# Rotation of Linear Polarization Plane and Circular Polarization from Cosmological Pseudo-Scalar Fields

**F Finelli†‡§, M Galaverni¶†§**

†INAF/IASF-BO, Istituto di Astrofisica Spaziale e Fisica Cosmica di Bologna via Gobetti 101, I-40129 Bologna - Italy
‡INAF/OAB, Osservatorio Astronomico di Bologna, via Ranzani 1, I-40127 Bologna - Italy
§INFN, Sezione di Bologna, Via Irnerio 46, I-40126 Bologna - Italy
¶Dipartimento di Fisica, Università di Ferrara, Via Saragat 1, I-44100 Ferrara - Italy

E-mail: `finelli@iasfbo.inaf.it`, `galaverni@iasfbo.inaf.it`

**Abstract.** We show that a time evolving pseudo-scalar field coupled to the electromagnetic tensor generates circular polarization and does not only rotate the plane of linear polarization. We compute analytically and numerically the propagation of the Stokes parameters from the last scattering surface for an oscillating and a monotonic decreasing pseudo-scalar field acting as dark matter.

PACS numbers: CMBR polarization theory, axions, dark matter

**Introduction**

In 1977 R. Peccei and H. Quinn [1] suggested a solution to the strong CP-problem of QCD introducing a new symmetry breaking at $f_a$ energy scale. The boson associated with this broken global symmetry was called *axion*. All the physical properties of this pseudo-scalar field strongly depend on the energy scale $f_a$ at which the new symmetry is broken: the particle mass and the coupling constants with other particles are inversely proportional to $f_a$. Pseudo-Goldstone bosons also arise in many particle physics scenarios [2].

Axions and, in general, other pseudo-scalar particles are among the most favoured particle physics candidates for the cold dark matter (CDM) [3, 4, 5]. They interact with photons according to the lagrangian:

$$\mathcal{L}_{int} = -\frac{g_\phi}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu} \,, \tag{1}$$

where $g_\phi$ is the coupling constant, $F^{\mu\nu}$ is the electromagnetic tensor and $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ its dual. Many constraints on axion derive from this interaction with photons: laboratory experiments (photon-axion conversion experiments) and astrophysical arguments (stellar evolution of red giants) constrain $g_\phi$ to be small. One of the most stringent experimental bound ($g_\phi < 8.8 \times 10^{-20}$ eV$^{-1}$ for $m_a < 0.02$ eV) is obtained by the CAST experiment [6] constraining the axion-photon conversion for solar axions. This limit supersedes the one obtained from the duration of the helium burning time in (horizontal-branch (HB)) stars in globular clusters: $g_\phi \lesssim 10^{-19}$ eV$^{-1}$ [4, 7].

In this paper we wish to study in detail the interaction of such a pseudo-scalar field with photons. The interaction in Eq. (1) modify the polarization of an electromagnetic wave [8]:

- along intervening magnetic fields;
- through a slowly varying background field $\phi$.

Here we are interested in the second case, which does not require the presence of a magnetic field (note that in the first case the polarization is modified also in absence of axions). We consider the time dependent pseudo-scalar condensate as dark matter or part of it and study the impact of its time derivative on the polarization of the photons. As a consequence of its coupling with a pseudo-scalar field, the plane of linear polarization of light is rotated (*cosmological birefringence*) [9, 10].

In the case of Cosmic Microwave Background (CMB) photons, we pay attention to the rotation along the path between the last scattering surface (LSS) and the observer, modifying the polarization pattern generated by Thomson scattering at LSS [11, 12, 13, 14]. This rotation induced by a cosmological pseudo-scalar generates a correlation between the gradient and curl of the polarization pattern ($E$ and $B$ following [15]), which would be otherwise vanishing for the standard Gaussian cosmological case [16, 17]. Such $EB$ correlation is already constrained by present data sets [18, 19].

We study two representative examples for the dynamics of a pseudo-Goldstone field behaving as dark matter: the oscillating and a monotonic decreasing behavior. In the latest case we study analytically the problem, whereas in the former numerically and analytically.

The case of a field growing linearly in time has been studied in [20].

Our paper is organized as follows. We review the relevant equations for pseudo-scalar field dynamics and Stokes parameters in sections 1 and 2, respectively. In section 3 we write the Stokes parameters in terms of the left and right polarizations gauge potential and solve the differential equations for the latter for oscillating behaviour of the pseudo-scalar field. In a similar way section 4 is dedicated to the monotonic behaviour of the pseudo-scalar field. We conclude in section 5.
We work in units where the speed of light is equal to one ($c = 1$).

## 1. Electrodynamics Coupled to a Pseudo-Scalar Field

The lagrangian density $\mathcal{L}$ for the photons and the pseudo-scalar field $\phi$ is [21] (we follow the notation of [22]):

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \nabla_\mu \phi \nabla^\nu \phi - V(\phi) - \frac{g_\phi}{4} \phi F_{\mu\nu} \tilde{F}^{\mu\nu} \,, \tag{2}$$

The Euler-Lagrange equations resulting from this lagrangian are:

$$\Box \phi \equiv \nabla_\mu \nabla^\mu \phi = \frac{dV}{d\phi} + \frac{g_\phi}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \,, \tag{3}$$

$$\frac{1}{4\pi} \nabla_\mu F^{\mu\nu} = - g_\phi (\nabla_\mu \phi) \tilde{F}^{\mu\nu} \,, \tag{4}$$

$$\nabla_\mu \tilde{F}^{\mu\nu} = 0 \,. \tag{5}$$

Using the definition of the electromagnetic tensor $F^{\mu\nu} \equiv \nabla^\mu A^\nu - \nabla^\nu A^\mu$ the equation (4) becomes:

$$\Box A_\nu - \nabla_\nu \left( \nabla_\mu A^\mu \right) - R^\mu{}_\nu A_\mu = -2\pi g_\phi (\nabla_\mu \phi) \epsilon^\mu{}_\nu{}^{\rho\sigma} F_{\rho\sigma} \,. \tag{6}$$

The complete antisymmetric tensor contain the determinant of the metric $g$ and $[\cdots]$ guarantees anti-symmetry in the four indexes [23]:

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\,[\alpha\beta\gamma\delta] \qquad \text{and} \qquad \epsilon^{\alpha\beta\gamma\delta} = -\left(\sqrt{-g}\right)^{-1} [\alpha\beta\gamma\delta] \,. \tag{7}$$

For a spatially flat Friedmann-Robertson-Walker universe the metric is:

$$ds^2 = -dt^2 + a^2(t) \boldsymbol{x}^2 = a^2(\eta) \left[ -d\eta^2 + d\boldsymbol{x}^2 \right] \,, \tag{8}$$

where $t$ is the cosmic time, $\eta$ is conformal time and $\boldsymbol{x}$ denote the space coordinates. We consider a plane wave propagating along $\hat{\boldsymbol{n}}$ in Coulomb gauge ($\nabla \cdot \boldsymbol{A} = 0$). Directing $\hat{\boldsymbol{n}}$ along the $z$ axis for simplicity and neglecting the spatial variation of the pseudo-scalar field $\phi = \phi(\eta)$, from the equation (6) we obtain two independent equations for the relevant components of electromagnetic potential:

$$A''_x(\eta, z) - \frac{\partial^2 A_x(\eta, z)}{\partial z^2} = 4\pi g_\phi \phi' \frac{\partial A_y(\eta, z)}{\partial z} \,, \tag{9}$$

$$A''_y(\eta, z) - \frac{\partial^2 A_y(\eta, z)}{\partial z^2} = - 4\pi g_\phi \phi' \frac{\partial A_x(\eta, z)}{\partial z} \,. \tag{10}$$

Defining Fourier transform as $\tilde{A}_{x,y}(\eta, k) = (2\pi)^{-1} \int e^{ikz} A_{x,y}(\eta, z) dz$ the previous equations become:

$$\tilde{A}''_x(\eta, k) + k^2 \tilde{A}_x(\eta, k) + 4\pi g_\phi \phi' ik \tilde{A}_y(\eta, k) = 0 \,, \tag{11}$$

$$\tilde{A}''_y(\eta, k) + k^2 \tilde{A}_y(\eta, k) - 4\pi g_\phi \phi' ik \tilde{A}_x(\eta, k) = 0 \,. \tag{12}$$

These equations can be decoupled introducing $\tilde{A}_\pm(\eta, k) = \tilde{A}_x(\eta, k) \pm i\tilde{A}_y(\eta, k)$, left and right components of the electro-magnetic vector potential

$$\tilde{A}''_\pm(\eta, k) + \left[ k^2 \pm 4\pi g_\phi \phi' k \right] \tilde{A}_\pm(\eta, k) = 0 \,. \tag{13}$$

## 2. Standard Review of Stokes Parameters

The complex electric field vector for a plane wave propagating along $\hat{\boldsymbol{z}}$ direction at a point $(x, y)$ in some transverse plane $z = z_0$ is:

$$\boldsymbol{E} = (E_x(t)\,, E_y(t)) = \left[\hat{\boldsymbol{e}}_{\boldsymbol{x}}\varepsilon_x(t)e^{i\varphi_x(t)} + \hat{\boldsymbol{e}}_{\boldsymbol{y}}\varepsilon_y(t)e^{i\varphi_y(t)}\right] e^{-ikt}\,, \qquad (14)$$

where the physical quantity is the real part of $\boldsymbol{E}$. For a spatially flat Friedmann-Robertson-Walker metric the relation between the electromagnetic tensor and the physical fields is:

$$F_{\mu\nu} = a(\eta)\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \qquad (15)$$

In general we consider quasi-monochromatic waves: the amplitudes ($\varepsilon_x(t)$ and $\varepsilon_y(t)$) and the phases ($\varphi_x(t)$ and $\varphi_y(t)$) are slowly varying functions of time respect to the inverse frequency of the wave.

According to [24, 25] we introduce the *covariance* or *equal time coherence matrix*:

$$\mathbf{J} = a^2\begin{pmatrix} \langle E_x^*(t)E_x(t)\rangle & \langle E_x^*(t)E_y(t)\rangle \\ \langle E_y^*(t)E_x(t)\rangle & \langle E_y^*(t)E_y(t)\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix}, \qquad (16)$$

where $\langle\cdots\rangle$ denote the *ensemble average*, the average over all possible realizations of a given quasi-monochromatic wave. Each element of the coherence matrix is related to a particular combination of the Stokes parameters $I$, $Q$, $U$ and $V$.:

$$I \equiv \frac{1}{a^2}\left(\langle E_x^*(t)E_x(t)\rangle + \langle E_y^*(t)E_y(t)\rangle\right)\,, \qquad (17)$$

$$Q \equiv \frac{1}{a^2}\left(\langle E_x^*(t)E_x(t)\rangle - \langle E_y^*(t)E_y(t)\rangle\right)\,, \qquad (18)$$

$$U \equiv \frac{1}{a^2}\left(\langle E_x^*(t)E_y(t)\rangle + \langle E_y^*(t)E_x(t)\rangle\right) = \frac{2}{a^2}\langle \varepsilon_x\varepsilon_y\cos(\varphi_x - \varphi_y)\rangle\,, \qquad (19)$$

$$V \equiv -\frac{i}{a^2}\left(\langle E_x^*(t)E_y(t)\rangle - \langle E_y^*(t)E_x(t)\rangle\right) = \frac{2}{a^2}\langle \varepsilon_x\varepsilon_y\sin(\varphi_x - \varphi_y)\rangle\,. \qquad (20)$$

For a pure monochromatic wave ensemble averages can be omitted and the wave is completely polarized:

$$4\det\mathbf{J} = I^2 - Q^2 - U^2 - V^2 = 0\,. \qquad (21)$$

The parameter $I$ gives the total intensity of the radiation, $Q$ and $U$ describe linear polarization and $V$ circular polarization. Linear polarization can also be characterized through a vector of modulus:

$$P_L \equiv \sqrt{Q^2 + U^2}\,, \qquad (22)$$

and an angle $\theta$, defined as:

$$\theta \equiv \frac{1}{2}\arctan\frac{U}{Q}\,. \qquad (23)$$

Sometimes it is also useful define the degrees of linear and circular polarization:

$$\Pi_L \equiv \frac{P_L}{I} \quad \text{and} \quad \Pi_C \equiv \frac{|V|}{I} \, . \tag{24}$$

Given the intensity of the polarized part $P^2 = Q^2 + U^2 + V^2 = P_L^2 + V^2$ and the total intensity $I$ the intensity of the non polarized part is:

$$I_{NP} \equiv \sqrt{I^2 - P^2} \quad \Longrightarrow \quad \Pi_{NP} \equiv \sqrt{1 - \Pi_L^2 - \Pi_V^2} = \frac{2\sqrt{\det \mathbf{J}}}{\mathrm{tr}\mathbf{J}} \, . \tag{25}$$

It is important to underline that $I$ and $V$ are physical observables, since they are independent on the particular orientation of the reference frame in the plane perpendicular to the direction of propagation $\hat{\boldsymbol{n}}$, while $Q$ and $U$ depend on the orientation of this basis [26]. After a rotation of the reference frame of an angle $\alpha$ ($R(\alpha)$) they transform according to:

$$\begin{aligned} Q &\xrightarrow{R(\alpha)} Q\cos(2\alpha) + U\sin(2\alpha) \, , \\ U &\xrightarrow{R(\alpha)} -Q\sin(2\alpha) + U\cos(2\alpha) \, . \end{aligned} \tag{26}$$

In a similar way it is possible to describe the electric vector field in the $x - y$ plane through a superposition of left and right circular polarized waves defining:

$$\hat{\boldsymbol{e}}_+ \equiv \frac{\hat{\boldsymbol{e}}_{\boldsymbol{x}} + i\hat{\boldsymbol{e}}_{\boldsymbol{y}}}{\sqrt{2}} \qquad \text{and} \qquad \hat{\boldsymbol{e}}_- \equiv \frac{\hat{\boldsymbol{e}}_{\boldsymbol{x}} - i\hat{\boldsymbol{e}}_{\boldsymbol{y}}}{\sqrt{2}} \, . \tag{27}$$

In this new basis:

$$\mathbf{J} = a^2 \begin{pmatrix} \left\langle E_+^*(t)E_+(t)\right\rangle & \left\langle E_+^*(t)E_-(t)\right\rangle \\ \left\langle E_-^*(t)E_+(t)\right\rangle & \left\langle E_-^*(t)E_-(t)\right\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I+V & Q-iU \\ Q+iU & I-V \end{pmatrix} . \tag{28}$$

and

$$I \equiv \frac{1}{a^2}\left(\left\langle E_+^*(t)E_+(t)\right\rangle + \left\langle E_-^*(t)E_-(t)\right\rangle\right) \, , \tag{29}$$

$$\begin{aligned} Q &\equiv \frac{1}{a^2}\left(\left\langle E_+^*(t)E_-(t)\right\rangle + \left\langle E_-^*(t)E_+(t)\right\rangle\right) \\ &= \frac{2}{a^2}\left\langle \varepsilon_+\varepsilon_- \cos\left(\varphi_+ - \varphi_-\right)\right\rangle \, , \end{aligned} \tag{30}$$

$$\begin{aligned} U &\equiv -\frac{i}{a^2}\left(\left\langle E_+^*(t)E_-(t)\right\rangle - \left\langle E_-^*(t)E_+(t)\right\rangle\right) \\ &= \frac{2}{a^2}\left\langle \varepsilon_+\varepsilon_- \sin\left(\varphi_+ - \varphi_-\right)\right\rangle \, , \end{aligned} \tag{31}$$

$$V \equiv \frac{1}{a^2}\left(\left\langle E_+^*(t)E_+(t)\right\rangle - \left\langle E_-^*(t)E_-(t)\right\rangle\right) \, . \tag{32}$$

A change in the relative phase between left and right polarized waves ($\varphi_+ - \varphi_- + \delta$) corresponds to a rotation of the plane of linear polarization:

$$\Delta\theta \equiv \theta - \theta_0 = \frac{\delta}{2} \, . \tag{33}$$

The elements of the coherence matrix in the Fourier space are defined:

$$\tilde{J}_{ik} = a^2 \left\langle \tilde{E}_i^*\left(\eta, k\right) \tilde{E}_k^*\left(\eta, k\right)\right\rangle \, , \tag{34}$$

where $i, k = \{+, -\}$. Stokes parameters and the degrees of polarization are defined starting from the elements of $\tilde{\mathbf{J}}$ as in real space.

The relation between the vector potential and the electric field for a wave propagating in a charge-free region is:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\boldsymbol{A}'}{a}\,, \tag{35}$$

denoting with $t$ the proper time ($dt = a d\eta$). According to definition given in the previous section the Stokes Parameters in terms of the vector potential are:

$$I = \frac{1}{a^4}\left(\left\langle A_+'^* A_+'\right\rangle + \left\langle A_-'^* A_-'\right\rangle\right)\,, \tag{36}$$

$$Q = \frac{1}{a^4}\left(\left\langle A_+'^* A_-'\right\rangle + \left\langle A_-'^* A_+'\right\rangle\right) = \frac{2}{a^4}\Re\left(\left\langle A_+'^* A_-'\right\rangle\right)\,, \tag{37}$$

$$U = -\frac{i}{a^4}\left(\left\langle A_+'^* A_-'\right\rangle - \left\langle A_-'^* A_+'\right\rangle\right) = \frac{2}{a^4}\Im\left(\left\langle A_+'^* A_-'\right\rangle\right)\,, \tag{38}$$

$$V = \frac{1}{a^4}\left(\left\langle A_+'^* A_+'\right\rangle - \left\langle A_-'^* A_-'\right\rangle\right)\,. \tag{39}$$

## 3. Axion as Dark Matter

We assume that dark matter is given by massive axions ($V(\phi) = m^2\phi^2/2$) and therefore $\phi(t)$ satisfies the equation:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0\,. \tag{40}$$

When $m > 3H$ the filed begins to oscillate, and the solution in a matter dominated universe ($\dot{a}/a = 2/3t$) is [27]:

$$\phi(t) = t^{-1/2}\left[aJ_{1/2}(mt) + bJ_{-1/2}(mt)\right] \tag{41}$$

$$\overset{mt\gg 1}{\simeq} \frac{\phi_0}{mt}\sin(mt)\,, \tag{42}$$

The energy density and pressure associated with the field are:

$$\rho_\phi = \frac{\overline{\dot{\phi}^2}}{2} + \frac{1}{2}m^2\overline{\phi}^2 \overset{mt\gg 1}{\simeq} \frac{\phi_0^2}{2t^2}\left[1 + \mathcal{O}\left(\frac{1}{(mt)^2}\right)\right]\,, \tag{43}$$

$$\tag{44}$$

where $\bar{}$ denotes the average over an oscillation period of the axion condensate. Note that we are implicitly assuming that the pseudo-scalar field is homogeneous. In the context of axion physics, this means that in our observable universe we have just one value for the misalignment angle, which means that the PQ symmetry has occurred before or during inflation.

We fix the constant $\phi_0$ comparing $\rho_\phi$ with the energy density in a matter dominated universe:

$$\rho_M = \frac{M_{pl}^2}{6\pi t^2} \quad\Longrightarrow\quad \phi_0 = \frac{M_{pl}}{\sqrt{3\pi}}\,. \tag{45}$$

where $M_{pl} \simeq 1.22 \times 10^{-19}$ GeV is the Planck mass.

Using the relation between cosmic and conformal time in a universe of matter ($t \propto \eta^3$) we find the following approximation for $\phi(\eta)$:

$$\phi(\eta) \simeq \sqrt{\frac{3}{\pi}}\frac{M_{pl}}{m\eta_0\left(\frac{\eta}{\eta_0}\right)^3}\sin\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right]\,, \tag{46}$$

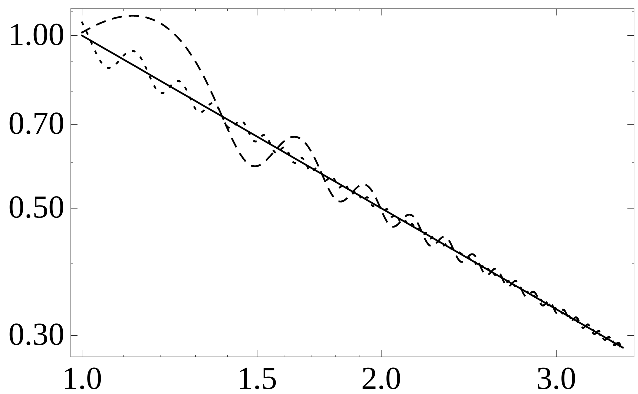


**Figure 1.** Evolution of $\mathcal{H}/\mathcal{H}_{\rm rec}$ in function of conformal time for $m = 10^{-28}$eV (dashed line), $m = 5\times10^{-27}$eV (dotted line) and for a matter dominated universe (continuous line), from recombination ($\eta_{\rm rec}$) to $3.5\eta_{\rm rec}$. Actual time corresponds to $\eta_0 = \eta_{\rm rec}\sqrt{1+z_{\rm rec}} \simeq 33.18\eta_{\rm rec}$.

and

$$\phi'(\eta) \simeq \sqrt{\frac{3}{\pi}}\frac{M_{pl}}{\eta}\left\{\cos\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right] - \frac{3\eta_0^2}{m\eta^3}\sin\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right]\right\} . \quad (47)$$

If $m$ is not too small the value of $\mathcal{H} \equiv a'/a$ obtained with the scalar field density in the Friedmann equation coincides with that of a matter dominated universe $\mathcal{H} = 2/\eta$ once the average through oscillations is performed [28]. (see Fig. 1). The derivative can be replaced in equation (13) for the evolution of the electromagnetic vector:

$$\tilde{A}''_\pm(\eta, k) + k^2\left[1 \pm \Delta(\eta; g_\phi, m, k, \eta_0)\right]\tilde{A}_\pm(\eta, k) = 0\ , \quad (48)$$

introduced the function:

$$\Delta(\eta; g_\phi, m, k, \eta_0) \equiv \quad (49)$$

$$4\sqrt{3\pi}\frac{g_\phi M_{pl}}{k\eta}\left\{\cos\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right] - \frac{3\eta_0^2}{m\eta^3}\sin\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right]\right\} . \quad (50)$$

This term, induced by axion-photon coupling, oscillates with frequency proportional to the mass of the axion and its amplitude decreases with time.

We now choose two reference Fourier modes which are physically relevant. First, we consider a wavenumber $k_1$ relevant for the angular correlation pattern of CMB anisotropies (e.g. $k_1 = 0.01$ Mpc$^{-1}$ $= 6.4\times10^{-32}$eV$= 9.73\times10^{-17}$ Hz); by requiring that the axion-photon coupling is subleading, we obtain a limit for the coupling constant:

$$4\sqrt{3\pi}\frac{g_\phi M_{pl}}{k_1\eta_{rec}} < 1 \quad \Longrightarrow \quad g_\phi < 1.67\times10^{-29}\ \mathrm{eV}^{-1}\ . \quad (51)$$

Second, for a much harder wavenumber $k_2$ ($k_2 = 10$ GHz$= 6.58\times10^{-6}$eV$=33.36\,$m$^{-1}$) the inequality becomes:

$$4\sqrt{3\pi}\frac{g_\phi M_{pl}}{k_2\eta_{rec}} < 1 \quad \Longrightarrow \quad g_\phi < 1.57\times10^{-3}\ \mathrm{eV}^{-1}\ . \quad (52)$$

### *3.1. Adiabatic solution*

Adiabatic solutions of Eq. (48) are:

$$\tilde{A}_s = \frac{1}{\sqrt{2\omega_s}} e^{\pm i \int \omega_s d\eta}\,, \tag{53}$$

where $\omega_s(\eta) = k\sqrt{1 \pm \frac{4\pi g_\phi \phi'(\eta)}{k}} = k\sqrt{1 \pm \Delta(\eta)}$ and $s = \pm$.
The second derivative respect to conformal time is:

$$\tilde{A}''_s = \tilde{A}_s \left( -\omega_s^2 + \frac{3\omega_s'^2}{4\omega_s^2} - \frac{\omega_s''}{2\omega_s^3} \right)\,. \tag{54}$$

The adiabatic solution (53) is a good approximation for $A_\pm$ when the terms $\frac{3\omega_s'^2}{4\omega_s^2}$ and $\frac{\omega_s''}{2\omega_s^3}$ are small compared to $\omega_s^2$:

$$\frac{3\omega_s'^2}{4\omega_s^4} = \frac{3\Delta'^2}{16k^2\,(1 \pm \Delta)^3} \ll 1\,, \tag{55}$$

$$\frac{\omega_s''}{2\omega_s^3} = \frac{\pm 2(1 \pm \Delta)\Delta'' - \Delta'^2}{8k^2\,(1 \pm \Delta)^3} \ll 1\,. \tag{56}$$

If both condition are satisfied and $\Delta \ll 1$:

$$\tilde{A}_\pm \simeq \frac{1}{\sqrt{2k\,(1 \pm \Delta/4)}} \exp\left[ \pm ik \left( \eta \pm \frac{1}{2} \int \Delta d\eta \right) \right] \tag{57}$$

$$= \frac{1}{\sqrt{2k\,(1 \pm \pi g \phi' k)}} \exp\left[ \pm i \left( k\eta \pm 2\pi g \phi \right) \right]\,. \tag{58}$$

For frequencies of the order of $k_2$ this is a good approximation for the electro magnetic potential in all the region of the plane $(m, g_\phi)$ allowed by CAST (see Fig. 2).

In the adiabatic regime the coupling between photons and axions produces a frequency independent shift $\delta$ between the two polarized waves, which corresponds (see eq. 33) to rotation of the plane of linear polarization.

$$\Delta\theta = 2\pi g \left[ \phi(\eta_{\rm rec}) - \phi(\eta_0) \right]\,. \tag{59}$$

This result agrees with the one obtained by D. Harari and P. Sikivie in [8]: the result obtained in [8] holds therefore in the adiabatic regime for the frequency $\omega_s = \sqrt{k^2 \pm 4\pi g_\phi \phi'(\eta) k}$.

Typically $\phi(\eta_{\rm rec}) \gg \phi(\eta_0)$; from last scattering to nowadays $\rho \simeq m^2 \overline{\phi}^2$ so:

$$\overline{\phi}(\eta) \simeq \sqrt{\frac{3}{8\pi}} \frac{M_{pl} \mathcal{H}(\eta)}{m a(\eta)} \simeq \sqrt{\frac{3}{2\pi}} \frac{M_{pl} \eta_0^2}{m \eta^3}\,. \tag{60}$$

A rough estimate of the angle $\Delta\theta$ is;

$$\Delta\theta \simeq g_\phi \sqrt{6\pi} \frac{M_{pl}}{m\eta_0} \left[ (1 + z_{\rm rec})^{3/2} - 1 \right]\,. \tag{61}$$

Note the dependence of $\Delta\theta$ on the coupling constant and on the mass of the pseudo-scalar field: fixed $g_\phi$ the effect is bigger for smaller masses. Otherwise the amplitude of the electromagnetic field changes according to:

$$\left| \tilde{\boldsymbol{E}} \right|^2 = \frac{\left| \tilde{\boldsymbol{A}}' \right|^2}{a^2} \simeq \frac{\omega_s}{2a^2}\,, \tag{62}$$

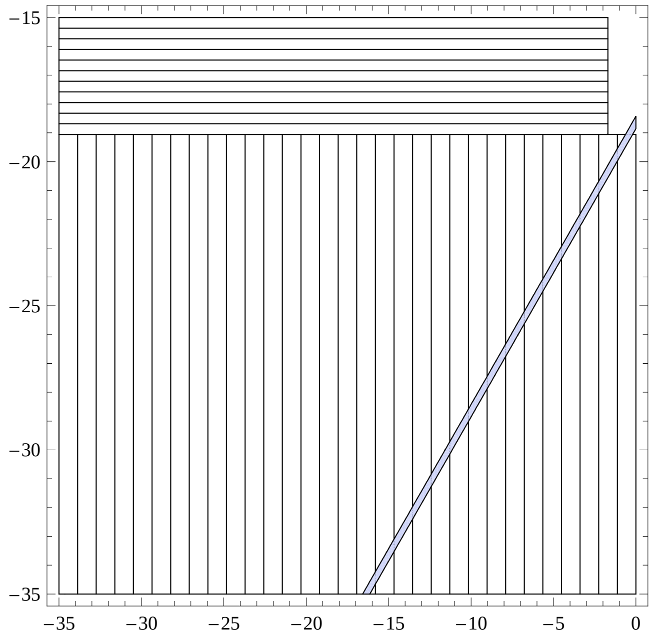


**Figure 2.** Plane $\left(\log_{10} m\ [\mathrm{eV}], \log_{10} g_\phi\ [\mathrm{eV}^{-1}]\right)$ for frequency of the electromagnetic wave $k_2 = 10$ GHz. Region excluded by CAST (region with horizontal lines), region allowed by CAST where adiabatic approximation hypothesis are verified (region with vertical lines), band of $(m, g_\phi)$ values expected for QCD axion models (colored region).

so the degree of circular polarization evolves according:

$$\tilde{\Pi}_C = \frac{\left|\tilde{A}'_+\right|^2 - \left|\tilde{A}'_-\right|^2}{\left|\tilde{A}'_+\right|^2 + \left|\tilde{A}'_-\right|^2} = \frac{\sqrt{1+\Delta} - \sqrt{1-\Delta}}{\sqrt{1+\Delta} + \sqrt{1-\Delta}} \simeq \frac{\Delta}{2} = \frac{2\pi g \phi'}{k}\,. \tag{63}$$

The degree of circular polarization generated for values of $g_\phi$ allowed by CAST it is always very small and weakly depends on the mass $m$.

In Tab. 1 we show some values of $\Delta\theta$ and $\tilde{\Pi}_C$ for different frequencies

| | $k = 1$ GHz | $k = 10$ GHz | $k = 100$ GHz |
|---|---|---|---|
| $\Delta\theta$ [rad] | 1.64 | 1.64 | 1.64 |
| $\Pi_{C,0}$ | $8.46 \times 10^{-18}$ | $8.46 \times 10^{-19}$ | $8.46 \times 10^{-20}$ |

**Table 1.** Rotation of linear polarization and degree of circular polarization at present time for three different frequencies, fixed $g_\phi = 8.8 \times 10^{-20}$ eV$^{-1}$ and $m = 10^{-19}$ eV.

### *3.2. Numerical solution*

For frequencies of the order of $k_1 = 0.01$ Mpc$^{-1}$ the adiabatic approximation is not valid in all the region of the plane $(m, g_\phi)$ allowed by CAST, but only in a small subregion (see Fig. 3). Therefore Eq. (48) is integrated numerically, in the frequency range relevant for CMB, for different values of the parameters $g_\phi$ and $m$:

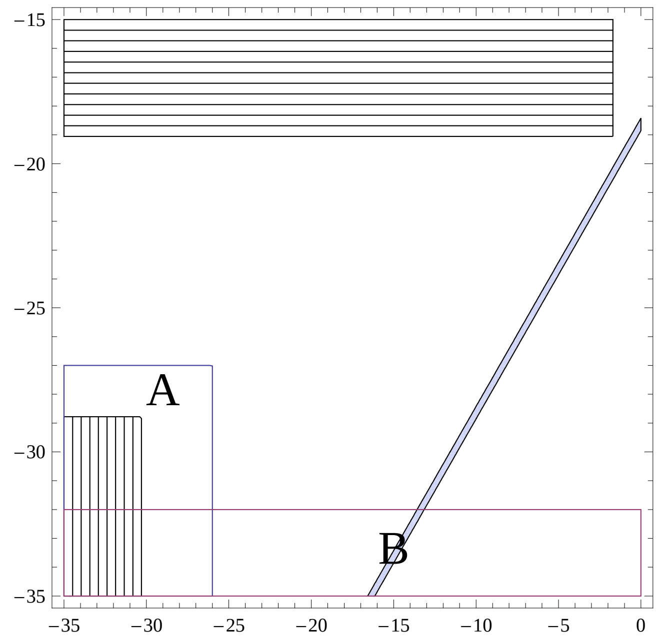


**Figure 3.** Plane $(\log_{10} m\ [\mathrm{eV}], \log_{10} g_\phi\ [\mathrm{eV}^{-1}])$ for frequency of the electromagnetic wave $k_1 = 0.01\ \mathrm{Mpc}^{-1}$. Region excluded by CAST (region with horizontal lines), region allowed by CAST where adiabatic approximation hypothesis are verified (region with vertical lines), band of $(m, g_\phi)$ values expected for QCD axion models (colored region). Regions where we solve numerically the equation for the electromagnetic potential are marked with A and B.

- in *region A* where $|\Delta(\eta_{\mathrm{rec}})| < \mathcal{O}(10)$ and the axion does not oscillate to quickly ($m \lesssim 10^{-26}$ eV)
- in *region B* where $|\Delta(\eta_{\mathrm{rec}})| \ll 1$, without restrictions on the axion mass.

Figs. 4, 5, 7, 6 show the modifications produced during the first oscillation of the electromagnetic wave after recombination for some values of $m$ and $g_\phi$.

Assuming a fully linear polarized wave at last scattering surface ($\Pi_{L,\mathrm{rec}} = 1$) we numerically evolve the equation for left and right components of the vector potential. Fixed the axion mass and the coupling constant we estimate, at present time, the shift generated between the two polarizations ($\delta_0$), the degree of circular polarization ($\Pi_{C,0}$) and the change in total intensity ($I_0/I_{\mathrm{rec}}$) for different frequencies and initial conditions of the electromagnetic wave (see Tabs. 2, 3, 5, 4, 6).

In *region A* Eq. (48) is integrated numerically for:

- $g_\phi = 5 \times 10^{-29}\ \mathrm{eV}^{-1}$, $m = 10^{-27}$ eV (see Fig.4 and Tab.2);
- $g_\phi = 5 \times 10^{-29}\ \mathrm{eV}^{-1}$, $m = 10^{-26}$ eV (see Fig.5 and Tab.3);
- $g_\phi = 5 \times 10^{-28}\ \mathrm{eV}^{-1}$, $m = 10^{-27}$ eV (see Fig.6 and Tab.4);
- $g_\phi = 5 \times 10^{-28}\ \mathrm{eV}^{-1}$, $m = 10^{-26}$ eV (see Fig.7 and Tab.5).

In *region B* equation (48) is integrated numerically for:

- $g_\phi = 5 \times 10^{-32}\ \mathrm{eV}^{-1}$, $m = 10^{-6}$ eV (see Fig.8 and Tab.6);

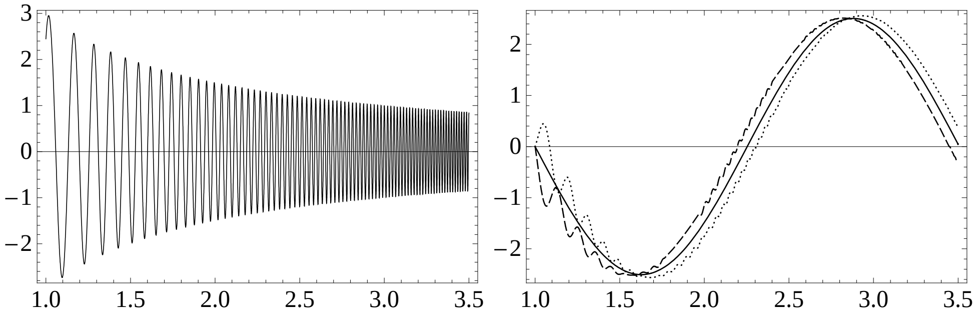


**Figure 4.** $g_\phi = 5 \times 10^{-29}$ eV$^{-1}$, $m = 10^{-27}$ eV and $k_1 = 0.01$, for $\tilde{A}_{rec} = 1$ $\tilde{A}'_{rec} = 0$.*Left:* $\Delta(\eta)$. *Right:* Continuous line: with $g_\phi = 0$, dashed line: $\tilde{A}_+$, dotted line: $\tilde{A}_-$.

| | $k = 0.1$ Mpc$^{-1}$ | $k = 0.01$ Mpc$^{-1}$ | $k = 0.001$ Mpc$^{-1}$ |
|---|---|---|---|
| $\Delta\theta$ [rad] | 0.408 | 0.273 | 0.271 |
| | 0.0213 | 0.00344 | $4.46 \times 10^{-5}$ |
| $\Pi_{C,0}$ | 0.740 | 0.0178 | 0.00290 |
| | 0.769 | 0.0197 | 0.00195 |
| $I_0/I_{\rm rec}$ | 1.604 | 1.023 | 1.016 |
| | 1.553 | 0.995 | 1.000 |

**Table 2.** Fixed $g_\phi = 5 \times 10^{-29}$ eV$^{-1}$ and $m = 10^{-27}$ eV rotation of linear polarization, degree of circular polarization and change in total intensity at present time for three frequencies and for different initial condition of the electromagnetic potential: $\tilde{A}_{\rm rec} = 1$ and $\tilde{A}'_{\rm rec} = 0$ (upper lines), $\tilde{A}_{\rm rec} = 0$ and $\tilde{A}'_{\rm rec} = 1$ (lower lines).

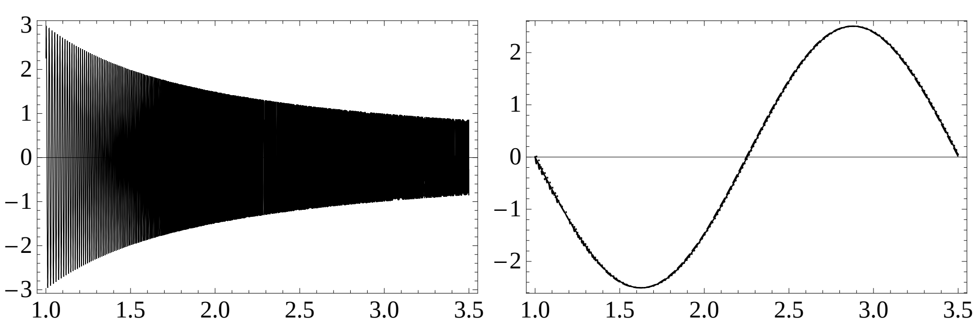


**Figure 5.** $g_\phi = 5 \times 10^{-29}$ eV$^{-1}$, $m = 10^{-26}$ eV and $k_1 = 0.01$, for $\tilde{A}_{rec} = 1$ $\tilde{A}'_{rec} = 0$. Continuous line: with $g_\phi = 0$, dashed line: $\tilde{A}_+$, dotted line: $\tilde{A}_-$.

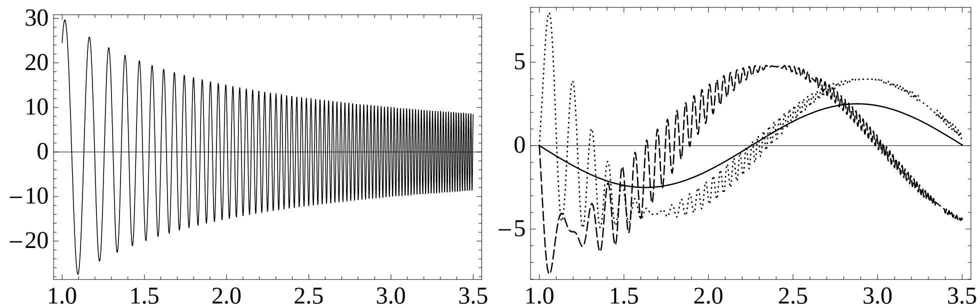

**Figure 6.** $g_\phi = 5 \times 10^{-28}$ eV$^{-1}$, $m = 10^{-27}$ eV and $k_1 = 0.01$, for $\tilde{A}_{rec} = 1$ $\tilde{A}'_{rec} = 0$. *Left:* $\Delta(\eta)$. *Right:* continuous line: with $g_\phi = 0$, dashed line: $\tilde{A}_+$, dotted line: $\tilde{A}_-$.

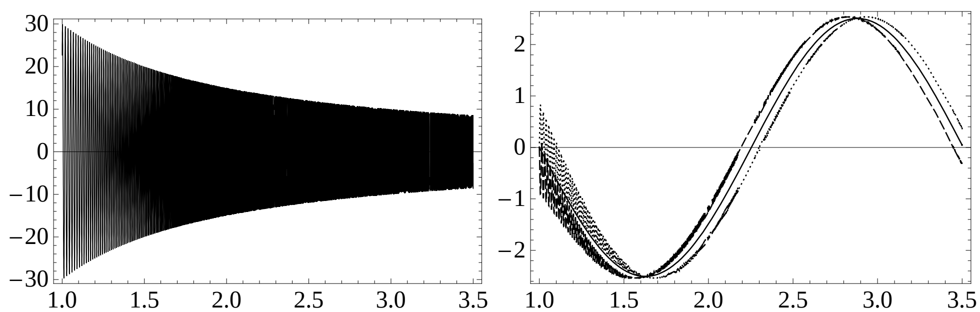

**Figure 7.** $g_\phi = 5 \times 10^{-28}$ eV$^{-1}$, $m = 10^{-26}$ eV and $k_1 = 0.01$, for $\tilde{A}_{rec} = 1$ $\tilde{A}'_{rec} = 0$. Continuous line: with $g_\phi = 0$, dashed line: $\tilde{A}_+$, dotted line: $\tilde{A}_-$.

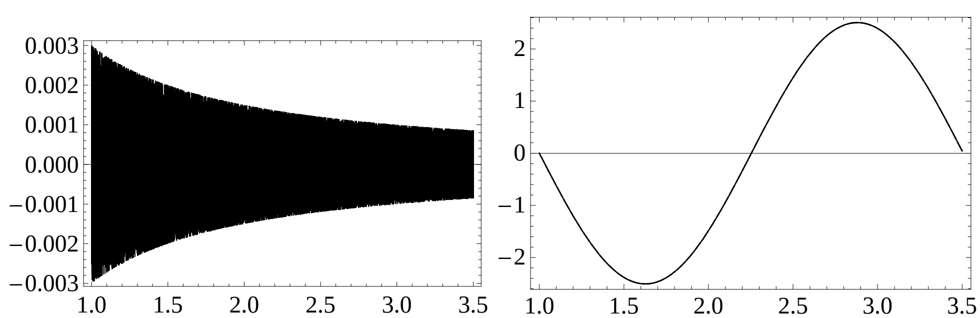

**Figure 8.** $g_\phi = 5 \times 10^{-32}$ eV$^{-1}$, $m = 10^{-6}$ eV and $k_1 = 0.01$, for $\tilde{A}_{rec} = 1$ $\tilde{A}'_{rec} = 0$. *Left:* $\Delta(\eta)$. *Right:* continuous line: with $g_\phi = 0$, dashed line: $\tilde{A}_+$, dotted line: $\tilde{A}_-$.

| | $k = 0.1$ Mpc$^{-1}$ | $k = 0.01$ Mpc$^{-1}$ | $k = 0.001$ Mpc$^{-1}$ |
|---|---|---|---|
| $\Delta\theta$ [rad] | 0.0282 | 0.279 | 0.280 |
| | 0.000300 | $1.84 \times 10^{-5}$ | $7.34 \times 10^{-4}$ |
| $\Pi_{C,0}$ | 0.00225 | 0.000240 | 0.0014 |
| | 0.00225 | 0.000233 | $2.7 \times 10^{-4}$ |
| $I_0/I_{\rm rec}$ | 1.000 | 1.000 | 0.998 |
| | 1.000 | 1.000 | 1.000 |

**Table 3.** Fixed $g_\phi = 5 \times 10^{-29}$ eV$^{-1}$ and $m = 10^{-26}$ eV rotation of linear polarization, degree of circular polarization and change in total intensity at present time for three frequencies and for different initial condition of the electromagnetic potential: $\tilde{A}_{\rm rec} = 1$ and $\tilde{A}'_{\rm rec} = 0$ (upper lines), $\tilde{A}_{\rm rec} = 0$ and $\tilde{A}'_{\rm rec} = 1$ (lower lines).

| | $k = 0.1$ Mpc$^{-1}$ | $k = 0.01$ Mpc$^{-1}$ | $k = 0.001$ Mpc$^{-1}$ |
|---|---|---|---|
| $\Delta\theta$ [rad] | 1.570 | 1.350 | 1.857 |
| | 1.570 | 0.057 | $7.55 \times 10^{-5}$ |
| $\Pi_{C,0}$ | 0.108 | 0.174 | 0.096 |
| | 0.847 | 0.210 | 0.0196 |
| $I_0/I_{\rm rec}$ | 64498 | 3.041 | 2.817 |
| | 23850 | 0.641 | 0.986 |

**Table 4.** Fixed $g_\phi = 5 \times 10^{-28}$ eV$^{-1}$ and $m = 10^{-27}$ eV rotation of linear polarization, degree of circular polarization and change in total intensity at present time for three frequencies and for different initial condition of the electromagnetic potential: $\tilde{A}_{\rm rec} = 1$ and $\tilde{A}'_{\rm rec} = 0$ (upper lines), $\tilde{A}_{\rm rec} = 0$ and $\tilde{A}'_{\rm rec} = 1$ (lower lines).

| | $k = 0.1$ Mpc$^{-1}$ | $k = 0.01$ Mpc$^{-1}$ | $k = 0.001$ Mpc$^{-1}$ |
|---|---|---|---|
| $\Delta\theta$ [rad] | 0.277 | 0.276 | 0.277 |
| | 0.00296 | $1.22 \times 10^{-4}$ | $2.75 \times 10^{-4}$ |
| $\Pi_{C,0}$ | 0.0242 | $1.23 \times 10^{-3}$ | 0.00211 |
| | 0.0247 | 0.00219 | 0.000176 |
| $I_0/I_{\rm rec}$ | 1.028 | 1.023 | 1.014 |
| | 0.989 | 0.994 | 0.999 |

**Table 5.** Fixed $g_\phi = 5 \times 10^{-28}$ eV$^{-1}$ and $m = 10^{-26}$ eV rotation of linear polarization, degree of circular polarization and change in total intensity at present time for three frequencies and for different initial condition of the electromagnetic potential: $\tilde{A}_{\rm rec} = 1$ and $\tilde{A}'_{\rm rec} = 0$ (upper lines), $\tilde{A}_{\rm rec} = 0$ and $\tilde{A}'_{\rm rec} = 1$ (lower lines).

- $g_\phi = 5 \times 10^{-32}$ eV$^{-1}$, $m = 10^{-13}$ eV (see Tab.7)

In both regions the angle of rotation $\Delta\theta$ and the degree of circular polarization $\Pi_C$ are directly proportional to the coupling constant and decrease when the mass of the pseudo-scalar field increases.

| | $k = 0.1$ Mpc$^{-1}$ | $k = 0.01$ Mpc$^{-1}$ | $k = 0.001$ Mpc$^{-1}$ |
|---|---|---|---|
| $\Delta\theta$ [rad] | $1.41 \times 10^{-6}$ | $4.22 \times 10^{-6}$ | $6.10 \times 10^{-4}$ |
| | $2.51 \times 10^{-4}$ | $4.73 \times 10^{-6}$ | $2.21 \times 10^{-7}$ |
| $\Pi_{C,0}$ | $1.85 \times 10^{-4}$ | $6.28 \times 10^{-6}$ | $2.85 \times 10^{-6}$ |
| | $2.66 \times 10^{-4}$ | $1.02 \times 10^{-6}$ | $1.39 \times 10^{-6}$ |
| $I_0/I_{\rm rec}$ | 1.000 | 1.000 | 1.000 |
| | 1.000 | 1.000 | 1.000 |

**Table 6.** Fixed $g_\phi = 5 \times 10^{-32}$ eV$^{-1}$ and $m = 10^{-6}$ eV rotation of linear polarization, degree of circular polarization and change in total intensity at present time for three frequencies and for different initial condition of the electromagnetic potential: $\tilde{A}_{\rm rec} = 1$ and $\tilde{A}'_{\rm rec} = 0$ (upper lines), $\tilde{A}_{\rm rec} = 0$ and $\tilde{A}'_{\rm rec} = 1$ (lower lines).

| | $k = 0.1$ Mpc$^{-1}$ | $k = 0.01$ Mpc$^{-1}$ | $k = 0.001$ Mpc$^{-1}$ |
|---|---|---|---|
| $\Delta\theta$ [rad] | $2.54 \times 10^{-6}$ | $3.41 \times 10^{-6}$ | $2.97 \times 10^{-5}$ |
| | $4.38 \times 10^{-5}$ | $1.16 \times 10^{-5}$ | $1.35 \times 10^{-5}$ |
| $\Pi_{C,0}$ | $1.40 \times 10^{-5}$ | $8.65 \times 10^{-6}$ | $6.18 \times 10^{-6}$ |
| | $4.04 \times 10^{-6}$ | $6.58 \times 10^{-6}$ | $1.59 \times 10^{-6}$ |
| $I_0/I_{\rm rec}$ | 1.000 | 1.000 | 1.000 |
| | 1.000 | 1.000 | 1.000 |

**Table 7.** Fixed $g_\phi = 5 \times 10^{-32}$ eV$^{-1}$ and $m = 10^{-13}$ eV rotation of linear polarization, degree of circular polarization and change in total intensity at present time for three frequencies and for different initial condition of the electromagnetic potential: $\tilde{A}_{\rm rec} = 1$ and $\tilde{A}'_{\rm rec} = 0$ (upper lines), $\tilde{A}_{\rm rec} = 0$ and $\tilde{A}'_{\rm rec} = 1$ (lower lines).

### *3.3. Comments for axion cosmology*

For axions the coupling constant with photons $g_\phi$ and the energy scale $f_a$ at which the new symmetry is broken are related [4]:

$$|g_\phi| = \frac{\alpha_{EM}}{2\pi f_a}\frac{3}{4}\xi \quad \text{with} \quad 0.1 \lesssim \xi \lesssim 1\,, \tag{64}$$

where the value for $\xi$ depends on the particular model considered for the axion. By using this relation a limit on the coupling constant is turned into a limit on the energy of symmetry breaking. The critical density associated with the misalignment production of axions strongly depend on this angle through the following relation [3, 4]:

$$\Omega_{mis}h^2 \sim 0.23 \times 10^{\pm 0.6}\left(\frac{f_a}{10^{12}\ \text{GeV}}\right)^{1.175}\Theta_i^2 F\left(\Theta_i\right)\,, \tag{65}$$

where $h$ encodes the actual value of the Hubble parameter ($H_0 = 100h$ km s$^{-1}$ Mpc$^{-1}$) and $F(\Theta_i)$ accounts for anharmonic effects if the initial misalignment angle associated with the axion field $\Theta_i \gg 1$. The demand $\Omega_{mis} \leq \Omega_{DM}$ provides an upper bound on $f_a^{1.175}\Theta_i^2$ (assuming $F(\Theta_i) \simeq 1$) [27, 29, 30]:

$$f_a\Theta_i^{1.7} \leq 2 \times 10^{11\div 12}\ \text{GeV}\,. \tag{66}$$

This condition becomes also an upper bound for $f_a$ under the assumption that inflation occurred before the breaking of PQ-symmetry ($f_a \leq f_{INF}$) [3]: in this scenario different regions have different values for $\Theta_i$, so averaging over all observable universe the value of $\Theta_i$ in equation can replaced be by its *rms* value $(\pi/\sqrt{3})$ and the limit $f_a \leq 10^{11 \div 12}$ GeV is obtained. In the case where inflation occurred before PQ-symmetry breaking the number of patches with different values for the pseudo-scalar field at recombination is of the order of:

$$N = \frac{a_{\rm rec}}{a_{PQ}} \simeq \frac{f_a}{T_{\rm rec}} \simeq 4 \times 10^{21} \,, \tag{67}$$

assumed $f_a = 10^{12}$ GeV and $T_{\rm rec} = 0.26$ eV. The expressions for the rotation of linear polarization and for the evolution of circular polarization must be reduced of a factor $\sqrt{N} \simeq 6 \times 10^{10}$. Therefore we believe that the effects we have discussed so far are negligible if the symmetry breaking occurred after inflation.

## 4. Exponential potential

We consider in this section a pseudo-scalar field with an exponential potential:

$$V(\phi) = V_0 \exp(-\lambda\kappa\phi) \,, \tag{68}$$

with $\kappa^2 \equiv 8\pi G$. It is known [31] that such potential with $\lambda^2 > 3(1 + w_{\rm F})$ leads to a component which tracks the dominant background fluid with equation of state $p_{\rm f} = w_{\rm F}\rho_{\rm F}$. In order to satisfy the nucleosynthesis bound we choose $\lambda = 4.5$. During the matter dominated era the scalar field behaves as:

$$\rho_\phi = \frac{\dot\phi^2}{2} + V_0 \exp(-\lambda\kappa\phi) = f\,\rho_{\rm MAT} \equiv f\frac{\rho_{\rm MAT,0}}{a^3} \,, \tag{69}$$

$$P_\phi = \frac{\dot\phi^2}{2} - V_0 \exp(-\lambda\kappa\phi) \,, \tag{70}$$

where $\rho_{\rm MAT} = \rho_{\rm DM} + \rho_{\rm baryons} + \rho_\phi$.

For $\lambda = 4.5$ the contribution of the pseudo-scalar field to universe energy density is shown in Fig. 9. The value of $\Omega_\phi$ changes with time, but it is almost constant ($\Omega_\phi \simeq \Omega_{phi\,,0} = 0.0.148$) from recombination ($\log a_{\rm rec} \simeq -7$) to nowadays.

The derivative of the pseudo-scalar field respect to conformal time is proportional to $a^{-1/2}$ and the evolution of the scale factor in the matter dominated phase is $a(\eta) = (\eta/\eta_0)^2$ so:

$$\phi' = \sqrt{f\,\rho_{\rm MAT,0}}\frac{\eta_0}{\eta} \,. \tag{71}$$

Substituting this relation in equation (13) we obtain the following expression for the evolution of the electromagnetic potential:

$$\tilde A''_\pm + \left(k^2 \pm 4\pi g_\phi \sqrt{f\,\rho_{\rm MAT,0}}\frac{\eta_0}{\eta}k\right)\tilde A_\pm = 0 \,. \tag{72}$$

This is a particular differential equation, called Coulomb wave equation; defining $q_\pm \equiv \mp 2\pi g_\phi \sqrt{f\,\rho_{\rm MAT,0}}\eta_0 = \mp q$ and $x \equiv k\eta$ it becomes:

$$\frac{d^2\tilde A_\pm}{dx^2} + \left(1 - \frac{2q_\pm}{x}\right)\tilde A_\pm = 0 \,. \tag{73}$$

The solution of this particular equation can be written in terms of regular $(F_0(q,x))$ and irregular $(G_0(q,x))$ Coulomb wave function [32, 33]:

$$\tilde A_+ = f_+F_0(q_+,x) + g_+G_0(q_+,x) = f_+F_0(-q,x) + g_+G_0(-q,x) \,,$$
$$\tilde A_- = f_-F_0(q_-,x) + g_-G_0(q_-,x) = f_-F_0(q,x) + g_-G_0(q,x) \,,$$

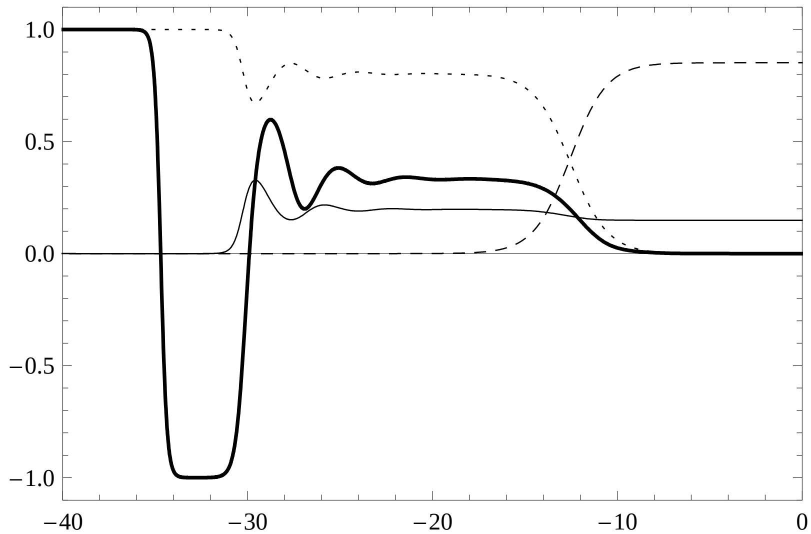


**Figure 9.** For $\lambda = 4.5$ Dashed line: $\Omega_{\rm MAT} + \Omega_{\rm baryons}$, dotted line: $\Omega_{\rm RAD}$, thin continuous line: $\Omega_\phi$, thick continuous line: $w_\phi$, in terms of the natural logarithm of the scale factor (from $\log a \simeq -40$ to nowadays $\log a_0 = 1$). Here $\Omega_{\rm DM,0} + \Omega_{\rm baryons,0} = 0.852$ and $\Omega_{\phi,0} = 0.148$.

where $f_+, f_-, g_+, g_- \in \mathbb{C}$; in a compact notation:

$$\tilde{A}_\pm(q,x) = f_\pm F_0(\mp q,x) + g_\pm G_0(\mp q,x)\,. \tag{74}$$

The Stokes parameters contain the derivative respect to conformal time $\eta$, so we evaluate:

$$\tilde{A}'_\pm(q,x) = k\left[f_\pm \frac{\partial F_0(\mp q,x)}{\partial x} + g_\pm \frac{\partial G_0(\mp q,x)}{\partial\, x}\right]\,. \tag{75}$$

The solution given in equation (74) verifies the *Wronskian condition* ($\tilde{A}_\pm \tilde{A}'^*_\pm - \tilde{A}'_\pm \tilde{A}^*_\pm = i$) if the following relation holds:

$$f^*_\pm g_\pm - f_\pm g^*_\pm = \frac{i}{k} \quad\Longrightarrow\quad \Im\left(f^*_\pm g_\pm\right) = \frac{1}{2k}\,. \tag{76}$$

### *4.1. Vanishing coupling ($g_\phi = 0$)*

If there is no coupling between the electromagnetic tensor and the pseudo-scalar field ($g_\phi = 0$) the equation (13) reduces to:

$$\tilde{A}''_\pm + k^2 \tilde{A}_\pm = 0\,, \tag{77}$$

Obviously the solution is:

$$\tilde{A}_\pm(x) = f_\pm \sin(x) + g_\pm \cos(x)\ , \tag{78}$$

and the derivative respect conformal time $\eta$ is:

$$\tilde{A}'_\pm(x) = k\left[f_\pm \cos(x) - g_\pm \sin(x)\right] \tag{79}$$

$$= \frac{k}{2}\left[e^{ix}\left(f_\pm + i g_\pm\right) + e^{-ix}\left(f_\pm - i g_\pm\right)\right]\ , \tag{80}$$

The elements of the coherence matrix $\tilde{\mathbf{J}}$ are:

$$\tilde{J}_{\pm\pm} = \frac{k^2}{2a^4}\left\langle |f_\pm|^2 + |g_\pm|^2 + \left(|f_\pm|^2 - |g_\pm|^2\right)\cos 2x - \left(f_\pm^* g_\pm + f_\pm g_\pm^*\right)\sin 2x \right\rangle ,$$
$$\tilde{J}_{\pm\mp} = \frac{k^2}{2a^4}\left\langle f_\pm^* f_\mp + g_\pm^* g_\mp + \left(f_\pm^* f_\mp - g_\pm^* g_\mp\right)\cos 2x - \left(f_\mp g_\pm^* + f_\pm^* g_\mp\right)\sin 2x \right\rangle ,$$

For a pure monochromatic wave ensemble averages can be omitted and the elements oscillate with period $2x$ due to the presence of both forward ($\tilde{A}' \propto e^{-ix}$) and backward ($\tilde{A}' \propto e^{+ix}$) moving waves.

In order to describe the propagation of an electromagnetic wave in a region with no coupling between photons and field $\phi$ (vacuum) we can use only forward moving waves, therefore setting $f_\pm = -ig_\pm$ in eq. (80) we obtain:

$$\tilde{A}'_\pm(x) = -ikg_\pm e^{-ix} . \tag{81}$$

The dependence from $g_\pm$ can be replaced by the value of at $\tilde{A}'_\pm$ at a particular time (e.g. recombination time):

$$\tilde{A}'_\pm(x) = \tilde{A}'_\pm(x_{\rm rec})e^{-i(x-x_{\rm rec})} . \tag{82}$$

In this particular situation the elements of the coherence matrix does not depend on $x$:

$$\begin{aligned}\tilde{\mathbf{J}} &= \frac{1}{a^4}\begin{pmatrix} \left\langle \tilde{A}'^*_+(x)\tilde{A}'_+(x)\right\rangle & \left\langle \tilde{A}'^*_+(x)\tilde{A}'_-(x)\right\rangle \\ \left\langle \tilde{A}'^*_-(x)\tilde{A}'_+(x)\right\rangle & \left\langle \tilde{A}'^*_-(x)\tilde{A}'_-(x)\right\rangle \end{pmatrix} \\ &= \frac{1}{a^4}\begin{pmatrix} \left\langle \tilde{A}'^*_+(x_{\rm rec})\tilde{A}'_+(x_{\rm rec})\right\rangle & \left\langle \tilde{A}'^*_+(x_{\rm rec})\tilde{A}'_-(x_{\rm rec})\right\rangle \\ \left\langle \tilde{A}'^*_-(x_{\rm rec})\tilde{A}'_+(x_{\rm rec})\right\rangle & \left\langle \tilde{A}'^*_-(x_{\rm rec})\tilde{A}'_-(x_{\rm rec})\right\rangle \end{pmatrix} \\ &= \left(\frac{a_{\rm rec}}{a}\right)^4 \tilde{\mathbf{J}}_{\rm rec} ,\end{aligned} \tag{83}$$

So the Stokes parameters evolve according to:

$$\tilde{I} = \left(\frac{a_{\rm rec}}{a}\right)^4 \tilde{I}_{\rm rec} , \quad \tilde{Q} = \left(\frac{a_{\rm rec}}{a}\right)^4 \tilde{Q}_{\rm rec} ,$$
$$\tilde{U} = \left(\frac{a_{\rm rec}}{a}\right)^4 \tilde{U}_{\rm rec} , \quad \tilde{V} = \left(\frac{a_{\rm rec}}{a}\right)^4 \tilde{V}_{\rm rec} .$$

Obviously the degrees of polarization are constant ($\tilde{\Pi}_L = \tilde{\Pi}_{L{\rm rec}}$ and $\tilde{\Pi}_C = \tilde{\Pi}_{C{\rm rec}}$) and also the angle of linear polarization does not change with time. If there is no coupling between the pseudo-scalar field and photons the electromagnetic wave propagates freely and the polarization does not change.

### *4.2. Non-vanishing coupling ($g_\phi \neq 0$)*

In the general case, when the coupling does not vanishes ($g_\phi \neq 0$), we expand the solution (74) for large value of $x$ neglecting terms proportional to $\mathcal{O}(x^{-2})$ (see Appendix):

$$\begin{aligned}\tilde{A}_\pm(q,x) \simeq\ & f_\pm\left[\frac{q^2}{2x}\cos\left(x \pm \alpha\left(q,x\right)\right) + \left(1 \mp \frac{q}{2x}\right)\sin\left(x \pm \alpha\left(q,x\right)\right)\right] \\ &+ g_\pm\left[\left(1 \mp \frac{q}{2x}\right)\cos\left(x \pm \alpha\left(q,x\right)\right) - \frac{q^2}{2x}\sin\left(x \pm \alpha\left(q,x\right)\right)\right] ,\end{aligned} \tag{84}$$

where $\alpha\left(q,x\right)\equiv q\log 2x-\arg\Gamma(1+iq)$. The derivative respect to conformal time is:

$$\begin{aligned}
\tilde{A}'_{\pm}(q,x) &\simeq k\left\{f_{\pm}\left[\left(1\pm\frac{q}{2x}\right)\cos\left(x\pm\alpha\left(q,x\right)\right)-\frac{q^2}{2x}\sin\left(x\pm\alpha\left(q,x\right)\right)\right]\right.\\
&\left.+g_{\pm}\left[-\frac{q^2}{2x}\cos\left(x\pm\alpha\left(q,x\right)\right)-\left(1\pm\frac{q}{2x}\right)\sin\left(x\pm\alpha\left(q,x\right)\right)\right]\right\}\\
&=\frac{k}{2}\left[e^{i(x\pm\alpha(q,x))}\left(1\pm\frac{q}{2x}+i\frac{q^2}{2x}\right)\left(f_{\pm}+ig_{\pm}\right)\right.\\
&\left.e^{-i(x\pm\alpha(q,x))}\left(1\pm\frac{q}{2x}-i\frac{q^2}{2x}\right)\left(f_{\pm}-ig_{\pm}\right)\right],
\end{aligned}\tag{85}$$

Using this asymptotic expansion, the elements of the coherence matrix result:

$$\begin{aligned}
\tilde{J}_{\pm\pm} &\simeq \frac{k^2}{2a^4}\left\langle\left(\left|f_{\pm}\right|^2+\left|g_{\pm}\right|^2\right)\left(1\pm\frac{q}{x}\right)\right.\\
&+\left[\left(\left|f_{\pm}\right|^2-\left|g_{\pm}\right|^2\right)\left(1\pm\frac{q}{x}\right)-\left(f^*_{\pm}g_{\pm}+f_{\pm}g^*_{\pm}\right)\frac{q^2}{x}\right]\cos\left[2\left(x\pm\alpha(q,x)\right)\right]\\
&\left.-\left[\left(f^*_{\pm}g_{\pm}+f_{\pm}g^*_{\pm}\right)\left(1\pm\frac{q}{x}\right)+\left(\left|f_{\pm}\right|^2-\left|g_{\pm}\right|^2\right)\frac{q^2}{x}\right]\sin\left[2\left(x\pm\alpha(q,x)\right)\right]\right\rangle,\\
\tilde{J}_{\pm\mp} &\simeq \frac{k^2}{2a^4}\left\langle\left(f^*_{\pm}f_{\mp}+g^*_{\pm}g_{\mp}\right)\cos 2\alpha(q,x)-\left(f^*_{\pm}g_{\mp}-g^*_{\pm}f_{\mp}\right)\sin 2\alpha(q,x)\right.\\
&+\left[f^*_{\pm}f_{\mp}-g^*_{\pm}g_{\mp}-\left(f^*_{\pm}g_{\mp}+f_{\mp}g^*_{\pm}\right)\frac{q^2}{x}\right]\cos 2x\\
&\left.-\left[f^*_{\pm}g_{\mp}+g^*_{\pm}f_{\mp}+\left(f^*_{\pm}f_{\mp}-g^*_{\pm}g_{\mp}\right)\frac{q^2}{x}\right]\sin 2x\right\rangle.
\end{aligned}$$

In general both forward moving waves ($\tilde{A}_{\pm}\propto e^{-ik\eta}$) and backward moving waves ($\tilde{A}_{\pm}\propto e^{ik\eta}$) must be taken into account for propagation of light in a medium. Chosen a particular value for the constants $f_{\pm}$ and $g_{\pm}$ that verifies the Wronskian relation (76) the evolution of polarization is fixed. In particular we consider a pure monochromatic wave, so averages can be neglected, with $f_{\pm}=k^{-1/2}\left(\sqrt{3}+i\right)/2$ and $g_{\pm}=-k^{-1/2}$ and describe the evolution from recombination to present time of:

- the degree of linear polarization (see Fig. 10,a):

$$\tilde{\Pi}_L=\frac{2\sqrt{\tilde{J}_{+-}\tilde{J}_{-+}}}{\tilde{J}_{++}+\tilde{J}_{--}};$$

- the angle of linear polarization (see Fig. 10,b):

$$\theta=\frac{1}{2}\arctan\left[i\frac{\tilde{J}_{+-}-\tilde{J}_{-+}}{\tilde{J}_{+-}+\tilde{J}_{-+}}\right];$$

- the degree of circular polarization (see Fig. 10,c):

$$\tilde{\Pi}_C=\frac{\left|\tilde{J}_{++}-\tilde{J}_{--}\right|}{\tilde{J}_{++}+\tilde{J}_{--}}.$$

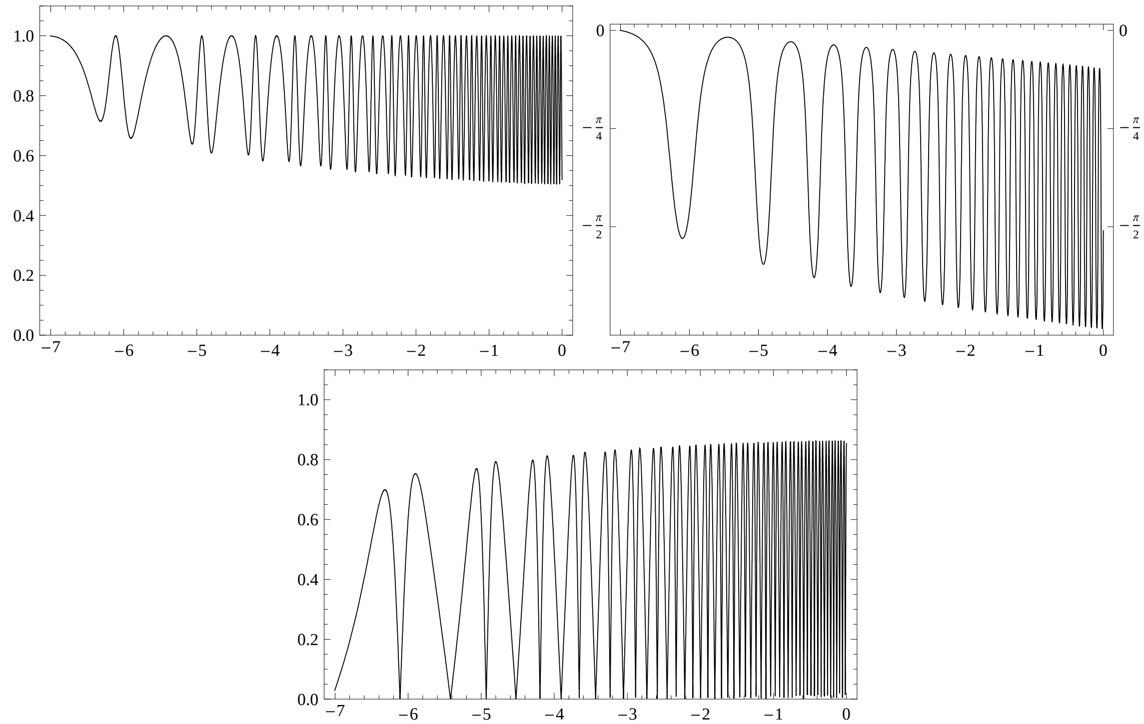


**Figure 10.** Evolution of the degree of linear polarization $\tilde{\Pi}_L$, angle of linear polarization $\theta$ and degree of circular polarization $\tilde{\Pi}_C$ for a completely linearly polarized monochromatic wave in terms of the natural logarithm of the scale factor, from recombination ($\log a_{\rm rec} \simeq -7$) to present ($\log a_0 = 0$). Fixed $k = 0.01\,{\rm Mpc}^{-1}$ and a particular value of the coupling constant ($g_\phi = 5\times10^{-29}$ eV$^{-1}$). Here we chose $f_\pm = k^{-1/2}\left(\sqrt{3}+i\right)/2$ and $g_\pm = -k^{-1/2}$.

When the coupling constant $g_\phi$ vanishes ($q = 0$) the degrees of polarization and the angle $\theta$ remain constant, as we discuss in the previous section, but when pseudo-scalar photon interaction is non zero they all vary with time.

If we assume, according with [34, 35], that the photon-pseudo-scalar conversion is a small effect due to low energy of CMB photons, the production of backward moving waves can be neglected (see [36] for the use of this approximation). The equation (85) setting $f_\pm = -ig_\pm$ becomes:

$$\tilde{A}'_\pm(q,x) \simeq -ikg_\pm\left(1 \pm \frac{q}{2x} - i\frac{q^2}{2x}\right)e^{-i(x\pm\alpha(q,x))}\,, \tag{86}$$

and in terms of the value at recombination time:

$$\begin{aligned}\tilde{A}'_\pm(q,x) \simeq \tilde{A}'_\pm(q,x_{\rm rec})&\left[1 \pm \frac{q}{2}\left(\frac{1}{x} - \frac{1}{x_{\rm rec}}\right) - i\frac{q^2}{2}\left(\frac{1}{x} - \frac{1}{x_{\rm rec}}\right)\right]\\ &\exp\left\{i\left[x - x_{\rm rec} \pm \Delta\alpha\right]\right\}\,.\end{aligned} \tag{87}$$

defined: $\Delta\alpha \equiv \alpha(q,x) - \alpha(q,x_{\rm rec}) = q\log\left(\eta/\eta_{\rm rec}\right)$.

The elements of the coherence matrix, always at the first order in $1/x$, are:

$$\tilde{\mathbf{J}} = \frac{1}{a^4}\begin{pmatrix} \left\langle \tilde{A}'^*_+(q,x)\,\tilde{A}'_+(q,x)\right\rangle & \left\langle \tilde{A}'^*_+(q,x)\,\tilde{A}'_-(q,x)\right\rangle \\ \left\langle \tilde{A}'^*_-(q,x)\,\tilde{A}'_+(q,x)\right\rangle & \left\langle \tilde{A}'^*_-(q,x)\,\tilde{A}'_-(q,x)\right\rangle \end{pmatrix} \tag{88}$$

$$= \left(\frac{a_{\rm rec}}{a}\right)^4 \begin{pmatrix} \tilde{J}_{++,\rm rec}\left(1+q\left(\frac{1}{x}-\frac{1}{x_{\rm rec}}\right)\right) & \tilde{J}_{+-,\rm rec}e^{2i\Delta\alpha} \\ \tilde{J}_{-+,\rm rec}e^{-2i\Delta\alpha} & \tilde{J}_{--,\rm rec}\left(1-q\left(\frac{1}{x}-\frac{1}{x_{\rm rec}}\right)\right) \end{pmatrix},$$

The relations between Stokes parameters now and at recombination are the following:

$$\tilde{I} \equiv \tilde{J}_{++}+\tilde{J}_{--} \simeq \left(\frac{a_{\rm rec}}{a}\right)^4\left[\tilde{I}_{\rm rec}+\tilde{V}_{rec}q\left(\frac{1}{x}-\frac{1}{x_{\rm rec}}\right)\right],$$

$$\tilde{Q} \equiv \tilde{J}_{+-}+\tilde{J}_{-+} \simeq \left(\frac{a_{\rm rec}}{a}\right)^4\left[\tilde{Q}_{\rm rec}\cos(2\Delta\alpha)+\tilde{U}_{\rm rec}\sin(2\Delta\alpha)\right],$$

$$\tilde{U} \equiv i\left(\tilde{J}_{+-}+\tilde{J}_{-+}\right) \simeq \left(\frac{a_{\rm rec}}{a}\right)^4\left[-\tilde{Q}_{\rm rec}\sin(2\Delta\alpha)+\tilde{U}_{\rm rec}\cos(2\Delta\alpha)\right],$$

$$\tilde{V} \equiv \tilde{J}_{++}-\tilde{J}_{--} \simeq \left(\frac{a_{\rm rec}}{a}\right)^4\left[\tilde{V}_{\rm rec}+\tilde{I}_{rec}q\left(\frac{1}{x}-\frac{1}{x_{\rm rec}}\right)\right].$$

Comparing this relation for $\tilde{Q}_0$ and $\tilde{U}_0$ with equation (26) we see that $\Delta\alpha$ describes the rotation of the plane of linear polarization. Degrees of linear and circular polarization evolves according to:

$$\tilde{\Pi}_{L,0} \simeq \tilde{\Pi}_{L,rec} - q\tilde{\Pi}_{L,\rm rec}\tilde{\Pi}_{C,\rm rec}\left(\frac{1}{x}-\frac{1}{x_{\rm rec}}\right), \tag{89}$$

$$\tilde{\Pi}_{C,0} \simeq \left|\tilde{\Pi}_{C,\rm rec} + q\left(\frac{1}{x}-\frac{1}{x_{\rm rec}}\right)\left(1-\tilde{\Pi}^2_{C,rec}\right)\right|. \tag{90}$$

We consider three different situations:

- if the wave is linearly polarized at recombination time ($\tilde{\Pi}_{L,\rm rec}=1$, $\tilde{\Pi}^2_{C,\rm rec}=0$):

  $$\tilde{\Pi}_{L,0} \simeq 1,$$
  $$\tilde{\Pi}_{C,0} \simeq \frac{q}{k}\left(\frac{1}{\eta_{\rm rec}}-\frac{1}{\eta}\right),$$

  so a small degree of circular polarization is generated. The angle of linear polarizarion rotates of $\Delta\theta = \Delta\alpha$.
- if the wave is circular polarized at recombination time ($\tilde{\Pi}_{L,\rm rec}=0$, $\tilde{\Pi}^2_{C,\rm rec}=1$):

  $$\tilde{\Pi}_{L,0} \simeq 0,$$
  $$\tilde{\Pi}_{C,0} \simeq 1,$$

  the polarization does not change at the first order in $1/k\eta$.
- if the wave is partially linearly polarized at recombination time ($\tilde{\Pi}_{L,\rm rec}=0.05$, $\tilde{\Pi}_{C,\rm rec}=0$): In this particular case, CMB , a certain amount of circular polarization is generated and the angle of linear polarization rotates of $\Delta\theta=\Delta\alpha$.

Summarizing the effect of coupling between the pseudo-scalar filed and the electromagnetic tensor, when backward moving waves can be neglected, is double:

- the plane of linear polarization is rotated of an angle $\Delta\theta$ independent from $k$ and from the degree of linear polarization $\tilde{\Pi}_L$ (see Fig. 11,b);
- a small degree of circular polarization, dependent from $k$, is generated (see Fig. 11,c).

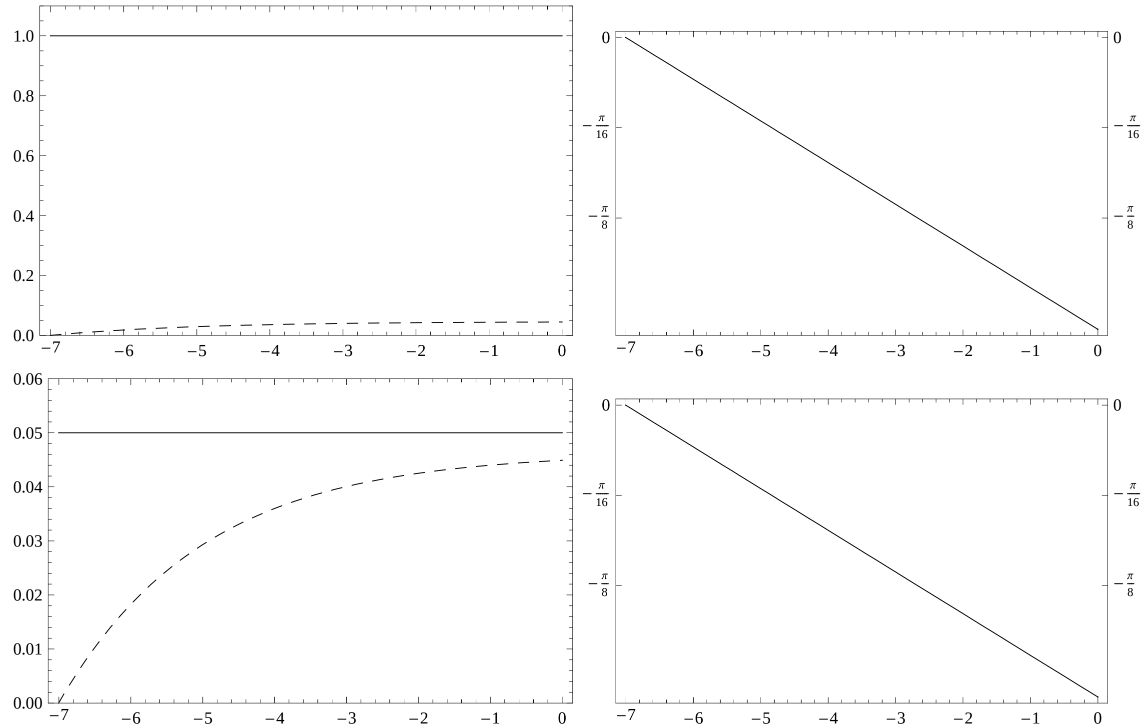


**Figure 11.** Evolution of degree of linear polarization $\tilde{\Pi}_L$ (solid), degree of circular polarization $\tilde{\Pi}_C$ (dashed) and angle of linear polarization $\theta$ in terms of the natural logarithm of the scale factor, from recombination ($\log a_{\rm rec} \simeq -7$) to present ($\log a_0 = 0$), neglecting the production of backward moving waves. The first two plots refers to a wave completely linearly polarized at recombination time ($\tilde{\Pi}_{L,\rm rec} = 1$, $\tilde{\Pi}_{C,\rm rec} = 0$), the last two plot are for a partially linearly polarized wave ($\tilde{\Pi}_{L,\rm rec} = 0.05$, $\tilde{\Pi}_{C,\rm rec} = 0$). The figures are for $k1$ and $g_\phi = 3\times 10^{-29}\ {\rm eV}^{-1}$.

*4.2.1. CMBP constraints on the coupling constant* In the particular approximation in which we neglect backward moving waves we can constraint the parameter $q$ using the upper limits on isotropic frequency-independent rotation of the linear polarization plane of CMBP. According to [19] measures and constraints on the polarization pattern of CMB anisotropies produces a two-sigma bound on this angle:

$$\Delta\theta = -2.5 \pm 6.0\ {\rm deg}\,. \tag{91}$$

This constraint updates previous results [18]. We now use the constraint (91) and our analytic expression:

$$\Delta\theta = -\frac{q}{2}\log(1 + z_{\rm rec})\,, \tag{92}$$

to obtain an upper bound for $q$, which can be turned into a upper bound on $g_\phi$:

$$\frac{|q|}{2}\log(1 + z_{\rm rec}) \le 6\ {\rm deg} = \frac{\pi}{30}\,,$$
$$\pi\,|g_\phi|\,\sqrt{\rho_{\phi,0}}\eta_0\log(1 + z_{\rm rec}) \le \frac{\pi}{30}\,,$$
$$|g_\phi| \le \frac{1}{M_{\rm pl}}\sqrt{\frac{2\pi}{3}\Omega_{\phi,0}}\frac{1}{30\log\left(z_{\rm rec} + 1\right)}\,,$$

$$\boxed{|g_\phi| \leq 2.17 \times 10^{-31}\ \text{eV}^{-1}\,,} \tag{93}$$

where we have assumed: $\Omega_{\phi,0} \simeq 0.148$ and $z_{\rm rec} \simeq 1100$.

This constraint improves, although for a different model, the limit obtained by CAST [6]: $g_\phi < 8.8 \times 10^{-20}$ eV$^{-1}$ for $m_a < 0.02$ eV (see regions with horizontal lines in Fig. 2 and Fig. 3)

## 5. Conclusions

We have studied the impact of a pseudo-scalar field acting as dark matter on CMBP. We have shown that such pseudo-scalar interaction with photons generates circular polarization in general and does not simply rotate the plane of linear polarization. In absence of measures for the $V$ mode of CMBP, the existing upper limits on an isotropic $E\,B$ correlation (the imprint of a rotation of the plane of linear polarization independent of the wave number) can constrain the coupling constants of photons with the pseudo-scalar field.

We have examinated two representative examples for the dynamics of a pseudo-Goldstone field behaving as dark matter: the oscillating and the monotonic decreasing behavior. In the monotonic decreasing behavior, by neglecting backward moving waves, we have shown that present CMB observations can constrain the coupling constant $g_\phi$ to small values as $\mathcal{O}\left(10^{-31}\right)$ eV.

The impact of pseudo-scalar interaction with photons CMBP is therefore complementary to its impact on the initial conditions for dark matter inhomogeinities. If the PQ-symmetry breaking would have occurred before or during inflation, initial conditions for dark matter perturbations are essentially adiabatic and the rotation of CMBP gives a constraint on $g_\phi$. If the PQ-symmetry breaking would have occurred after inflation, the rotation of linear polarization does not produce a stringent constraint, but axion dark matter start with isocurvature initial conditions leading to constraints on $f_a$ (see [37] for an updated study of this topic).

**Acknowledgments:** We wish to thank Günter Sigl and Guido Zavattini for discussions. FF and MG are partially supported by INFN IS PD51. FF is partially supported by INFN IS BO11.

## Appendix

The Coulomb wave equation is [32]:

$$\frac{d^2 w}{dx^2} - \left[1 - \frac{2q}{x} - \frac{L(L+1)}{x^2}\right] w = 0\,, \tag{94}$$

with $x > 0, -\infty < q < \infty, L$ a non negative integer. Here, in order to solve equation (73), we are particular interested to the particular case when L=0.

The solution can be written in terms of regular ($F_L(q,x)$) and irregular ($G_L(q,x)$) Coulomb wave function:

$$w = c_1 F_L(q,x) + c_2 G_L(q,x)\,. \tag{95}$$

The Coulomb functions can be expanded for large values of $x$ [32]:

$$F_0 = g\cos\theta + f\sin\theta\,, \tag{96}$$

$$G_0 = f\cos\theta - g\sin\theta\,, \tag{97}$$

similarly for the first derivative respect to $x$

$$F_0' = g^* \cos\theta + f^* \sin\theta\,, \tag{98}$$

$$G_0' = f^* \cos\theta - g^* \sin\theta\,, \tag{99}$$

with $\theta \equiv x - q\log 2x + \arg\Gamma(1+iq)$ and:

$$f = \sum_{k=0}^{\infty} f_k\,, \quad g = \sum_{k=0}^{\infty} g_k\,, \quad f^* = \sum_{k=0}^{\infty} f_k^*\,, \quad g^* = \sum_{k=0}^{\infty} g_k^*\,, \tag{100}$$

where:

$$f_0 = 1\,, \quad f_{k+1} = a_k f_k - b_k g_k\,; \tag{101}$$

$$g_0 = 0\,, \quad g_{k+1} = a_k g_k + b_k f_k\,; \tag{102}$$

$$f_0^* = 0\,, \quad f_{k+1}^* = a_k f_k^* - b_k g_k^* - \frac{f_{k+1}}{x}\,; \tag{103}$$

$$g_0^* = 1 - \frac{q}{x}\,, \quad f_{k+1}^* = a_k g_k^* + b_k f_k^* - \frac{g_{k+1}}{x}\,; \tag{104}$$

$$a_k = \frac{(2k+1)q}{2(k+1)x}\,, \quad b_k = \frac{q^2 - k(k+1)}{2(k+1)x}\,. \tag{105}$$

$$\tag{106}$$

Restricting to the first order:

$$f = 1 + \frac{q}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right)\,, \tag{107}$$

$$g = \frac{q^2}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right)\,, \tag{108}$$

$$f^* = -\frac{q^2}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right)\,, \tag{109}$$

$$g^* = 1 - \frac{q}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right)\,. \tag{110}$$

Summarizing the asymptotic expansion of $F_L(q,x)$ and $F_L(q,x)$ for large values of $x$ is:

$$F_0(q,x) \simeq \frac{q^2}{2x}\cos\theta + \left(1 + \frac{q}{2x}\right)\sin\theta\,, \tag{111}$$

$$G_0(q,x) \simeq \left(1 + \frac{q}{2x}\right)\cos\theta - \frac{q^2}{2x}\sin\theta\,, \tag{112}$$

and for the first derivative:

$$F_0'(q,x) \simeq \left(1 - \frac{q}{2x}\right)\cos\theta - \frac{q^2}{2x}\sin\theta\,, \tag{113}$$

$$G_0'(q,x) \simeq -\frac{q^2}{2x}\cos\theta - \left(1 - \frac{q}{2x}\right)\sin\theta\,. \tag{114}$$

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