta(T_i^\dagger)U,\end{aligned}$$

where we used that $c_{k^{-1}}(z^{-1}) = c_k(z)$ in the first equality, while the second equality follows from Lemma 3.3.2(i) and the third equality from Lemma 2.4.3.
(ii) Unfortunately it is not possible to derive (ii) directly from (i) and from (3.3.5). Instead, one has to repeat the steps of the proof of (i). It again amounts to verifying (3.3.7) for $h = \pi \in H(k)$ and for $h = T_i \in H(k)$ ($1 \le i < N$). We show the second case, the first case is left to the reader.

Let $1 \le i < N$. Then we have for $T_i \in H(k)$,

$$\begin{aligned}\vartheta_{\mathcal{T}}(D^y_{T_i})U &= (k - c_k(y_i/y_{i+1}))U + c_k(y_i/y_{i+1})\vartheta_{\mathcal{T}}((e, s_i))U\\ &= (k - c_k(y_i/y_{i+1}))U + c_k(y_i/y_{i+1})C^{-1}_{(e,s_i)}U.\end{aligned} \tag{3.3.8}$$

Since

$$C_{(e,s_i)} = C_\iota \iota(C_{(s_i,e)})C_\iota$$

by the cocycle condition (recall Remark 2.3.5) and since $C_\iota^2 = \mathrm{id}$ and $C_{(s_i,e)}(t,\gamma)^{-1} = C_{(s_i,e)}(s_i t,\gamma)$, Lemma 2.4.3 implies that

$$C_{(e,s_i)}^{-1} = c_k(y_i/y_{i+1})^{-1}(C_\iota \iota(\eta(T_i))C_\iota - k) + 1.$$

Substituting in (3.3.8) gives $\vartheta_{\mathcal{T}}(D^y_{T_i})U = C_\iota \iota(\eta(T_i^\ddagger))C_\iota U$, as desired. □

The following lemma plays an important role in the bispectral version of the correspondence. Recall that the center $Z(H)$ of the affine Hecke algebra H is given by $\mathbb{C}_Y[T]^{S_N}$ (Bernstein, see [38]).

Lemma 3.3.7. *For $p \in \mathbb{C}[T]^{S_N}$ we have*

$$p(Y)^\dagger = p(Y^{-1}), \qquad p(Y)^\ddagger = p(Y^{-1}).$$

Proof. By (2.2.4) it immediately follows that $Y_i^\dagger = Y_i^{-1}$ for $1 \le i \le N$. This implies the first formula.

For the second formula, it suffices to show that

$$Y_i^\ddagger = T_{w_0} Y_{N-i+1}^{-1} T_{w_0}^{-1} \tag{3.3.9}$$

in $H(k)$ for $1 \le i \le N$, since we then have, for $p \in \mathbb{C}[T]^{S_N}$,

$$p(Y)^\ddagger = T_{w_0} p(Y_N^{-1},\ldots,Y_1^{-1})T_{w_0}^{-1} = T_{w_0} p(Y^{-1})T_{w_0}^{-1} = p(Y^{-1}),$$

where the last equality follows from the fact that $p(Y^{-1}) \in Z(H(k))$. To prove (3.3.9), note that $T_{w_0}T_i^{-1} = T_{N-i}^{-1}T_{w_0}$, and $Y_{i+1} = T_i^{-1}Y_iT_i^{-1}$ by (2.2.4), for $1 \le i < N$. Hence (3.3.9) holds for Y_{i+1} if it is true for Y_i. It thus remains to prove (3.3.9) for $i = 1$. We will use the following observation. Write $\sigma_i = s_i s_{i+1}\cdots s_{N-1}$ ($1 \le i < N$) and $\tau_j = s_j \cdots s_{N-2}$ ($1 \le j < N-1$), which are reduced expressions in S_N. Then the longest Weyl group element $w_0 \in S_N$ can be written as

$$\begin{aligned} w_0 &= \sigma_{N-1}\sigma_{N-2}\cdots\sigma_1 \\ &= \sigma_1(\tau_{N-2}\tau_{N-3}\cdots\tau_1), \end{aligned} \tag{3.3.10}$$

and $\ell(w_0)$ is the sum of the lengths of the factors in the respective products in (3.3.10).

By (2.2.4), formula (3.3.9) for $i = 1$ will be valid if

$$T_{\sigma_1}\pi^{-1} = T_{w_0}\pi^{-1}T_{\sigma_1}T_{w_0}^{-1} \tag{3.3.11}$$

in $H(k)$. By the first expression in (3.3.10) and the fact that $\pi^{-1}T_{i+1}\pi = T_i$ for $1 \le i < N-1$, we have in $H(k)$,

$$\pi^{-1}T_{\sigma_1}T_{w_0}^{-1} = \pi^{-1}T_{\sigma_2}^{-1}T_{\sigma_3}^{-1}\cdots T_{\sigma_{N-1}}^{-1} = T_{\tau_1}^{-1}T_{\tau_2}^{-1}\cdots T_{\tau_{N-2}}^{-1}\pi^{-1},$$

so that (3.3.11) will follow from

$$T_{\sigma_1} = T_{w_0} T_{\tau_1}^{-1} T_{\tau_2}^{-1} \cdots T_{\tau_{N-2}}^{-1}$$

in $H(k)$. But this is a direct consequence of the second expression of w_0 in (3.3.10). □

Corollary 3.3.8. *For $p \in \mathbb{C}[T]^{S_N}$, we have $p(Y)^\circ = p(Y)$, where $^\circ\colon H(k^{-1}) \to H(k)$ is the algebra isomorphism defined in Remark 3.3.5.*

Proof. This follows from the previous lemma and the fact that $\dagger = \ddagger \circ {}^\circ$. □

Definition 3.3.9. **(i)** Define

$$L_p^x := \mathrm{Res}(D^x_{p(Y)}) \in \mathbb{D}, \qquad p \in \mathbb{C}[T]^{S_N},$$

where $p(Y)$ is the corresponding element in $\mathbb{C}_Y[T]^{S_N} = Z(H(k^{-1}))$.
(ii) Define

$$L_p^y := \mathrm{Res}(D^y_{p(Y)}) \in \mathbb{D}, \qquad p \in \mathbb{C}[T]^{S_N},$$

where $p(Y)$ is the corresponding element in $Z(H(k))$.

By Corollary 3.3.8 and (3.3.5) we have

$$L_p^y = \iota L_p^x \iota, \qquad \forall\, p \in \mathbb{C}[T]^{S_N}.$$

Furthermore, it is well-known (see [10] and [42]) that the $L_p^x \in \mathbb{C}(T \times \{1\})\#(\mathbb{Z}^N \times \{e\}) \subset \mathbb{D}$ are pairwise commuting and $S_N \times S_N$-invariant,

$$\mathrm{w} L_p^x \mathrm{w}^{-1} = L_p^x, \qquad \forall\, \mathrm{w} \in S_N \times S_N.$$

Similarly, the $L_p^y = \iota L_p^x \iota \in \mathbb{C}(\{1\} \times T)\#(\{e\} \times \mathbb{Z}^N) \subset \mathbb{D}$ are pairwise commuting and $S_N \times S_N$-invariant. Clearly also $[L_p^x, L_{p'}^y] = 0$ for all $p, p' \in \mathbb{C}[T]^{S_N}$ in $\mathbb{D}$.

For the elementary symmetric functions $e_i \in \mathbb{C}[T]^{S_N}$ $(1 \leq i \leq N)$ given by

$$e_i(t) = \sum_{\substack{I \subseteq \{1,\ldots,N\} \\ \#I = i}} \prod_{j \in I} t_j,$$

the corresponding $L_{e_i}^x$, viewed as elements in

$$\mathbb{C}(T)\#_q \mathbb{Z}^N \simeq \mathbb{C}(T \times \{1\})\#(\mathbb{Z}^N \times \{e\}) \subset \mathbb{D},$$

are explicitly given by

$$L_{e_i}^x = \sum_{\substack{I \subseteq \{1,\ldots,N\} \\ \#I = i}} \left(\prod_{\substack{r \in I \\ s \notin I}} \frac{k x_r - k^{-1} x_s}{x_r - x_s} \right) \sum_{r \in I} \epsilon_r \in \mathbb{C}(T)\#_q \mathbb{Z}^N, \qquad 1 \leq i \leq N; \qquad (3.3.12)$$

see, e.g., [10, §1.3.5] and [32]. Hence, the $L_{e_i}^x$ $(1 \leq i \leq N)$ are, under their natural interpretation as q-difference operators on $\mathcal{M}(T)$, Ruijsenaars' commuting, trigonometric q-difference operators from [50].

Definition 3.3.10. Consider the bispectral problem

$$\begin{aligned}(L^x_p f)(t,\gamma) &= p(\gamma^{-1})f(t,\gamma), \qquad \forall\, p \in \mathbb{C}[T]^{S_N},\\ (L^y_p f)(t,\gamma) &= p(t)f(t,\gamma), \qquad \forall\, p \in \mathbb{C}[T]^{S_N}\end{aligned} \tag{3.3.13}$$

for $f \in \mathbb{K}$, where the equations (3.3.13) are viewed as identities between meromorphic functions in $(t,\gamma) \in T \times T$. We write BiSP $\subset \mathbb{K}$ for the set of solutions $f \in \mathbb{K}$ of (3.3.13).

Remark 3.3.11. The bispectral problem for ordinary linear differential operators was introduced by Duistermaat and Grünbaum in [13]. Many different types of bispectral problems have since been considered. In particular, in [22] the bispectral problem for ordinary linear second-order q-difference operators is investigated. For $N = 2$, our bispectral problem belongs to this class.

The preceding remarks on the invariance properties of the L^x_p and the L^y_p ($p \in \mathbb{C}[T]^{S_N}$) directly give

Lemma 3.3.12. BiSP *is an* $\mathbb{S}_N$*-invariant* $\mathbb{F}$*-subspace of* $\mathbb{K}$ *with respect to the usual* $\mathbb{S}_N$*-action* $(\mathrm{w}f)(t,\gamma) = f(\mathrm{w}^{-1}(t,\gamma))$ *on* $f \in \mathbb{K}$.

We can now prove the following bispectral version of the correspondence between solutions of the quantum KZ equations and the spectral problem of the L^x_p ($p \in \mathbb{C}[T]^{S_N}$).

Theorem 3.3.13. *The linear functional* $\chi_+ \in H_0^*$ *(see Lemma 3.3.3) defines a* $\mathbb{S}_N$*-equivariant* $\mathbb{F}$*-linear map*

$$\chi_+ : \mathrm{SOL} \to \mathrm{BiSP}.$$

Proof. The $\mathbb{K}$-linear extended linear functional χ_+ defines an $\mathbb{S}_N$-equivariant, $\mathbb{F}$-linear map $\chi_+ : H_0^{\mathbb{K}} \to \mathbb{K}$, since Lemma 3.3.3 implies that $\chi_+(\tau(\mathrm{w})f) = \mathrm{w}(\chi_+ f)$ for $\mathrm{w} \in \mathbb{S}_N$ and $f \in H_0^{\mathbb{K}}$. Hence χ_+ restricts to an $\mathbb{S}_N$-equivariant, $\mathbb{F}$-linear map $\chi_+ : \mathrm{SOL} \to \mathbb{K}$.

It remains to show that $\chi_+(f) \in$ BiSP if $f \in$ SOL. Let $f \in$ SOL. By Proposition 2.5.13 and $\mathbb{F}$-linearity, it suffices only to consider f of the form $f = Uv$ for $v \in H_0$. Then $\chi_+(f) = \chi_+(Uv) = \phi^U_{\chi_+,v}$. For $p \in \mathbb{C}[T]^{S_N}$ we have

$$\begin{aligned}\left(L^x_p \phi^U_{\chi_+,v}\right)(t,\gamma) &= \left(\mathrm{Res}(D^x_{p(Y)})(\phi^U_{\chi_+,v})\right)(t,\gamma) = \phi^{\vartheta_\tau(D^x_{p(Y)})U}_{\chi_+,v}(t,\gamma)\\ &= \phi^{\eta(p(Y)^\dagger)U}_{\chi_+,v}(t,\gamma) = p(\gamma^{-1})\phi^U_{\chi_+,v}(t,\gamma)\end{aligned}$$

as meromorphic functions in $(t,\gamma) \in T \times T$, where the last equality follows from Lemma 3.3.7, (2.4.3) and the fact $p \in \mathbb{C}[T]^{S_N}$. Similarly,

$$\begin{aligned}\left(L^y_p \phi^U_{\chi_+,v}\right)(t,\gamma) &= \left(\mathrm{Res}(D^y_{p(Y)})(\phi^U_{\chi_+,v})\right)(t,\gamma) = \phi^{\vartheta_\tau(D^y_{p(Y)})U}_{\chi_+,v}(t,\gamma)\\ &= \phi^{C_\iota \iota(\eta(p(Y)^\ddagger))C_\iota U}_{\chi_+,v}(t,\gamma) = p(t)\phi^U_{\chi_+,v}(t,\gamma)\end{aligned}$$

as meromorphic functions in $(t,\gamma) \in T \times T$, hence $f = \phi^U_{\chi_+,v} \in$ BiSP. □

3.4 Bispectral Harish-Chandra series

Definition 3.4.1. We call $\Phi_\kappa^+ := \chi_+(\Phi_\kappa) \in \mathrm{BiSP}$ the basic Harish-Chandra series solution of the bispectral problem.

Corollary 3.4.2. *The solution* $\Phi_\kappa^+ \in \mathrm{BiSP}$ *of the bispectral problem is selfdual, i.e.,*

$$\Phi_\kappa^+(t,\gamma) = \Phi_\kappa^+(\gamma^{-1}, t^{-1})$$

as meromorphic functions in $(t,\gamma) \in T \times T$.

Proof. By Theorem 2.5.6, we have

$$\Phi_\kappa^+(t,\gamma) = \chi_+(C_\iota \Phi_\kappa(\gamma^{-1}, t^{-1})).$$

But $\chi_+ C_\iota = \chi_+$, hence the result. □

Remark 3.4.3. In [18], for special values of k, the function Φ_κ^+ is constructed as formal power series in terms of generalized characters of Verma modules over the quantum group $\mathcal{U}_q(\mathfrak{sl}_N)$ (see also [16], [17]). The quantum group approach also leads to the self-duality of Φ_κ^+; see [18, Theorem 5.6] (see [17]).

Note that $\Phi_\kappa^+ = W_\kappa \Psi^+$ with $\Psi^+ = \chi_+(\Psi)$. For $\alpha \in Q_+$ set

$$\Gamma_\alpha^+(\gamma) = \chi_+(\Gamma_\alpha(\gamma))$$

as meromorphic function in $\gamma \in T$. By Lemma 2.5.7 and Theorem 2.5.10, Γ_α^+ is analytic at $T \setminus \mathcal{S}_+$ and

$$\Gamma_0^+(\gamma) = k^{\binom{N}{2}} K(\gamma)$$

with K given by (2.5.17). Recall that the solution space BiSP of the bispectral problem is $\mathbb{S}_N$-stable. In particular, we have solutions $\Phi_w^+ \in \mathrm{BiSP}$ given by

$$\Phi_w^+(t,\gamma) := \Phi_\kappa^+(t, w^{-1}\gamma). \tag{3.4.1}$$

These are solutions of the bispectral problem which are asymptotically free in the asymptotic sector $\{t \in T \,|\, |t^{\alpha_i}| \gg 0 \;\forall\, 1 \leq i < N\}$ in the following sense: by Lemma 2.5.7 we have $\Phi_w^+(t,\gamma) = W_\kappa(t, w^{-1}\gamma)\Psi_w^+(t,\gamma)$ with $\Psi_w^+(t,\gamma) := \Psi^+(t, w^{-1}\gamma)$ admitting, for $\epsilon > 0$ sufficiently small, the power series expansion

$$\Psi_w^+(t,\gamma) = \sum_{\alpha \in Q_+} \Gamma_\alpha^+(w^{-1}\gamma) t^{-\alpha} \tag{3.4.2}$$

for $(t,\gamma) \in B_\epsilon^{-1} \times T \setminus w(\mathcal{S}_+)$, converging normally in compacta of $B_\epsilon^{-1} \times T \setminus w(\mathcal{S}_+)$.

Proposition 3.4.4. *The set of asymptotic solutions* $\{\Phi_w^+\}_{w \in S_N} \subset \mathrm{BiSP}$ *of the bispectral problem is* $\mathbb{F}$*-linearly independent.*

Proof. Suppose that

$$\sum_{w\in S_N} a_w(t,\gamma)\Phi_w^+(t,\gamma)=0$$

as meromorphic functions in $(t,\gamma)\in T\times T$ with coefficients $a_w\in\mathbb{F}$ ($w\in S_N$). Replacing t by $q^{-m\delta}t$ ($m\in\mathbb{N}$) and using (2.5.5) we obtain

$$\sum_{w\in S_N} k^{-m\langle\delta,\delta\rangle}\gamma^{-mww_0(\delta)}a_w(t,\gamma)W_\kappa(t,w^{-1}\gamma)\Psi_w^+(q^{-m\delta}t,\gamma)=0 \tag{3.4.3}$$

as meromorphic functions in $(t,\gamma)\in T\times T$. Fix $u\in S_N$. We are going to derive from (3.4.3) that $a_u=0$. For this we will use the fact that for $w\neq u$,

$$\lim_{m\to\infty}\zeta^{m(uw_0(\delta)-ww_0(\delta))}=\lim_{m\to\infty}(w_0u^{-1}\zeta)^{m(\delta-w_0u^{-1}ww_0(\delta))}=0 \tag{3.4.4}$$

if $\zeta\in uw_0(B_1)$.

Recall the W-invariant subset $\mathcal{S}\subset T$ (see (2.4.6)), which contains $\mathcal{S}_+$. For generic $\zeta\in T$ (concretely, $\zeta\notin\mathcal{S}$, and $a_w(t,\gamma)$ and $W_\kappa(t,w^{-1}\gamma)$ specializable at $\gamma=\zeta$ for all $w\in S_N$), it follows from Proposition 2.5.9 and (3.4.3) that, for all $m\in\mathbb{N}$,

$$\sum_{w\in S_N}\zeta^{m(uw_0(\delta)-ww_0(\delta))}a_w(t,\zeta)W_\kappa(t,w^{-1}\zeta)\Psi_w^+(q^{-m\delta}t,\zeta)=0 \tag{3.4.5}$$

as meromorphic function in $t\in T$. Using (3.4.4) and the power series expansion (3.4.2), the limit $m\to\infty$ of (3.4.5) yields, for generic $\zeta\in uw_0(B_1)$,

$$a_u(t,\zeta)W_\kappa(t,u^{-1}\zeta)\Gamma_0^+(u^{-1}\zeta)=0$$

as meromorphic function in $t\in T$. This implies $a_u=0$, as desired. □

Corollary 3.4.5. *The map* $\chi_+:\mathrm{SOL}\to\mathrm{BiSP}$ *is injective.*

Proof. Note that $\chi_+\big(\tau(e,w)\Phi_\kappa\big)=\Phi_w^+$ ($w\in S_N$). The statement follows now directly from Proposition 3.4.4 and Proposition 2.5.13. □

3.5 Specialized central character and Harish-Chandra series

We write

$$\mathrm{SP}_\zeta=\{f\in\mathcal{M}(T)\mid L_p^xf=p(\zeta^{-1})f\quad\forall\,p\in\mathbb{C}[T]^{S_N}\}$$

for the spectral problem of the Ruijsenaars q-difference operators with fixed spectral parameter $\zeta\in T$. Note that $\mathrm{SP}_\zeta\subset H_0^{\mathcal{M}(T)}$ is S_N-stable, with S_N-action on $H_0^{\mathcal{M}(T)}$ given by $(wf)(t)=f(w^{-1}t)$ for $f\in H_0^{\mathcal{M}(T)}$ and $w\in S_N$.

By [14, Proposition 5.2], the quantum KZ equations (2.4.10) are consistent for all values $\zeta\in T$ of the central character. The arguments from Subsection 3.3, applied to the quantum KZ equations (2.4.10) for fixed ζ and with the role of U taken over by an invertible matrix solution U_ζ of (2.4.10), result in the following special case of the Cherednik-Matsuo correspondence from [7, 8] (concretely, in the notations of [8], take the principal series module $V = M_\zeta$ in [8, Theorem 4.2] and let τ be the projection from M_ζ, along the direct sum decomposition of M_ζ in H_0-isotypical components, onto the trivial component).

Proposition 3.5.1. *Let $\zeta \in T$. Then χ_+ defines an $\mathcal{E}(T)$-linear S_N-equivariant map $\chi_+ \colon \mathrm{SOL}_\zeta \to \mathrm{SP}_\zeta$.*

For a further analysis of the map $\chi_+ : \mathrm{SOL}_\zeta \to \mathrm{SP}_\zeta$, we refer to [8] and [10, §1.3.4].

Harish-Chandra type series solutions of the spectral problem of the Ruijsenaars q-difference operators L_p^x ($p \in \mathbb{C}[T]^{S_N}$) with fixed spectral parameter $\zeta \in T$ were studied in, e.g., [16] and [31] (see also [36] for arbitrary root systems). The results of the previous section allow us to reobtain these solutions by specialization of the basic Harish-Chandra series Φ_κ^+. It leads to new results on the convergence and singularities of these solutions, which we state now explicitly.

By Subsection 2.5.4, for generic $\kappa \in \mathbb{C}^\times$ the basic Harish-Chandra series $\Phi_\kappa^+(t,\gamma)$ is specializable at $\gamma = \zeta$ when $\zeta \in T \setminus \mathcal{S}_+$. Concretely, for $\zeta \in T \setminus \mathcal{S}_+$ and generic $\kappa \in \mathbb{C}^\times$, we can write

$$\Phi_\kappa^+(t,\zeta) = W_\kappa(t,\zeta)\Psi^+(t,\zeta)$$

as meromorphic function in $t \in T$, where $\Psi^+ = \chi_+(\Psi)$ (see Section 3.4). Due to the results in Subsection 2.5.4 (see Proposition 2.5.8) we obtain the following result.

Corollary 3.5.2. *For $\zeta \in T \setminus \mathcal{S}_+$, the meromorphic function $\Psi^+(t,\zeta)$ in $t \in T$ is analytic at $t \in T \setminus \mathcal{S}_+^{-1}$.*

Let $\zeta \in T \setminus \mathcal{S}$, where $\mathcal{S} \subset T$ is the W-invariant set (2.4.6). For $\kappa \in \mathbb{C}^\times$ such that $W_\kappa(t, w^{-1}\gamma)$ may be specialized at $\gamma = \zeta$ for all $w \in S_N$, the asymptotic solutions $\Phi_w^+(t,\gamma)$ ($w \in S_N$) of the bispectral problem (see (3.4.1)) may thus be specialized at $\gamma = \zeta$, giving rise to solutions $\Phi_w^+(\cdot;\zeta) \in \mathrm{SP}_\zeta$ ($w \in S_N$); see Corollary 2.5.14 and Proposition 3.5.1. Observe that for $\epsilon > 0$ sufficiently small,

$$\Phi_w^+(t,\zeta) = W_\kappa(t, w^{-1}\zeta) \sum_{\alpha \in Q_+} \Gamma_\alpha^+(w^{-1}\zeta) t^{-\alpha}$$

for $t \in B_\epsilon^{-1}$, with normal convergence of the power series on compacta of B_ϵ^{-1}. Since $\zeta \notin \mathcal{S}$ we furthermore have

$$\Gamma_0^+(w^{-1}\zeta) = k^{\binom{N}{2}} K(w^{-1}\zeta) \neq 0,$$

with K given by (2.5.17).

Definition 3.5.3. Let $\zeta \in T \setminus \mathcal{S}$. The $\Phi_w^+(\cdot;\zeta) \in \mathrm{SP}_\zeta$ ($w \in S_N$) are the Harish-Chandra series solutions of the spectral problem $L_p^x f = p(\zeta^{-1}) f$ ($p \in \mathbb{C}[T]^{S_N}$).

Remark 3.5.4. In [16] (and [36]) the Harish-Chandra series are investigated as formal power series solutions to the spectral problem of the Ruijsenaars operators. The advantage of the present approach is the fact that it implies the convergence of the formal power series, basically as a consequence of a general statement about convergence of formal power series solutions of holonomic systems of q-difference equations (see the appendix). Chalykh's [5] Baker-Akhiezer functions arise as Harish-Chandra series solutions for special values of k; see [36, §4.4]. In [31], the Harish-Chandra series solutions of the Ruijsenaars operators are constructed as matrix coefficients of products of vertex operators. By this approach, one obtains an explicit integral representation of the Harish-Chandra series.

Remark 3.5.5. Observe that

$$\lim_{\substack{\lambda\in\Lambda:\\ \lambda\to\infty}} \frac{\Phi_\kappa^+(t,q^\lambda k^{-\delta})}{W_\kappa(t,q^\lambda k^{-\delta})} = \Gamma_0^+(t^{-1}) = k^{\binom{N}{2}} K(t^{-1}), \tag{3.5.1}$$

with $\lambda\to\infty$ meaning $\lambda_i-\lambda_{i+1}\to\infty$ for all $1\le i<N$. Thus, K (see (2.5.17)) is a normalized limit of the asymptotic solutions $\Phi_\kappa^+(\cdot,q^\lambda k^{-\delta})\in \mathrm{SP}_{q^\lambda k^{-\delta}}$. The solution space $\mathrm{SP}_{q^\lambda k^{-\delta}}$ contains the symmetric Macdonald polynomial of degree $\lambda\in\Lambda$. It turns out though that $\Phi_\kappa^+(\cdot,q^\lambda k^{-\delta})$ is not a multiple of the Macdonald polynomial of degree $\lambda\in\Lambda$, but $\Phi_\kappa^+(\cdot,q^{w_0(\lambda)}k^{\delta})$ is (this will become apparent in the next section). On the other hand, the leading coefficient K (see (2.5.17)) also naturally appears as a normalized limit of the Macdonald polynomial when the degree $\lambda\in\Lambda$ of the polynomial tends to infinity; see [11, Lemma 4.3] (this limit was proven in [51] in the L^2-sense).

Chapter 4

Polynomial solutions of quantum KZ equations and Macdonald polynomials

4.1 Introduction

In view of Proposition 2.4.9, the cocycle matrices corresponding to the dual quantum KZ equations of the bispectral quantum KZ equations can be viewed as shift operators, mapping solutions of qKZ_ζ (for fixed central character $\zeta \in T$; see (2.4.10)) to solutions of $\mathrm{qKZ}_{\zeta'}$ where $\zeta' \in T$ is a certain shift of ζ. In this chapter we use this fact to create Laurent polynomial solutions of quantum KZ equations, starting from a constant solution of the quantum KZ equations. Exploiting the correspondence with the spectral problem of the Ruijsenaars q-difference operators (see Proposition 3.5.1), this leads to a new construction of the symmetric self-dual Macdonald polynomials. From the opposite perspective, we could say that we have found analogs of the symmetric Macdonald polynomials as solutions of quantum KZ equations. Anyway, together with the results from the previous chapters, these observations yield a new approach to the well-known duality and evaluation formulas for the symmetric Macdonald polynomials ([41, Chapter VI]).

Let us give a detailed description of the contents of this chapter. In Section 4.2, we start with constructing a constant solution of the quantum KZ equations. From this constant solution we obtain Laurent polynomial solutions Q_λ (with λ running over $\Lambda \subset \mathbb{Z}^N$; see (2.5.13)) by means of the cocycle matrices of the dual quantum KZ equations.

In Section 4.3, we prove that the Q_λ satisfy a certain duality property between $\lambda \in \Lambda$ and specific points of evaluation. The Laurent polynomial Q_λ can be related to the basic asymptotic solution Φ_κ of BqKZ (see Subsection 2.5.2) and this happens in Section 4.4.

In the final section, we use the correspondence (Proposition 3.5.1) between the quantum KZ equations and the spectral problem of the Ruijsenaars q-difference operators to derive from Q_λ a symmetric self-dual Laurent polynomial eigenfunction of the Ruijsenaars operators, which we prove to be the normalized symmetric Macdonald polynomial of degree λ. We reobtain the well-known duality and evaluation formulas for the Macdonald polynomials as consequences of the properties of the Laurent polynomials Q_λ.

This chapter agrees with Section 7 of [45].

Convention

We adopt the notations from the previous chapters. In particular, we still have fixed $0 < q < 1$. For various reasons, which we address specifically when appropriate, we need to impose some generic conditions on the Hecke algebra parameter k. Concretely, we assume throughout this chapter that $k \in \mathbb{C}^\times$ satisfies

$$\begin{aligned} k^{2j} &\notin q^{\mathbb{Z}}, \qquad \forall\, 1 \leq j \leq N, \\ k^{\langle \delta, \varpi_j - w(\varpi_j)\rangle} &\notin q^{\mathbb{Z}}, \qquad \forall\, 1 \leq j < N, \forall\, w \in S_N : w(\varpi_j) \neq \varpi_j. \end{aligned} \tag{4.1.1}$$

4.2 Constructing polynomial solutions of qKZ

We are going to use a special case of Proposition 2.4.9 to create S_N-invariant (with respect to the S_N-action ς on SOL_ζ; see (2.4.11)) polynomial solutions of the quantum KZ equations.

Lemma 4.2.1. *Let $\lambda \in \Lambda$. The possible poles of the $\mathbb{C}[T]\otimes\mathrm{End}(H_0)$-valued rational function*

$$\gamma \mapsto C_{(e,-\lambda)}(\cdot, q^\lambda \gamma) = C_{(e,\lambda)}(\cdot, \gamma)^{-1}$$

in $\gamma \in T$ are at $\gamma^\alpha \in k^2 q^{-\mathbb{N}}$ for some $\alpha \in R^+$. The possible poles of

$$\gamma \mapsto C_{(e,\lambda)}(\cdot, \gamma)$$

are at $\gamma^\alpha \in k^{-2} q^{-\mathbb{N}}$ for some $\alpha \in R^+$.

Proof. Since $R_i(z)$ has only a (simple) pole at $z = k^{-2}$, this follows from (2.5.14) and the cocycle property of C; see Lemma 2.5.7. □

Since k satisfies $k^{2j} \notin q^{\mathbb{Z}}$ for $1 \leq j \leq N$ by (4.1.1), the spectrum of $\eta_{q^\lambda k^{-\delta}}(\mathbb{C}_Y[T])$ is simple and the $\xi_w(q^\lambda k^{-\delta})$ $(w \in S_N)$ form a $\mathbb{C}$-basis of H_0 for all $\lambda \in \Lambda$. Furthermore, for such k we have that $\gamma \mapsto C_{(e,\lambda)}(\cdot, \gamma)^{\pm 1}$ is regular at $\gamma = k^{-\delta}$ for all $\lambda \in \Lambda$; see Lemma 4.2.1. The additional conditions on k in (4.1.1) will play a role in Section 4.4 and Section 4.5.

Proposition 2.4.9 now immediately implies the following result.

Corollary 4.2.2. *Let $\lambda \in \Lambda$. Then $f \mapsto C_{(e,\lambda)}(\cdot, k^{-\delta})^{-1} f$ defines an S_N-equivariant isomorphism* $\mathrm{SOL}_{k^{-\delta}} \to \mathrm{SOL}_{q^\lambda k^{-\delta}}$.

The special interest in the quantum KZ equations for the particular central characters $\gamma = q^\lambda k^{-\delta}$ ($\lambda \in \Lambda$) comes from the fact that it admits S_N-invariant polynomial solutions. The key step in deriving this result is the following lemma.

Lemma 4.2.3. *The element $v_+ := \sum_{w \in S_N} k^{\ell(w)} T_w \in H_0$ is a constant S_N-invariant solution of the quantum KZ equation with central character $k^{-\delta}$. In other words,*

$$C_\lambda^{k^{-\delta}}(t) v_+ = v_+, \qquad \forall\, \lambda \in \mathbb{Z}^N.$$

Proof. Note that $R_i(z) v_+ = v_+$, so by Lemma 2.4.1 and Lemma 2.4.5(ii), for any $\zeta \in T$,

$$C_{\varpi_i}^{\zeta}(t) v_+ = \eta_\zeta(\pi)^i v_+ = \sum_{w \in S_N} k^{\ell(w)} \zeta^{w^{-1} w_0 \varpi_i} T_{\sigma^i w} = \sum_{w \in S_N} k^{\ell(\sigma^{-i} w)} \zeta^{w^{-1}\varpi_i} T_w.$$

Then use $\ell(\sigma^{-i} w) - \ell(w) = \langle \delta, w^{-1} \varpi_i \rangle$ for $1 \le i \le N$ (for the proof of this formula, it suffices to prove it for $i = 1$. In that case, look at the positive roots that are mapped to negative roots by $\sigma^{-1} w$). It implies that

$$C_{\varpi_i}^{\zeta}(t) v_+ = \sum_{w \in S_N} k^{\ell(w)} (k^\delta \zeta)^{w^{-1}\varpi_i} T_w.$$

In particular, $C_{\varpi_i}^{k^{-\delta}}(t) v_+ = v_+$ for all i. Note, furthermore, that $R_i(z) v_+ = v_+$ implies that $\varsigma(s_i) v_+ = C_{s_i}^{k^{-\delta}}(t) v_+ = v_+$ for all $1 \le i < N$ (with ς given by (2.4.11)). Hence, $v_+ \in \mathrm{SOL}_{k^{-\delta}}$ is S_N-invariant. □

Proposition 4.2.4. *For $\lambda \in \Lambda$, the nonzero S_N-invariant solution*

$$Q_\lambda := C_{(e,\lambda)}(\cdot, k^{-\delta})^{-1} v_+ \in \mathrm{SOL}_{q^\lambda k^{-\delta}}$$

of the quantum KZ equation is an H_0-valued Laurent polynomial on T satisfying

$$Q_\lambda(t) = \sum_{\alpha \in Q_+} K_\alpha(\lambda) t^{\lambda - \alpha}, \tag{4.2.1}$$

with $K_\alpha(\lambda) \in H_0$ (all but finitely many terms zero).

Proof. Note that Corollary 4.2.2 and Lemma 4.2.3 imply that $0 \neq Q_\lambda \in \mathrm{SOL}_{q^\lambda k^{-\delta}}$ and that Q_λ is S_N-invariant. The triangularity property (4.2.1) follows from the cocycle property, (2.5.14), the explicit form of the $R_i(z)$ and the fact that

$$\eta(\pi)(t^{-1})^{-i} T_w = t^{w^{-1}\varpi_i} T_{\sigma^{-i} w}, \qquad w \in S_N,$$

which is a direct consequence of Lemma 2.4.1. □

4.3 Duality

Lemma 4.3.1. *For $\lambda \in \Lambda$, we have $Q_\lambda(k^\delta) = v_+$.*

Proof. Since $v_+ \in \mathrm{SOL}_{k^{-\delta}}$ and $C_\iota v_+ = v_+$, we obtain

$$\begin{aligned} Q_\lambda(k^\delta) &= C_{(e,-\lambda)}(k^\delta, q^\lambda k^{-\delta})v_+ \\ &= C_\iota C_{(-\lambda,e)}(q^{-\lambda}k^\delta, k^{-\delta})C_\iota v_+ = v_+, \end{aligned}$$

for $\lambda \in \Lambda$. □

The polynomial solutions Q_λ of the quantum KZ equations are self-dual in the following sense.

Proposition 4.3.2. *For $\lambda, \mu \in \Lambda$, we have*

$$Q_\lambda(q^{-\mu}k^\delta) = C_\iota Q_\mu(q^{-\lambda}k^\delta).$$

Proof. For $\lambda, \mu \in \Lambda$ we have, using $v_+ \in \mathrm{SOL}_{k^{-\delta}}$ and the previous lemma,

$$\begin{aligned} Q_\lambda(q^{-\mu}k^\delta) &= C_{(e,-\lambda)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta})v_+ \\ &= C_{(e,-\lambda)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta})C_{(-\mu,e)}(q^{-\mu}k^\delta, k^{-\delta})v_+ \\ &= C_{(-\mu,-\lambda)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta})v_+. \end{aligned} \tag{4.3.1}$$

Since $C_{(-\mu,-\lambda)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta}) = C_\iota C_{(-\lambda,-\mu)}(q^{-\lambda}k^\delta, q^\mu k^{-\delta})C_\iota$ and $C_\iota v_+ = v_+$, we conclude from (4.3.1) that $Q_\lambda(q^{-\mu}k^\delta) = C_\iota Q_\mu(q^{-\lambda}k^\delta)$. □

4.4 Relation to the basic asymptotically free solution

In this section, we relate the polynomial solutions Q_λ ($\lambda \in \Lambda$) of the quantum KZ equations to the basic asymptotic solution Φ_κ. Some care is needed though: it is not possible to specialize all the asymptotic solutions $C_{(e,w)}(t,\gamma)\Phi_\kappa(t, w^{-1}\gamma)$ ($w \in S_N$) to $\gamma = q^\lambda k^{-\delta}$ ($\lambda \in \Lambda$) since $q^\lambda k^{-\delta} \in \mathcal{S}$; see Corollary 2.5.14. We shall see that $C_{(e,w_0)}(t,\gamma)\Phi_\kappa(t, w_0\gamma)$ can be specialized at $\gamma = q^\lambda k^{-\delta}$, which is sufficient for our purposes.

Lemma 4.4.1. *Let $\lambda \in \Lambda$. There exists a unique $\Xi_\lambda \in \mathrm{SOL}_{q^\lambda k^{-\delta}}$ such that, for $\epsilon > 0$ sufficiently small, we have an H_0-valued power series expansion*

$$\Xi_\lambda(t) = \sum_{\alpha \in Q^+} \widetilde{\Gamma}_\alpha(\lambda) t^{\lambda - \alpha}$$

converging normally on compacta of B_ϵ^{-1} and with leading coefficient

$$\widetilde{\Gamma}_0(\lambda) = \eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_{w_0}(q^\lambda k^{-\delta}).$$

Proof. Consider the gauged quantum KZ equations for $1 \leq i \leq N$,

$$\widetilde{A}_i(t)\widetilde{\Xi}(q^{-\varpi_i}t) = \widetilde{\Xi}(t), \qquad \widetilde{\Xi} \in H_0^{\mathcal{M}(T)}, \tag{4.4.1}$$

with cocycle matrices $\widetilde{A}_i(t) = q^{-\langle\lambda,\varpi_i\rangle}C_{(\varpi_i,e)}(t, q^\lambda k^{-\delta})$. Note that $\widetilde{A}_N(t) = \mathrm{id}$; see Lemma 2.5.2. Observe that $\widetilde{\Xi}$ is a solution of the holonomic system (4.4.1) of q-difference equations if and only if $x^\lambda\widetilde{\Xi} \in \mathrm{SOL}_{q^\lambda k^{-\delta}}$. By Corollary 2.4.6, we have $\widetilde{A}_i \in Q_0(\mathcal{A}) \otimes \mathrm{End}(H_0)$ and

$$\widetilde{A}_i^{(0)} = q^{-\langle\lambda,\varpi_i\rangle}k^{\langle\delta,\varpi_i\rangle}\eta_{q^\lambda k^{-\delta}}\big(T_{w_0}Y^{w_0(\varpi_i)}T_{w_0}^{-1}\big). \tag{4.4.2}$$

The $\widetilde{A}_i^{(0)}$ $(1 \leq i < N)$ are semisimple endomorphisms of H_0. A basis of simultaneous eigenvectors of H_0 is given by $\eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_w(q^\lambda k^{-\delta})$ $(w \in S_N)$. In fact,

$$\widetilde{A}_i^{(0)}\big(\eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_w(q^\lambda k^{-\delta})\big) = \gamma_{w,i}\eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_w(q^\lambda k^{-\delta})$$

for all $1 \leq i < N$ and $w \in S_N$ with

$$\gamma_{w,i} = q^{\langle\lambda,w^{-1}w_0(\varpi_i)-\varpi_i\rangle}k^{\langle\delta,\varpi_i-w^{-1}w_0(\varpi_i)\rangle};$$

see (2.5.10). Note, in particular, that $\gamma_{w,i} \notin q^{-\mathbb{N}}$ for all $w \in S_N$ and all $1 \leq i < N$ by the generic conditions (4.1.1) on k, and that $\gamma_{w_0,i} = 1$ for all $1 \leq i < N$. Hence, Theorem A.6 in the appendix, applied to (4.4.1) by taking $M = N - 1$, $q_i = q$ and variables $z_i = x^{-\alpha_i}$ $(1 \leq i < N)$ shows that there exists a unique $\widetilde{\Xi} \in \mathcal{M}(T) \otimes H_0$ satisfying (4.4.1) and admitting an H_0-valued power series expansion

$$\widetilde{\Xi}(t) = \sum_{\alpha\in Q^+}\widetilde{\Gamma}_\alpha(\lambda)t^{-\alpha}$$

converging normally on compacta of B_ϵ^{-1} for some $\epsilon > 0$ small enough, and having as leading coefficient $\widetilde{\Gamma}_0(\lambda) = \eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_{w_0}(q^\lambda k^{-\delta})$. This directly implies the lemma. ☐

Recall that the cocycle values $C_{(e,w)}(t,\gamma)$ $(w \in S_N)$ are independent of $t \in T$. We suppress t from the notation and simply write $C_{(e,w)}(\gamma)$. Recall that $C_{(e,w)}(\gamma)$ for $w \in S_N$ is an $\mathrm{End}(H_0)$-valued regular function in $\gamma \in T$.

Theorem 4.4.2. *Fix $\lambda \in \Lambda$. For $\kappa \notin q^{\mathbb{Z}}$, the basic asymptotic solution $\Phi_\kappa(t,\gamma)$ of BqKZ can be specialized at $\gamma = q^{w_0(\lambda)}k^\delta$, giving rise to an H_0-valued meromorphic function $\Phi_\kappa(t, q^{w_0(\lambda)}k^\delta)$ in $t \in T$. Then*

$$Q_\lambda(t) = r_\kappa C_{(e,w_0)}(q^\lambda k^{-\delta})\Phi_\kappa(t, q^{w_0(\lambda)}k^\delta) \tag{4.4.3}$$

with

$$r_\kappa = \theta(\kappa)^N k^{-\binom{N}{2}} \prod_{1\leq i<j\leq N}\frac{\big(k^{2(j-i+1)};q\big)_\infty}{\big(k^{2(j-i)};q\big)_\infty}. \tag{4.4.4}$$

Proof. We first show that both Q_λ and the right-hand side of (4.4.3) are nonzero scalar multiples of Ξ_λ.

We start with the right-hand side of (4.4.3). Since Φ is $\mathbb{S}_N$-stable, we have

$$\Phi_{w_0} := \tau(e, w_0)\Phi_\kappa \in \mathrm{SOL}.$$

Concretely, it is given by

$$\Phi_{w_0}(t,\gamma) = C_{(e,w_0)}(\gamma)\Phi_\kappa(t, w_0(\gamma)) = W_\kappa(t, w_0(\gamma))C_{(e,w_0)}(\gamma)\Psi(t, w_0(\gamma)).$$

Since $w_0(q^\lambda k^{-\delta}) = q^{w_0(\lambda)}k^\delta \notin \mathcal{S}_+$ by (4.1.1), we may, in view of Proposition 2.5.9, specialize $\Phi_{w_0}(t,\gamma)$ at $\gamma = q^\lambda k^{-\delta}$, obtaining $\Phi_{w_0}(\cdot, q^\lambda k^{-\delta}) \in \mathrm{SOL}_{q^\lambda k^{-\delta}}$. By (2.5.3) we have

$$W_\kappa(t, w_0(q^\lambda k^{-\delta})) = k^{\langle\delta,\lambda\rangle}\theta(\kappa)^{-N}t^\lambda, \tag{4.4.5}$$

hence by Proposition 2.5.9,

$$\Phi_{w_0}(t, q^\lambda k^{-\delta}) = k^{\langle\delta,\lambda\rangle}\theta(\kappa)^{-N}\sum_{\alpha\in Q_+}\Gamma_\alpha^{w_0}t^{\lambda-\alpha}$$

with $\Gamma_\alpha^{w_0} = C_{(e,w_0)}(q^\lambda k^{-\delta})\Gamma_\alpha(q^{w_0(\lambda)}k^\delta)$. From the definitions of $C_{(e,w_0)}$, d_w, η and ξ_{w_0} (see Subsections 2.2.3, 2.3.1 and 2.4.1) we have

$$\begin{aligned} C_{(e,w_0)}(\gamma)T_{w_0} &= d_{w_0}(\gamma^{-1})^{-1}\eta(T_{w_0})\xi_{w_0}(\gamma) \\ &= \left(\prod_{\alpha\in R^+}\frac{1}{k - k^{-1}\gamma^\alpha}\right)\eta(T_{w_0})\xi_{w_0}(\gamma) \end{aligned} \tag{4.4.6}$$

as H_0-valued regular functions in $\gamma \in T$. By Theorem 2.5.10, the leading coefficient $\Gamma_0^{w_0}$ thus simplifies to

$$\begin{aligned} \Gamma_0^{w_0} &= K(q^{w_0(\lambda)}k^\delta)C_{(e,w_0)}(q^\lambda k^{-\delta})T_{w_0} \\ &= K(q^{w_0(\lambda)}k^\delta)d_{w_0}(q^{-\lambda}k^\delta)^{-1}\eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_{w_0}(q^\lambda k^{-\delta}), \end{aligned}$$

where K is given by (2.5.17). Combined with the previous lemma, we conclude that

$$\Phi_{w_0}(t, q^\lambda k^{-\delta}) = k^{\langle\delta,\lambda\rangle}\theta(\kappa)^{-N}K(q^{w_0(\lambda)}k^\delta)d_{w_0}(q^{-\lambda}k^\delta)^{-1}\Xi_\lambda(t). \tag{4.4.7}$$

In view of (4.1.1), $\Phi_{w_0}(t, q^\lambda k^{-\delta})$ thus is a nonzero constant multiple of $\Xi_\lambda(t)$.

Next, we consider $0 \neq Q_\lambda \in \mathrm{SOL}_{q^\lambda k^{-\delta}}$. By Lemma 4.4.1 and (4.2.1), it suffices to note that $K_0(\lambda)$ is a constant multiple of $\eta_{q^\lambda k^{-\delta}}(T_{w_0})\xi_{w_0}(q^\lambda k^{-\delta})$, which follows directly from the fact that $K_0(\lambda) \in H_0$ satisfies

$$\widetilde{A}_i^{(0)}K_0(\lambda) = K_0(\lambda), \qquad \forall 1 \leq i \leq N,$$

where $\widetilde{A}_i$ is given by (4.4.2); see the proof Lemma of 4.4.1. Thus, $Q_\lambda(t)$ is a nonzero constant multiple of $\Xi_\lambda(t)$, and we conclude that

$$Q_\lambda(t) = r_\kappa(\lambda)\Phi_{w_0}(t, q^\lambda k^{-\delta}),$$

for some $r_\kappa(\lambda) \in \mathbb{C}^\times$. We first show that $r_\kappa(\lambda)$ is independent of $\lambda \in \Lambda$.

For $w \in S_N$, we write $C_{(w,e)}(t)$ for the γ-independent value $C_{(w,e)}(t,\gamma)$ of the cocycle. Let $\lambda, \mu \in \Lambda$. By the S_N-invariance of Q_λ, we then have, on the one hand,

$$\begin{aligned} Q_\lambda(q^{-\mu}k^\delta) &= C_{(w_0,e)}(q^{-\mu}k^\delta)Q_\lambda(q^{-w_0(\mu)}k^{-\delta}) \\ &= r_\kappa(\lambda)C_{(w_0,e)}(q^{-\mu}k^\delta)C_{(e,w_0)}(q^\lambda k^{-\delta})\Phi_\kappa(q^{-w_0(\mu)}k^{-\delta}, q^{w_0(\lambda)}k^\delta) \\ &= r_\kappa(\lambda)C_{(w_0,w_0)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta})\Phi_\kappa(q^{-w_0(\mu)}k^{-\delta}, q^{w_0(\lambda)}k^\delta). \end{aligned}$$

On the other hand, using the self-duality of Q_λ (see Proposition 4.3.2) and of Φ_κ (see Theorem 2.5.6),

$$\begin{aligned} Q_\lambda(q^{-\mu}k^\delta) &= C_\iota Q_\mu(q^{-\lambda}k^\delta) = C_\iota C_{(w_0,e)}(q^{-\lambda}k^\delta)Q_\mu(q^{-w_0(\lambda)}k^{-\delta}) \\ &= r_\kappa(\mu)C_\iota C_{(w_0,e)}(q^{-\lambda}k^\delta)C_{(e,w_0)}(q^\mu k^{-\delta})\Phi_\kappa(q^{-w_0(\lambda)}k^{-\delta}, q^{w_0(\mu)}k^\delta) \\ &= r_\kappa(\mu)C_\iota C_{(w_0,w_0)}(q^{-\lambda}k^\delta, q^\mu k^{-\delta})\Phi_\kappa(q^{-w_0(\lambda)}k^{-\delta}, q^{w_0(\mu)}k^\delta) \\ &= r_\kappa(\mu)C_{(w_0,w_0)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta})C_\iota\Phi_\kappa(q^{-w_0(\lambda)}k^{-\delta}, q^{w_0(\mu)}k^\delta) \\ &= r_\kappa(\mu)C_{(w_0,w_0)}(q^{-\mu}k^\delta, q^\lambda k^{-\delta})\Phi_\kappa(q^{-w_0(\mu)}k^{-\delta}, q^{w_0(\lambda)}k^\delta). \end{aligned}$$

We conclude that $r_\kappa(\lambda) = r_\kappa(\mu)$ if $Q_\lambda(q^{-\mu}k^\delta) \neq 0$. In particular, since $Q_\lambda(k^\delta) = v_+ \neq 0$, we have $r_\kappa(\lambda) = r_\kappa(0)$ for all $\lambda \in \Lambda$.

It remains to compute $r_\kappa := r_\kappa(0)$. Using the fact that $C_{(e,s_i)}(\gamma) = C_\iota R_i(\gamma^{-\alpha_i})C_\iota$, with $R_i(z) = c_k(z)^{-1}(\eta(T_i^{-1}) - k^{-1}) + 1$ for $1 \leq i < N$, as well as that $C_\iota(T_{w^{-1}}^{-1}T_{w_0}) = T_{w_0w^{-1}}$ for all $w \in S_N$, we get $C_{(e,w_0)}(\gamma)T_{w_0} = \sum_{w \leq w_0} e_w(\gamma)T_{w_0w^{-1}}$ as H_0-valued regular function in $\gamma \in T$ with $e_w \in \mathbb{C}[T]$ and with

$$e_{w_0}(\gamma) = \prod_{\beta \in R^+} c_k(\gamma^{-\beta})^{-1}.$$

Taking the T_e-coefficient in the expansion of the formula

$$v_+ = Q_0 = r_\kappa \Phi_{w_0}(\cdot, k^{-\delta}) = r_\kappa \theta(\kappa)^{-N}K(k^\delta)C_{(e,w_0)}(k^{-\delta})T_{w_0}$$

with respect to the $\mathbb{C}$-basis $\{T_w\}_{w \in S_N}$ of H_0, we conclude that

$$r_\kappa = \theta(\kappa)^N K(k^\delta)^{-1} \prod_{\beta \in R^+} c_k(k^{\langle \delta, \beta \rangle}).$$

Substituting the explicit expressions (2.2.5) and (2.5.17) of c_k and K, respectively, we get the desired formula (4.4.4) for r_κ. □

The following formula is an analog for the Q_λ ($\lambda \in \Lambda$) of the evaluation formula for the self-dual symmetric Macdonald polynomials (see Section 4.5).

Corollary 4.4.3. *Let $\lambda \in \Lambda$ and write $Q_\lambda(t) = \sum_{\alpha\in Q_+} K_\alpha(\lambda) t^{\lambda-\alpha}$ with $K_\alpha(\lambda) \in H_0$ (see Proposition 4.2.4). The leading coefficient $K_0(\lambda)$ is given by*

$$K_0(\lambda) = k^{\langle\delta,\lambda\rangle} \left(\prod_{1\le i<j\le N} \prod_{m=0}^{\lambda_i-\lambda_j-1} \frac{1-q^{-m}k^{2(j-i)}}{1-q^{-m}k^{2(j-i+1)}} \right) k^{-\binom{N}{2}} P(k^2) C_{(e,w_0)}(q^\lambda k^{-\delta}) T_{w_0}$$

with

$$P(k^2) = \prod_{1\le i<j\le N} \frac{1-k^{2(j-i+1)}}{1-k^{2(j-i)}}. \tag{4.4.8}$$

Proof. By (4.4.3) and Theorem 2.5.10, we have for $\lambda \in \Lambda$,

$$K_0(\lambda) = r_\kappa k^{\langle\delta,\lambda\rangle} \theta(\kappa)^{-N} K(q^{w_0(\lambda)} k^\delta) C_{(e,w_0)}(q^\lambda k^{-\delta}) T_{w_0},$$

with K given by (2.5.17) and r_κ given by (4.4.4). Substituting the explicit expressions for K and r_κ we get the desired expression. □

The following consequence should be compared with the general expansion formula of $v_+ = \sum_{w\in S_N} k^{\ell(w)} T_w \in H_0$ in terms of the $\xi_w(\gamma)$ ($w \in S_N$); see [47, Lemma 2.27 (2)].

Corollary 4.4.4. *The element $v_+ = \sum_{w\in S_N} k^{\ell(w)} T_w \in H_0$ can be written as*

$$\begin{aligned} v_+ &= k^{-\binom{N}{2}} P(k^2) C_{(e,w_0)}(k^{-\delta}) T_{w_0} \\ &= \left(\prod_{1\le i<j\le N} \frac{1}{1-k^{2(i-j)}} \right) \eta_{k^{-\delta}}(T_{w_0}) \xi_{w_0}(k^{-\delta}). \end{aligned} \tag{4.4.9}$$

Proof. We have $v_+ = Q_0 = K_0(0)$, hence the previous corollary gives the first equality of (4.4.9). The second equality then follows from (4.4.6). □

Applying the map χ_+ to the first line of (4.4.9) gives

$$\sum_{w\in S_N} k^{2\ell(w)} = P(k^2)$$

with $P(k^2)$ given by (4.4.8), which is a well-known product formula for the Poincaré series of S_N; see [39, Corollary (2.5)].

4.5 Relation to symmetric self-dual Macdonald polynomials

In this section we collect various consequences of the previous sections for the symmetric Laurent polynomials $\chi_+(Q_\lambda) \in \mathbb{C}[T]^{S_N}$ ($\lambda \in \Lambda$). We keep the generic conditions (4.1.1) on $k \in \mathbb{C}^\times$. We define

$$E_\lambda := P(k^2)^{-1} \chi_+(Q_\lambda) \in \mathbb{C}[T]^{S_N}, \qquad \lambda \in \Lambda.$$

By Proposition 4.2.4, we have

$$E_\lambda(t) = \sum_{\alpha \in Q_+} K_\alpha^+(\lambda) t^{\lambda-\alpha},$$

with $K_\alpha^+(\lambda) = P(k^2)^{-1}\chi_+(K_\alpha(\lambda)) \in \mathbb{C}$ all but finitely many zero, and with leading coefficient $K_0^+(\lambda) \neq 0$ by Corollary 4.4.3 and (4.1.1).

Theorem 4.5.1. *The $E_\lambda \in \mathbb{C}[T]^{S_N}$ ($\lambda \in \Lambda$) are the symmetric self-dual Macdonald polynomials. In other words, the E_λ are the unique symmetric regular functions on T satisfying*

$$L_p^x(E_\lambda) = p(q^{-\lambda}k^\delta)E_\lambda \quad \forall\, p \in \mathbb{C}[T]^{S_N} \tag{4.5.1}$$

and $E_\lambda(k^\delta) = 1$ for all $\lambda \in \Lambda$.

Proof. By Proposition 3.5.1, $E_\lambda \in \mathbb{C}[T]^{S_N}$ satisfies (4.5.1). Because the S_N-orbits $S_N(q^{-\lambda}k^\delta)$ ($\lambda \in \Lambda$) in T are pairwise different by (4.1.1), the eigenvalue equations (4.5.1) uniquely characterize $E_\lambda \in \mathbb{C}[T]^{S_N}$ up to a nonzero constant multiple. Now

$$E_\lambda(k^\delta) = P(k^2)^{-1}\chi_+(Q_\lambda(k^\delta)) = 1$$

by Lemma 4.3.1, which fixes the normalization of the solution $E_\lambda \in \mathbb{C}[T]^{S_N}$ of (4.5.1) uniquely. □

The duality property of Q_λ (see Proposition 4.3.2) immediately gives the well-known duality property of the Macdonald polynomials.

Corollary 4.5.2. *The Macdonald polynomials E_λ ($\lambda \in \Lambda$) are self-dual, in the sense that*

$$E_\lambda(q^{-\mu}k^\delta) = E_\mu(q^{-\lambda}k^\delta)$$

for all $\lambda, \mu \in \Lambda$.

Remark 4.5.3. The self-duality of (the suitably normalized) Macdonald polynomials was initially proved by Koornwinder using Pieri formulas in an unpublished manuscript (the argument is reproduced in [41, VI (6.6)]). Cherednik ([10, Theorem 1.4.6] and [9, Theorem 3.2]) reproduced the self-duality of the Macdonald polynomials using the anti-involution $*$ (see Theorem 2.2.8) on the double affine Hecke algebra.

We also immediately reobtain the well-known evaluation formula for the symmetric Macdonald polynomials; see [41, VI (6.11)] (the parameters (n, q, t) in [41, Chapter VI] correspond to (N, q^{-1}, k^2) in our notations).

Corollary 4.5.4. *For $\lambda \in \Lambda$ let $P_\lambda := K_0^+(\lambda)^{-1}E_\lambda \in \mathbb{C}[T]^{S_N}$ be the monic symmetric Macdonald polynomial of degree λ. Then*

$$P_\lambda(k^\delta) = k^{-\langle \delta, \lambda \rangle} \prod_{1 \le i < j \le N} \prod_{m=0}^{\lambda_i - \lambda_j - 1} \frac{1 - q^{-m}k^{2(j-i+1)}}{1 - q^{-m}k^{2(j-i)}}. \tag{4.5.2}$$

Proof. By the previous theorem we have $P_\lambda(k^\delta) = K_0^+(\lambda)^{-1}$. Corollary 4.4.3 gives

$$K_0^+(\lambda) = k^{\langle \delta, \lambda \rangle} \prod_{1 \le i < j \le N} \prod_{m=0}^{\lambda_i - \lambda_j - 1} \frac{1 - q^{-m} k^{2(j-i)}}{1 - q^{-m} k^{2(j-i+1)}},$$

which implies the desired result. □

Chapter 5

Bispectral quantum KZ equations for arbitrary root systems

5.1 Introduction

In this chapter we extend the theory of the bispectral quantum KZ equations and their solutions, as developed in Chapter 2, to arbitrary root systems. Apart from the case of GL_N, there are three cases to consider in the Macdonald-Cherednik theory, namely the twisted and untwisted affine root systems and the nonreduced root system of type $C^\vee C$ (see [42, (1.4.1)-(1.4.3)]). In this chapter we consider the twisted case ([42, (1.4.2)]); the untwisted case is expected to allow for a similar treatment. The construction of BqKZ for $C^\vee C$ (along the lines of [45]) appeared in a recent preprint by Takeyama [56].

Though the general constructions are more or less the same as for GL_N, various technical results require a different approach. This becomes apparent, for example, in Section 5.4 when computing the cocycle values, in Section 5.5 determining the asymptotic behavior of the cocycle matrices and their singularities, and in Section 5.7 finding the leading term of Φ. An important difference with the case of GL_N, complicating some of the proofs, is the fact that the affine Weyl group of type GL_N (and the corresponding affine Hecke algebra) allows a rather convenient presentation in terms of the finite Weyl group (respectively finite Hecke algebra) and an affine Dynkin diagram automorphism (see Section 2.2 or [10, Lemma 1.3.4]), which is lacking for affine Weyl groups (respectively affine Hecke algebras) of arbitrary type. In this paper we give all the main constructions and provide those proofs that are substantially different from the proofs for GL_N.

The chapter is organized as follows. Section 5.2 serves to introduce the notations and to recall Cherednik's construction of the double affine Hecke algebra associated with an irreducible reduced root system. Just as for GL_N, it is the main ingredient in the construction of the bispectral quantum Knizhnik-Zamolodchikov equations, which will be done in Section 5.3. Next, in Section 5.4, we describe the generic principal series representation, needed, in particular, to express the (asymptotic) values of the cocycle matrices $C_{(\lambda,\mu)}(t,\gamma)$.

In Section 5.5, the latter are used to construct an asymptotically free self-dual meromorphic solution Φ of BqKZ. The set of solutions SOL of BqKZ allows an action of W_0, and the orbit $W_0\Phi$ constitutes a basis of SOL viewed as a vector space over the field of q-dilation invariant meromorphic functions on $T \times T$.

The correspondence (Theorem 3.3.13) between solutions of BqKZ of type GL_N and solutions of the bispectral problem (3.3.13) involving Ruijsenaars' commuting trigonometric q-difference operators was derived as a bispectral incarnation of Cherednik's [7, Theorem 4.4] embedding of the solutions of the quantum affine KZ equations (for GL_N) into the solutions of the Ruijsenaars eigenvalue problem. The latter has been generalized to an embedding of the solution space of the quantum affine KZ equations for an arbitrary root system into the solution space of a system of q-difference operators involving the Macdonald q-difference operator. In Section 5.6, we give the analog of the bispectral correspondence Theorem 3.3.13 in the setting of arbitrary root systems.

Finally, in Section 5.7, we apply the correspondence to Φ to obtain a self-dual Harish-Chandra series solution of the bispectral problem (see Subsection 3.4 for GL_N). It is a bispectral analog of (difference) Harish-Chandra series solutions of the spectral problem for Macdonald's q-difference operators, which were studied in [16] and [31] for root systems of type A and in [36] for arbitrary root systems. We will obtain new results on the convergence and singularities of the Harish-Chandra series from the corresponding results for Φ.

The notations we use in this chapter are built up from scratch. In particular, they are independent of the notations used in the previous chapters.

This chapter is a slightly adapted version of [44].

5.2 The double affine Hecke algebra

5.2.1 Root data

Let $(V, \langle\, , \rangle)$ be a real Euclidean space of dimension $N > 0$. Let $\widehat{V}$ be the space of affine linear real functions on V. Consider the 1-dimensional vector space $\mathbb{R}c$. There is a natural isomorphism of real vector spaces $V \oplus \mathbb{R}c \simeq \widehat{V}$ via $v + rc \mapsto (u \mapsto \langle v, u\rangle + r)$ for $u, v \in V$ and $r \in \mathbb{R}$. We will use this isomorphism to identify $\widehat{V}$ and $V \oplus \mathbb{R}c$, thus regarding $c \in \widehat{V}$ as the constant function equal to 1.

The map $D\colon \widehat{V} \to V$ defined by $D(v+rc) = v$ ($v \in \mathbb{R}$, $r \in \mathbb{R}$) is called the gradient map. We extend the inner product $\langle\, , \rangle$ to a positive semi-definite bilinear form on $\widehat{V}$ by

$$\langle f, g\rangle := \langle Df, Dg\rangle,$$

for $f, g \in \widehat{V}$. For $f \in \widehat{V}$ with $Df \neq 0$, we set $f^\vee := 2f/\langle f, f\rangle \in \widehat{V}$.

Let $R \subset V$ be a reduced irreducible finite root system in V and assume that the scalar product is normalized such that long roots have squared length 2. The Weyl group $W_0 \subset O(V)$ associated to R is the group generated by the orthogonal reflections s_α in the hyperplanes $\alpha^\perp$ ($\alpha \in R$). Explicitly, we have

$$s_\alpha(v) = v - \langle v, \alpha\rangle \alpha^\vee,$$

for $\alpha \in R$, $v \in V$. Fix a basis of simple roots $\{\alpha_1, \ldots, \alpha_N\}$ of R. Write R_+ for the set of positive roots, $R_- := -R_+$ for the set of negative roots, and ϕ for the highest root with respect to this basis. Note that $\phi \in R_+$ is a long root (and so $\phi^\vee = \phi$).

We use the standard notations for the (co)root and (co)weight lattices, that is,

$$\begin{aligned} Q &:= \mathbb{Z}\text{-span of } R, \\ Q^\vee &:= \mathbb{Z}\text{-span of } R^\vee, \\ P &:= \{\lambda \in V \mid \langle \lambda, \alpha^\vee\rangle \in \mathbb{Z},\ \forall \alpha \in R\}, \\ P^\vee &:= \{\mu \in V \mid \langle \mu, \alpha\rangle \in \mathbb{Z},\ \forall \alpha \in R\}. \end{aligned}$$

Note that $Q \subseteq P$ and $Q^\vee \subseteq P^\vee$. Furthermore, since $\|\alpha\|^2 = 2$ for $\alpha \in R$ a long root and thus $\|\alpha\|^2 \in \{1, 2/3\}$ for $\alpha \in R$ short, we have $\alpha^\vee = \frac{2}{\|\alpha\|}\alpha \in \{\alpha, 2\alpha, 3\alpha\} \subset Q$ for any $\alpha \in R$. Hence $Q^\vee \subseteq Q$ and therefore also $P^\vee \subseteq P$.

Let $L \subset V$ be any W_0-invariant lattice. The canonical action of W_0 on V extends to a faithful action of the semi-direct product group $W_L := W_0 \ltimes L$ on V such that elements of L act as translations. If we want to stress that we view $\lambda \in L$ as an element of W_L, we write $\mathsf{t}(\lambda)$. In this notation, $L \subset W_L$ acts on V by

$$\mathsf{t}(\lambda)v = v + \lambda,$$

for $\lambda \in L$ and $v \in V$. Transposing the action of W_L on V gives an action of W_L on $\widehat{V}$. It is given by

$$\begin{aligned} w(v + rc) &= w(v) + rc, && w \in W_0, \\ \mathsf{t}(\lambda)(v + rc) &= v + (r - \langle v, \lambda\rangle)c, && \lambda \in L, \end{aligned}$$

for $v \in V$, $r \in \mathbb{R}$. Note that $\langle w(f), w(g)\rangle = \langle f, g\rangle$ for all $f, g \in \widehat{V}$ and $w \in W_L$. In the case that $L = Q^\vee$, $W_L = W_{Q^\vee} = W_0 \ltimes Q^\vee$ is the affine Weyl group. The extended affine Weyl group is $W_{P^\vee} = W_0 \ltimes P^\vee$ and we will simply denote it by W.

Associated to the reduced irreducible finite root system R there is a reduced irreducible affine root system $S = S(R) := \{\alpha + rc \mid \alpha \in R, r \in \mathbb{Z}\}$ in $\widehat{V}$. For $a \in S$, let $s_a \colon V \to V$ be the reflection in the hyperplane $a^{-1}(\{0\})$, given by

$$s_a(v) = v - a(v)Da^\vee,$$

for $v \in V$. Then $s_a = s_{Da}\mathsf{t}(a(0)Da^\vee) \in W_{Q^\vee}$. Note that $S \subset \widehat{V}$ is W-invariant. We define an ordered basis $(a_0, \ldots, a_N)$ of S by setting

$$(a_0, a_1, \ldots, a_N) := (-\phi + c, \alpha_1, \ldots, \alpha_N).$$

Write S_+ and S_- for the associated sets of positive and negative affine roots respectively. Note that

$$S_+ := \{\alpha + rc \mid \alpha \in R, r \geq \chi(\alpha)\},$$

where χ is the characteristic function of R_-, i.e., $\chi(\alpha) = 1$ if $\alpha \in R_-$, and $\chi(\alpha) = 0$ if $\alpha \in R_+$.

We put $s_i := s_{a_i} \in W_{Q^\vee} \subseteq W$ for $i = 0, \ldots, N$. The affine Weyl group $W_{Q^\vee}$ is a Coxeter group with Coxeter generators the simple reflections s_i. For $w \in W$ write $S(w) := S_+ \cap w^{-1}S_-$. The length function ℓ on W is defined by

$$\ell(w) := \#S(w), \qquad w \in W.$$

The unique element with maximal length in W_0 is denoted by w_0.

The finite abelian subgroup $\Omega := \{w \in W \mid \ell(w) = 0\}$ of W is isomorphic to $P^\vee/Q^\vee$ and we have

$$W \simeq W_{Q^\vee} \rtimes \Omega.$$

The action of Ω on $\widehat{V}$ restricts to a faithful action on the set $\{a_0, \ldots, a_N\}$ of simple roots of S, so we can view Ω as a group of permutations on the set of indices $\{0, \ldots, N\}$. We write $\mathbb{C}[\Omega]$ for the group algebra of Ω.

The Bruhat order $\leq$ on $W_{Q^\vee}$ extends to a partial order on W, referred to as the Bruhat order on W (cf. [42, §2.3]). It is defined as follows. For $w = \omega u$ and $w' = \omega' u'$ with $\omega, \omega' \in \Omega$ and $u, u' \in W_{Q^\vee}$ we have by definition

$$w \leq w' \iff \omega = \omega' \text{ and } u \leq u'. \tag{5.2.1}$$

5.2.2 Algebra of q-difference reflection operators

Consider the complex torus $T := \mathrm{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{C}^\times)$. By transposition, the natural action of W_0 on $P^\vee$ gives rise to an action of W_0 on T. Fix $0 < q < 1$. For $\lambda \in P^\vee$, let $q^\lambda \in T$ be defined by

$$q^\lambda(\mu) := q^{\langle \lambda, \mu \rangle}, \qquad \mu \in P^\vee.$$

The action of W_0 on T extends to an action of $W = W_0 \ltimes P^\vee$ on T by letting $\lambda \in P^\vee$ act via $t \mapsto q^\lambda t$. Let the evaluation of $t \in T$ in a point $\lambda \in P^\vee$ be denoted by $t^\lambda \in \mathbb{C}^\times$. Then, summarizing, we have an action of W on T given by

$$(wt)^\mu = t^{w^{-1}\mu},$$
$$(\mathsf{t}(\lambda)t)^\mu = q^{\langle \lambda, \mu \rangle} t^\mu,$$

for $t \in T$, $w \in W_0$ and $\lambda, \mu \in P^\vee$.

Let $\{\varpi_i^\vee\}_{i=1}^N$ be the set of fundamental coweights in $P^\vee$ with respect to $\{\alpha_j\}_{=1}^N$, so $\langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq N$. We identify $T \simeq (\mathbb{C} \setminus \{0\})^N$ via $t \leftrightarrow (t_1, \ldots, t_N)$ defined by

$$t_i := t^{\varpi_i^\vee}$$

for $i = 1, \dots, N$. Under this identification, the action of $P^\vee$ on T reads

$$\mathrm{t}(\lambda)t = q^\lambda t = (q^{\langle \lambda, \varpi_1^\vee \rangle} t_1, \dots, q^{\langle \lambda, \varpi_N^\vee \rangle} t_N) \tag{5.2.2}$$

for $\lambda \in P^\vee$ and $t = (t_1, \dots, t_N) \in T$.

The algebra of complex-valued regular functions on T is

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_N^{\pm 1}] = \mathrm{span}_{\mathbb{C}}\{x^\lambda\}_{\lambda \in P^\vee},$$

where x_i is the coordinate function $x_i(t) := t^{\varpi_i^\vee}$ $(i = 1, \dots, N)$ and $x^\lambda(t) := t^\lambda$ for $\lambda \in P^\vee$. Clearly, it is isomorphic to the group algebra $\mathbb{C}[P^\vee]$ of $P^\vee$. We write $\mathbb{C}[T] = \mathbb{C}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$ and we let $\mathbb{C}(T)$ denote the field of rational functions on T, $\mathcal{O}(T)$ the ring of analytic functions on T, and $\mathcal{M}(T)$ the field of meromorphic functions on T. The W-action on T gives rise to a W-action by algebra automorphisms on each of these function algebras, via

$$(wf)(t) = f(w^{-1}t),$$

for $w \in W$, $t \in T$ and f a (regular, rational or meromorphic) function on T. Note that for $\lambda \in P^\vee$ and $r \in \mathbb{R}$, we have

$$w(x^{\lambda + rc}) = x^{w(\lambda + rc)},$$

where $x^{\lambda + rc} := q^r x^\lambda \in \mathbb{C}[T]$.

By means of this W-action by field automorphisms on $\mathbb{C}(T)$, we can form the smash product algebra $\mathbb{C}(T)\#_q W$ (see Subsection 2.2.1), which we call the algebra of q-difference reflection operators with coefficients in $\mathbb{C}(T)$, since it acts canonically on $\mathbb{C}(T)$ and $\mathcal{M}(T)$ as q-difference reflection operators. For $f \in \mathbb{C}(T)$ we will write $f(X) \in \mathbb{C}(T)\#_q W$ for the operator on $\mathcal{M}(T)$ (or $\mathbb{C}(T)$) defined as multiplication by f. We will also write $X^{\lambda + rc} = q^r X^\lambda$ for $\lambda \in P^\vee$ and $r \in \mathbb{R}$.

Remark 5.2.1. Note that since $(\mathrm{t}(\lambda)f)(t) = f(q^{-\langle \varpi_1^\vee, \lambda \rangle} t_1, \dots, q^{-\langle \varpi_N^\vee, \lambda \rangle} t_N)$ $(\lambda \in P^\vee$, $f \in \mathcal{M}(T))$, $\mathbb{C}(T)\#_q W$ actually depends on a choice for $q^{\frac{1}{m}}$, where $m \in \mathbb{N}$ is determined by $m\langle P^\vee, P^\vee \rangle = \mathbb{Z}$. Our global convention concerning real powers of positive real numbers justifies the apparent abuse of notation writing q instead of $q^{1/m}$.

5.2.3 The extended affine Hecke algebra and Cherednik's basic representation

Let k_i $(i = 0, \dots, N)$ be nonzero complex numbers such that $k_i = k_j$ if s_i and s_j are conjugate in W. Write $\underline{k}$ for the corresponding multiplicity label $\underline{k}\colon S \to \mathbb{C} \setminus \{0\}$, so $\underline{k}(a) = k_i$ for all $a \in W(a_i)$ $(i = 0, \dots, N)$. We set $k_a := \underline{k}(a)$ for $a \in S$. Furthermore, for $w \in W$ we define

$$k(w) := \prod_{a \in S(w)} k_a.$$

A coweight $\lambda \in P^\vee$ is called dominant if $\langle \lambda, \alpha_i \rangle \geq 0$ for $i = 1, \dots, N$. Let $P_+^\vee$ denote the set of dominant coweights.

Lemma 5.2.2. *For $\lambda \in P_+^\vee$, we have*

$$k(\mathsf{t}(\lambda)) = \prod_{\alpha \in R_+} k_\alpha^{\langle \lambda, \alpha \rangle} = \delta_{\underline{k}}^{\lambda}, \tag{5.2.3}$$

where $\delta_{\underline{k}} \in T$ is defined by $(\delta_{\underline{k}})_i = \prod_{\alpha \in R_+} k_\alpha^{\langle \varpi_i^\vee, \alpha \rangle}$ ($i = 1, \ldots, N$).

Proof. For $\lambda \in P_+^\vee$ we have

$$S(\mathsf{t}(\lambda)) = \{\alpha + rc \mid \alpha \in R_+,\ 0 \leq r < \langle \lambda, \alpha \rangle\},$$

cf. [42, §2.4]. Note that $k_{\alpha+rc} = k_\alpha$ for $\alpha \in R$ and $r \in \mathbb{Z}$ since $\alpha + rc$ and α are conjugate under the action of W. Indeed, for $\mu \in P^\vee$ we have $\mathsf{t}(\mu)(\alpha + rc) = \alpha + (r - \langle \mu, \alpha \rangle)c$ and for any $\alpha \in R$ there exists some $\nu \in P^\vee$ such that $\langle \nu, \alpha \rangle = 1$, so that we can take $\mu = r\nu$. Therefore

$$k(\mathsf{t}(\lambda)) = \prod_{\substack{\alpha \in R_+ \\ 0 \leq r < \langle \lambda, \alpha \rangle}} k_{\alpha + rc} = \prod_{\alpha \in R_+} k_\alpha^{\langle \lambda, \alpha \rangle}.$$

The second equality in (5.2.3) follows from the definitions. □

Definition 5.2.3. The affine Hecke algebra $H_{Q^\vee}$ associated with the Coxeter system $(W_{Q^\vee}, \{s_0, \ldots, s_N\})$ and the multiplicity label $\underline{k}$, is the unital complex associative algebra generated by elements $T_0, \ldots, T_N$, such that

(i) $T_0, \ldots, T_N$ satisfy the braid relations, i.e. if for $i \neq j$, we have

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots,$$

with m_{ij} factors on each side, then

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots,$$

with m_{ij} factors on each side;

(ii) $(T_j - k_j)(T_j + k_j^{-1}) = 0$, for $j = 0, \ldots, N$.

Note that since $\underline{k}$ is W-invariant, the group Ω acts on $H_{Q^\vee}$ by algebra automorphisms via $T_i \mapsto T_{\omega(i)}$ for $i = 0, \ldots, N$.

Definition 5.2.4. The extended affine Hecke algebra $H = H(\underline{k})$ is the smash product $H := H_{Q^\vee} \# \Omega$.

For $w \in W$ and a reduced expression $w = \omega s_{i_1} \cdots s_{i_{\ell(w)}}$ with $\omega \in \Omega$ and $i_k \in \{0, \ldots, N\}$, we define

$$T_w := \omega T_{i_1} \cdots T_{i_{\ell(w)}} \in H,$$

which is independent of the reduced expression chosen. The set $\{T_w \mid w \in W\}$ is a linear basis of H. Note that for $\underline{k} \equiv 1$ the extended affine Hecke algebra is just the group algebra $\mathbb{C}[W]$ of W. The finite Hecke algebra is the subalgebra $H_0 = H_0(\underline{k})$ of H, generated by $T_1, \ldots, T_N$.

For $\lambda \in P_+^\vee$, put

$$Y^\lambda := T_{\mathrm{t}(\lambda)} \in H,$$

and for arbitrary $\lambda \in P^\vee$ put

$$Y^\lambda := Y^\mu (Y^\nu)^{-1},$$

if $\lambda = \mu - \nu$ with $\mu, \nu \in P_+^\vee$. Then the Y^λ ($\lambda \in P^\vee$) are well-defined and we have $Y^0 = 1$ and $Y^\lambda Y^\mu = Y^{\lambda+\mu} = Y^\mu Y^\lambda$ for all $\lambda, \mu \in P^\vee$. Set $Y_i := Y^{\varpi_i^\vee}$ for $i = 1, \ldots, N$.

For $\kappa \in \mathbb{C} \setminus \{0\}$ we define the functions $b(z, \kappa)$ and $c(z, \kappa)$ by

$$b(z;\kappa) := \frac{\kappa - \kappa^{-1}}{1 - z},$$
$$c(z;\kappa) := \frac{\kappa^{-1} - \kappa z}{1 - z},$$

as rational functions in z. Then for $a \in S$, we define $b_{a;\underline{k},q} = b_a \in \mathbb{C}(T)$ and $c_{a;\underline{k},q} = c_a \in \mathbb{C}(T)$ by

$$b_a(t) := b(t^{a^\vee}; k_a)$$
$$c_a(t) := c(t^{a^\vee}; k_a).$$

Remark 5.2.5. The q-dependence of $b_{a,\underline{k},q}$ and $c_{a;\underline{k},q}$ comes from our convention $t^{\alpha+rc} = q^r t^\alpha$ for $\alpha \in R$ and $r \in \mathbb{R}$. Note that

$$c_{a;\underline{k},q}(t^{-1}) = c_{a;\underline{k}^{-1},q^{-1}}(t) \tag{5.2.4}$$

for all $a \in S$ and $t \in T$. We leave out the subscripts $\underline{k}$ and q as long as there is no chance of confusion (which is until Section 5.6).

Note that $b_a(t) = k_a - c_a(t)$ and $(wc_a)(t) = c_{w(a)}(t)$ for all $w \in W$. It is convenient to introduce the notations $b_j := b_{a_j}$ and $c_j := c_{a_j}$ for $j = 0, \ldots, N$. The following characterization of H is due to Bernstein and Zelevinsky (see, e.g., [42, §4.2]).

Theorem 5.2.6. *The affine Hecke algebra $H = H(\underline{k})$ is the unique complex associative algebra, such that*
(i) *$H_0 \otimes \mathbb{C}[T] \simeq H$ as complex vector spaces, via $h \otimes f \mapsto hf(Y)$ for $h \in H_0$, $f \in \mathbb{C}[T]$, where $f(Y) = \sum_\lambda a_\lambda Y^\lambda$ if $f = \sum_\lambda a_\lambda x^\lambda \in \mathbb{C}[T]$;*
(ii) *the canonical maps $H_0, \mathbb{C}[T] \hookrightarrow H$ are algebra embeddings; we will also write $\mathbb{C}_Y[T] = \mathrm{span}_{\mathbb{C}}\{Y^\lambda\}_{\lambda \in P^\vee}$ for the image of $\mathbb{C}[T]$ in H;*
(iii) *Lusztig's relations are satisfied, that is,*

$$f(Y)T_j = T_j(s_j f)(Y) + b_j(Y^{-1})\left(f(Y) - (s_j f)(Y)\right) \tag{5.2.5}$$

for $j = 1, \ldots, N$ and $f \in \mathbb{C}[T]$.

Remark 5.2.7. Note that $b_j(Y^{-1})\,(f(Y)-(s_jf)(Y)) \in \mathbb{C}_Y[T]$ although $b_j(Y^{-1})$ by itself is not defined as an element of H.

We end this section with the definition of the double affine Hecke algebra and state some of its key results. All of this is due to Cherednik; see [10]. It starts with the realization of the affine Hecke algebra inside the algebra $\mathbb{C}(T)\#_qW$ of q-difference reflection operators.

Theorem 5.2.8. *There exists a unique injective algebra homomorphism* $\rho = \rho_{\underline{k},q}\colon H \to \mathbb{C}(T)\#_qW$ *satisfying*

$$\begin{aligned}\rho(T_i) &= k_i + c_i(X)(s_i-1), \qquad & i &= 0,\ldots,N,\\ \rho(\omega) &= \omega, & \omega &\in \Omega.\end{aligned}$$

Remark 5.2.9. The image $\rho(H)$ preserves $\mathbb{C}[T]$, viewed as a subspace of the canonical $\mathbb{C}(T)\#_qW$-module $\mathbb{C}(T)$. The resulting faithful representation of H on $\mathbb{C}[T]$ is called the basic representation of H.

Definition 5.2.10. The double affine Hecke algebra $\mathbb{H} = \mathbb{H}(\underline{k}, q)$ is the subalgebra of $\mathbb{C}(T)\#_qW$ generated by H (i.e. by $\rho_{\underline{k},q}(H)$) and by the multiplication operators $f(X)$ ($f \in \mathbb{C}[T]$).

Remark 5.2.11. Note that $\rho = \rho_{\underline{k},q}$ and $\mathbb{H} = \mathbb{H}(\underline{k}, q)$ actually depend on $q^{\frac{1}{m}}$ (see Remark 5.2.1).

We view $\mathbb{H}$ as a left $\mathbb{C}[T]$-module by $(f,h) \mapsto f(X)h$ ($f \in \mathbb{C}[T]$, $h \in \mathbb{H}$). The rule $f \otimes h \mapsto f(X)h$ ($h \in H$, $f \in \mathbb{C}[T]$) induces an isomorphism of $\mathbb{C}[T]$-modules

$$\mathbb{C}[T] \otimes H \simeq \mathbb{H}, \tag{5.2.6}$$

Similarly to Theorem 5.2.6, the algebra structure of $\mathbb{H}$ can be described in terms of the left-hand side of (5.2.6), allowing for an abstract definition of $\mathbb{H}$:

Theorem 5.2.12. *The double affine Hecke algebra* $\mathbb{H}$ *can be characterized as the unique associative algebra satisfying*
(i) $\mathbb{C}[T] \otimes H \simeq \mathbb{H}$ *as complex vector spaces;*
(ii) *the canonical maps* $H, \mathbb{C}[T] \hookrightarrow \mathbb{H}$ *are algebra embeddings;*
(iii) *the following cross relations are satisfied: for* $f \in \mathbb{C}[T]$

$$T_jf(X) = (s_jf)(X)T_j + b_j(X)\,(f(X)-(s_jf)(X))\,, \quad j = 0,\ldots,N, \tag{5.2.7}$$

$$\omega f(X) = (\omega f)(X)\omega, \quad \omega \in \Omega. \tag{5.2.8}$$

A crucial ingredient in the construction of the bispectral quantum KZ equations is Cherednik's duality anti-involution on $\mathbb{H}$ (see [10, Theorem 1.4.8]).

Theorem 5.2.13. *There exists a unique anti-algebra involution* $*\colon \mathbb{H} \to \mathbb{H}$ *determined by*

$$\begin{aligned} T_w^* &= T_{w^{-1}}, & w &\in W_0, \\ (Y^\lambda)^* &= X^{-\lambda}, & \lambda &\in P^\vee, \\ (X^\lambda)^* &= Y^{-\lambda}, & \lambda &\in P^\vee. \end{aligned}$$

5.3 Bispectral quantum KZ equations

In this section we extend the construction of the bispectral quantum KZ equations (2.3.7) for GL_N to arbitrary root systems. First we recall Cherednik's construction of the quantum affine Knizhnik-Zamolodchikov equation [6] associated to a finite-dimensional H-module. Since this takes place in terms of a certain extension (localization) of the double affine Hecke algebra, we start with a short intermezzo on Ore localization.

5.3.1 Ore localization

Let R be a unital ring and S a subset of R. A ring homomorphism $R \to R'$ is called S-inverting if it maps S into the group of units of R'. For any R and S as above there exists a (up to isomorphism) unique ring ${}_SR$ and S-inverting homomorphism $\lambda\colon R \to {}_SR$ satisfying the following universal property: for any S-inverting homomorphism $f\colon R \to R'$ there is a unique homomorphism $g\colon {}_SR \to R'$ such that $f = g \circ \lambda$. The ring ${}_SR$, which is called the universal S-inverting ring for the pair (R, S), is constructed from a presentation of R by adjoining an element s' for each element $s \in S$ with defining relation $ss' = s's = 1$ and then λ is defined by sending each $r \in R$ to the corresponding element in ${}_SR$.

Now suppose that S is a multiplicative subset of R, that is, S satisfies $S \cdot S \subset R$, $1 \in S$ and $0 \notin S$. One could ask under what circumstances the universal S-inverting ring ${}_SR$ is a (left) ring of fractions. By definition, this means that, besides being S-inverting, each of its elements can be written as a fraction $s^{-1}a$ ($s \in S$, $a \in R$) and that $\ker\lambda = \{a \in R \mid sa = 0 \text{ for some } s \in S\}$, so that if in particular S does not possess any zero divisors, λ is a ring embedding $\lambda\colon R \hookrightarrow {}_SR$. The following result, due to Ore, provides the answer.

Theorem 5.3.1. *Let* R *be a unital ring and* S *a multiplicative subset satisfying*

(i) *for all* $a \in R$ *and* $s \in S$ *we have* $Sa \cap Rs \neq \emptyset$*;*

(ii) *for* $a \in R$ *and* $s \in S$*, if* $as = 0$ *then* $ta = 0$ *for some* $t \in S$*.*

Then the relation $\sim$ *on* $S \times R$*, defined by* $(s, a) \sim (s', a')$ *if and only if there exist* $x, x' \in R$ *such that* $xs = x's' \in S$ *and* $xa = x'a'$*, defines an equivalence relation on* $S \times R$*. Writing* $s^{-1}a$ *for the equivalence class of* $(s, a) \in (S \times R)/{\sim}$*, we define addition on* $(S \times R)/{\sim}$ *by*

$$s_1^{-1}a_1 + s_2^{-1}a_2 = (ss_1)^{-1}(sa_1 + ra_2),$$

where, using (i), $r \in R$ *and* $s \in S$ *are chosen such that* $rs_2 = ss_1$*. Multiplication is defined by*

$$s_1^{-1}a_1 \cdot s_2^{-1}a_2 = (ss_1)^{-1}ra_2,$$

where, again using (i), $r \in R$ *and* $s \in S$ *are chosen such that* $sa_1 = rs_2$*.*
This defines a ring structure on $(S \times R)/\!\sim$ *and we have*

$$_SR \simeq (S \times R)/\!\sim .$$

In particular, $_SR$ *a left ring of fractions.*

We call a multiplicative subset S satisfying conditions (i) and (ii) of the theorem a left Ore set and $(S \times R)/\!\sim$ the left Ore localization of R with respect to S. Of course there is also a notion of right Ore localization and it is clear what the appropriate definitions should be. For more details we refer to [34, §10] (which is in fact concerned with right Ore localizations).

5.3.2 The quantum affine KZ equations

Consider the multiplicative subset $\mathbb{C}[T] \setminus \{0\}$ of the double affine Hecke algebra $\mathbb{H}$ (cf. (5.2.6)). It is a left Ore set in $\mathbb{H}$. Indeed, consider $a = g(X)\omega T_w \in \mathbb{H}$ ($g \in \mathbb{C}[T]$, $\omega \in \Omega$, $w \in W_{Q^\vee}$) and $s = f(X) \in \mathbb{C}[T] \setminus \{0\}$. We will use induction on $\ell(w)$ to verify condition (i) of Theorem 5.3.1. Suppose (i) is true for $u \in W_{Q^\vee}$ with $\ell(u) < \ell(w)$. The cross relations (5.2.7) imply that

$$T_w f(X) = (wf)(X)T_w + \sum_{u<w} f_u(X)T_u \tag{5.3.1}$$

for certain $f_u \in \mathbb{C}[T]$. By the induction hypothesis, for a given $u < w$, we can find $h_u \in \mathbb{H}$ and $p_u \in \mathbb{C}[T] \setminus \{0\}$ such that $p_u(X)f_u(X)T_u = h_u f(X)$. So by multiplying (5.3.1) from the left by $p_u(X)$, and repeating this for each $u < w$, we arrange that

$$p(X)T_w f(X) = p(X)(wf)(X)T_w + hf(X)$$

for certain $p \in \mathbb{C}[T] \setminus \{0\}$ and $h \in \mathbb{H}$. It follows that

$$g(X)\omega p(X)(wf)(X)T_w = g(X)\omega\big(p(X)T_w f(X) - hf(X)\big),$$

and hence $(\omega p)(X)(\omega wf)(X)a = g(X)\omega(p(X)T_w - h)s$. Condition (ii) is satisfied as well, for, in view of (5.2.6), the elements of $\mathbb{C}[T] \setminus \{0\}$ are clearly no zero divisors.

Let $\widehat{\mathbb{H}}$ denote the left Ore localization of $\mathbb{H}$ with respect to $\mathbb{C}[T] \setminus \{0\}$. We can view $\mathbb{C}(T)$ and H as subalgebras of $\widehat{\mathbb{H}}$ and then $\widehat{\mathbb{H}} \simeq \mathbb{C}(T) \otimes H$ as complex vector spaces, where the isomorphism is given by the multiplication map, and the algebra structure is characterized by the cross relations (5.2.7) and (5.2.8) for $f \in \mathbb{C}(T)$.

The injective map ρ of Theorem 5.2.8 extends to an injective algebra homomorphism

$$\rho\colon \widehat{\mathbb{H}} \to \mathbb{C}(T)\#_q W$$

by setting $\rho(f(X)) = f(X)$ for $f \in \mathbb{C}(T)$. Note that $\rho\big(c_j(X)^{-1}(T_j - b_j(X))\big) = s_j$ for $0 \leq j \leq N$, which shows that ρ is surjective and therefore establishes an isomorphism $\widehat{\mathbb{H}} \simeq \mathbb{C}(T)\#_q W$. Restricting the inverse ρ^{-1} to W gives a realization of W inside $\widehat{\mathbb{H}}^{\times}$.

The left multiplication map turns H into a left module over itself. The action of $\widehat{\mathbb{H}}$ on the induced module $\mathrm{Ind}_H^{\widehat{\mathbb{H}}}(H) = \widehat{\mathbb{H}} \otimes_H H$ can be pushed forward along the linear isomorphism $\widehat{\mathbb{H}} \otimes_H H \simeq \mathbb{C}(T) \otimes H$ to obtain an algebra homomorphism

$$\pi = \pi_{\underline{k}} \colon \widehat{\mathbb{H}} \to \mathrm{End}(\mathbb{C}(T) \otimes H).$$

We regard $\mathbb{C}(T)\#_q W \otimes H$ as a subalgebra of $\mathrm{End}(\mathbb{C}(T) \otimes H)$ by letting $\mathbb{C}(T)\#_q W$ act on $\mathbb{C}(T)$ as in Subsection 5.2.2, and H on H by left multiplication. Then the pullback $\tau_x = \tau_{x,\underline{k}} := \pi \circ \rho^{-1}$ of π along ρ^{-1} is an algebra homomorphism

$$\tau_x \colon \mathbb{C}(T)\#_q W \to \mathbb{C}(T)\#_q W \otimes H \subset \mathrm{End}(\mathbb{C}(T) \otimes H),$$

which is explicitly given by

$$\begin{aligned} \tau_x(f) &= f(X) \otimes 1, && f \in \mathbb{C}(T), \\ \tau_x(s_j) &= (c_j(X)^{-1} \otimes 1)(s_j \otimes T_j - b_j(X)s_j \otimes 1), && 0 \leq j \leq N, \\ \tau_x(\omega) &= \omega \otimes \omega, && \omega \in \Omega, \end{aligned}$$

as can be verified by a direct computation using the formula for ρ^{-1} and the cross relations (5.2.7).

Remark 5.3.2. The reason for the subscript x in τ_x will become apparent in the next subsection when we discuss the bispectral story. Then two copies of T will play a role and x will denote the set of coordinate functions on one of them.

Note that $\tau_x(s_j) = (F_{s_j}(X) \otimes 1)(s_j \otimes 1)$ with $F_{s_j}(X) = c_j(X)^{-1}(1 \otimes T_j - b_j(X) \otimes 1) \in \mathbb{C}(T) \otimes H$, and trivially also $\tau_x(\omega) = F_\omega(X)(\omega \otimes 1)$ with $F_\omega = 1 \otimes \omega \in \mathbb{C}(T) \otimes H$. In fact, more generally, we have

$$\tau_x(w) = F_w(X)(w \otimes 1), \qquad w \in W,$$

where F_w are H-valued rational functions on T satisfying

$$F_e(t) = 1, \quad F_{vw}(t) = F_v(t)F_w(v^{-1}t) \tag{5.3.2}$$

for all $v, w \in W$ and $t \in T$. Viewed as elements of $\mathrm{End}(\mathbb{C}(T) \otimes H)$ the $F_w(X)$ ($w \in W$) are $\mathbb{C}(T)$-linear and invertible (indeed $F_w^{-1}(X) = (w^{-1} \otimes 1)\tau_x(w^{-1})$). In the language of non-abelian group cohomology, (5.3.2) asserts that $w \mapsto F_w(X)$ constitutes a cocycle $W \to \mathrm{GL}_{\mathbb{C}(T)}(\mathbb{C}(T) \otimes H)$, where $\mathrm{GL}_{\mathbb{C}(T)}(\mathbb{C}(T) \otimes H)$ is a W-group via the usual action of W on the first tensor leg of $\mathbb{C}(T) \otimes \mathrm{End}(H) \simeq \mathrm{GL}_{\mathbb{C}(T)}(\mathbb{C}(T) \otimes H)$.

Now let M be a left module over the affine Hecke algebra H. Then $M^{\mathcal{M}(T)} = \mathcal{M}(T) \otimes M$ is a module over $\mathbb{C}(T)\#_q W \otimes H$, where $\mathbb{C}(T)\#_q W$ acts on $\mathcal{M}(T)$ as described in subsection 5.2.2. Consequently, τ_x gives rise to a representation

$$\tau_x^M : W \to \mathrm{GL}(M^{\mathcal{M}(T)}),$$

defining $\tau_x^M(w)$ $(w \in W)$ to be $\tau_x(w) \in \mathbb{C}(T)\#_q W \otimes H$ acting on $M^{\mathcal{M}(T)}$. Let F_w^M $(w \in W)$ denote the corresponding functions $F_w \in \mathbb{C}(T) \otimes H$ acting on $M^{\mathcal{M}(T)}$. For simplicity we write $F_\lambda^M = F_{\mathrm{t}(\lambda)}^M$ for $\lambda \in P^\vee$.

Definition 5.3.3 (Cherednik [6])**.** The q-difference equations

$$F_\lambda^M(t) f(q^{-\lambda} t) = f(t), \qquad \lambda \in P^\vee \tag{5.3.3}$$

for $f \in \mathcal{M}(T) \otimes M$, are called the quantum affine KZ (qKZ) equations for the H-module M.

From the cocycle condition (5.3.2) and the fact that $P^\vee$ is an abelian subgroup of W, it follows immediately that the quantum KZ equation is a holonomic system of q-difference equations, that is,

$$F_\lambda^M(t) F_\mu^M(q^{-\lambda} t) = F_\mu^M(t) F_\lambda^M(q^{-\mu} t)$$

for all $\lambda, \mu \in P^\vee$.

In this paper we will restrict our attention to a particular representation of H. Recall that $H \simeq H_0 \otimes \mathbb{C}_Y[T]$ (cf. Theorem 5.2.6). Fix $\zeta \in T$ and let $\chi_\zeta : \mathbb{C}_Y[T] \to \mathbb{C}$ be the evaluation character $f(Y) \mapsto f(\zeta)$ for $f \in \mathbb{C}[T]$. We define M_ζ to be the induced H-module $M_\zeta := \mathrm{Ind}_{\mathbb{C}_Y[T]}^H(\chi_\zeta) = H \otimes_{\chi_\zeta} \mathbb{C}$. It is the minimal principal series representation of H with central character ζ. As complex vector spaces we identify $M_\zeta \simeq H_0$ via

$$T_w \otimes_{\chi_\zeta} 1 \mapsto T_w, \qquad (w \in W_0, f \in \mathbb{C}[T]). \tag{5.3.4}$$

The qKZ equation corresponding to M_ζ thus can be viewed as a holonomic system of q-difference equations for meromorphic functions $f(t)$ on T with values in H_0. Now $\mathbb{H} \simeq \mathbb{C}[T] \otimes H$, so that since $H \simeq H_0 \otimes \mathbb{C}[T]$, the double affine Hecke algebra $\mathbb{H}$ contains another copy of $\mathbb{C}[T]$. In view of Cherednik's duality anti-isomorphism one might ask, when ζ is considered as a variable γ on the second torus, whether one can find a set of q-difference equations acting on this central character γ, such that together with the original qKZ equations it makes up a holonomic system of q-difference equations for meromorphic functions $f(t, \gamma)$ on $T \times T$ with values in H_0. As a first step, one could try to localize $\widehat{\mathbb{H}}$ once again, this time with respect to the multiplicative subset $\mathbb{C}_Y[T] \setminus \{0\}$, but we will take a different approach, based on the following idea.

The construction of qKZ depended on the realization of W inside the localization of $\mathbb{H}$ by sending the w to the so-called normalized intertwiners $\rho^{-1}(w)$. Of course, we can multiply these intertwiners by appropriate factors from $\mathbb{C}[T]$ to obtain elements $\widetilde{S}_w$ which do live in $\mathbb{H}$. Clearly, the map $W \to \mathbb{H}^\times$, $w \mapsto \widetilde{S}_w$ will no longer be a group homomorphism (like ρ^{-1}), but the $\widetilde{S}_w$ still serve as intertwining elements from which a cocycle can be constructed. Then Cherednik's duality anti-isomorphism can be invoked to obtain Y-intertwining elements and extend the cocycle to a 'double cocycle' which will give rise to the bispectral quantum KZ equations. This is explained in the following subsection.

5.3.3 Bispectral quantum KZ equations

The construction of the bispectral quantum KZ equations in the present setting is more or less the same as in the GL_N case, which was done in Section 2.3. Here we repeat the construction, but, since it is a matter of simply adapting the notations from Section 2.3 we omit the proofs.

In view of the last paragraph of the previous subsection we should first renormalize the intertwiners so that they become members of $\mathbb{H}$. We put

$$\widetilde{S}_i := (k_i - k_i^{-1}X^{-a_i^\vee})s_i \in \mathbb{C}(T)\#_q W, \quad i = 0, \ldots, N$$
$$\widetilde{S}_\omega := \omega \in \mathbb{C}(T)\#_q W, \quad \omega \in \Omega,$$

giving rise to the renormalized intertwiners $\widetilde{S}_w$ ($w \in W$), defined in the following proposition (see also [10, §1.3]).

Proposition 5.3.4. *Let $w = s_{i_1} \cdots s_{i_r}\omega$ be a reduced expression for $w \in W$ ($i_1, \ldots, i_r \in \{0, \ldots, N\}$, $\omega \in \Omega$). Then*

- **(i)** $\widetilde{S}_w := \widetilde{S}_{i_1} \cdots \widetilde{S}_{i_r}\widetilde{S}_\omega$ *is a well-defined element of* $\mathbb{C}(T)\#W$*;*
- **(ii)** $\widetilde{S}_w \in \mathbb{H}$*, in particular* $\widetilde{S}_i = (1 - X^{-a_i^\vee})T_i + (k_i - k_i^{-1})X^{-a_i^\vee}$ $(0 \le i \le N)$*;*
- **(iii)** *the* $\widetilde{S}_i$ $(i = 0, \ldots, N)$ *satisfy the braid relations (cf. Definition 5.2.3(i));*
- **(iv)** $\widetilde{S}_w f(X) = (wf)(X)\widetilde{S}_w$ *for* $w \in W$, $f \in \mathbb{C}[T]$*;*
- **(v)** $\widetilde{S}_i\widetilde{S}_i = (k_i - k_i^{-1}X^{a_i^\vee})(k_i - k_i^{-1}X^{-a_i^\vee})$ *for* $i = 0, \ldots N$.

For $0 \le i \le N$ define $d_i \in \mathbb{C}[T]$ by $d_i(t) := (k_i - k_i^{-1}t^{-a_i^\vee})$. Then for $w \in W$ as in the proposition we have

$$\widetilde{S}_w = d_{i_1}(X)(s_{i_1}d_{i_2})(X)\cdots(s_{i_1}\cdots s_{i_{r-1}}d_{i_r})(X)w.$$

The proof of part (i) of the proposition relies on the fact that

$$d_w := d_{i_1}(s_{i_1}d_{i_2})\cdots(s_{i_1}\cdots s_{i_{r-1}}d_{i_r})$$

is independent of the reduced expression for w.

Now the 'double cocycle' we are going to construct is a cocycle of $W \times W$. In fact, it turns out to be convenient to anticipate the role that the anti-involution of $\mathbb{H}$ will play and extend $W \times W$ as follows. Note that the two-group $\mathbb{Z}_2$ acts on $W \times W$ by $\iota(w, w') = (w', w)$, where $\iota \in \mathbb{Z}_2$ denotes the nontrivial element. Then we put

$$\mathbb{W} := \mathbb{Z}_2 \ltimes (W \times W).$$

Furthermore, the cocycle will act on H_0-valued meromorphic functions on $T \times T$. Let us write $\mathbb{K} := \mathcal{M}(T \times T)$ for the field of meromorphic functions on $T \times T$. Moreover, write $\mathbb{L} := \mathbb{C}[T] \otimes \mathbb{C}[T] \simeq \mathbb{C}[T \times T]$ for the ring of complex-valued regular functions on $T \times T$. It acts on $\mathbb{H}$ via

$$(f \otimes g) \cdot h := f(X)hg(Y) \tag{5.3.5}$$

for $f, g \in \mathbb{C}[T]$ and $h \in \mathbb{H}$. We will usually write (t, γ) for a typical point of $T \times T$. Let $x = (x_1, \ldots, x_N)$ denote the coordinate functions of the first copy of T in $T \times T$ and $y = (y_1, \ldots, y_N)$ the coordinate functions of the second copy. For $f \in \mathbb{C}[T]$ we define $f(x) \in \mathbb{L}$ by the rule $(t, \gamma) \mapsto f(t)$, and $f(y) \in \mathbb{L}$ by $(t, \gamma) \mapsto f(\gamma)$. We use the same conventions for $f(x), f(y) \in \mathbb{K}$ when $f \in \mathcal{M}(T)$.

An intermediate step in the construction of a $\mathbb{W}$-action on $H^{\mathbb{K}} = \mathbb{K} \otimes H_0$ are the complex linear endomorphisms $\sigma_{(w,w')}$ $(w, w' \in W)$ of $\mathbb{H}$ defined by

$$\sigma_{(w,w')}(h) = \widetilde{S}_w h \widetilde{S}^*_{w'},$$
$$\sigma_\iota(h) = h^*$$

for $h \in \mathbb{H}$. As a corollary of Proposition 5.3.4 we have the following lemma.

Lemma 5.3.5. *The complex linear endomorphisms $\sigma_{(w,w')}$ and σ_ι of $\mathbb{H}$ satisfy:*

- **(i)** *the $\sigma_{(s_i,e)}$ $(i = 0, \ldots, N)$ satisfy the braid relations;*
- **(ii)** $\sigma^2_{(s_i,e)} = d_{s_i}(x)(s_i d_{s_i})(x) \cdot \mathrm{id}_{\mathbb{H}}$ *for $i = 0, \ldots, N$;*
- **(iii)** $\sigma_{(\omega,e)}\sigma_{(s_i,e)}\sigma_{(\omega^{-1},e)} = \sigma_{(s_{\omega(i)},e)}$ *for $i = 0, \ldots, N$ and $\omega \in \Omega$;*
- **(iv)** $\sigma^2_\iota = \mathrm{id}_{\mathbb{H}}$ *and* $\sigma_{(e,w)} = \sigma_\iota \sigma_{(w,e)} \sigma_\iota$ *for $w \in W$;*
- **(v)** $\sigma_{(w,e)}\sigma_{(e,w')} = \sigma_{(w,w')} = \sigma_{(e,w')}\sigma_{(w,e)}$ *for $w, w' \in W$.*

Let us investigate the behavior of these maps under the action of $\mathbb{L}$. First consider the group involution ${}^\diamond \colon W \to W$ given by $w^\diamond = w$ for $w \in W_0$ and $\lambda^\diamond = -\lambda$ for $\lambda \in P^\vee$. Then $\mathbb{W}$ acts on $T \times T$ by

$$(w, w')(t, \gamma) = (wt, w'^\diamond \gamma),$$
$$\iota(t, \gamma) = (\gamma^{-1}, t^{-1})$$

for $w, w' \in W$, where $t^{-1} := (t_1^{-1}, \ldots, t_N^{-1}) \in T$. Transposition yields an action of $\mathbb{W}$ on $\mathbb{K}$ by field automorphisms and is given by

$$(\mathrm{w}f)(t,\gamma) = f(\mathrm{w}^{-1}(t,\gamma)), \qquad \mathrm{w} \in \mathbb{W}. \tag{5.3.6}$$

Note that $\mathbb{L} = \mathbb{C}[T \times T]$ is a $\mathbb{W}$-subalgebra of $\mathbb{K}$. The following lemma is a consequence of the intertwining properties of the $\widetilde{S}_w$.

Lemma 5.3.6. *For $h \in \mathbb{H}$ and $f \in \mathbb{L}$ we have*

$$\begin{aligned} \sigma_{(w,w')}(f \cdot h) &= ((w,w')f) \cdot \sigma_{(w,w')}(h), \\ \sigma_\iota(f \cdot h) &= (\iota f) \cdot \sigma_\iota(h) \end{aligned} \tag{5.3.7}$$

for $w, w' \in W$.

As $\mathbb{L}$-modules we have $H_0^{\mathbb{K}} \simeq \mathbb{K} \otimes_{\mathbb{L}} \mathbb{H}$, so the lemma enables us to extend the maps $\sigma_{(w,w')}$ ($w, w' \in W$) and σ_ι to complex linear endomorphisms of $H_0^{\mathbb{K}}$ for which (5.3.7) holds for all $f \in \mathbb{K}$ and $h \in H_0^{\mathbb{K}}$. Note that the properties of $\sigma_{(w,w')}$ and σ_ι as described in Lemma 5.3.5 also hold true as identities between endomorphisms of $H_0^{\mathbb{K}}$.

We come to the main result of this subsection. It follows from the previous observations in the same way as the corresponding result for GL_N (Theorem 2.3.3).

Theorem 5.3.7. *There is a unique group homomorphism*

$$\tau\colon \mathbb{W} \to \mathrm{GL}_{\mathbb{C}}(H_0^{\mathbb{K}})$$

satisfying

$$\begin{aligned} \tau(w,w')(f) &= d_w(x)^{-1} d^{\diamond}_{w'}(y)^{-1} \cdot \sigma_{(w,w')}(f), \\ \tau(\iota)(f) &= \sigma_\iota(f) \end{aligned} \tag{5.3.8}$$

for $w, w' \in W$ and $f \in H_0^{\mathbb{K}}$. It satisfies $\tau(\mathrm{w})(g \cdot f) = \mathrm{w}g \cdot \tau(\mathrm{w})(f)$ for $g \in \mathbb{K}$, $f \in H_0^{\mathbb{K}}$ and $\mathrm{w} \in \mathbb{W}$.

Remark 5.3.8. Fix $\zeta \in T$. Let $w \in W$ and recall that we write $\tau_x^{M_\zeta}(w)$ for $\tau_x(w) \in \mathbb{C}(T)\#_q W$ viewed as endomorphism of $\mathcal{M}(T) \otimes M_\zeta$ as explained in subsection 5.3.2. Then for $w \in W$, $f \in \mathcal{M}(T)$ and $h \in H_0 \simeq M_\zeta$ (see (5.3.4)), we have

$$\tau_x^{M_\zeta}(w)(f \otimes h) = \tau(w,e)(f(x) \otimes h)(\cdot,\zeta)$$

as H_0-valued meromorphic functions on T.

We are in position to define the $\mathbb{W}$-cocycle with values in $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$, which is a $\mathbb{W}$-group by the action of $\mathbb{W}$ on the first tensor leg of $\mathbb{K} \otimes \mathrm{GL}(H_0) \simeq \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ (cf. Subsection 5.3.2). This $\mathbb{W}$-action on $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ is denoted without mentioning the representation map (just as we do for the $\mathbb{W}$-action on $\mathbb{K}$, cf. (5.3.6)).

Corollary 5.3.9. *The map* $\mathrm{w} \mapsto C_{\mathrm{w}} := \tau(\mathrm{w})\mathrm{w}^{-1}$ *is a cocycle of* $\mathbb{W}$ *with values in the* $\mathbb{W}$*-group* $\mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$. *In other words,* $C_{\mathrm{w}} \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$ *and*

$$C_{\mathrm{ww}'} = C_{\mathrm{w}}\mathrm{w}C_{\mathrm{w}'}\mathrm{w}^{-1}$$

for all $\mathrm{w}, \mathrm{w}' \in \mathbb{W}$.

In the same way as the cocycle F_w ($w \in W$) in subsection 5.3.2 gave rise to the quantum KZ equations, the cocycle C_{w} ($\mathrm{w} \in \mathbb{W}$) gives rise to a holonomic system of q-difference equations for meromorphic functions on $T \times T$ with values in H_0. By construction we have

$$(\tau(\mathrm{w})f)(t,\gamma) = C_{\mathrm{w}}(t,\gamma)f(\mathrm{w}^{-1}(t,\gamma)) \tag{5.3.9}$$

for $\mathrm{w} \in \mathbb{W}$ and $f \in H_0^{\mathbb{K}}$. For the sake of simplicity, write $C_{(\lambda,\mu)} := C_{(\mathrm{t}(\lambda),\mathrm{t}(\mu))}$ for $\lambda, \mu \in P^\vee$.

Definition 5.3.10. We call the q-difference equations

$$C_{(\lambda,\mu)}(t,\gamma)f(q^{-\lambda}t, q^{\mu}\gamma) = f(t,\gamma) \qquad \forall\, \lambda,\mu \in P^\vee, \tag{5.3.10}$$

the bispectral quantum KZ (BqKZ) equations. We write SOL for the set of solutions $f \in H_0^{\mathbb{K}}$ of (5.3.10).

Let $\mathbb{F} \subset \mathbb{K}$ denote the subfield consisting of $f \in \mathbb{K}$ satisfying $(\mathrm{t}(\lambda),\mathrm{t}(\mu))f = f$ for all $\lambda, \mu \in P^\vee$. Furthermore let $\mathbb{W}_0$ denote the subgroup $\mathbb{Z}_2 \ltimes (W_0 \times W_0)$ of $\mathbb{W}$.

Corollary 5.3.11. (i) *The BqKZ equations* (5.3.10) *form a holonomic system of q-difference equations, that is*

$$C_{(\lambda,\mu)}(t,\gamma)C_{(\nu,\xi)}(q^{-\lambda}t, q^{\mu}\gamma) = C_{(\nu,\xi)}(t,\gamma)C_{(\lambda,\mu)}(q^{-\nu}t, q^{\xi}\gamma) \tag{5.3.11}$$

for $\lambda, \mu, \nu, \xi \in P^\vee$, *as* $\mathrm{End}(H_0)$*-valued meromorphic functions in* $(t,\gamma) \in T \times T$.
(ii) *The solution space* SOL *of BqKZ is a* $\tau(\mathbb{W}_0)$*-invariant* $\mathbb{F}$*-subspace of* $H_0^{\mathbb{K}}$.

Now fix $\zeta \in T$. By construction, BqKZ (in some sense) contains Cherednik's qKZ equation associated to the principal series module M_ζ. Concretely, in view of Remark 5.3.8, Cherednik's quantum KZ equation (5.3.3) for $M = M_\zeta$ is just

$$C_{(\lambda,e)}(t,\zeta)f(q^{-\lambda}t) = f(t), \qquad \forall \lambda \in P^\vee, \tag{5.3.12}$$

for H_0-valued meromorphic functions f on T. In analogy with BqKZ, we write $\mathrm{SOL}_\zeta \subset H_0^{\mathcal{M}(T)}$ for the set of solutions of (5.3.12). Regarding $H_0^{\mathcal{M}(T)}$ as a vector space over $\mathcal{E}(T) := \{f \in \mathcal{M}(T) \mid \mathrm{t}(\lambda)f = f,\ \forall \lambda \in P^\vee\}$, SOL_ζ is a $\tau_x^{M_\zeta}(W_0)$-invariant subspace of $H_0^{\mathcal{M}(T)}$.

5.4 Generic principal series representation and the cocycle values

In this section we investigate the principal series representation M_ζ of H, when the (fixed) central character $\zeta \in T$ is regarded as a meromorphic variable. This allows us to give explicit expressions for the cocycle values of the simple reflections.

5.4.1 Generic principal series representation

Recall that $M_\zeta = \mathrm{Ind}_{\mathbb{C}_Y[T]}^H(\chi_\zeta)$. Now we view $\mathbb{C}_Y[T]$ as a left $\mathbb{C}_Y[T]$-module by left multiplication and we put $M := \mathrm{Ind}_{\mathbb{C}_Y[T]}^H(\mathbb{C}_Y[T])$. We regard M as a left H-module over $\mathbb{C}[T] \simeq \mathbb{C}[\{1\} \times T] \subset \mathbb{L}$ via

$$f \cdot (h \otimes_{\mathbb{C}_Y[T]} g(Y)) = h \otimes_{\mathbb{C}_Y[T]} (fg)(Y) \qquad f, g \in \mathbb{C}[T],\ h \in H.$$

Note that $M \simeq \mathbb{C}[\{1\} \times T] \otimes H_0 = H_0^{\mathbb{C}[\{1\}\times T]}$ as modules over $\mathbb{C}[\{1\} \times T]$, hence the representation map can be regarded as an algebra homomorphism

$$\eta\colon H \to \mathrm{End}_{\mathbb{C}[\{1\}\times T]}\big(H_0^{\mathbb{C}[\{1\}\times T]}\big).$$

Also note that $\mathrm{End}_{\mathbb{C}[\{1\}\times T]}(H_0^{\mathbb{C}[\{1\}\times T]}) \simeq \mathbb{C}[\{1\} \times T] \otimes \mathrm{End}(H_0)$, so we can and sometimes will regard $\eta(h)$ ($h \in H$) as an $\mathrm{End}(H_0)$-valued regular function on T denoted by $\gamma \mapsto \eta(h)(\gamma)$. By extending the ground ring $\mathbb{C}[\{1\} \times T]$ to $\mathbb{K}$ we can extend η to an algebra homomorphism

$$\eta\colon H \to \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}}).$$

Similarly, $\eta(h)$ can be viewed as an $\mathrm{End}(H_0)$-valued function in $(t, \gamma) \in T \times T$. As such it is constant in t, and in case $h \in H_0$ it is also constant in γ.

Before being more specific about η, we need the following concept (cf. [42, §2.6]). A subset X of $P^\vee$ is said to be saturated if for each $\lambda \in X$ and $\alpha \in R$ we have $\lambda - r\alpha^\vee \in X$ for all $0 \le r \le \langle \lambda, \alpha \rangle$. For $\lambda \in P^\vee$, let $\Sigma(\lambda)$ denote the smallest saturated subset of $P^\vee$ that contains λ.

Lemma 5.4.1. *For $w \in W_0$ and $1 \le i \le N$ we have*

$$\eta(T_i)T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) = \ell(w) + 1, \\ (k_i - k_i^{-1})T_w + T_{s_i w} & \text{if } \ell(s_i w) = \ell(w) - 1, \end{cases} \tag{5.4.1}$$

and for $p \in \mathbb{C}[T]$ we have

$$\eta(p(Y))(\gamma)T_e = p(\gamma)T_e \tag{5.4.2}$$

as regular H_0-valued functions in γ. Moreover, for $\lambda \in P^\vee$ and $w \in W_0$, we have

$$\eta(Y^\lambda)(\gamma)T_w = \sum_{u \le w} p_{u,w}^\lambda(\gamma)T_u, \tag{5.4.3}$$

where $p_{u,w}^\lambda(\gamma) \in \mathrm{span}_{\mathbb{C}}\{\gamma^\mu\}_{\mu \in \Sigma(\lambda_+)}$ and $p_{w,w}^\lambda(\gamma) = \gamma^{w^{-1}(\lambda)}$.

Proof. Only (5.4.3) requires proof. We use induction with respect to the length $\ell(w)$ of w, the case $\ell(w) = 0$ being (5.4.2). Next, consider $T_{s_i w}$ with $\ell(s_i w) = \ell(w) + 1$. Using (5.2.5), we find

$$\begin{aligned}\eta(Y^\lambda)(\gamma)T_{s_i w} &= \eta(Y^\lambda T_i)(\gamma)T_w \\ &= \eta(T_i Y^{s_i(\lambda)})(\gamma)T_w + (k_i - k_i^{-1})\eta\Big(\frac{Y^\lambda - Y^{s_i(\lambda)}}{1 - Y^{-\alpha_i^\vee}}\Big)(\gamma)T_w.\end{aligned}$$

Considering the first term we use the induction hypothesis to find

$$\eta(T_i Y^{s_i(\lambda)})(\gamma)T_w = T_i \sum_{u \le w} \tilde{p}^{s_i(\lambda)}_{u,w}(\gamma)T_u = \sum_{u \le w} \tilde{p}^{s_i(\lambda)}_{u,w}(\gamma)T_i T_u,$$

with $\tilde{p}^{s_i(\lambda)}_{u,w}(\gamma) \in \mathrm{span}_{\mathbb{C}}\{\gamma^\mu\}_{\mu \in \Sigma(s_i(\lambda)_+)}$ and $\tilde{p}^{s_i(\lambda)}_{w,w}(\gamma) = \gamma^{w^{-1}(s_i(\lambda))}$. Since $\Sigma(s_i(\lambda)_+) = \Sigma(\lambda_+)$ and $w^{-1}(s_i(\lambda)) = (s_i w)^{-1}(\lambda)$, we can rewrite this as

$$\eta(T_i Y^{s_i(\lambda)})(\gamma)T_w = \sum_{u \le s_i w} p^\lambda_{u,s_i w}(\gamma)T_u,$$

with $p^\lambda_{u,s_i w}(\gamma) \in \mathrm{span}_{\mathbb{C}}\{\gamma^\mu\}_{\mu \in \Sigma(\lambda_+)}$ and $p^\lambda_{s_i w,s_i w}(\gamma) = \gamma^{(s_i w)^{-1}(\lambda)}$.

We deal with the second term, the expansion of which will consist of terms only involving T_u with $u < s_i w$. Set $n := \langle \lambda, \alpha_i \rangle$. Note that

$$\frac{Y^\lambda - Y^{s_i(\lambda)}}{1 - Y^{-\alpha_i^\vee}} = \begin{cases} Y^\lambda + Y^{\lambda - \alpha_i^\vee} + \cdots + Y^{\lambda - (n-1)\alpha_i^\vee}, & n > 0, \\ 0, & n = 0, \\ -Y^{\lambda - n\alpha_i^\vee} - Y^{\lambda - (n+1)\alpha_i^\vee} - \cdots - Y^{\lambda + \alpha_i^\vee}, & n < 0, \end{cases}$$

which is in $\mathrm{span}_{\mathbb{C}}\{Y^\mu\}_{\mu \in \Sigma(\lambda_+)}$ in all three cases. We can apply the induction hypothesis to each of the Y^μ ($\mu \in \Sigma(\lambda_+)$) to obtain

$$\eta(Y^\mu)(\gamma)T_w = \sum_{u \le w} \check{p}^\mu_{u,w}(\gamma)T_u,$$

with coefficients $\check{p}^\mu_{u,w}(\gamma) \in \mathrm{span}_{\mathbb{C}}\{\gamma^\nu\}_{\nu \in \Sigma(\mu_+)}$. Since for each $\mu \in \Sigma(\lambda_+)$ we have $\mu_+ \in \Sigma(\lambda_+)$, and then by [42, (2.6.3)] $\Sigma(\mu_+) \subset \Sigma(\lambda_+)$, we obtain the desired expansion. □

We end this subsection by introducing a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ consisting of common eigenfunctions of $\eta(\mathbb{C}_Y[T])$. Note that $\widetilde{S}^*_w \in H$ for $w \in W_0$. Define

$$\xi_w := \eta(\widetilde{S}^*_{w^{-1}})T_e, \qquad w \in W_0.$$

Just as we view $\eta(h)$ as $\mathrm{End}(H_0)$-valued function in different ways, we will regard ξ_w both as regular H_0-valued function in $\gamma \in T$ and as a meromorphic H_0-valued function in $(t, \gamma) \in T \times T$ (constant in t).

Lemma 5.4.2. *$\{\xi_w\}_{w\in W_0}$ is a $\mathbb{K}$-basis of $H_0^{\mathbb{K}}$ consisting of common eigenfunctions for the η-action of $\mathbb{C}_Y[T]$ on $H_0^{\mathbb{K}}$. For $p\in\mathbb{C}[T]$ and $w\in W_0$, we have*

$$\eta(p(Y))(\gamma)\xi_w(\gamma)=(w^{-1}p)(\gamma)\xi_w(\gamma) \tag{5.4.4}$$

as H_0-valued regular functions in $\gamma\in T$.

5.4.2 The cocycle values

Write

$$R_i(z;\gamma)=c(z;k_i)^{-1}(\eta(T_i)(\gamma)-b(z;k_i)),\qquad 0\leq i\leq N, \tag{5.4.5}$$

viewed as a $\mathrm{End}(H_0)$-valued function which depends rationally on z and rationally on $\gamma\in T$ for $i=0$ and is otherwise γ-independent.

Lemma 5.4.3. **(i)** *We have*

$$C_{(s_i,e)}(t,\gamma)=R_i(t^{a_i^\vee};\gamma),\qquad 0\leq i\leq N,$$
$$C_{(\omega,e)}(t,\gamma)=\eta(\omega)(\gamma),\qquad \omega\in\Omega,$$

and C_ι is the $\mathbb{K}$-linear extension of the anti-algebra involution of H_0 determined by

$$C_\iota(T_w)=T_{w^{-1}},\qquad w\in W_0.$$

(ii) $R_i(z;\gamma)R_i(z^{-1};\gamma)=\mathrm{id}$ *for $0\leq i\leq N$.*

Remark 5.4.4. Note that

$$C_{(e,w)}(t,\gamma)=C_\iota C_{(w,e)}(\gamma^{-1},t^{-1})C_\iota,\qquad w\in W,$$

so part (i) of the previous lemma uniquely determines C_{w} for all $\mathrm{w}\in\mathbb{W}$.

5.5 Solutions of the bispectral quantum KZ equations

The main result of this section is the construction of a particular meromorphic solution Φ of BqKZ called the basic asymptotically free solution. The idea is as follows. We first look for $v\in H_0$ and $G\in\mathbb{K}$ such that Gv will be the leading term of a solution of BqKZ in some asymptotic region. These are obtained by looking for a solution of an asymptotic version of BqKZ, that is, BqKZ in which the cocycle matrices are replaced by their limit values in the asymptotic region.

Next, we gauge BqKZ by G and look for a power series solution Ψ of the gauged BqKZ equation converging deep inside the asymptotic region and which has constant term v. By meromorphic continuation Ψ can be extended to a meromorphic solution of the gauged BqKZ equation yielding the desired solution $\Phi=G\Psi\in H_0^{\mathbb{K}}$ of BqKZ. Apart from the construction itself we will derive various properties of Φ and give an explicit $\mathbb{F}$-basis of SOL, but we start with the computation of the leading term.

5.5.1 The leading term

In order to find these v and G, we first need to compute the asymptotic leading terms of the cocycle matrices $C_{(\lambda,e)}(t,\gamma)$ ($\lambda \in P^\vee$) as $|t^{-\alpha_i^\vee}| \to 0$ ($1 \le i \le N$).

We define the subring $\mathcal{A} := \mathbb{C}[x^{-\alpha_1^\vee}, \ldots, x^{-\alpha_N^\vee}]$ of $\mathbb{C}[T \times \{1\}] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}] \subset \mathbb{C}[T \times T]$. Let $Q(\mathcal{A})$ denote its quotient field and write $Q_0(\mathcal{A})$ for the subring of $Q(\mathcal{A})$ consisting of rational functions which are regular at the point $x^{-\alpha_i^\vee} = 0$ ($1 \le i \le N$). We consider $Q_0(\mathcal{A}) \otimes \mathbb{C}[T]$ as subring of $\mathbb{C}(T \times T)$ in the natural way.

Lemma 5.5.1. *Let $\lambda \in P^\vee$. We have*

$$C_{(\lambda,e)} \in (Q_0(\mathcal{A}) \otimes \mathbb{C}[T]) \otimes \mathrm{End}(H_0). \tag{5.5.1}$$

If we write $C^{(0)}_{(\lambda,e)} = C_{(\lambda,e)}|_{x^{-\alpha_1^\vee}=0,\ldots,x^{-\alpha_N^\vee}=0} \in \mathbb{C}[T] \otimes \mathrm{End}(H_0)$, we have

$$C^{(0)}_{(\lambda,e)} = \delta_{\underline{k}}^{\lambda} \eta(T_{w_0} Y^{w_0(\lambda)} T_{w_0}^{-1}). \tag{5.5.2}$$

Proof. First we consider $\lambda \in P_+^\vee$. Suppose we have a reduced expression $\mathrm{t}(\lambda) = s_{i_1} \cdots s_{i_r} \omega$ ($0 \le i_1, \ldots, i_r \le N$, $\omega \in \Omega$). Then

$$C_{(\lambda,e)}(t,\gamma) = R_{i_1}(t^{a_{i_1}^\vee};\gamma) R_{i_2}(t^{s_{i_1}(a_{i_2}^\vee)};\gamma) \cdots R_{i_r}(t^{s_{i_1}\cdots s_{i_{r-1}}(a_{i_r}^\vee)};\gamma)\eta(\omega)(\gamma). \tag{5.5.3}$$

It follows that $C_{(\lambda,e)} \in (Q(\mathcal{A}) \otimes \mathbb{C}[T]) \otimes \mathrm{End}(H_0)$. Expanding $C_{(-\lambda,e)}$ along the reduced expression $\mathrm{t}(-\lambda) = \omega^{-1} s_{i_r} \cdots s_{i_1}$ gives an expression similar to (5.5.3), from which we conclude that also $C_{(-\lambda,e)} \in (Q(\mathcal{A}) \otimes \mathbb{C}[T]) \otimes \mathrm{End}(H_0)$. Since the $R_i(z;\gamma)$ are analytic at $z = 0$ and $z = \infty$, we have $C_{(\lambda,e)}, C_{(-\lambda,e)} \in (Q_0(\mathcal{A}) \otimes \mathbb{C}[T]) \otimes \mathrm{End}(H_0)$. Writing an arbitrary weight as the difference of two dominant weights and using the cocycle property we conclude (5.5.1) for any $\lambda \in P^\vee$.

To prove (5.5.2) we will first compute the limit of $C_{(\lambda,e)}(t,\gamma)$ as $|t^{\alpha_i^\vee}| \to 0$ for $1 \le i \le N$ and then use this together with the cocycle property to find $C^{(0)}_{(\lambda,e)}(\gamma)$, which is the limit as $|t^{-\alpha_i^\vee}| \to 0$ ($1 \le i \le N$). Similarly as in the proof of (5.5.1), it suffices to consider only dominant weights. Assume we have $\lambda \in P_+^\vee$ a reduced expression for $\mathrm{t}(\lambda)$ as above and put $u = s_{i_1} \cdots s_{i_r}$. By formulas (2.2.9) and (2.2.5) from [42] we have $\{a_{i_1}, s_{i_1}(a_{i_2}), \ldots, s_{i_1} \cdots s_{i_{r-1}}(a_{i_r})\} = S(u^{-1}) = S(\omega^{-1}u^{-1}) = S(\mathrm{t}(-\lambda))$. Because $\lambda \in P_+^\vee$ we have

$$S(\mathrm{t}(-\lambda)) = \{\alpha + mc \mid \alpha \in R_-,\ 1 \le m \le -\langle \lambda, \alpha \rangle\}$$

(cf. [42, §2.4]), and thus, since $w(a^\vee) = (wa)^\vee$ ($a \in S$, $w \in W$), we have $|t^{b^\vee}| \to \infty$ ($b \in S(\mathrm{t}(-\lambda))$) as $|t^{\alpha_i^\vee}| \to 0$ ($1 \le i \le N$). Observe that $\lim_{z\to\infty} R_i(z;\gamma) = k_i^{-1}\eta(T_i)(\gamma)$ for $0 \le i \le N$. It follows that

$$C_{(\lambda,e)}(t,\gamma) \to k_{i_1}^{-1} \cdots k_{i_r}^{-1} \eta(Y^\lambda)(\gamma) = k(\mathrm{t}(\lambda))^{-1}\eta(Y^\lambda)(\gamma)$$

as $|t^{\alpha_i^\vee}| \to 0$ for all $1 \le i \le N$. More generally, we conclude that

$$C_{(\lambda,e)}(t,\gamma) \to \delta_{\underline{k}}^{-\lambda} \eta(Y^\lambda)(\gamma), \qquad \lambda \in P^\vee \tag{5.5.4}$$

as $|t^{\alpha_i^\vee}| \to 0$ for all $1 \leq i \leq N$. In order to find $C^{(0)}_{(\lambda,e)}$ we use the cocycle property to write

$$C_{(\lambda,e)}(t,\gamma) = C_{(w_0,e)}(t,\gamma)C_{(w_0(\lambda),e)}(w_0t,\gamma)C_{(w_0,e)}(q^{-w_0(\lambda)}w_0t,\gamma)$$

and consider the limit as $|t^{-\alpha_i^\vee}| \to 0$ for $1 \leq i \leq N$. Note that $C_{(w_0,e)}(t,\gamma) \to k(w_0)^{-1}\eta(T_{w_0})$ as $|t^{-\alpha_i^\vee}| \to 0$ for $1 \leq i \leq N$. Hence, using (5.5.4),

$$C^{(0)}_{(\lambda,e)} = \delta_{\underline{k}}^{-w_0(\lambda)}\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1}) = \delta_{\underline{k}}^{\lambda}\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1}),$$

where the last equality follows from (5.2.3). □

The previous lemma implies that the asymptotic form of the quantum KZ equations

$$C_{(\lambda,e)}(t,\gamma)f(q^{-\lambda}t,\gamma) = f(t,\gamma), \qquad \lambda \in P^\vee$$

in the asymptotic region $|t^{\alpha_i^\vee}| \gg 0$ $(1 \leq i \leq N)$ is

$$\delta_{\underline{k}}^{\lambda}\eta(T_{w_0}Y^{w_0(\lambda)}T_{w_0}^{-1})(\gamma)f(q^{-\lambda}t,\gamma) = f(t,\gamma), \qquad \lambda \in P^\vee. \tag{5.5.5}$$

Let $\theta_q \in \mathcal{O}(T)$ denote the theta function associated to the root system R (see [37]), defined by

$$\theta_q(t) := \sum_{\lambda \in P^\vee} q^{\frac{1}{2}\langle\lambda,\lambda\rangle}t^\lambda, \tag{5.5.6}$$

for $t \in T$. Note that θ_q is invariant under the action of W_0 on $\mathcal{O}(T)$. Furthermore, it satisfies $\theta_q(t^{-1}) = \theta_q(t)$ and

$$\theta_q(q^\mu t) = q^{-\frac{1}{2}\langle\mu,\mu\rangle}t^{-\mu}\theta_q(t), \tag{5.5.7}$$

for all $\mu \in P^\vee$.

Let $G \in \mathbb{K}$ be given by

$$G(t,\gamma) := \frac{\theta_q(tw_0(\gamma)^{-1})}{\theta_q(\delta_{\underline{k}}t)\theta_q(\delta_{\underline{k}}^{-1}w_0(\gamma)^{-1})}. \tag{5.5.8}$$

Proposition 5.5.2. *We have:*
(i) $\iota(G) = G$.
(ii) $G(t,\gamma)$ *satisfies the q-difference equations*

$$G(q^{-\lambda}t,q^\mu\gamma) = \delta_{\underline{k}}^{-\lambda-\mu}q^{-\langle w_0(\lambda),\mu\rangle}t^{w_0(\mu)}\gamma^{-w_0(\lambda)}G(t,\gamma) \tag{5.5.9}$$

for $\lambda,\mu \in P^\vee$.
(iii) $f^{(0)}(t,\gamma) := G(t,\gamma)T_{w_0}$ *is a solution of* (5.5.5) *and* $\tau(\iota)f^{(0)} = f^{(0)}$.

Proof. By construction we have (i). From (5.5.7) it follows that G satisfies $G(q^{-\lambda}t,\gamma) = \delta_{\underline{k}}^{-\lambda}\gamma^{-w_0(\lambda)}G(t,\gamma)$ for all $\lambda \in P^\vee$. In view of (i) this suffices to prove (ii). (iii) easily follows from (i) and (ii). □

5.5.2 The basic asymptotically free solution Φ

As indicated in the introduction of this section we are now going to gauge BqKZ by G. We obtain the gauged cocycle matrices

$$\begin{aligned}D_{(\lambda,\mu)}(t,\gamma) &= G(t,\gamma)^{-1}C_{(\lambda,\mu)}(t,\gamma)G(q^{-\lambda}t,q^{\mu}\gamma)\\ &= \delta_{\underline{k}}^{-\lambda-\mu}q^{-\langle\mu,w_0(\lambda)\rangle}\gamma^{-w_0(\lambda)}t^{w_0(\mu)}C_{(\lambda,\mu)}(t,\gamma),\end{aligned} \tag{5.5.10}$$

for $\lambda,\mu\in P^\vee$. It is clear that for $f\in H_0^{\mathbb{K}}$ we have $f\in\mathrm{SOL}$ if and only if $g:=G^{-1}f\in H_0^{\mathbb{K}}$ satisfies the holonomic system of q-difference equations

$$D_{(\lambda,\mu)}(t,\gamma)g(q^{-\lambda}t,q^{\mu}\gamma)=g(t,\gamma),\qquad \lambda,\mu\in P^\vee \tag{5.5.11}$$

as H_0-valued meromorhic functions in $(t,\gamma)\in T\times T$.

We write $\mathcal{B}$ for the analog of $\mathcal{A}$ corresponding to the second copy of T in $T\times T$. That is, $\mathcal{B}$ is the subring $\mathcal{B}:=\mathbb{C}[y^{\alpha_1^\vee},\ldots,y^{\alpha_N^\vee}]$ of $\mathbb{C}[\{1\}\times T]=\mathbb{C}[y_1^{\pm1},\ldots,y_N^{\pm1}]$. Similarly, we write $Q(\mathcal{B})$ for its quotient field and $Q_0(\mathcal{B})$ for the subring of $Q(\mathcal{B})$ consisting of rational functions which are regular at the point $y^{\alpha_j^\vee}=0$ $(1\le j\le N)$. We consider $Q_0(\mathcal{A})\otimes\mathcal{B}$ and $\mathcal{A}\otimes Q_0(\mathcal{B})$ as subrings of $\mathbb{C}(T\times T)$ in the natural way.

In the proof of the lemma below, we will need a partial order $\succeq$ on $P^\vee$. First recall the dominance partial order $\ge$ on $P_+^\vee$, which is defined by

$$\lambda\ge\mu\iff\lambda-\mu\in Q_+^\vee,$$

for $\lambda,\mu\in P_+^\vee$. We can extend this to a partial order on $P^\vee$ as follows. For $\lambda\in P^\vee$ write λ_+ for the unique dominant coweight in the orbit $W_0\lambda$ and let $\overline{v}_\lambda$ be the shortest $w\in W_0$ such that $w(\lambda_+)=\lambda$. For $\lambda,\mu\in P^\vee$ we say that $\lambda\succeq\mu$ if either

(i) $\lambda_+>\mu_+$, or

(ii) $\lambda_+=\mu_+$ and $\overline{v}_\lambda\ge\overline{v}_\mu$ (in the Bruhat order).

Note that with respect to this order, the anti-dominant coweight $w_0(\lambda_+)$ is the largest element in the orbit $W_0\lambda$. More details can be found in [42, §2.7].

The following lemma describes the asymptotic behavior of the gauged cocycle matrices. It allows us to put them in the context of the general theory of solutions of q-difference equations as described in the appendix and is therefore a key ingredient in the construction of Φ.

Lemma 5.5.3. *Set* $A_i=D_{(\varpi_i^\vee,e)}$ *and* $B_i=D_{(e,\varpi_i^\vee)}$ *for* $1\le i\le N$.
(i) $A_i\in(Q_0(\mathcal{A})\otimes\mathcal{B})\otimes\mathrm{End}(H_0)$ *and* $B_j\in(\mathcal{A}\otimes Q_0(\mathcal{B}))\otimes\mathrm{End}(H_0)$.
(ii) *Write* $A_i^{(0,0)}\in\mathrm{End}(H_0)$ *and* $B_i^{(0,0)}\in\mathrm{End}(H_0)$ *for the value of* A_i *and* B_i *at* $x^{-\alpha_r^\vee}=0=y^{\alpha_s^\vee}$ $(1\le r,s\le N)$. *For* $w\in W_0$ *we have*

$$A_i^{(0,0)}(T_{w_0}T_w)=\begin{cases}0 & \text{if } w^{-1}w_0(\varpi_i^\vee)\neq w_0(\varpi_i^\vee),\\ T_{w_0}T_w & \text{if } w^{-1}w_0(\varpi_i^\vee)=w_0(\varpi_i^\vee)\end{cases} \tag{5.5.12}$$

and

$$B_i^{(0,0)}(T_{w_0}T_w)=\begin{cases}0 & \text{if } w(\varpi_i^\vee)\neq\varpi_i^\vee,\\ T_{w_0}T_w & \text{if } w(\varpi_i^\vee)=\varpi_i^\vee.\end{cases} \tag{5.5.13}$$

Proof. We give the proof of (i), which differs substantially from the GL_N case (cf. Lemma 2.5.2), and omit the proof of (ii) which is similar. By (5.5.10) we have

$$A_i(t,\gamma) = \delta_{\underline{k}}^{-\varpi_i^\vee} \gamma^{-w_0(\varpi_i^\vee)} C_{(\varpi_i^\vee,e)}(t,\gamma).$$

Because of (5.5.1) we only need to worry about the γ-dependence of $A_i(t,\gamma)$.

Let $\mathsf{t}(\varpi_i^\vee) = \omega s_{i_1}\cdots s_{i_r}$ ($\omega \in \Omega$, $0 \leq i_1,\ldots,i_r \leq N$) be a reduced expression. Then, in view of the cocycle condition, Lemma 5.4.3 and formula (5.4.5),

$$C_{(\varpi_i^\vee,e)}(t,\gamma) = \eta(\omega)(\gamma) C_{(s_{i_1}\cdots s_{i_r},e)}(\omega^{-1}t,\gamma) = \sum_{w\leq \mathsf{t}(\varpi_i^\vee)} a_w(t)\eta(T_w)(\gamma)$$

for certain $a_w \in Q_0(\mathcal{A})$. Now consider such $w \in W$ with $w \leq \mathsf{t}(\varpi_i^\vee)$. We have a unique decomposition $w = \mathsf{t}(\lambda)\widetilde{w}$, with $\lambda = w(0) \in P^\vee$ and $\widetilde{w} \in W_0$. Then

$$\mathsf{t}(\lambda) = \mathsf{t}(\overline{v}_\lambda(\lambda_+)) = \overline{v}_\lambda \mathsf{t}(\lambda_+)\overline{v}_\lambda^{-1},$$

hence $w = \overline{v}_\lambda \mathsf{t}(\lambda_+)\overline{v}_\lambda^{-1}\widetilde{w}$. Multiple use of [42, (3.1.7)] yields $T_w = hT_{\mathsf{t}(\lambda_+)}h' = hY^{\lambda_+}h'$ for some $h, h' \in H_0$, hence

$$\eta(T_w)(\gamma) = \eta(h)\eta(Y^{\lambda_+})(\gamma)\eta(h').$$

It remains to show that $\gamma^{-w_0(\varpi_i^\vee)}\eta(Y^{\lambda_+})(\gamma) \in \mathcal{B}\otimes \mathrm{End}(H_0)$. We can use (5.4.3) to write

$$\eta(Y^{\lambda_+})(\gamma)T_w = \sum_{u\leq w} p_{u,w}^{\lambda_+}(\gamma)T_u$$

with $p_{u,w}^{\lambda_+}(\gamma) \in \mathrm{span}_{\mathbb{C}}\{\gamma^\mu\}_{\mu\in\Sigma(\lambda_+)}$ and $p_{w,w}^{\lambda_+}(\gamma) = \gamma^{w^{-1}(\lambda_+)}$. Thus we need to show that

$$\gamma^{-w_0(\varpi_+)+\mu} \in \mathcal{B} \qquad \forall \mu \in \Sigma(\lambda_+),$$

i.e., that $-w_0(\varpi_i^\vee)+\mu \in Q_+^\vee$ for all $\mu \in \Sigma(\lambda_+)$. Since $\Sigma(\lambda_+)$ is W_0-invariant and $w_0(Q_+^\vee) = -Q_+^\vee$, this is equivalent to showing that $-\varpi_i^\vee+\mu \in -Q_+^\vee$ for all $\mu \in \Sigma(\lambda_+)$, or

$$\varpi_i^\vee - \mu \in Q_+^\vee \qquad \forall \mu \in \Sigma(\lambda_+).$$

Now the fact that $w \leq \mathsf{t}(\varpi_i^\vee)$ in the Bruhat order on W, implies that $\lambda \preceq \varpi_i^\vee$ (cf. [42, (2.7.11)]), and hence either $\lambda_+ = \varpi_i^\vee$ or $\lambda_+ < \varpi_i^\vee$. Fix $\mu \in \Sigma(\lambda_+)$. In the first case, if $\lambda_+ = \varpi_i^\vee$, we have $\mu \in \varpi_i^\vee - Q_+^\vee$, since

$$\Sigma(\varpi_i^\vee) = \bigcap_{v\in W_0} v(\varpi_i^\vee - Q_+^\vee)$$

by [42, (2.6.2)]. Hence $\varpi_i^\vee - \mu \in Q_+^\vee$. In the second case, if $\lambda_+ < \varpi_i^\vee$, then $\Sigma(\lambda_+) \subset \Sigma(\varpi_i^\vee)$ by [42, (2.6.3)], and again $\mu \in \varpi_i^\vee - Q_+^\vee$. This concludes the proof for A_i. For B_i, use that $C_{(e,\varpi_i^\vee)}(t,\gamma) = C_\iota C_{(\varpi_i^\vee,e)}(\gamma^{-1},t^{-1})C_\iota$. □

Part (ii) of the previous lemma asserts that the endomorphisms $A_i^{(0,0)}$ and $B_i^{(0,0)}$ are semisimple. Similarly as for GL_N, the main theorem follows from the lemma together with the general theory of solutions of q-difference equations as described in the appendix (in particular Theorem A.6).

For $\epsilon > 0$, put $B_\epsilon := \{t \in T \mid |t^{\alpha_i^\vee}| < \epsilon \text{ for } 1 \leq i \leq N\}$ and $B_\epsilon^{-1} := \{t \in T \mid t^{-1} \in B_\epsilon\}$.

Theorem 5.5.4. *There exists a unique solution $\Psi \in H_0^{\mathbb{K}}$ of the gauged equations* (5.5.11) *such that, for some $\epsilon > 0$,*
(i) *$\Psi(t,\gamma)$ admits an H_0-valued power series expansion*

$$\Psi(t,\gamma) = \sum_{\alpha,\beta \in Q_+^\vee} K_{\alpha,\beta} t^{-\alpha} \gamma^\beta, \qquad (K_{\alpha,\beta} \in H_0) \tag{5.5.14}$$

for $(t,\gamma) \in B_\epsilon^{-1} \times B_\epsilon$ which is normally convergent on compacta of $B_\epsilon^{-1} \times B_\epsilon$. In particular, $\Psi(t,\gamma)$ is analytic at $(t,\gamma) \in B_\epsilon^{-1} \times B_\epsilon$;
(ii) $K_{0,0} = T_{w_0}$.

Proof. We only remark that in order to match the present situation with the one considered in the appendix, one should take in the appendix: $M = 2N$, $A_i = A_i^{(0,0)}$, $A_{N+i} = B_i^{(0,0)}$ and $q_i = q^{2/\|\alpha_i\|^2}$ for $1 \leq i \leq N$ and variables $z_i = x^{-\alpha_i^\vee}$ and $z_{N+j} = y^{\alpha_j^\vee}$ for $1 \leq i,j \leq N$. □

Definition 5.5.5. We call $\Phi := G\Psi \in \mathrm{SOL}$ the basic asymptotically free solution of BqKZ.

The $\tau(\iota)$-invariance of SOL, the ι-invariance of G, and the uniqueness part of Theorem 5.5.4 imply that Φ enjoys the following duality property.

Theorem 5.5.6 (Duality)**.** *The basic asymptotically free solution Φ of BqKZ is self-dual, in the sense that*

$$\tau(\iota)\Phi = \Phi.$$

5.5.3 Singularities

In this subsection we have a closer look at the analytic properties of Ψ. Write $q_\alpha := q^{2/\|\alpha\|^2}$ for $\alpha \in R$ and set

$$\mathcal{S}_+ := \{t \in T \mid t^{\alpha^\vee} \in k_\alpha^{-2} q_\alpha^{-\mathbb{N}} \text{ for some } \alpha \in R_+\}.$$

Proposition 5.5.7. *The H_0-valued meromorphic function Ψ is analytic on $T \setminus \mathcal{S}_+^{-1} \times T \setminus \mathcal{S}_+$.*

Proof. Let $\lambda, \mu \in P_+^\vee$. By (5.5.10) and the cocycle property, $D_{(\lambda,\mu)}(t,\gamma)$ is regular at $(t,\gamma) = (s,\zeta)$ if $C_{(\varpi_i^\vee,\varpi_j^\vee)}(q^{-\nu}t, q^\xi\gamma)$ is regular at $(t,\gamma) = (s,\zeta)$ for all $1 \leq i,j \leq N$ and $\xi,\nu \in P_+^\vee$. This in turn holds, again by virtue of the cocycle property together with (5.5.1), if $C_{(\omega_j^\vee,e)}(q^{-\nu}t,\gamma)$ is regular at $(t,\gamma) = (s,\zeta)$ for all $\nu \in P_+^\vee$ and $1 \leq j \leq N$.

Suppose we have a reduced expression $\mathsf{t}(\varpi_j^\vee) = s_{i_1}\cdots s_{i_r}\omega$ $(1 \le j \le N)$. Similarly as in the proof of Lemma 5.5.1, we have

$$C_{(\varpi_j^\vee,e)}(t,\gamma) = R_{i_1}(t^{a_{i_1}^\vee};\gamma)\cdots R_{i_r}(t^{s_{i_1}\cdots s_{i_{r-1}}(a_{i_r}^\vee)};\gamma)\eta(\omega)(\gamma),$$

and

$$\begin{aligned}\{a_{i_1}, s_{i_1}(a_{i_2}),\ldots,s_{i_1}\cdots s_{i_{r-1}}(a_{i_r})\} &= S(\mathsf{t}(-\varpi_j^\vee)) \\ &= \{\alpha + mc \mid \alpha \in R_-,\ 1 \le m \le -\langle \varpi_j^\vee, \alpha\rangle\}\end{aligned}$$

Now $R_i(z;\gamma)$ has only a simple pole at $z = k_i^{-2}$, so $C_{(\varpi_j^\vee,e)}(t,\gamma)$ has possibly poles at

$$t^{a^\vee} = k_a^{-2}, \qquad a \in S(\mathsf{t}(-\varpi_j^\vee)).$$

Note that

$$t^{(\alpha+mc)^\vee} = t^{\alpha^\vee + (2m/\|\alpha\|^2)c} = q_\alpha^m t^{\alpha^\vee},$$

hence there are possibly poles at

$$q_\alpha^m t^{\alpha^\vee} = k_\alpha^{-2}, \qquad \alpha \in R_-,\ 1 \le m \le -\langle \varpi_j^\vee, \alpha\rangle,$$

or, equivalently, at

$$t^{-\alpha^\vee} = q_\alpha^{-m} k_\alpha^{-2}, \qquad \alpha \in R_+,\ 1 \le m \le \langle \varpi_j^\vee, \alpha\rangle.$$

Consequently, $C_{(\varpi_j^\vee,e)}(q^{-\nu}t,\gamma)$ is regular at $t \in T \setminus \mathcal{S}_+^{-1}$ for all $\nu \in P_+^\vee$. By the considerations in the previous paragraph we conclude that $D_{(\lambda,\mu)}(t,\gamma)$ is regular at $(t,\gamma) \in T \setminus \mathcal{S}_+^{-1} \times T \setminus \mathcal{S}_+$ for all $\lambda, \mu \in P_+^\vee$.

Let $U \times V$ be a relatively compact open subset of $T \setminus \mathcal{S}_+^{-1} \times T \setminus \mathcal{S}_+$. Choose $\lambda, \mu \in P_+^\vee$ such that the closure of $q^{-\lambda}U \times q^\mu V$ is contained in $B_\epsilon^{-1} \times B_\epsilon$. Then as meromorphic H_0-valued function in $(t,\gamma) \in U \times V$, we have

$$\Psi(t,\gamma) = D_{(\lambda,\mu)}(t,\gamma)\Psi(q^{-\lambda}t, q^\mu\gamma), \tag{5.5.15}$$

and by Theorem 5.5.4(i) the proof is now complete. □

Remark 5.5.8. The previous proposition gives, in particular, information about the singularities of the basic asymptotic solution $\Phi = G\Psi$. Unfortunately, it is not possible to precisely pinpoint the singularities of G. To overcome this issue we could choose a different theta function in the definition of G, namely one for which we have a product formula available. The price we pay is that we have to enlarge the torus T. Let $\vartheta_q \in \mathcal{M}(T)$ denote the renormalized Jacobi theta function

$$\vartheta_q(z) := \prod_{m\ge 0}(1-q^m z)(1-q^{m+1}/z) \tag{5.5.16}$$

for $z \in \mathbb{C}^\times$. It satisfies

$$\vartheta_q(q^m z) = (-z)^{-m} q^{-\frac{1}{2}m(m-1)} \vartheta_q(z), \qquad m \in \mathbb{Z}. \tag{5.5.17}$$

Let $e \in \mathbb{N}$ be the unique positive integer such that $e\langle P^\vee, P^\vee\rangle = \mathbb{Z}$. For all $a \in S$, fix $k_a^{1/6e}$ such that $k_a^{1/6e} = k_{w(a)}^{1/6e}$ for all $w \in W$. Now put $T' := \mathrm{Hom}_{\mathbb{Z}}(\frac{1}{6e}P^\vee, \mathbb{C}^\times)$. The canonical map $T' \twoheadrightarrow T$ gives rise to an embedding $\mathcal{M}(T \times T) \hookrightarrow \mathcal{M}(T' \times T')$. Now define $\widehat{G} \in \mathcal{M}(T' \times T')$ by

$$\widehat{G}(t,\gamma) := \prod_{i,j=1}^{N} \left(\frac{\vartheta_{q^{1/e}}(\kappa_j^{-1/e} t^{\alpha_i/e}) \vartheta_{q^{1/e}}(\kappa_i^{-1/e} \gamma^{w_0(\alpha_j)/e})}{\vartheta_{q^{1/e}}(t^{\alpha_i/e} \gamma^{w_0(\alpha_j)/e})} \right)^{e\langle \varpi_i^\vee, \varpi_j^\vee\rangle}, \tag{5.5.18}$$

where $\kappa_j^{1/e} := \prod_{\alpha \in R_+} k_\alpha^{\langle \alpha_j, \alpha\rangle/e}$. Then $\widehat{G}$ satisfies the properties of Proposition 5.5.2.

Corollary 5.5.9. **(i)** *Write $\Psi(t,\gamma) = \sum_{\alpha \in Q_+^\vee} \Gamma_\alpha(\gamma) t^{-\alpha}$ for $(t,\gamma) \in B_\epsilon^{-1} \times B_\epsilon$, with Γ_α ($\alpha \in Q_+^\vee$) the analytic H_0-valued function $\Gamma_\alpha(\gamma) := \sum_{\beta \in Q_+^\vee} K_{\alpha,\beta}\gamma^\beta$ on B_ϵ. Then each Γ_α can uniquely be extended to a meromorphic H_0-valued function on T, analytic on $T \setminus \mathcal{S}_+$, such that for $(t,\gamma) \in B_\epsilon^{-1} \times T \setminus \mathcal{S}_+$*

$$\Psi(t,\gamma) = \sum_{\alpha \in Q_+^\vee} \Gamma_\alpha(\gamma) t^{-\alpha},$$

converging normally on compacta of $B_\epsilon^{-1} \times T \setminus \mathcal{S}_+$.
(ii) *The leading term Γ_0 satisfies*

$$\Gamma_0(\gamma) = K(\gamma) T_{w_0}, \tag{5.5.19}$$

for some $K \in \mathcal{M}(T)$.

Proof. (i) See Lemma 2.5.7.
(ii) This is also similar as in Chapter 2, but for the convenience of the reader we provide the details. Ψ satisfies $A_i(t,\gamma)\Psi(q^{-\varpi_i^\vee} t, \gamma) = \Psi(t,\gamma)$ for $1 \le i \le N$. Considering the limit $|t^{-\alpha_j^\vee}| \to 0$, we obtain

$$\gamma^{-w_0(\varpi_i^\vee)} \eta(T_{w_0} Y^{w_0(\varpi_i^\vee)} T_{w_0}^{-1})(\gamma) \Gamma_0(\gamma) = \Gamma_0(\gamma)$$

for $1 \le i \le N$, and in view of Lemma 5.4.2 this forces

$$\Gamma_0(\gamma) = K(\gamma)\eta(T_{w_0})\xi_e(\gamma) = K(\gamma) T_{w_0}$$

for some $K \in \mathcal{M}(T)$. □

Remark 5.5.10. In the following section we will give an explicit formula for $K(\gamma)$. It will follow immediately from an explicit formula for the leading term of the so-called Harish-Chandra series solution of a bispectral problem corresponding to BqKZ. In Chapter 2, for GL_N, it was exactly the other way around. There, the latter was found as a consequence of an explicit formula for $K(\gamma)$, which in turn is due to rather explicit expressions for the cocycle matrices of BqKZ.

From Proposition 5.5.7 and its corollary we obtain the following result for specialized spectral parameter.

Corollary 5.5.11. *Fix $\zeta \in T \setminus \mathcal{S}_+$.*
(i) *The H_0-valued meromorphic function $\Psi(t,\gamma)$ in $(t,\gamma) \in T \times T$ can be specialized at $\gamma = \zeta$, giving rise to a meromorphic H_0-valued function $\Psi(t,\zeta)$ in $t \in T$, which is regular at $t \in T \setminus \mathcal{S}_+^{-1}$.*
(ii) *For $t \in B_\epsilon^{-1}$ we have the power series expansion*

$$\Psi(t,\zeta) = \sum_{\alpha \in Q_+^\vee} \Gamma_\alpha(\zeta) t^{-\alpha},$$

converging normally on compacta of B_ϵ^{-1}.
(iii) *$\Psi(t,\zeta)$ satisfies the system of q-difference equations*

$$D_{(\lambda,e)}(t,\zeta)\Psi(q^{-\lambda}t,\zeta) = \Psi(t,\zeta), \qquad \forall \lambda \in P^\vee. \tag{5.5.20}$$

5.5.4 Consistency

BqKZ is a holonomic system of first-order q-difference equations with cocycle matrices depending rationally on $(t,\gamma) \in T \times T$ and therefore it is consistent (see [14, Proposition 5.2]). This means that $\dim_{\mathbb{F}}(\mathrm{SOL}) = \dim_{\mathbb{C}}(H_0)$, or, equivalently, that BqKZ allows a so-called fundamental matrix solution U. In [14], such a fundamental matrix solution was found by algebraic geometric arguments. A different approach, using the asymptotic solution Φ, was taken in [45]. Here we shortly repeat this latter approach for arbitrary root systems. The advantage of this approach is that it produces a basis of SOL in terms of asymptotically free solutions. For details we refer to Chapter 3.

We say that $F \in \mathrm{End}(H_0)^{\mathbb{K}} = \mathbb{K} \otimes \mathrm{End}(H_0)$ is an $\mathrm{End}(H_0)$-valued solution of BqKZ, if

$$C_{(\lambda,\mu)}(t,\gamma)F(q^{-\lambda}t, q^{\mu}\gamma) = F(t,\gamma), \qquad \lambda,\mu \in P^\vee,$$

as $\mathrm{End}(H_0)$-valued meromorphic functions in $(t,\gamma) \in T \times T$.

Define $U \in \mathrm{End}(H_0)^{\mathbb{K}}$ by

$$U\big(k(w)^{-1}T_{w_0}T_{w^{-1}}\big) := \tau(e,w)\Phi \tag{5.5.21}$$

for $w \in W_0$.

Proposition 5.5.12. *We have*
(i) *$U \in \mathrm{End}(H_0)^{\mathbb{K}}$ is an invertible solution of BqKZ with values in $\mathrm{End}(H_0)$. In particular, identifying $\mathrm{End}(H_0)^{\mathbb{K}} \simeq \mathrm{End}_{\mathbb{K}}(H_0^{\mathbb{K}})$ as $\mathbb{K}$-algebras, we have $U \in \mathrm{GL}_{\mathbb{K}}(H_0^{\mathbb{K}})$.*
(ii) *$U' \in \mathrm{End}(H_0)^{\mathbb{K}}$ is an $\mathrm{End}(H_0)$-valued meromorphic solution of BqKZ if and only if $U' = UF$ for some $F \in \mathrm{End}(H_0)^{\mathbb{F}}$.*
(iii) *U, viewed as $\mathbb{K}$-linear endomorphism of $H_0^{\mathbb{K}}$, restricts to an $\mathbb{F}$-linear isomorphism $U\colon H_0^{\mathbb{F}} \to \mathrm{SOL}$.*
(iv) *$\{\tau(e,w)\Phi\}_{w \in W_0}$ is an $\mathbb{F}$-basis of* SOL.

Remark 5.5.13. The quantum KZ equations (5.3.12) form a consistent system of q-difference equations as well. For generic $\zeta \in T$ (that is, for $\zeta \in T$ where $\Phi(t,\gamma)$ can be specialized in $\gamma = \zeta$ and moreover $U(\cdot,\zeta)$ is invertible), this follows along the same line as above, but of course one can use [14, Proposition 5.2] again, which applies for all $\zeta \in T$.

5.6 Correspondence with bispectral problems

For the principal series representation M_ζ (ζ generic) of H, Cherednik [7, Theorem 3.4] constructed a map which embeds the associated solution space of the quantum affine KZ equation (5.3.3) into the solution space of a system of q-difference equations, involving the Macdonald q-difference operator. This is a special case of a correspondence between the solutions of the quantum affine KZ equations associated with an arbitrary finite-dimensional H-module M and a more general system of q-difference equations (see [8]).

We will consider the map when M is the generic principal series module $M := \mathrm{Ind}_{\mathbb{C}_Y[T]}^{H}(\mathbb{C}_Y[T])$ (see Subsection 5.4.1). In this case Cherednik's correspondence yields an embedding χ_+ of SOL into the solution space of a bispectral problem for the Macdonald q-difference operators.

5.6.1 The bispectral problem for the Macdonald q-difference operators

Using the action of $\mathbb{W}$ on $\mathbb{C}(T \times T)$ given by (5.3.6), we can form the smash product algebra $\mathbb{C}(T \times T)\#\mathbb{W}$. It contains $\mathbb{C}(T)\#_q W \simeq \mathbb{C}(T \times \{1\})\#(W \times \{e\})$ and $\mathbb{C}(T)\#_{q^{-1}} W \simeq \mathbb{C}(\{1\} \times T)\#(\{e\} \times W)$ as subalgebras. In this interpretation, Cherednik's algebra homomorphism $\rho_{\underline{k}^{-1},q} \colon H(k^{-1}) \to \mathbb{C}(T)\#_q W$ (see Theorem 5.2.8) gives rise to an algebra homomorphism

$$\rho^x_{\underline{k}^{-1},q} \colon H(\underline{k}^{-1}) \to \mathbb{C}(T \times T)\#\mathbb{W},$$

considered as q-difference reflection operators in the first torus variable, and similarly $\rho_{\underline{k},q^{-1}} \colon H(\underline{k}) \to \mathbb{C}(T)\#_{q^{-1}} W$ to an algebra homomorphism

$$\rho^y_{\underline{k},q^{-1}} \colon H(\underline{k}) \to \mathbb{C}(T \times T)\#\mathbb{W},$$

considered as q-difference reflection operators in the second torus variable. Note that the images of $\rho^x_{\underline{k}^{-1},q}$ and $\rho^y_{\underline{k},q^{-1}}$ in $\mathbb{C}(T \times T)\#\mathbb{W}$ commute, so we can form the algebra homomorphism

$$\rho^x_{\underline{k}^{-1},q} \otimes \rho^y_{\underline{k},q^{-1}} \colon H(\underline{k}^{-1}) \otimes H(\underline{k}) \to \mathbb{C}(T \times T)\#\mathbb{W}.$$

The maps $\rho^x_{\underline{k}^{-1},q}$ and $\rho^y_{\underline{k},q^{-1}}$ are related as follows.

Lemma 5.6.1. *Let* $^\circ\colon H(\underline{k}^{-1}) \to H(\underline{k})$ *be defined as the unique algebra isomorphism satisfying*

$$T_i^\circ = T_i^{-1}, \quad \omega^\circ = \omega,$$

for $0 \le i \le N$ *and* $\omega \in \Omega$*. Then we have*

$$\rho^y_{\underline{k},q^{-1}}(h^\circ) = \iota\rho^x_{\underline{k}^{-1},q}(h)\iota \tag{5.6.1}$$

for all $h \in H(\underline{k}^{-1})$*.*

Proof. Since $\rho^x_{\underline{k}^{-1},q}$, $\rho^y_{\underline{k},q^{-1}}$ and $^\circ$ are algebra homomorphisms, the lemma follows by verifying (5.6.1) for T_i ($0 \le i \le N$) and $\omega \in \Omega$. Let $0 \le i \le N$ and $f \in \mathbb{K}$. In $H(\underline{k})$, we have $T_i^{-1} = T_i + k_i^{-1} - k_i$, hence, on the one hand,

$$\big(\rho^y_{\underline{k},q^{-1}}(T_i^{-1})f\big)(t,\gamma) = k_i^{-1}f(t,\gamma) + c_{a_i;\underline{k},q^{-1}}(\gamma)\big(f(t,s_i^\diamond\gamma) - f(t,\gamma)\big).$$

On the other hand,

$$\begin{aligned}\big(\iota\rho^x_{\underline{k}^{-1},q}(T_i)\iota f\big)(t,\gamma) &= \big(\rho^x_{\underline{k}^{-1},q}(T_i)\iota f\big)(\gamma^{-1},t^{-1})\\ &= k_i^{-1}(\iota f)(\gamma^{-1},t^{-1}) + c_{a_i;\underline{k}^{-1},q}(\gamma^{-1})\big((\iota f)(s_i\gamma^{-1},t^{-1}) - (\iota f)(\gamma^{-1},t^{-1})\big)\\ &= k_i^{-1}f(t,\gamma) + c_{a_i;\underline{k},q^{-1}}(\gamma)\big(f(t,s_i^\diamond\gamma) - f(t,\gamma)\big),\end{aligned}$$

where we used (5.2.4) for the last equality. The verification for $\omega \in \Omega$ is easier and left to reader. □

By means of the canonical action of $\mathbb{C}(T\times T)\#\mathbb{W}$ on $\mathbb{C}(T\times T)$, the subalgebra $\mathbb{D} := \mathbb{C}(T\times T)\#(P^\vee\times P^\vee) \subset \mathbb{C}(T\times T)\#\mathbb{W}$ can be identified with the algebra of q-difference operators on $T\times T$ with rational coefficients. Any element $D \in \mathbb{C}(T\times T)\#\mathbb{W}$ has an expansion

$$D = \sum_{\mathrm{w}\in\mathbb{W}_0} D_{\mathrm{w}}\mathrm{w}, \tag{5.6.2}$$

with $D_{\mathrm{w}} \in \mathbb{D}$. Since this expansion is unique, we have a well-defined $\mathbb{C}(T\times T)$-linear map $\mathrm{Res}\colon \mathbb{C}(T\times T)\#\mathbb{W} \to \mathbb{D}$, determined by

$$\mathrm{Res}(D) := \sum_{\mathrm{w}\in\mathbb{W}_0} D_{\mathrm{w}},$$

with $D \in \mathbb{C}(T\times T)\#\mathbb{W}$ given as in (5.6.2). Let $\mathbb{C}(T\times T)^{\mathbb{W}_0}$ denote the field of $\mathbb{W}_0$-invariant rational functions on $T\times T$. Restricted to $\mathbb{C}(T\times T)^{\mathbb{W}_0}$, we have $D|_{\mathbb{C}(T\times T)^{\mathbb{W}_0}} = \mathrm{Res}(D)|_{\mathbb{C}(T\times T)^{\mathbb{W}_0}}$ for all $D \in \mathbb{C}(T\times T)\#\mathbb{W}$.

It is well-known (see, e.g., [42, (4.2.10)]) that the center $Z(H)$ of the affine Hecke algebra H is $\mathbb{C}_Y[T]^{W_0}$. For $p \in \mathbb{C}[T]^{W_0}$, set

$$L^x_p := \mathrm{Res}(\rho^x_{\underline{k}^{-1},q}(p(Y))) \in \mathbb{D},$$

where $p(Y)$ is considered as element of $Z(H(\underline{k}^{-1}))$, and set

$$L_p^y := \mathrm{Res}(\rho^y_{\underline{k},q^{-1}}(p(Y))) \in \mathbb{D},$$

where $p(Y)$ is considered as element of $Z(H(\underline{k}))$. It is well-known that the operators L_p^x (and hence L_p^y) are pairwise commuting and $(W_0 \times W_0)$-invariant, and by construction $[L_p^x, L_{p'}^y] = 0$ in $\mathbb{D}$ for all $p, p' \in \mathbb{C}[T]^{W_0}$. The operators L_p^x and L_p^y are related as follows.

Lemma 5.6.2. *For $p \in \mathbb{C}[T]^{W_0}$, we have*

$$L_p^y = \iota L_p^x \iota. \tag{5.6.3}$$

Proof. Similarly as for GL_N (see Section 3.3), the lemma follows from (5.6.1) together with the fact that

$$p(Y)^\circ = p(Y), \qquad p \in \mathbb{C}[T]^{W_0} \tag{5.6.4}$$

with $^\circ\colon H(\underline{k}^{-1}) \to H(\underline{k})$ as defined in Lemma 5.6.1. We elaborate on the proof of (5.6.4), which is different than for GL_N. Note that since $p \in \mathbb{C}[T]^{W_0}$, the result follows if we can prove that $(Y^\lambda)^\circ = T_{w_0} Y^{w_0(\lambda)} T_{w_0}^{-1}$ for $\lambda \in P^\vee$. Moreover, it suffices to show this only for specific elements of $P^\vee$, as we demonstrate first. For any $\lambda \in P^\vee$, let v_λ be the shortest element of W_0 such that $v_\lambda(\lambda) = w_0(\lambda)$, and put $u_\lambda := \mathsf{t}(\lambda) v_\lambda^{-1}$. Then by [42, (2.5.4)] $\Omega = \{e\} \cup \{u_{\varpi_j^\vee}\}_{j \in J}$ with $J := \{i \in 1, \ldots, N \mid \langle \varpi_i^\vee, \phi \rangle = 1\}$. If $\lambda \in P^\vee \setminus Q^\vee$, we can write $\mathsf{t}(\lambda) = u_{\varpi_j^\vee} w$ for some $j \in J$ and $w \in W_{Q^\vee}$ (using $W = \Omega \ltimes W_{Q^\vee}$), and then $\mathsf{t}(\lambda) = \mathsf{t}(\varpi_j^\vee) v_{\varpi_j^\vee}^{-1} w = \mathsf{t}(\varpi_j^\vee)\mathsf{t}(\alpha) w'$ for some $\alpha \in Q^\vee$ and $w' \in W_0$ (using $W_{Q^\vee} = Q^\vee \rtimes W_0$). But then $w' = e$ and $\lambda = \varpi_j^\vee + \alpha$. In particular, $\{0\} \cup \{\varpi_j^\vee\}_{j \in J}$ is a complete set of representatives of $P^\vee/Q^\vee$. Since $Q^\vee = \mathrm{span}_{\mathbb{Z}}\{w(\phi^\vee) \mid w \in W_0\}$, it thus suffices to show $(Y^\lambda)^\circ = T_{w_0} Y^{w_0(\lambda)} T_{w_0}^{-1}$ only for $\lambda = \varpi_j^\vee$ with $j \in J$ and for $\lambda = w(\phi^\vee)$ ($w \in W_0$).

Let $j \in J$ and write $u_j := u_{\varpi_j^\vee}$ and $v_j := v_{\varpi_j^\vee}$. By [42, (3.3.3)], we have $u_j = T_w Y^{w^{-1}(\varpi_j^\vee)} T_{v_j w}^{-1}$ for all $w \in W_0$. Let $^\bullet\colon H(\underline{k}) \to H(\underline{k}^{-1})$ denote the inverse of $^\circ$. It follows that

$$\begin{aligned}
(Y^{w_0(\varpi_j^\vee)})^\bullet &= (T_{w_0}^{-1} u_j T_{v_j w_0})^\bullet = T_{w_0} u_j T^{-1}_{w_0 v_j^{-1}} \\
&= T_{w_0} u_j (T_{w_0} T_{v_j}^{-1})^{-1} = T_{w_0} u_j T_{v_j} T_{w_0}^{-1} \\
&= T_{w_0} Y^{\varpi_j^\vee} T_{w_0}^{-1},
\end{aligned}$$

since $u_j T_{v_j} = T_{u_j v_j} = T_{\mathsf{t}(\varpi_j^\vee)} = Y^{\varpi_j^\vee}$. Hence $(Y^{\varpi_j^\vee})^\circ = T_{w_0} Y^{w_0(\varpi_j^\vee)} T_{w_0}^{-1}$. Similarly, we can use [42, (3.3.6)] to obtain $(Y^{w(\phi^\vee)})^\circ = T_{w_0} Y^{w_0 w(\phi^\vee)} T_{w_0}^{-1}$ for $w \in W_0$, and the proof is complete. □

In order to give more explicit formulas for L_p^x and L_p^y, we need to introduce some notation. For $\lambda \in P^\vee$, write $W_{0,\lambda}$ for the isotropy subgroup of λ in W_0, and W_0^λ for a complete set of representatives of $W_0/W_{0,\lambda}$. We may assume that $e \in W_0^\lambda$. Let $m_\lambda \in \mathbb{C}[T]^{W_0}$ be the associated monomial symmetric function, that is, $m_\lambda(t) := \sum_{\mu \in W_0\lambda} t^\mu$. Finally, set $\Sigma^0(\lambda) := \Sigma(\lambda) - W_0\lambda$ (recall that $\Sigma(\lambda)$ is the smallest saturated subset of $P^\vee$ that contains λ, cf. Subsection 5.4.1).

Now fix $\lambda \in P_-^\vee = -P_+^\vee$. By [42, (4.4.12)], we have for $f \in \mathbb{K}$

$$\begin{aligned}(L^x_{m_\lambda} f)(t,\gamma) = \sum_{w \in W_0^\lambda} \prod_{a \in S(\mathrm{t}(-\lambda))} & c_{w(a),\underline{k},q}(t^{-1}) f(q^{-w(\lambda)}t,\gamma) + \\ \sum_{\mu \in \Sigma^0(\lambda)} & g_\mu(t) f(q^{-\mu}t,\gamma)\end{aligned} \tag{5.6.5}$$

for some $g_\mu \in \mathcal{M}(T)$ (here we used (5.2.4)). In view of (5.6.3), one immediately obtains a similar formula for $L^y_{m_\lambda}$.

Remark 5.6.3. For $\lambda = w_0(\varpi_j^\vee)$ with $\varpi_j^\vee$ minuscule (that is, $\langle \varpi_j^\vee, \alpha\rangle \in \{0,1\}$ for all $\alpha \in R_+$), we have $\Sigma^0(\lambda) = \emptyset$, while for $\lambda = -\phi^\vee$ we have $\Sigma^0(\lambda) = \{0\}$. In both cases one obtains an explicit formula for $L^x_{m_\lambda}$ and the resulting operators are the Macdonald q-difference operators [40].

We now define the following bispectral version of Macdonald's eigenvalue problem.

Definition 5.6.4. We define BiSP as the set of solutions $f \in \mathbb{K}$ of the following bispectral problem:

$$\begin{aligned}(L^x_p f)(t,\gamma) &= p(\gamma^{-1}) f(t,\gamma), \qquad \forall p \in \mathbb{C}[T]^{W_0},\\ (L^y_p f)(t,\gamma) &= p(t) f(t,\gamma), \qquad \forall p \in \mathbb{C}[T]^{W_0}.\end{aligned} \tag{5.6.6}$$

Remark 5.6.5. Note that BiSP is a $\mathbb{W}_0$-invariant $\mathbb{F}$-linear subspace of $\mathbb{K}$.

5.6.2 The correspondence

Consider the linear map $\chi_+ \colon H_0 \to \mathbb{C}$ defined by $\chi_+(T_w) = k(w)$. By $\mathbb{K}$-linear extension we obtain a $\mathbb{K}$-linear map $\chi_+ \colon H_0^{\mathbb{K}} \to \mathbb{K}$. It gives rise to the following correspondence between SOL and BiSP.

Theorem 5.6.6. *The $\mathbb{K}$-linear functional $\chi_+ \colon H_0^{\mathbb{K}} \to \mathbb{K}$ restricts to an injective $\mathbb{W}_0$-equivariant $\mathbb{F}$-linear map*

$$\chi_+ \colon \mathrm{SOL} \to \mathrm{BiSP}.$$

The theorem follows by restricting Cherednik's correspondence mentioned in the introduction of this section (for M the generic principal series module) to SOL. Indeed, if $f \in$ SOL, then for fixed $\gamma \in T$, $f(t,\gamma)$ can be viewed as a solution of qKZ for the H-module M_γ, and then by Cherednik's correspondence $\chi_+(f)$ satisfies the first system of equations of (5.6.6). This holds for all $\gamma \in T$. By (5.6.3) and the ι-invariance of SOL, it then follows that

$$(L_p^y f)(t,\gamma) = (\iota L_p^x \iota f)(t,\gamma) = (L_p^x \iota f)(\gamma^{-1}, t^{-1}) = p(t)(\iota f)(\gamma^{-1}, t^{-1}) = p(t) f(t,\gamma),$$

so also the second equation of (5.6.6) is satisfied.

For GL_N, a detailed proof can be found in Section 3 and the arguments used there can also be applied in the present setting.

5.7 Harish-Chandra series

Application of χ_+ to the basic asymptotic solution Φ leads to a meromorphic solution Φ_+ of the bispectral problem, which can be viewed as a bispectral analogue of the difference Harish-Chandra solutions of the Macdonald difference equations ([36]). For root systems of type A, Harish-Chandra series solutions were studied before in [16] and [31]. In Section 3.5, the Harish-Chandra series solution of type A was reobtained from $\Phi_+(t,\gamma)$, by specializing $\gamma \in T$, yielding new results on the convergence and singularities of these solutions as a consequence of corresponding results for Φ. In the final subsection we extend this to arbitrary root systems.

5.7.1 Bispectral Harish-Chandra series

As announced, we apply the map χ_+ to the basic asymptotically free solution Φ of BqKZ to obtain a special meromorphic solution of the bispectral problem (see Section 3.4 for GL_N).

Definition 5.7.1. We call $\Phi^+ := \chi_+(\Phi) \in \mathrm{BiSP}$ the basic Harish-Chandra series solution of the bispectral problem.

Put $\Psi^+ := \chi_+(\Psi)$. Then $\Phi^+ = G\Psi^+$ and as a consequence of Proposition 5.5.7 and Corollary 5.5.9, Ψ^+ is analytic on $T \setminus \mathcal{S}_+^{-1} \times T \setminus \mathcal{S}_+$, and for $(t,\gamma) \in B_\epsilon^{-1} \times T \setminus \mathcal{S}_+$ we have

$$\Psi^+(t,\gamma) = \sum_{\alpha \in Q_+^\vee} \Gamma_\alpha^+(\gamma) t^{-\alpha},$$

where $\Gamma_\alpha^+ := \chi_+(\Gamma_\alpha) \in \mathcal{M}(T)$ for all $\alpha \in Q_+^\vee$. Recall that $\Gamma_0(\gamma) = K(\gamma) T_{w_0}$ for some $K \in \mathcal{M}(T)$ (see (5.5.19)).

Theorem 5.7.2. *We have*

$$\Gamma_0^+(\gamma) = k(w_0) K(\gamma), \tag{5.7.1}$$

with $K \in \mathcal{M}(T)$ given by

$$K(\gamma) = \prod_{\alpha \in R_+} \frac{(q_\alpha \gamma^{\alpha^\vee}; q_\alpha)_\infty}{(q_\alpha k_\alpha^2 \gamma^{\alpha^\vee}; q_\alpha)_\infty}, \tag{5.7.2}$$

where $q_\alpha = q^{2/\|\alpha\|^2}$ for $\alpha \in R_+$, as before.

Proof. The definition of χ_+ and the preceding remarks imply (5.7.1). Let $L(\gamma)$ denote the right-hand side of (5.7.2). Then $L \in \mathcal{M}(T)$ is uniquely characterized by the following properties.

(i) There exists an $\epsilon > 0$ such that for $\gamma \in B_\epsilon$, L admits a power series expansion

$$L(\gamma) = \sum_{\alpha \in Q_+^\vee} l_\alpha \gamma^\alpha,$$

converging normally on compacta of B_ϵ.

(ii) $l_0 = 1$.

(iii) $L(\gamma)$ satisfies the following system of q-difference equations:

$$\left(\prod_{\alpha \in R_+} \prod_{r=1}^{\langle \lambda, \alpha \rangle} \frac{1 - q_\alpha^r \gamma^{\alpha^\vee}}{1 - q_\alpha^r k_\alpha^2 \gamma^{\alpha^\vee}} \right) L(q^\lambda \gamma) = L(\gamma), \qquad \lambda \in P_+^\vee.$$

From Theorem 5.5.4 it follows that K satisfies (i), and since $K_{0,0} = T_{w_0}$, K also satisfies (ii). It thus suffices to show that K solves the q-difference equations in (iii).

Recall that in order to show that $\Gamma_0(\gamma) = K(\gamma)T_{w_0}$ for some $K \in \mathcal{M}(T)$, we exploited the fact that Φ is a solution of the quantum KZ equation in t and investigated what taking the limit $|t^{-\alpha_j^\vee}| \to 0$ had to mean for $\Gamma_0(\gamma)$. We are now going to exploit the fact that Φ^+ satisfies the spectral problem

$$(L_p^y \Phi^+)(t, \gamma) = p(t)\Phi^+(t, \gamma), \qquad p \in \mathbb{C}[T]^{W_0}, \tag{5.7.3}$$

and consider the limit $|t^{-\alpha_i^\vee}| \to 0$ to obtain the desired q-difference equations for Γ_0^+, and hence for K.

Fix $\lambda \in P_-^\vee$. From formula (5.6.5) we deduce

$$(L_{m_\lambda}^y \Phi^+)(t, \gamma) = \sum_{w \in W_0^\lambda} \prod_{a \in S(\mathrm{t}(-\lambda))} c_{w(a), \underline{k}, q^{-1}}(\gamma) \Phi^+(t, q^{w(\lambda)}\gamma) + \sum_{\mu \in \Sigma^0(\lambda)} g_\mu(\gamma^{-1}) \Phi^+(t, q^\mu \gamma)$$

with $g_\mu \in \mathcal{M}(T)$. Plugging in $\Phi^+ = G\Psi^+$, using (5.5.9) and dividing both sides by $G(t, \gamma)$, the equality $(L_{m_\lambda}^y \Phi^+)(t, \gamma) = m_\lambda(t)\Phi^+(t, \gamma)$ gives

$$\begin{aligned} m_\lambda(t)\Psi^+(t, \gamma) = &\sum_{w \in W_0^\lambda} \prod_{a \in S(\mathrm{t}(-\lambda))} c_{w(a), \underline{k}}(\gamma) \delta_{\underline{k}}^{-w(\lambda)} t^{w_0 w(\lambda)} \Psi^+(t, q^{w(\lambda)}\gamma) + \\ &\sum_{\mu \in \Sigma^0(\lambda)} g_\mu(\gamma^{-1}) \delta_{\underline{k}}^{-\mu} t^{w_0(\mu)} \Psi^+(t, q^\mu \gamma). \end{aligned}$$

Now we multiply both sides by $t^{-w_0(\lambda)}$ and consider the limit $|t^{-\alpha_j^\vee}| \to 0$. By (5.7.1) this will result in a q-difference equation for K. Note that:

(1) $t^{-w_0(\lambda)}m_\lambda(t) = \sum_{\mu\in W_0\lambda} t^{-w_0(\lambda)+\mu} \to 1$ since $w_0(\lambda) \in P_+^\vee$ and $\nu - w(\nu) \in Q_+^\vee$ for all $\nu \in P_+^\vee$ and $w \in W_0$.

(2) $t^{-w_0(\lambda)}t^{w_0 w(\lambda)} = t^{-w_0(\lambda)+w_0 w(\lambda)}$ which is equal to 1 if $w(\lambda) = \lambda$ and tends to 0 otherwise. Considering $w \in W_0^\lambda$, we have $w(\lambda) = \lambda$ only for $w = e$.

(3) $t^{-w_0(\lambda)}t^{w_0(\mu)} \to 0$ for all $\mu \in \Sigma^0(\lambda)$. Indeed, by [42, (2.6.3)] we have

$$\mu_+ \in \Sigma(w_0(\lambda)) \Leftrightarrow w_0(\lambda) - \mu_+ \in Q_+^\vee$$

and hence also $w_0(\lambda) - w_0(\mu) \in Q_+^\vee$ for $\mu \in \Sigma^0(\lambda) \subset \Sigma(w_0(\lambda))$. Moreover, $w_0(\lambda) \neq w_0(\mu)$ since $\mu \notin W_0\lambda$.

Consequently, K satisfies the following set of q-difference equations:

$$\left(\prod_{a\in S(\mathrm{t}(-\lambda))} c_{a;\underline{k},q^{-1}}(\gamma)\right)\delta_{\underline{k}}^{-\lambda}K(q^\lambda\gamma) = K(\gamma), \qquad \lambda \in P_-^\vee.$$

Equivalently, also setting $\mu := -\lambda \in P_+^\vee$,

$$\left(\prod_{a\in S(\mathrm{t}(\mu))} \frac{k_a^{-1} - k_a(q^\mu\gamma)^{a^\vee}}{1-(q^\mu\gamma)^{a^\vee}}\right)\delta_{\underline{k}}^{\mu}K(\gamma) = K(q^\mu\gamma), \qquad \mu \in P_+^\vee. \tag{5.7.4}$$

Note that $L^y_{m_\lambda} \in \mathbb{C}(T)\#_{q^{-1}}W \simeq \mathbb{C}(\{1\}\times T)\#(\{e\}\times W)$, so $\gamma^{(\alpha+rc)^\vee} = q_\alpha^{-r}\gamma^{\alpha^\vee}$ for $\alpha \in R$ and $r \in \mathbb{Z}$. Using

$$\begin{aligned}\prod_{a\in S(\mathrm{t}(\mu))} \frac{k_a^{-1} - k_a(q^\mu\gamma)^{a^\vee}}{1-(q^\mu\gamma)^{a^\vee}} &= \prod_{\alpha\in R_+}\prod_{r=0}^{\langle\mu,\alpha\rangle-1} \frac{k_\alpha^{-1} - k_\alpha q_\alpha^{\langle\mu,\alpha\rangle}q_\alpha^{-r}\gamma^{\alpha^\vee}}{1-q_\alpha^{\langle\mu,\alpha\rangle}q_\alpha^{-r}\gamma^{\alpha^\vee}}\\ &= \prod_{\alpha\in R_+}\prod_{r=1}^{\langle\mu,\alpha\rangle} \frac{k_\alpha^{-1} - k_\alpha q_\alpha^{r}\gamma^{\alpha^\vee}}{1-q_\alpha^{r}\gamma^{\alpha^\vee}}\end{aligned}$$

and $\delta_{\underline{k}}^\mu = \prod_{\alpha\in R_+} k_\alpha^{\langle\mu,\alpha\rangle}$, we obtain from (5.7.4)

$$\left(\prod_{\alpha\in R_+}\prod_{r=1}^{\langle\mu,\alpha\rangle} \frac{1 - k_\alpha^2 q_\alpha^{r}\gamma^{\alpha^\vee}}{1-q_\alpha^{r}\gamma^{\alpha^\vee}}\right)K(\gamma) = K(q^\mu\gamma), \qquad \mu \in P_+^\vee,$$

and the proof is complete. □

In view of Remark 5.6.5, we obtain solutions $\Phi_w^+ \in \mathrm{BiSP}$ ($w \in W_0$), given by

$$\Phi_w^+(t,\gamma) := \Phi^+(t, w^{-1}\gamma).$$

Setting $\Psi_w^+(t,\gamma) := \Psi^+(t, w^{-1}\gamma)$, we have $\Phi_w^+(t,\gamma) = G(t, w^{-1}\gamma)\Psi_w^+(t,\gamma)$ and by Corollory 5.5.11(ii), for $\epsilon > 0$ sufficiently small, Ψ_w^+ has a power series expansion

$$\Psi_w^+(t,\gamma) = \sum_{\alpha \in Q_+^\vee} \Gamma_\alpha^+(w^{-1}\gamma) t^{-\alpha}$$

for $(t,\gamma) \in B_\epsilon \times T \setminus w(\mathcal{S}_+)$, converging normally on compacta of $B_\epsilon \times T \setminus w(\mathcal{S}_+)$. The next result follows along the same line as Proposition 3.4.4.

Proposition 5.7.3. *The set* $\{\Phi_w^+\}_{w\in W_0} \subset \mathrm{BiSP}$ *is* $\mathbb{F}$*-linearly independent.*

We expect that the set $\{\Phi_w^+\}_{w\in W_0}$ is in fact a basis of BiSP over $\mathbb{F}$. This would follow, for example, if we could prove that χ_+ is an $\mathbb{F}$-linear isomorphism SOL $\to$ BiSP (rather than only an embedding). Both are still open problems.

5.7.2 Application to Harish-Chandra series solutions of Macdonald's difference equations

Let $\zeta \in T$. The spectral problem of the Macdonald q-difference operators with spectral parameter ζ is

$$L_p^x f = p(\zeta^{-1}) f, \qquad \forall p \in \mathbb{C}[T]^{W_0}, \tag{5.7.5}$$

for meromorphic functions f on T. Let $\mathrm{SP}_\zeta \subset \mathcal{M}(T)$ denote the set of solutions of (5.7.5). It is a vector space over $\mathcal{E}(T)$, invariant under the usual action of W_0 on $M(T)$.

Recall the solution space $\mathrm{SOL}_\zeta \subset H_0^{\mathcal{M}(T)}$ of the quantum KZ equation (5.3.12) associated to M_ζ, also W_0-invariant, but with respect to the $\tau_x^{M_\zeta}(W_0)$-action on $H_0^{\mathcal{M}(T)}$. We have the following special case of Cherednik's correspondence from [7, 8] (see Proposition 3.5.1).

Proposition 5.7.4. *For each* $\zeta \in T$*,* χ_+ *defines an* W_0*-equivariant* $\mathcal{E}(T)$*-linear map*

$$\chi_+ \colon \mathrm{SOL}_\zeta \to \mathrm{SP}_\zeta.$$

Remark 5.7.5. In an upcoming paper by Stokman it is shown that χ_+ is an isomorphism if $\zeta^{\alpha^\vee} \neq k_\alpha^2, 1$ for all $\alpha \in R$ (see [55]).

Recall that $\Psi^+ = \chi_+(\Psi)$ with Ψ, as usual, the solution of the gauged bispectral BqKZ equations (5.5.11) obtained in Theorem (5.5.4). It follows from Corollary 5.5.11 that $\Psi^+(t,\gamma)$ may be specialized at $\gamma = \zeta$ for $\zeta \in T \setminus \mathcal{S}_+^{-1}$, yielding a meromorphic function $\Psi^+(\cdot,\zeta) \in \mathcal{M}(T)$ with poles at $t \in \mathcal{S}_+^{-1}$. Define $\widetilde{G} \in \mathbb{K}$ by

$$\widetilde{G}(t,\gamma) := \frac{\theta_q(t w_0(\gamma)^{-1})}{\theta_q(\delta_{\underline{k}} t)}.$$

Remark 5.7.6. Note that $\widetilde{G}(t,\gamma) = \theta_q(\delta_{\underline{k}}^{-1}w_0(\gamma)^{-1})G(t,\gamma)$ and that $\widetilde{G}(t,\gamma)$ can be specialized in $\gamma=\zeta$. Lacking the factor $\theta_q(\delta_{\underline{k}}^{-1}w_0(\gamma)^{-1})$ in the denominator, $\widetilde{G}$ does not satisfy $\iota(\widetilde{G})=\widetilde{G}$. Therefore, $\widetilde{G}\Psi \notin$ SOL, but we do have $\widetilde{G}(\cdot,\zeta)\Psi(\cdot,\zeta) \in \mathrm{SOL}_\zeta$.

It follows that $\widetilde{G}(\cdot,\zeta)\Psi^+(\cdot,\zeta) \in \mathrm{SP}_\zeta$ and hence $\Psi^+(\cdot,\zeta)$ is a solution of the spectral problem for the gauged Macdonald q-difference operators with spectral parameter ζ, that is, a solution of

$$(\widetilde{L}_p^x f)(t) = p(\zeta^{-1})f(t), \qquad \forall p \in \mathbb{C}[T]^{W_0}, \tag{5.7.6}$$

with

$$\widetilde{L}_p^x := \widetilde{G}(\cdot,\zeta)^{-1}\, L_p^x\, \widetilde{G}(\cdot,\zeta).$$

At the end of the previous subsection we introduced $\Psi_w^+(t,\gamma)=\Psi^+(t,w^{-1}\gamma)$ for $w \in W_0$. Put $\mathcal{S} := \bigcup_{w\in W_0} w(\mathcal{S}_+)$. The considerations of this section imply the following.

Theorem 5.7.7. *Fix $\zeta \in T \setminus S$.*
(i) *For $\epsilon > 0$ sufficiently small, $\Psi_w^+(\cdot,\zeta)$ has a power series expansion*

$$\Psi_w^+(t,\zeta) = \sum_{\alpha\in Q_+^\vee} \Gamma_\alpha^+(w^{-1}\zeta)t^{-\alpha}$$

for $t \in B_\epsilon$, converging normally on compacta of B_ϵ and with $\Gamma_0^+(w^{-1}\zeta)\neq 0$ explicitly given by (5.7.1).
(ii) *$\Psi_w^+(t,\zeta)$ ($w \in W_0$) is analytic in $t \in T \setminus \mathcal{S}_+^{-1}$.*
(iii) *The function $\widetilde{\Psi}_w^+(\cdot,\zeta) \in \mathcal{M}(T)$ ($w\in W_0$) defined by*

$$\widetilde{\Psi}_w^+(t,\zeta) := \frac{\widetilde{G}(t,w^{-1}(\zeta))}{\widetilde{G}(t,\zeta)}\Psi_w^+(t,\zeta) = \frac{\theta_q(t(w_0w^{-1})(\zeta)^{-1})}{\theta_q(tw_0(\zeta)^{-1})}\Psi_w^+(t,\zeta),$$

is a nonzero solution of the spectral problem (5.7.6) *for the gauged Macdonald q-difference operators for all $w \in W_0$.*

The functions $\widetilde{\Psi}_w^+(\cdot,\zeta)$ ($w \in W_0$) are the Harish-Chandra series solutions of the spectral problem (5.7.6). As already mentioned in the introduction of this section, formal Harish-Chandra series solutions of Macdonald's spectral problem were already obtained in [36], and earlier for the root system of type A in [16] and [31]. The upshot here is that we obtain the Harish-Chandra series solutions as meromorphic functions and are able to explicitly determine the leading term and the pole locations of $\Psi_w^+(\cdot,\zeta)$.

Appendix: Holonomic systems of q-difference equations

In the appendix we detail the construction of power series solutions of holonomic systems of q-difference equations. Special cases have been investigated in, e.g., [2], [21] and [15, §12]. Many arguments go back to classical works [1], [3], [4], [61] on ordinary linear q-difference equations.

We begin with the construction of formal asymptotic solutions to holonomic systems of q-difference equations. Let $\mathbb{C}[[z]] = \mathbb{C}[[z_1, \ldots, z_M]]$ denote the ring of formal power series in M indeterminates $z_1, \ldots, z_M$ over the complex numbers. Let V be a finite-dimensional complex vector space and let

$$A_i \in \mathbb{C}[[z]] \otimes \mathrm{End}(V)$$

for $i = 1, \ldots, M$. Since $\mathbb{C}[[z]] \otimes \mathrm{End}(V)$ is isomorphic to $\mathrm{End}_{\mathbb{C}[[z]]}(\mathbb{C}[[z]] \otimes V)$ as $\mathbb{C}[[z]]$-module, we can view the A_i as $\mathbb{C}[[z]]$-linear endomorphisms of $\mathbb{C}[[z]] \otimes V$. Fix $0 < q_i < 1$ for $1 \leq i \leq M$. Define the q_i-dilation operators

$$\mathcal{T}_i \colon \mathbb{C}[[z]] \to \mathbb{C}[[z]]$$

for $i = 1, \ldots, M$ as the complex linear maps

$$\mathcal{T}_i(\sum_{\mathbf{m}} d_{\mathbf{m}} z^{\mathbf{m}}) := \sum_{\mathbf{m}} q_i^{m_i} d_{\mathbf{m}} z^{\mathbf{m}} \qquad (d_{\mathbf{m}} \in \mathbb{C}),$$

where we use multi-index notation $z^{\mathbf{m}} = z_1^{m_1} \cdots z_M^{m_M}$ for $\mathbf{m} = (m_1, \ldots, m_M)$ with $m_j \in \mathbb{Z}_{\geq 0}$. We also view $\mathcal{T}_i$ as operators on $\mathbb{C}[[z]] \otimes V$ and on $\mathbb{C}[[z]] \otimes \mathrm{End}(V)$. Consider the system of first-order linear q-difference equations

$$A_i \mathcal{T}_i f = f, \qquad (i = 1, \ldots, M) \tag{A.7.7}$$

for $f \in \mathbb{C}[[z]] \otimes V$.

For $f \in \mathbb{C}[[z]] \otimes V$ and $A \in \mathbb{C}[[z]] \otimes \mathrm{End}(V)$, we introduce the notations

$$f^{(m)} := f|_{z_{m+1} = \ldots = z_M = 0} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$$
$$A^{(m)} := A|_{z_{m+1} = \ldots = z_M = 0} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes \mathrm{End}(V)$$

for $0 \leq m \leq M$, with the convention that $f^{(M)} = f$ and $A^{(M)} = A$. We make the following assumptions on the system of q-difference equations (A.7.7):

(a) The system (A.7.7) is holonomic, that is

$$A_i \mathcal{T}_i(A_j) = A_j \mathcal{T}_j(A_i) \tag{A.7.8}$$

for all $1 \leq i, j \leq M$. Note that the holonomy implies that the leading coefficients $A_i^{(0)} \in \mathrm{End}(V)$ of A_i mutually commute, i.e.,

$$[A_i^{(0)}, A_j^{(0)}] = 0$$

for all $1 \leq i, j \leq M$.

(b) The complex linear endomorphisms $A_1^{(0)}, \ldots, A_M^{(0)}$ of V are semisimple. Combined with **(a)** we thus have

$$V = \bigoplus_{\gamma \in S} V[\gamma]$$

with $V[\gamma] := \{v \in V \mid A_i^{(0)} v = \gamma_i v \;\; \forall i\}$ ($\gamma \in \mathbb{C}^M$) and $S := \{\gamma \in \mathbb{C}^M \mid V[\gamma] \neq \{0\}\}$.

(c) $(1^M) := (1, \ldots, 1) \in \mathbb{C}^M$ belongs to S.

(d) $\gamma_k \notin q_k^{-\mathbb{N}}$ for all $\gamma \in S$ and $1 \leq k \leq M$.

Proposition A.1. *Fix $v \in V[(1^M)]$. Consider the system* (A.7.7) *of q-difference equations and suppose that* **(a)-(d)** *are satisfied. Then there exists a unique solution $\Phi_v \in \mathbb{C}[[z]] \otimes V$ of* (A.7.7) *such that*

$$\Phi_v^{(0)} = v.$$

Proof. The proposition is a consequence of the following lemma.

Lemma A.2. *Let $0 \leq m < M$. Suppose one has a solution*

$$f_m \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$$

of the system of equations

$$\begin{aligned} A_r^{(m)} \mathcal{T}_r f_m &= f_m, \qquad 1 \leq r \leq m, \\ A_s^{(m)} f_m &= f_m, \qquad m < s \leq M. \end{aligned} \tag{A.7.9}$$

Then there exists a unique

$$f_{m+1} = \sum_{n \geq 0} f_{m;n} z_{m+1}^n \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes V$$

with $f_{m;n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $f_{m;0} = f_m$ satisfying (A.7.9) *with the role of m replaced by $m+1$:*

$$\begin{aligned} A_r^{(m+1)} \mathcal{T}_r f_{m+1} &= f_{m+1}, \qquad 1 \leq r \leq m+1, \\ A_s^{(m+1)} f_{m+1} &= f_{m+1}, \qquad m+1 < s \leq M. \end{aligned} \tag{A.7.10}$$

The proposition follows directly from the lemma as follows. Note that $f_0 := v \in V[(1^M)]$ is a solution of (A.7.9) for $m = 0$. The formal V-valued series $f_M \in \mathbb{C}[[z]]\otimes V$, obtained by repeated application of the lemma starting from $f_0 = v$, gives a formal V-valued series solution of (A.7.7) satisfying $f_M^{(0)} = v$. For uniqueness, assume that $f \in \mathbb{C}[[z]] \otimes V$ is another formal V-valued series satisfying $f^{(0)} = v$ and solving (A.7.7). We have $f^{(0)} = v = f_0$ and $f^{(m)}$ solves (A.7.9) for all $0 \leq m < M$. Hence, by the uniqueness part of the lemma, $f = f^{(M)} = f_M$.

We now proceed to prove the lemma. We assume that we have a formal power series solution $f_m \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ of (A.7.9) for some $0 \leq m < M$. We write

$$A_r^{(m+1)} = \sum_{n\geq 0} A_{r;n}^{(m)} z_{m+1}^n, \tag{A.7.11}$$

where $A_{r;n}^{(m)} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes \mathrm{End}(V)$ and $A_{r;0}^{(m)} = A_r^{(m)}$. By a direct computation one verifies that

$$f_{m+1} = \sum_{n\geq 0} f_{m;n} z_{m+1}^n \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes V$$

with $f_{m;n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $f_{m;0} = f_m$ satisfies the q-difference equation

$$A_{m+1}^{(m+1)} \mathcal{T}_{m+1} f_{m+1} = f_{m+1}$$

if and only if

$$\big(1 - q_{m+1}^n A_{m+1}^{(m)}\big) f_{m;n} = \sum_{l=1}^{n} q_{m+1}^{n-l} A_{m+1;l}^{(m)} f_{m;n-l} \tag{A.7.12}$$

for all $n \in \mathbb{Z}_{\geq 0}$. The recurrence relations (A.7.12) admit a unique solution $(f_{m;n})_{n\in\mathbb{Z}_{\geq 0}}$ with $f_{m;n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and with initial condition $f_{m;0} = f_m$. Indeed, note that (A.7.12) is valid for $n = 0$ since $f_{m;0} = f_m$ satisfies (A.7.9). For $n \geq 1$, we have

$$\det\big(1 - q_{m+1}^n A_{m+1}^{(m)}\big) \in \mathbb{C}[[z_1, \ldots, z_m]]^{\times},$$

since

$$\det\big(1 - q_{m+1}^n A_{m+1}^{(m)}\big)|_{z_1=\ldots=z_m=0} = \det\big(1 - q_{m+1}^n A_{m+1}^{(0)}\big) = \prod_{\gamma\in S}(1 - q_{m+1}^n \gamma_{m+1}^{\dim(V[\gamma])}) \neq 0$$

by assumption **(d)**. Cramer's rule then implies that (A.7.12) admits a unique solution $(f_{m;n})_{n\in\mathbb{Z}_{\geq 0}}$ with $f_{m;0} = f_m$.

We conclude that there exists a unique

$$f_{m+1} = \sum_{n\geq 0} f_{m;n} z_{m+1}^n$$

with $f_{m;n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $f_{m;0} = f_m$ satisfying the q-difference equation

$$A_{m+1}^{(m+1)} \mathcal{T}_{m+1} f_{m+1} = f_{m+1}. \tag{A.7.13}$$

It remains to show that f_{m+1} also satisfies (A.7.10) for $r = 1, \dots, m$ and for $s = m+2, \dots, M$.

Fix $1 \le r \le m$ and write $g_r := A_r^{(m+1)} \mathcal{T}_r f_{m+1}$. Its expansion in powers of z_{m+1} is written as

$$g_r = \sum_{n \ge 0} g_{r;n} z_{m+1}^n$$

with $g_{r;n} \in \mathbb{C}[[z_1, \dots, z_m]] \otimes V$ and $g_{r;0} = A_r^{(m)} \mathcal{T}_r f_m = f_m$, where the last equality follows from the fact that f_m is assumed to satisfy (A.7.9). Furthermore, using the holonomy (A.7.8) and the q-difference equation (A.7.13) in z_{m+1} satisfied by f_{m+1}, we have

$$\begin{aligned} A_{m+1}^{(m+1)} \mathcal{T}_{m+1} g_r &= A_{m+1}^{(m+1)} \mathcal{T}_{m+1} (A_r^{(m+1)}) \mathcal{T}_r \mathcal{T}_{m+1} f_{m+1} \\ &= A_r^{(m+1)} \mathcal{T}_r (A_{m+1}^{(m+1)}) \mathcal{T}_r \mathcal{T}_{m+1} f_{m+1} \\ &= A_r^{(m+1)} \mathcal{T}_r \big(A_{m+1}^{(m+1)} \mathcal{T}_{m+1} f_{m+1} \big) \\ &= A_r^{(m+1)} \mathcal{T}_r f_{m+1} = g_r. \end{aligned}$$

We conclude that g_r satisfies the characterizing properties of f_{m+1}. Hence $g_r = f_{m+1}$, i.e.,

$$A_r^{(m+1)} \mathcal{T}_r f_{m+1} = f_{m+1}.$$

Fix $m+1 < s \le M$ and write $g_s := A_s^{(m+1)} f_{m+1}$. By a similar argument as used in the previous paragraph, we now show that $g_s = f_{m+1}$. We write

$$g_s = \sum_{n \ge 0} g_{s;n} z_{m+1}^n$$

with $g_{s;n} \in \mathbb{C}[[z_1, \dots, z_m]] \otimes V$ and $g_{s;0} = A_s^{(m)} f_m = f_m$, where the last equality follows by the assumption that f_m satisfies (A.7.9). Using the holonomy (A.7.8), the q-difference equation (A.7.13), and the obvious fact that $\mathcal{T}_s(A_{m+1}^{(m+1)}) = A_{m+1}^{(m+1)}$ since $s > m+1$, we have

$$\begin{aligned} A_{m+1}^{(m+1)} \mathcal{T}_{m+1} g_s &= A_{m+1}^{(m+1)} \mathcal{T}_{m+1} (A_s^{(m+1)}) \mathcal{T}_{m+1} f_{m+1} \\ &= A_s^{(m+1)} \mathcal{T}_s (A_{m+1}^{(m+1)}) \mathcal{T}_{m+1} f_{m+1} \\ &= A_s^{(m+1)} A_{m+1}^{(m+1)} \mathcal{T}_{m+1} f_{m+1} \\ &= A_s^{(m+1)} f_{m+1} = g_s. \end{aligned}$$

We conclude that g_s satisfies the characterizing properties of f_{m+1}. Hence $g_s = f_{m+1}$, i.e.

$$A_s^{(m+1)} f_{m+1} = f_{m+1}.$$

This completes the proof of Lemma A.2, and hence the proof of Proposition A.1. □

We investigate the analytical properties of the solution Φ_v when the q-connection matrices A_i ($1 \leq i \leq M$) satisfy, besides the conditions **(a)-(d)**, the following analyticity condition:

(e) For some $\epsilon > 0$ the formal End(V)-valued series $A_i \in \mathbb{C}[[z]] \otimes \mathrm{End}(V)$ ($1 \leq i \leq M$) converges normally on compacta of the open polydisc $D_\epsilon^M := \{z \in \mathbb{C}^M \mid |z_i| < \epsilon \;\forall\, i\}$.

In other words, if we expand A_i along a basis of End(V), condition **(e)** requires its coefficients in $\mathbb{C}[[z]]$ to converge normally on compacta of D_ϵ^M.

Proposition A.3. *Suppose that the q-connection matrices $A_i \in \mathbb{C}[[z]] \otimes \mathrm{End}(V)$ ($1 \leq i \leq M$) satisfy* **(a)-(e)**. *Let $v \in V[(1^M)]$. There exists an $\epsilon > 0$ such that the formal V-valued series $\Phi_v \in \mathbb{C}[[z]] \otimes V$ converges normally on compacta of D_ϵ^M.*

Proof. For ease of notation, we will write Φ instead of Φ_v. By induction on $m = 0, \ldots, M$ we prove that there exists $\epsilon > 0$ such that $\Phi^{(m)} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ converges normally on compacta of D_ϵ^m.

For $m = 0$, there is nothing to prove. Fix $0 \leq m < M$ and suppose $\Phi^{(m)}$ converges normally on compacta of D_δ^m for some $\delta > 0$. Write

$$\Phi^{(m+1)} = \sum_{n \geq 0} \Phi_{m;n} z_{m+1}^n$$

with $\Phi_{m;n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ and $\Phi_{m;0} = \Phi^{(m)}$. Recall from the proof of Lemma A.2 that the formal V-valued power series $\Phi_{m;n}$ ($n \geq 1$) are unique characterized by the recurrence relations

$$\Phi_{m;n} = \sum_{l=1}^{n} q_{m+1}^{n-l} \big(1 - q_{m+1}^n A_{m+1}^{(m)}\big)^{-1} A_{m+1;l}^{(m)} \Phi_{m;n-l} \tag{A.7.14}$$

for all $n \geq 1$. We use this recurrence formula to find bounds for $\Phi_{m;n}$ in a neighborhood of $0 \in \mathbb{C}^m$.

Turn the finite-dimensional complex vector space V into an inner product space, with corresponding norm denoted by $\|\cdot\|$. We also write $\|\cdot\|$ for the operator norm of the associated finite-dimensional normed space End(V). We continue the proof of the proposition with two technical sublemmas. First we find a proper uniform bound for $A_{m+1;l}^{(m)}$ for all l (see (A.7.14)).

Lemma A.4. *There exists $\epsilon > 0$ and $M > 0$ such that $\|A_{m+1;l}^{(m)}\| \leq M\epsilon^{-l}$ on $\overline{D}_\epsilon^m$ for all $l \geq 0$.*

Proof. By **(e)** there exists an $\epsilon > 0$ such that $A_{m+1}^{(m+1)} \in \mathbb{C}[[z_1, \ldots, z_{m+1}]] \otimes \mathrm{End}(V)$ converges normally on compacta of the polydisc $D_{2\epsilon}^{m+1}$. Consequently, for $\epsilon < \epsilon' < 2\epsilon$ we have that $\|A_{m+1}^{(m+1)}\|$ is uniformly bounded on the polydisc $D_{\epsilon'}^{m+1}$, say by $M > 0$. In particular, we get

$$\|(\partial_{z_{m+1}}^l A_{m+1}^{(m+1)})(z_1, \ldots, z_{m+1})|_{z_{m+1}=0}\| \leq M\epsilon^{-l} l!$$

for all $(z_1, \ldots, z_m) \in \overline{D}_\epsilon^m$ and for all $l \geq 0$ (see, e.g., [25] Theorem 2.2.7). This proves the lemma in view of the definition (A.7.11) of $A^{(m)}_{m+1;l}$. $\square$

Lemma A.5. *There exists an $\epsilon > 0$ such that $\Phi_{m;n} \in \mathbb{C}[[z_1, \ldots, z_m]] \otimes V$ converges normally on compacta of D_ϵ^m for all $n \geq 0$. Furthermore, there exists a constant $C > 0$ (independent of n) such that*

$$\|\Phi_{m;n}\| \leq \frac{C}{1+C} \left(\frac{1+C}{q_{m+1}\epsilon} \right)^n \|\Phi^{(m)}\|$$

on D_ϵ^m for all $n \geq 1$.

Proof. In the proof of this lemma, we write q instead of q_{m+1}. By assumption, $\Phi_{m;0} = \Phi^{(m)}$ converges normally on compacta of D_ϵ^m if $0 < \epsilon < \delta$. We now use the recurrence relation (A.7.14) to obtain the desired results for $\Phi_{m;n}$ with $n \geq 1$.

By the proof of Lemma A.2 and since $0 < q < 1$, there exists some $\epsilon > 0$ (independent of $n \geq 1$) such that $\det(1 - q^n A^{(m)}_{m+1})^{-1}$ is analytic on D_ϵ^m for all $n \geq 1$ and such that $|\det(1 - q^n A^{(m)}_{m+1})^{-1}|$ is bounded on the closure $\overline{D}_\epsilon^m$ of D_ϵ^m, with bound independent of $n \geq 1$. For such ϵ, it follows from (A.7.14) that $\Phi_{m;n}$ converges normally on compacta of D_ϵ^m for all $n \geq 1$. Furthermore, by **(e)**, $0 < q < 1$, and Cramer's rule, it implies that for $\epsilon > 0$ small enough,

$$\|(1 - q^n A^{(m)}_{m+1})^{-1}\| \leq C'$$

on $\overline{D}_\epsilon^m$ for all $n \geq 1$, with $C' > 0$ also independent of n. By (A.7.14), $0 < q < 1$ and the previous lemma, we thus obtain for $\epsilon > 0$ small enough,

$$\|\Phi_{m;n}\| \leq C' \sum_{l=1}^{n} q^{n-l} \|A^{(m)}_{m+1;l}\| \, \|\Phi_{m;n-l}\| \leq C \sum_{l=1}^{n} \left(\frac{1}{q\epsilon} \right)^l \|\Phi_{m;n-l}\| \tag{A.7.15}$$

on $\overline{D}_\epsilon^m$ for all $n \geq 1$ with the constant $C = C'M > 0$ (independent of n).

Now, we have the following claim (cf. [15] §10.6): the recurrence relation

$$g_n = C \sum_{l=1}^{n} \left(\frac{1}{q\epsilon} \right)^l g_{n-l}, \quad (n > 0)$$

with $g_0 \in \mathbb{R}$ fixed is uniquely solved by

$$g_n = \frac{C}{C+1} \left(\frac{1+C}{q\epsilon} \right)^n g_0$$

for $n \geq 1$. Being obvious for $n = 1$, the claim follows using induction for $n > 1$ by

$$\begin{aligned} g_n &= C\left(\frac{1}{q\epsilon}\right)^n g_0 + C\sum_{l=1}^{n-1}\left(\frac{1}{q\epsilon}\right)^l g_{n-l} \\ &= C\left(\frac{1}{q\epsilon}\right)^n g_0 + \frac{C^2}{C+1}\sum_{l=1}^{n-1}\left(\frac{1}{q\epsilon}\right)^l\left(\frac{(1+C)}{q\epsilon}\right)^{n-l} g_0 \\ &= C\left(\frac{1}{q\epsilon}\right)^n g_0\left(1 + C\sum_{l=0}^{n-2}(1+C)^l\right) \\ &= C\left(\frac{1}{q\epsilon}\right)^n g_0\left(1 + C\left(\frac{(1+C)^{n-1}-1}{1+C-1}\right)\right) \\ &= \frac{C}{C+1}\left(\frac{1+C}{q\epsilon}\right)^n g_0. \end{aligned}$$

Combined with (A.7.15), the lemma now follows immediately. □

To conclude the proof of the proposition, note that the previous lemma shows that

$$\Phi^{(m+1)} = \sum_{n\geq 0}\Phi_{m;n} z_{m+1}^n \in \mathbb{C}[[z_1,\ldots,z_{m+1}]] \otimes \mathrm{End}(V)$$

converges normally on compacta of $D_{\epsilon'}^{m+1}$ if we take $\epsilon' > 0$ sufficiently small. This concludes the proof of the induction step. □

We interpret the q-dilation operators $\mathcal{T}_i$ as automorphisms of $\mathcal{M}(\mathbb{C}^M)$ by

$$(\mathcal{T}_i f)(z) = f(z_1,\ldots,z_{i-1},q_i z_i,z_{i+1},\ldots,z_M).$$

Theorem A.6. *Suppose $A_i \in \mathcal{M}(\mathbb{C}^M) \otimes \mathrm{End}(V)$ $(1 \leq i \leq M)$ satisfy the holonomy conditions* (A.7.8) *as meromorphic* $\mathrm{End}(V)$*-valued functions on $\mathbb{C}^M$. Suppose that the A_i are analytic at $0 \in \mathbb{C}^M$ and that their power series expansions at $0 \in \mathbb{C}^M$ satisfy the conditions* **(b)-(d)**.

Let $v \in V[(1^M)]$. There exists a unique $\Phi_v \in \mathcal{M}(\mathbb{C}^M) \otimes V$ solving the holonomic system (A.7.7) *of q-difference equations and coinciding, in a small neighborhood of $0 \in \mathbb{C}^M$, with the converging V-valued power series solution Φ_v from Proposition A.3.*

Proof. Since the A_i are assumed to be analytic at $0 \in \mathbb{C}^M$, their power series expansions at $0 \in \mathbb{C}^M$ are converging normally on compacta of some open polydisc D_ϵ^M $(\epsilon > 0)$. Hence, condition **(e)** is automatically satisfied.

Let $\Phi_v \in \mathbb{C}[[z]] \otimes V$ be the power series solution from Proposition A.3 and let $\epsilon > 0$ such that Φ_v converges normally on compacta of D_ϵ^M. Let $z' \in \mathbb{C}^M$ and $U \subset \mathbb{C}^M$ some open locally compact neighborhood of z'. Since $0 < q_i < 1$ $(1 \leq i \leq M)$, there exists a $\lambda \in \mathbb{Z}_{\geq 0}^M$ such that $q^\lambda U \subset D_\epsilon^M$, where $q^\lambda z = (q_1^{\lambda_1} z_1,\ldots,q_M^{\lambda_M} z_M)$. Define Φ_v as V-valued meromorphic function on $z \in U$ by

$$\Phi_v(z) = A_\lambda(z)\Phi_v(q^\lambda z), \tag{A.7.16}$$

where $A_\lambda \in \mathcal{M}(\mathbb{C}^M) \otimes \mathrm{End}(V)$ is defined inductively by

$$A_{\lambda+\mu}(z) = A_\lambda(z)A_\mu(q^\lambda z), \qquad \forall \lambda, \mu \in \mathbb{Z}_{\geq 0}^M,$$

and $A_{\epsilon_i} = A_i$ $(1 \leq i \leq M)$, where the ϵ_i $(1 \leq i \leq M)$ are the standard generators of the additive monoid $\mathbb{Z}_{\geq 0}^M$. Of course, the definition of $A_\lambda(z)$ makes sense by the holonomy conditions for the A_i. Furthermore, (A.7.16) together with the holonomy conditions for the A_i show that the power series solution Φ_v of (A.7.8) has a unique extension to a meromorphic V-valued solution on $\mathbb{C}^M$ of (A.7.8). $\square$

Bibliography

[1] C.R. ADAMS, On the linear ordinary q-difference equation, *Ann. Math.* **30** (1929), 195–205.

[2] K. AOMOTO, A note on holonomic q-difference systems, in: *Algebraic analysis*, Vol. I, 25–28, Academic Press, Boston, MA, 1988.

[3] G.D. BIRKHOFF, The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, *Proc. Amer. Acad.* **49** (1913), 521–568.

[4] R.D. CARMICHAEL, The general theory of linear q-difference equations, *Amer. J. Math.* **34** (1912), 147–168.

[5] O. CHALYKH, Macdonald polynomials and algebraic integrability, *Adv. Math.* **166** (2002), no. 2, 193–259.

[6] I. CHEREDNIK, Quantum Knizhnik-Zamolodchikov equations and affine root systems, *Commun. Math. Phys.* **150** (1992), 109–136.

[7] I. CHEREDNIK, Double affine Hecke algebras, Knizhnik-Zamolodchikov equations, and Macdonald's operators, *Int. Math. Res. Not.* **9** (1992), 171–80.

[8] I. CHEREDNIK, Induced representations of double affine Hecke algebras and applications, *Math. Res. Lett.* **1** (1994), 319–337.

[9] I. CHEREDNIK, Macdonald's evaluation conjectures and difference Fourier transform, *Invent. Math.* **122** (1995), 119–145.

[10] I. CHEREDNIK, *Double affine Hecke algebras*, London Mathematical Society Lecture Note Series 319, Cambridge University Press, Cambridge, 2005.

[11] I. CHEREDNIK, Whittaker limits of difference spherical functions, *Int. Math. Res. Not.* **2009** (2009), no. 20, 3793–842.

[12] P. DI FRANCESCO, P. ZINN-JUSTIN, Quantum Knizhnik-Zamolodchikov equation, generalized Razumov-Stroganov sum rules and extended Joseph polynomials, *J. Phys. A* **38** (2005), L815–L822.

[13] J.J. DUISTERMAAT, F.A. GRÜNBAUM, Differential equations in the spectral parameter, *Comm. Math. Phys.* **103** (1986), no. 2, 177–240.

[14] P.I. ETINGOF, Difference equations with elliptic coefficients and quantum affine algebras, preprint (1993), arXiv:hep-th/9312057.

[15] P.I. ETINGOF, I.B. FRENKEL, A.A. KIRILLOV, JR., Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations, Mathematical Surveys and Monographs, Vol. 58, Amer. Math. Soc., 1998.

[16] P.I. ETINGOF, A.A. KIRILLOV, JR., Macdonald's polynomials and representations of quantum groups, *Math. Res. Lett.* **1**, no. 3 (1994), 279–296.

[17] P. ETINGOF, A.A. KIRILLOV, JR., Representation-theoretic proof of the inner product and symmetry identities for Macdonald's polynomials, *Compos. Math.* **102** (1996), no. 2, 179–202.

[18] P. ETINGOF, K. STYRKAS, Algebraic integrability of Macdonald polynomials and representations of quantum groups, *Compos. Math.* **114** (1998), 125–152.

[19] P. ETINGOF, A. VARCHENKO, Dynamical Weyl groups and applications, *Adv. Math.* **167** (2002), no. 1, 74–127.

[20] G. FELDER, YA. MARKOV, V. TARASOV, A. VARCHENKO, Differential equations compatible with KZ equations, *Math. Phys. Anal. Geom.* **3** (2000), 139–177.

[21] I.B. FRENKEL, N.YU. RESHETIKHIN, Quantum affine algebras and holonomic difference equations, *Comm. Math. Phys.* **146** (1992), no. 1, 1–60.

[22] F.A. GRÜNBAUM, L. HAINE, Some functions that generalize the Askey-Wilson polynomials, *Comm. Math. Phys.* **184** (1997), no. 1, 173–202.

[23] J. HARNAD, Quantum isomonodromic deformations and the Knizhnik-Zamolodchikov equations, in: *Symmetries and integrability of difference equations* (Estérel, PQ, 1994), 155–161, CRM Proc. Lecture Notes **9**, Amer. Math. Soc., Providence, RI, 1996.

[24] J. HARNAD, A. KASMAN, editors, *The bispectral problem*, CRM Proceedings and lecture note series, American Mathematical Society, Providence, 1998.

[25] L. HÖRMANDER, *An introduction to complex analysis in several variables*, Second revised edition, North-Holland Mathematical Library, Vol. 7. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[26] J.E. HUMPHREYS, *Reflection groups and Coxeter groups*, Cambridge studies in advanced mathematics 29, Cambridge University Press, Cambridge, 1990.

[27] M. JIMBO, T. MIWA, *Algebraic analysis of solvable lattice models*, CBMS Regional Conference Series in Mathematics 85, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1995.

[28] M. JIMBO, T. MIWA, K. UENO, Monodromy preserving deformations of ordinary differential equations with rational coefficients I, *Physica* **2D** (1981), 306–352.

[29] M. KASATANI, Y. TAKEYAMA, The quantum Knizhnik-Zamolodchikov equation and non-symmetric Macdonald polynomials, *Funkcialaj Ekvacioj* **50** (2007), 491–509.

[30] S. KATO, R-matrix arising from affine Hecke algebras and its applications to Macdonald's difference operators, *Comm. Math. Phys.* **165** (1994), no. 3, 533-553.

[31] A. KAZARNOVSKI-KROL, Matrix elements of vertes operators of the deformed WA_n-algebra and the Harish-Chandra solutions to Macdonald's difference equations, *Sel. Math., New Ser.* **5** (1999), no. 2, 257–301.

[32] A.N. KIRILLOV, M. NOUMI, Affine Hecke algebras and raising operators for Macdonald polynomials, *Duke Math. J.* **93** (1998), no. 1, 1–39.

[33] V.G. KNIZHNIK, A.B. ZAMOLODCHIKOV, Current algebra and Wess-Zumino model in two dimensions, *Nuclear Phys. B* **247** (1984), 83–103.

[34] T.Y. LAM, *Lectures on modules and rings*, Graduate Texts in Mathematics 189, Springer-Verlag, New York, 1999.

[35] V.P. LEKSIN, Isomonodromic and bispectral properties of the generalized Knizhnik-Zamolodchikov equations, *J. Math. Sci.* (N.Y.) **129** (2005), no. 1, 3635–3641.

[36] G. LETZTER, J.V. STOKMAN, Macdonald difference operators and Harish-Chandra series, *Proc. Lond. Math. Soc.* **97** (2008), no. 1, 60–96.

[37] E. LOOIJENGA, Root systems and elliptic curves, *Invent. Math.* **38** (1976), 17–32.

[38] G. LUSZTIG, Affine Hecke Algebras and their Graded Version, *J. Amer. Math. Soc.* **2** (1989), no. 3, 599–635.

[39] I.G. MACDONALD, The Poincaré series of a Coxeter group, *Math. Ann.* **199** (1972), 161–174.

[40] I.G. MACDONALD, Orthogonal polynomials associated with root systems, preprint (1987); *Séminaire Lotharingien de Combinatoire* **45** (2000), 1–40.

[41] I.G. MACDONALD, *Symmetric functions and Hall polynomials*, Second edition, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

[42] I.G. MACDONALD, *Affine Hecke algebras and orthogonal polynomials*. Cambridge Tracts in Mathematics, Vol. 157, Cambridge University Press, Cambridge, 2003.

[43] A. MATSUO, Integrable connections related to zonal spherical functions, *Invent. Math.* **110** (1992), 96–121.

[44] M. VAN MEER, Bispectral quantum Knizhnik-Zamolodchikov equations for arbitrary root systems, preprint (2009), arXiv:0912.3784.

[45] M. VAN MEER, J.V. STOKMAN, Double affine Hecke algebras and bispectral quantum Knizhnik-Zamolodchikov equations, *Int. Math. Res. Not.* **2010** (2010), no. 6, 969–1040.

[46] K. MIMACHI, A solution to quantum Knizhnik-Zamolodchikov equations and its applications to eigenvalue problems of the Macdonald type, *Duke Math. J.* **85** (1996), no. 3, 635–658.

[47] E.M. OPDAM, A generating function for the trace of the Iwahori-Hecke algebra, in: *Studies in memory of Issai Schur* (Chevaleret/Rehovot, 2000), 301–323, Progress in Mathematics 210, Birkhäuser Boston, Boston, MA, 2003.

[48] M. VAN DER PUT, M.F. SINGER, *Galois theory of difference equations*. Lecture Notes in Mathematics 1666, Springer-Verlag, Berlin Heidelberg, 1997.

[49] N. RESHETIKHIN, The Knizhnik-Zamolodchikov system as a deformation of the isomonodromy problem, *Lett. Math. Phys.* **26** (1992), 167–177.

[50] S.N.M. RUIJSENAARS, Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, *Comm. Math. Phys.* **110** (1987), 191–213.

[51] S.N.M. RUIJSENAARS, Factorized weight functions vs. factorized scattering, *Comm. Math. Phys.* **228** (2002), no. 3, 467–494.

[52] J. SAULOY, Systemes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie, *Ann. Inst. Fourier (Grenoble)* **50** (2000), no. 4, 1021–1071.

[53] J.-P. SERRE, *Local Fields*, Graduate Texts in Mathematics 67, Springer-Verlag, New York-Berlin, 1979.

[54] F. SMIRNOV, General formula for soliton form factors in sine-Gordon model, *J. Phys. A* **19** (1986), no. 10, L575–L578.

[55] J.V. STOKMAN, Quantum affine Knizhnik-Zamolodchikov equations and quantum spherical functions I (2010), arXiv:1001.2645, to appear in *International Mathematics Research Notices*.

[56] Y. TAKEYAMA, Differential equations compatible with boundary rational qKZ equation, preprint (2009), arXiv:0909.2288v1.

[57] V. TARASOV, Duality for Knizhnik-Zamolodchikov and dynamical equations, and hypergeometric integrals, in: *Infinite dimensional algebras and quantum integrable systems*, 235–263, Progress in Mathematics 237, Birkhäuser, Basel, 2005.

[58] V. TARASOV, A. VARCHENKO, Difference equations compatible with trigonometric KZ differential equations, *Int. Math. Res. Not.* **2000** (2000), no. 15, 801–829.

[59] V. TARASOV, A. VARCHENKO, Duality for Knizhnik-Zamolodchikov and dynamical equations. The 2000 Twente conference on Lie groups (Enschede) *Acta Appl. Math.* **73** (2002), no. 1-2, 141–154.

[60] V. TOLEDANO LAREDO, A Kohno-Drinfeld theorem for quantum Weyl groups, *Duke Math. J.* **112** (2002), no. 3, 421–451.

[61] W.J. TRJITZINSKY, Analytic theory of linear q-difference equations, *Acta Math.* **61** (1933), 1–38.

[62] A. TSUCHIYA, Y. KANIE, Vertex operators in conformal field theory on $\mathbb{P}^1$ and monodromy representations of the braid group, *Lett. Math. Phys.* **13** (1987), 303–312.

Samenvatting

In deze samenvatting zullen we proberen een globale beschrijving te geven van de inhoud van dit proefschrift. We beginnen met het verduidelijken van een aantal veel gebruikte concepten aan de hand van een alledaags voorbeeld.

De hoofdrolspelers in dit proefschrift, zoals de titel wellicht al doet vermoeden, zijn de *bispectrale quantum Knizhnik-Zamolodchikov vergelijkingen*. Ze vormen een nogal niet-triviaal voorbeeld van een systeem van *differentievergelijkingen*. Vele fenomenen in de natuurkunde, scheikunde, biologie, maar ook in de financiële wereld, kunnen gemodelleerd worden met behulp van differentievergelijkingen. Wanneer men bijvoorbeeld een bepaald bedrag x_0 op een spaarrekening zet tegen een jaarlijkse rente van 4% en x_n stelt het bedrag voor dat men na n jaar bij elkaar heeft gespaard, dan wordt het verloop van het spaarproces vastgelegd door de eenvoudige differentievergelijking

$$x_{n+1} = 1,04x_n. \tag{1}$$

Als de begininleg €50 bedraagt, dat wil zeggen $x_0 = 50$, dan heeft men na 1 jaar dus €52 ($x_1 = 1,04x_0 = 52$), na 2 jaar €54,08 ($x_2 = 1,04x_1 = 54,08$), etc. We kunnen het rijtje $x_0, x_1, x_2, x_3, \ldots$ beschouwen als een functie f op de niet-negatieve gehele getallen, bepaald door $f(n) = x_n$. In deze notatie krijgt bovenstaande differentievergelijking de vorm $f(n+1) = 1,04f(n)$. De oplossingen van deze differentievergelijking worden gegeven door $f(n) = c \cdot 1,04^n$, waarbij c de begininleg is. In ons geval hebben we dus $f(n) = 50 \cdot 1,04^n$. Merk op dat deze uitdrukking niet alleen zinvol is voor gehele getallen n; $f(t) = 50 \cdot 1,04^t$ is gedefinieerd voor ieder reëel getal t en geeft de waarde van de spaarrekening als ware het een continu proces.

Indien we de rente niet gelijk aan 4% maar willekeurig kiezen, zouden we, meer algemeen, de differentievergelijking

$$f(t+1) = \lambda f(t) \tag{2}$$

kunnen beschouwen, met λ een gegeven constante. De oplossingen van (2) worden gegeven door $f(t) = c \cdot \lambda^t$ met c weer de begininleg, of, als we de financiële context even vergeten, gewoon een willekeurige constante. De vergelijking (2) laat zich weer verder generaliseren tot een vergelijking

$$f(t+1, \lambda) = A(t, \lambda)f(t, \lambda), \tag{3}$$

waarbij $A(t, \lambda)$ een gegeven functie is die afhangt van t en λ. Men zou zich nu de vraag kunnen stellen of er ook een differentievergelijking in de variabele λ bestaat, dus zeg $f(t, \lambda + 1) = B(t, \lambda)f(t, \lambda)$ voor een zekere functie $B(t, \lambda)$, zodanig dat het systeem van differentievergelijkingen

$$\begin{aligned} f(t+1, \lambda) &= A(t, \lambda)f(t, \lambda), \\ f(t, \lambda+1) &= B(t, \lambda)f(t, \lambda) \end{aligned} \tag{4}$$

compatibel is, wat ongeveer wil zeggen dat er een niet-triviale functie $f(t, \lambda)$ bestaat die aan *beide* vergelijkingen tegelijk voldoet. Als we nu (3) vervangen door een specifiek, belangrijk systeem van differentievergelijkingen, de zogeheten *quantum Knizhnik-Zamolodchikov (KZ) vergelijkingen* (dit zijn zekere vergelijkingen werkende in $t = (t_1, \ldots, t_N)$ voor *vectorwaardige* functies $f(t, \lambda)$ die afhangen van extra variabelen $\lambda = (\lambda_1, \ldots, \lambda_N)$), dan is de analoge vraag of er een systeem van differentievergelijkingen werkende in λ bestaat, zodanig dat het totale, 'verdubbelde' systeem compatibel is, de centrale vraag in dit proefschrift. Deze vraag kunnen we met "ja" beantwoorden.

De quantum KZ vergelijkingen hebben hun oorsprong in de theoretische fysica en zijn sinds hun eerste verschijning begin jaren '90 veelvuldig bestudeerd. Eigenlijk zijn de quantum KZ vergelijkingen zogenaamde *q-differentievergelijkingen*, maar voor de doeleinden van deze samenvatting, kan het niet zoveel kwaad om het verschil hier niet te duiden.

Het verdubbelde, compatibele systeem dat we construeren, noemen we de *bispectrale quantum KZ vergelijkingen*. De constructie van de bispectrale quantum KZ vergelijkingen en het bewijs dat ze daadwerkelijk een compatibel stel differentievergelijkingen vormen, berust op de eigenschappen van een geavanceerd algebraïsch object geïntroduceerd door Ivan Cherednik, de zogenaamde *dubbele affiene Hecke algebra*. Een van de belangrijkste toepassingen van de bispectrale quantum KZ vergelijkingen is dat het extra, compatibele stel vergelijkingen gebruikt kan worden om uit bestaande oplossingen van de quantum KZ vergelijkingen nieuwe oplossingen te creëren. Dit wordt uitgebreid in het proefschrift besproken.

Een andere toepassing van de bispectrale quantum KZ vergelijkingen is de volgende. Een door Cherednik gevonden correspondentie tussen de quantum KZ vergelijkingen en een spectraalprobleem (eigenwaardevergelijking) voor de zogenaamde Macdonald-Ruijsenaars operatoren, blijkt aanleiding te geven tot een correspondentie tussen de bispectrale quantum KZ vergelijkingen en een bispectraal probleem voor de Macdonald-Ruijsenaars operatoren. Oplossingen van de bispectrale quantum KZ vergelijkingen die we in dit proefschrift vinden, gaan via de correspondentie over in oplossingen voor het bispectrale probleem. Op die manier worden nieuwe resultaten verkregen omtrent de convergentie van speciale machtreeksoplossingen (Harish-Chandra machtreeksoplossingen) van het spectraalprobleem, alsook nieuwe inzichten in de theorie van zekere polynomiale oplossingen van het spectraalprobleem, de beroemde (symmetrische) *Macdonaldpolynomen*.

Er zijn verschillende typen van de dubbele affiene Hecke algebra en het blijkt dat we bispectrale quantum KZ vergelijkingen kunnen associëren met ieder type. In de hoofstukken 2 tot en met 4 geven we een uitgebreide behandeling van bovengenoemde theorie in het geval dat de dubbele affiene Hecke algebra van het type A_N is. In het laatste hoofdstuk beschrijven we de theorie corresponderend met de dubbele affiene Hecke algebra van een *willekeurig* type.

Dankwoord

Allereerst en bovenal wil ik mijn begeleider Jasper Stokman bedanken. Jasper, ik kan met volle overtuiging zeggen dat zonder jouw zeer deskundige, intensieve en prettige begeleiding dit proefschrift er nooit zou zijn geweest. Veel dank voor al je moeite en de fijne samenwerking! Ook mijn promotor Eric Opdam wil ik graag bedanken. Gedurende het promotietraject hebben we weliswaar niet heel veel contact gehad, maar mede door jouw interessante colleges en inspirerende gesprekken tijdens mijn studie, ben ik deze richting ingeslagen. De overige leden van de promotiecommissie wil ik bedanken voor het lezen van het proefschrift en voor een aantal nuttige opmerkingen na lezing van het voorlopige manuscript.

Het onderzoek voor dit proefschrift is verricht aan het Korteweg-de Vries Instituut voor Wiskunde van de Universiteit van Amsterdam. Ik wil iedereen van het KdVI van harte bedanken voor de gezellige sfeer de afgelopen vier jaar. In het bijzonder Abdel, Alejan, Andries, Enno, Evelien, Jan, Peter, Peter, Phyllis, Ricardo, Roland, Said en Sjors. Uiteraard wil ik ook onze portier annex praktisch vangnet René bedanken. In al onze gesprekken heeft hij veel van mij kunnen leren over voetbal. Een speciale vermelding verdienen mijn voormalige kamergenoten Ramon en Hicham. De vele conversaties, samen fitnessen, het organiseren van voetbalpoules en het door ons opgerichte zaalvoetbalteam Chuck Norris maakten jullie meer tot vrienden dan collega's. Bedankt daarvoor! (Chuck Norris schijnt inmiddels trouwens *drie* keer tot oneindig te hebben geteld.)

Tijdens de laatste fase van mijn promotieonderzoek ben ik als wiskundedocent gaan werken aan de Faculteit Economie en Bedrijfskunde van de UvA. Ik dank mijn collega's van de wiskundesectie, Barry, Brian, Dick, Herman, Jan, Joris, Pim en Wim, voor het feit dat ze mij hebben ontzien in perioden dat ik extra hard aan mijn proefschrift moest werken.

Graag wil ik ook mijn oud-studievrienden Bert, Bram en Ruben bedanken; onze afstudeerperiode in kamer P.208 staat in zekere zin aan de basis van dit proefschrift. Mooi dat we elkaar nog regelmatig zien. Dat houden we erin.

Een krappe twee jaar geleden heb ik het wijze besluit genomen te gaan basketballen bij US. Ik wil bij deze mijn team Heren 5 (getallen zeggen niets) en de twee coaches Jiska en Sanne bedanken voor het vele plezier in het veld, maar misschien nog wel meer voor de gezelligheid erbuiten. Nog een biertje, of toch niet, is iedere woensdagavond weer een beproeving voor mijn ruggengraat.

Essentieel in mijn leven, niet alleen de afgelopen periode, maar al vele jaren, is mijn illustere gezelschap beste vrienden. Marald, dank voor de geweldige dagjes Amsterdam vol ongebreideld associëren en onafgemaakte gesprekken. Rob, dank voor de ontzettende hoeveelheid lol altijd en voor het feit dat je mijn paranimf wilt zijn; ik kan je hier helaas niet omschrijven zoals jij mij omschreef in je 'getuigeverklaring' op je bruiloft. Ruud, niet alleen dank voor alle 'principe-afspraken'; de erg gezellige werkweekenden met veel koffie, een broodje makreel en op de achtergrond (of voorgrond) onze vrienden van Eredivisielive, waren cruciaal voor het totstandkomen van dit proefschrift. Dank ook daarvoor. Tristan, bedankt voor je oprechte belangstelling, het af en toe opfrissen van mijn zelfkennis en voor de biljartavondjes in de Windhoek (met goede gesprekken, tenzij het erom spande); eigenlijk niet voor het feit dat je mijn paranimf bent, want daar heb je ruim vier jaar geleden zelf al om gevraagd.

Natuurlijk mag ik mijn familie niet vergeten. Papa en mama, ik kan jullie niet genoeg bedanken voor alles wat jullie voor mij hebben gedaan en nog steeds voor me doen. Mijn lieve, grote broers, Mathieu en Serge, bedankt dat jullie mij al op jonge leeftijd probeerden uit te leggen wat x was; dat heeft wellicht geholpen. Ook Claudy, Susanne en jullie kinderen wil ik bedanken. Pasen, kerst en Sinterklaas zijn tegenwoordig dan wel niet ontspannen meer te noemen, gezellig zijn ze wel!

Tot slot wil ik mijn lieve vriendinnetje Thecla bedanken. Je bent in korte tijd heel belangrijk voor me geworden. Dankzij jou leken de laatste loodjes van aluminium. Ik hoop dat dit slechts het begin was.

Curriculum vitae

Michel van Meer werd op 3 januari 1980 geboren in Roosendaal, alwaar hij in 1998 aan het Gertudiscollege zijn gymnasiumdiploma behaalde. Vervolgens ging hij wiskunde studeren aan de Universiteit van Amsterdam. In 2005 studeerde hij cum laude af op een scriptie over de Drinfeld-Kohno stelling en dubbele affiene Hecke algebra's. Vrijwel direct hierna, februari 2006, begon hij met een promotieonderzoek onder begeleiding van Jasper Stokman. De resultaten hiervan zijn te vinden in dit proefschrift.

Gedurende de laatste fase van het promotietraject combineerde Michel zijn aanstelling als promovendus met die van wiskundedocent aan de Faculteit Economie en Bedrijfskunde van de Universiteit van Amsterdam. Sinds januari 2010 is hij hier fulltime docent.